A new model for statistics of the first recurrent moments and cycle lengths in discretizations of dynamical systems*

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Abstract

A new approach to analysis of a statistical law for the combinatorical characteristics of spatial discretizations of dynamical systems is suggested

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1 Introduction

Space discretizations, in particular computer realizations, of dynamical systems can usually be treated as mappings on finite sets. Information concerning combinatorical characteristics of such mappings is essential [1, 2, 12] for an understanding of the relationship between properties of the underlying dynamical system and its discretization. Unfortunately, for systems with complicated quasi-chaotic behaviour such characteristics are extremely sensitive to the discretization procedure; for this reason often only averaged characteristics over an ensemble of discretizations can be considered, but even here a rigorous theoretical investigation is not without difficulties.

Phenomenological models for the statistics of the first recurrent times, cycle lengths and others combinatorical characteristics of discretizations of continuous dynamical systems with complicated behaviour can provide useful and interesting information when a more rigorous analysis is not possible. One of such models [7, 9, 13] is based on the theory of completely random mappings [3], has been most successful in the situations when a "typical" discretization of the continuous system does not have a strong algebraic structure and the underlying continuous system has a stochastic attractor for which the Hausdorff dimension coincides with its correlation dimension (see details in [5, 9]). The condition concerning the dimension of attractor is rather restrictive and does not hold even for the simplest one-dimensional systems generated by the mappings from the family

$$f^{(\gamma)}(x) = 1 - 2|x - 0.5|^{1/\gamma}, \qquad x \in [0, 1], \quad 0 < \gamma < 1/2, \tag{1}$$

for which the Hausdorff dimension of the stochastic attractor $[0, 1]$ is equal to 1, whereas the correlation dimension is equal to 2γ. The first successful models for systems with different Hausdorff and correlation dimensions of stochastic attractors were suggested in [5, 6]; these models were based on the theory of random mappings with a single attracting centre [14]. While these models gave better results than those in [7, 9, 13], they are still not quite satisfactory quantitatively. In this paper a more appropriate class of models is presented and analized.

2 Special family of random mappings

Let N be a natural number, $\alpha > 0$ and $\gamma \in (0, 1/2)$. Consider the *random mapping* $T^{(\gamma)}_{\alpha,N}$ of the set $X(N) = \{0, \ldots, N\}$ into itself, which is defined by the following conditions. Define

$$q_0(\alpha, \gamma) = \alpha^\gamma \quad \text{and} \quad q_i(\alpha, \gamma) = (\alpha + i)^\gamma - (\alpha + i - 1)^\gamma, \quad i = 1, 2, \ldots, N.$$

Suppose that the point 0 is fixed for each realization $\widehat{T}$ of the random mapping $T^{(\gamma)}_{\alpha,N}$ and that the probability of the realization $\widehat{T}$ is equal to

$$\prod_{i=1}^{N} (\alpha + N)^{-\gamma} q_{\widehat{T}(i)}(\alpha, \gamma). \tag{2}$$

In other words, in the construction of a realization $\widehat{T}$ the images $\widehat{T}(i)$ of the points $i \in X(N)$ are chosen independently and equi-probably with the probability of the event $T^{(\gamma)}_{\alpha,N}(i) = j$ to be proportional to $q_j(\alpha,\gamma)$. This definition is a natural analog of that for random mappings with a single attracting centre [3, 4, 14].

Each realization $\widehat{T}$ of the random mapping $T^{(\gamma)}_{\alpha,N}$ is a deterministic mapping. Thus for each $i \in X(N)$ the trajectory $\mathrm{Tr}(i,\widehat{T})$, that is the sequence $i_0, i_1, \ldots, i_n, \ldots$ which satisfies the equalities

$$i_0 = i, \; i_n = \widehat{T}(i_{n-1}), \quad n = 1, 2, \ldots$$

it is uniquely defined. For each such trajectory $\mathrm{Tr}(i,\widehat{T})$ the *first recurrence time* $Q(i,\widehat{T})$ is defined to be the first n after which the trajectory is cyclic with the minimal period, say $C(i,\widehat{T})$.

Let $\aleph(X)$ denote the number of elements in a finite set X and let

$$\mathcal{Q}(x,\widehat{T}) = N^{-1}\aleph[\{i : Q(i,\widehat{T}) < xN^{\gamma}\}], \qquad x \geq 0, \tag{3}$$

$$\mathcal{C}(x,\widehat{T}) = N^{-1}\aleph[\{i : C(i,\widehat{T}) < xN^{\gamma}\}], \qquad x \geq 0 \tag{4}$$

denote the *scaled distribution functions* of the first recurrence moments and of the minimal periods for the totality of trajectories of the mapping $\widehat{T}$. Also denote by $\mathcal{P}(\widehat{T})$ the proportion of those i for which $C(i,\widehat{T}) = 1$. Note that for fixed α, γ, N the quantities (3), (4) are random functions; so let

$$Q^{(\gamma)}(x;\alpha,N), \; C^{(\gamma)}(x;\alpha,N), \quad x \geq 0$$

denote the corresponding mathematical expectations. Finally denote $x \in [0,1]$, the distribution function of the random variable $\mathcal{P}$ by $d^{(\gamma)}(x;\alpha,N)$.

The proof of the following theorem is similar to that of the main theorem in [10], so will be omitted.

Theorem 1. *For each $\alpha, \beta > 0$ and $0 < \gamma < 1/2$ there are valid the limit equalities*

$$\lim_{N\to\infty} Q^{(\gamma)}(x;\alpha,\beta N) = 1 - F^{(\gamma)}(\beta x;\alpha), \qquad x \geq 0,$$

$$\lim_{N\to\infty} C^{(\gamma)}(x;\alpha,\beta N) = H^{(\gamma)}(\beta x;\alpha), \qquad x > 0$$

and

$$\lim_{N\to\infty} \int_0^1 (1 - d^{(\gamma)}(x;\alpha,N))\,dx = \alpha^{\gamma} \int_0^{\infty} F^{(\gamma)}(x,\alpha)\,dx,$$

where

$$F^{(\gamma)}(x;\alpha) = e^{-\alpha^{\gamma}x} \prod_{i=1}^{\infty} \frac{1 + q_i(\alpha,\gamma)x}{e^{q_i(\alpha,\gamma)x}},$$

$$G^{(\gamma)}(x;\alpha) = F(x;\alpha,\gamma) \sum_{i=1}^{\infty} \frac{q_i(\alpha,\gamma)^2}{1 + q_i(\alpha,\gamma)x},$$

and

$$H^{(\gamma)}(x;\alpha) = 1 - F^{(\gamma)}(x;\alpha) + \alpha^{\gamma} \int_x^{\infty} F^{(\gamma)}(y;\alpha)\,dy + x \int_x^{\infty} G^{(\gamma)}(y;\alpha)\,dy.$$

3 Principle of Correspondence

Let us come back to the analysis of discretizations of continuous dynamical systems from the family (1). Let n be a natural number. Define the *n-discretization* of the mapping $f^{(\gamma)}$ by the equality $\varphi_n^{(\gamma)}(\ell) = [f^{(\gamma)}(\ell)]_n$, where $[x]_n$ is the round-off operator:

$$[x]_n = \frac{k}{n}, \quad \frac{k-0,5}{n} \le x < \frac{k+0,5}{n}$$

for some integer k. Each n-discretization thus maps a finite lattice

$$\mathrm{L}_n = \{0, 1/n, \ldots, (n-1)/n, 1\}$$

into itself, so for each $\ell \in \mathrm{L}_n$ the first recurrence moment $Q(\ell, \varphi_n^{(\gamma)})$, and the corresponding minimal period of a cyclic part of the trajectory $Q(\ell, \varphi_n^{(\gamma)})$, are well defined, as are the distribution function $\mathcal{Q}(x; \varphi_n^{(\gamma)})$, $\mathcal{C}(x; \varphi_n^{(\gamma)})$ and the number $\mathcal{P}(\varphi_n^{(\gamma)})$.

A theoretical analysis of the sequences

$$\{\mathcal{Q}(x, \varphi_n^{(\gamma)})\}_{n=1}^{\infty}, \{\mathcal{C}(x, \varphi_n^{(\gamma)})\}_{n=1}^{\infty}, \{\mathcal{P}(\varphi_n^{(\gamma)})\}_{n=1}^{\infty}$$

is substationally more complicated than that of the sequences

$$\{\mathcal{Q}(x; \widehat{T}_n)\}_{n=1}^{\infty}, \{\mathcal{C}(x; \widehat{T}_n)\}_{n=1}^{\infty}, \{\mathcal{P}(\widehat{T}_n)\}_{n=1}^{\infty}.$$

Nevertheless, this difficulty can be overcome by means of the *principle of correspondence*, to be formulated below. This principle is not a rigorous mathematical theorem, but admits an heuristic explanation analogously to the reasoning in [6], Section 2, which in turn is not dissimilar that in [9, 13]. Some numerical experiments, which will be discussed below, demonstrate that this principle hold with rather high accuracy

For each $x \ge 0$ introduce the functions

$$\mathbf{q}^{(\gamma)}(x; N, M) = \sum_{n=N+1}^{N+M} \mathcal{Q}(x; \varphi_n^{(\gamma)}), \qquad \mathbf{c}^{(\gamma)}(x; N, M) = \sum_{n=N+1}^{N+M} \mathcal{C}(x; \varphi_n^{(\gamma)})$$

and denote the distribution function of the set

$$\{\mathcal{P}(\varphi_n^{(\gamma)}) : n = N+1, \ldots, N+M\}$$

by $\mathbf{p}^{(\gamma)}(x; N, M)$, $x \in [0, 1]$. A family of positive integers $\mathcal{N}$ is said to be *dense* if

$$\lim_{n \to \infty} n^{-1} \aleph[\{m \in \mathcal{N} : m \le n\}] = 1.$$

Principle of Correspondence. *There exist constants $\alpha(\gamma), \beta(\gamma) > 0$ and and a dense set of integers $\mathcal{N}(M)$ depending on a positive integer M such that the functions*

$$\mathbf{q}^{(\gamma)}(x; N, M), \mathbf{c}^{(\gamma)}(x; N, M), \mathbf{p}^{(\gamma)}(x; N, M)$$

are close in the Levy metric [8] to the corresponding functions

$$Q^{(\gamma)}(x; \alpha, \beta N), C^{(\gamma)}(x; \alpha, \beta N), d^{(\gamma)}(x; \alpha, N)$$

for all sufficiently large M, $N \in \mathcal{N}(M)$.

4 Numerical experiments

The Principle of Correspondence for the sequences **q** and **c**, together with Theorem 1 suggest that for randomly chosen $1 \ll M \ll N$ the functions

$$\mathbf{q}^{(\gamma)}(x; M, N) \quad \text{and} \quad \mathbf{c}^{(\gamma)}(x; M, N)$$

should be similar to the functions

$$1 - F^{(\gamma)}(\beta x; \alpha) \quad \text{and} \quad H^{(\gamma)}(\beta x; \alpha).$$

This assertion can be tested numerically through simulation. Figure 1 graphs four curves for the value of parameter $\gamma = 1/3$. The two curves below graph the experimental results

$$\mathbf{q}^{(\gamma)}(x; M, N) \quad \text{for} \quad M = 10^4, N = 10^5$$

and the theoretical prediction $1 - F^{(\gamma)}(\beta x; \alpha)$ for $\alpha = 0.3$, $\beta = 0.6$. The curves can be distinguished only at $x < 1$; the experimental curve is the less smooth one. The two curves above plot the numerical results $\mathbf{c}^{(\gamma)}(x; M, N)$ against the theoretical prediction $H^{(\gamma)}(\beta x; \alpha)$ for the same N, M, α and β. Furthermore the Principle of Correspondence for the sequences $\mathcal{P}$ together with Theorem 1 suggest that for $1 \ll M \ll N$ the mean value

$$\frac{1}{M} \sum_{m=N+1}^{N+M} \mathcal{P}(\varphi_m^{(\gamma)}) \;=\; \int_0^1 (1 - \mathbf{p}^{(\gamma)}(x; M, N))\, dx$$

should be close to the value

$$\alpha^\gamma \int_0^\infty F^{(\gamma)}(x, \alpha(\gamma))\, dx.$$

Again the experimental result ≈ 0.675 appeared to be indeed quite close to the theoretical prediction ≈ 0.678 which was calculated for $\alpha = 0.3$. Similar experiments were carried out also for other values of γ, such as $\gamma = 2/5, 2/7$ etc., and also for other values of the parameters M, N. All of these experiments supported the Principle of correspondence. Recall that a scalar sequence $\{s_n\}$ has a *stable distribution function* $d(x)$ if

$$\lim_{n\to\infty} n^{-1} \aleph[\{k \le n : s_k < x\}] = d(x).$$

By [6] the sequence $\{\mathcal{P}(\varphi_n^{(\gamma)})\}_{n=1}^\infty$ should have a stable distribution function for $\gamma \in (0, 1/2)$. By the Principle of Correspondence the function $\mathbf{p}^{(\gamma)}(x; \gamma, M, N)$ should be close to the distribution of the random variable $d^{(\gamma)}(x; \alpha(\gamma), N)$ for typical $M, N \gg 1$. Therefore, $\mathbf{p}^{(\gamma)}(x; \gamma, M, N)$ should be close to the sample distribution

$$\widehat{d}^{(\gamma)}(x; \alpha; M_0, N) = M_0^{-1} \aleph[\{m < M_0 : \mathcal{P}(\widehat{T}_{N+m}) < x\}],$$

where $M_0 \gg 1$ and $\widehat{T}_n$ are independent realizations of the random mapping $M_0 \gg 1$, $T^{(\gamma)}_{\alpha(\gamma),N}$. Figure 6 graphs the results of an experimental test the validity this assertion. The two curves in the middle represent

$$\mathbf{p}^{(\gamma)}(x; M, N) \quad \text{and} \quad \widehat{d}^{(\gamma)}(x; \alpha; M_0, N)$$

for the same α, M, N and γ, as above and $M_0 = 10^3$. In numerics of the last function a random number generator was used. The function $\mathbf{p}^{(\gamma)}(x; M, N)$ here is the smoother one because $M_0 < M$. The lower and the upper pairs of curves in Figure 2 were obtained in much the same way for the values $\gamma_1 = 2/7 \quad \gamma_2 = 2/5$, with the corresponding values $\alpha_1 = 0.25, \alpha_2 = 0.4$ and the previous M, M_0, N. An agreement between the theory and experiment in Figure 2 is much better than for the model based on random mappings with a single attracting centre [6].

Let us formulate an unsolved question which seems to be important in view of results of the last experiment.

> *Prove for $\alpha > 0$, $\gamma \in (0, 1/2)$ that the sequence $d^{(\gamma)}(x; \alpha, n)$ convergences uniformly to a limit $d_\infty^{(\gamma)}(x; \alpha)$ at $n \to \infty$.*

5 Conclusion

A Principle of Correspondence was formulated above for three concrete combinatorical characteristics of the spatial discretizations of continuous dynamical systems. This Principle is also applicable to the investigation of other combinatorical characteristics, such as statistics of absolutely collapsing discretizations $n \to \infty$ ([6]), p. 566, basins of attractions etc.

It can be used without change for other quasi-chaotic systems which have an invariant stochastic attractor for which the corresponding invariant measure has a singularity on a preimage of a fixed point of the mapping. When the last condition does not hold, the Principle of correspondence should be modified slightly: in the definition of random mapping $T_{\alpha,N}^{(\gamma)}$ the requirement that 0 is a fixed point there must be omitted and, correspondingly, the expression (2) should be changed to the expression $\prod_{i=0}^{N} (\alpha + N)^{-\gamma} q_{\widehat{T}(i)}(\alpha, \gamma)$.

Finally note that the model can also be applied to the analysis of some characteristics of hysteretic system [11] following the scheme suggested in [7].

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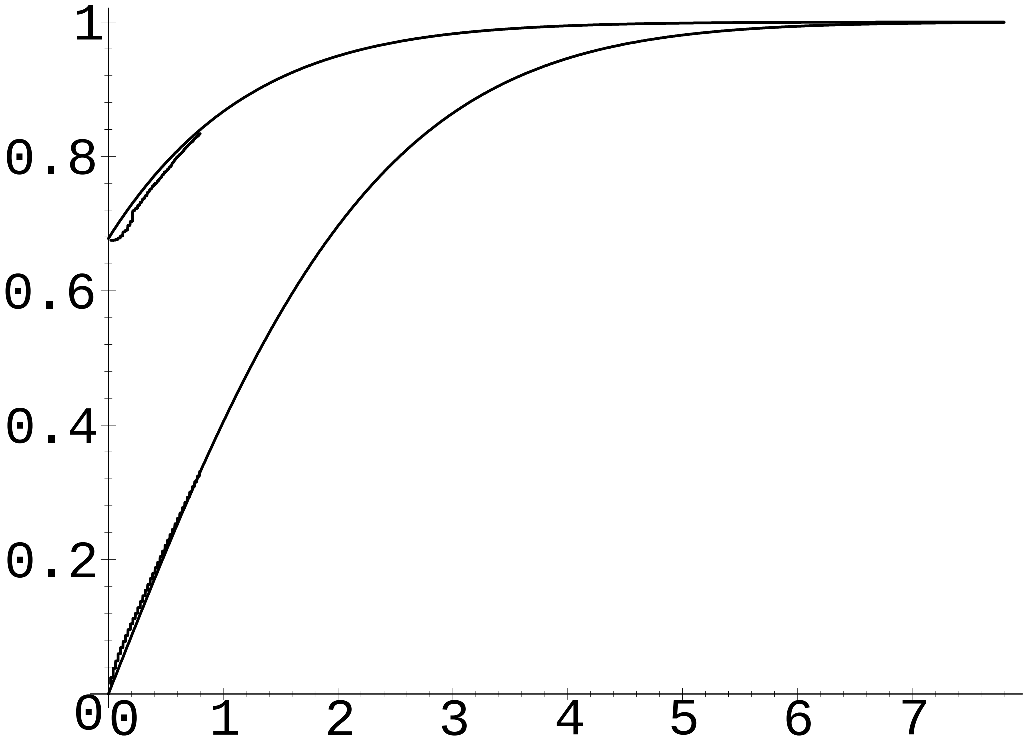

Figure 1. Below: experimental distribution $\mathbf{q}^{(1/3)}(x; 10^4, 10^5)$ against the theoretical prediction $1-F^{(1/3)}(0,6x; 0,3)$ for $\alpha = 0,3$, $\beta = 0,6$. Above: the distribution $\mathbf{c}^{(1/3)}(x; 10^4, 10^5)$ against $H^{(1/3)}(0,6x; 0,3)$.

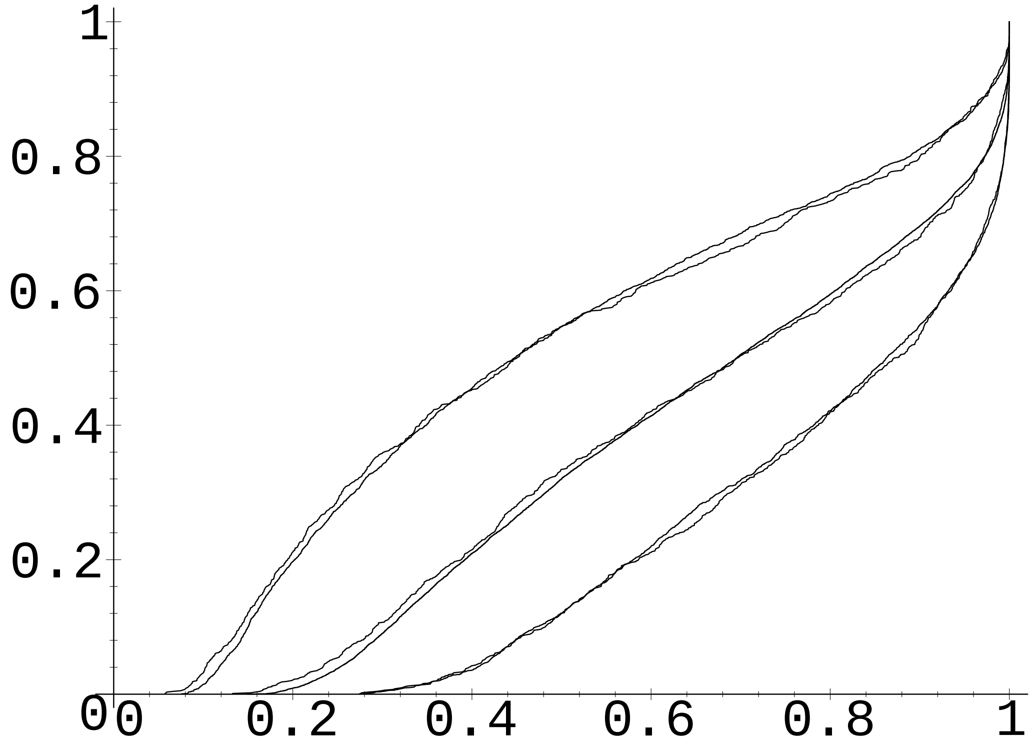

Figure 2. Below: the distributions $\mathbf{q}^{(2/7)}(x;10^4,10^5)$ and $\widehat{d}^{(2/7)}(x;0,25,10^3,10^5)$;
In the middle: $\mathbf{q}^{(1/3)}(x;10^4,10^5)$ and $\widehat{d}^{(1/3)}(x;0,3,10^3,10^5)$;
above: $\mathbf{q}^{(2/5)}(x;10^4,10^5)$ and $\widehat{d}^{(2/5)}(x;0,4,10^3,10^5)$.