A phenomenological model for discretizations of a a class of chaotic dynamical systems*

P. Diamond
Department of Mathematics
University of Queensland,
Brisbane, Qld 4072, Australia

P. Kloeden
School of Comp & Maths
Deakin University,
Geelong, Vic 3217, Australia

V. Kozyakin
IPPI, Russian Academy,
19 Ermolovoy str.,
Moscow 101447, Russia.

A. Pokrovskii†
Department of Mathematics
University of Queensland,
Brisbane, Qld 4072, Australia

Abstract

Computer simulations of dynamical systems contain discretizations*, where finite machine arithmetic replaces continuum state spaces. For chaotic dynamical systems main characteristics of those discretizations depend on parameters of both underlying continuous systems and discretization procedure in random way. To describe and analyze corresponding statistical regularities, some adequate phenomenological models of discretization process should be developed. Such a model is suggested for a family of mappings* $x \mapsto 1 - |1-2x|^{\ell}$*,* $x \in [0,1]$*,* $\ell > 2$*. Results of computer modeling are presented.*

*This research has been supported by the Australian Research Council Grant A 8913 2609.

†*Permanent address:* Institute of Information Transmission Problems Russian Academy of Science, Moscow.

1 Introduction

Let f be a given chaotic mapping. It means, in particular, that trajectories are exponentially sensitive to initial conditions and behave apparently randomly. Consequently, not much information can be gleamed from analysis of individual trajectories. Nevertheless, there exists a rich qualitative theory of these dynamical systems in terms of statistical properties, like Sinai–Ruelle–Bowen (SRB) invariant measures. The role of SRB invariant measures is determined by the fact that such a measure describes the properties of exact trajectories for almost all, with respect to Lebesgue measure, initial conditions.

Interesting questions arise in analysis of space discretizations of such chaotical systems. Many reasonable computer realizations of such systems can be treated as deterministic mappings φ of a certain finite subset L into itself and we will consider only realizations of such type. The central problem is the fact that such discretizations are also very sensitive to initial conditions and perturbations but each trajectory of a spatial discretization is eventually periodic and so is not apparently random as it is the case in a continuum. Consequently the main characteristics of discretization are somehow connected with their cycles. There are different characteristics of such type, for instance:

1. The maximal, or the average, length of cycles of discretization [12, 1];

2. The proportion of initial points $\xi \in \mathrm{L}$ which collapse on a very short cycle [8, 9];

3. The typical length of a transient, nonperiodic part, of a trajectory with a random initial condition.

We do not know nontrivial statistical characteristics which describe system behaviour for most initial conditions in the way SRB invariant measures do for the original system. To obtain meaningful results, one more level of averaging can be done. Rather than considering system behaviour only with respect to a collection of randomly selected initial conditions, one can study an ensemble of discretizations on different lattices, or an ensemble of discretizations of different mappings for the same lattices, or both. Instead of SRB invariant measures, other statistical properties can be investigated in detail. Certainly, such characteristics should be sufficiently robust in a natural sense. As an example let us mention scaling of average length of maximal cycle [1].

An of important questions in this area can be formulated as follows:

> *To suggest phenomenological models of discretizations which allow to predict some statistical properties of characteristics enumerated above in terms of original mappings and/or establish some connections between different characteristics.*

Usually some kind of completely random mapping [3] were used for this purpose. It is convenient to mention the basic ideas in constructing of such models. Let us consider a dynamical system f in $\mathbb{R}^d$ with chaotic behaviour. Suppose also that this system has a SRB invariant measure μ_f. That is μ_f is a weak limit of the sequence of measures

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} f_*^1 \delta_x$$

for almost all initial conditions x with respect to Lebesgue measure. Here δ_x is the Dirac measure concentrated at x and f_* is the mapping in the space of Borel measures generated by f. Consider the lattice $\mathrm{L}_\nu^d = \nu^{-1}\mathbf{Z}^d$ where $\mathbf{Z}^d$ is the standard integer lattice in $\mathbb{R}^d$ and ν is a large integer parameter. The L_ν^d-discretization f_ν of f is defined by equalities $f_\nu(\xi) = ([y^1]_\nu, \ldots, [y^d]_\nu)$, where $\xi = (\xi^1, \ldots \xi^d) \in \mathrm{L}_\nu^d$, $y = f(\xi) \in \mathbb{R}^d$ and $[\alpha]_\nu$ is a scalar raundoff operator defined by $[\alpha]_\nu = k/\nu$ if $(k-0.5)/\nu \le \alpha < (k+0.5)/\nu$, for an integer k. Define the quantity $H(\mathrm{L}_\nu^d, \mu_f) = \sum_{\xi \in \mathrm{L}_\nu^d} \mu_f(\xi + Q_\nu)$ where Q_ν is the ν^{-1}-cube in $\mathbb{R}^d$ centred at zero:

$$Q_\nu^d = \{\mathbf{x} = (x^1, \ldots, x^d) \in \mathbb{R}^d : \ -1/(2\nu) < x^i \le 1/(2\nu), \ i = 1, \ldots, d\}.$$

Consider as a phenomenological model of a discretization f_ν of a chaotic mapping $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ a completely random mapping [3] on the set X of $(H(\mathrm{L}_\nu^d, \mu_f))^{-1}$ points into itself. At $\nu \to 0$ it leads to modeling of discretization f_ν as a random mapping, defined on the set of $\gamma_f \nu^{\dim_c(\mu_f)}$ points where $\dim_f(\mu)$ is *correlation dimension* [10, 17] of μ and γ_f is a parameter. This idea was, as we know, firslty suggested in explicit form and successfully exploited in [11] It works especially well if the mapping f has no strong singuliarities.

Below another model is discussed for a classical family of chaotic mappings with singuliarities

$$f_\ell(x) = 1 - |2x-1|^\ell, \qquad 0 \le x \le 1 \tag{1}$$

where $\ell > 2$ is a parameter.

2 Description of a model

It is well known [3, 11, 12] that it is profitable to consider discretizations of mappings with chaotic behaviour as a realization of some random mapping. The main idea of these works may be formulated as follows: if we have an ensemble of discretizations of one and the same or of distinct mappings, and if there are no obvious reasons to consider these discretizations to be correlated, then the statistical characteristics of those discretizations are similar to the corresponding characteristics of a suitable ensemble of random mappings. The key question is: *which kind of random mapping should be chosen in a concrete situation?* This question is discussed in the next subsection for the family (1).

2.1 Random mappings with a single absorbing centre

Let $X = 0, 1, \ldots, \kappa$ and Δ be a positive number. Define *a random mapping* $T_{\Delta,\kappa}$*:*$X \mapsto X$*, with a single absorbing centre* 0, by formulas $T_{\Delta,\kappa}(0) = 0$ and

$$P(T_{\Delta,\kappa}(i) = j) = \begin{cases} \Delta/(\kappa+\Delta) & \text{if} \quad i \neq 0, j = 0, \\ 1/(\kappa+\Delta) & \text{if} \quad i, j \neq 0 \end{cases}$$

where $P(\cdot)$ denotes the probability of corresponding event and where the image of an element i, $i = 1, \ldots, \kappa$ is chosen independently of those of other elements i.

Random mappings with single absorbing centre are similar to, though differ from, mappings with single attracting centre [3, 4].

Now formulate a main claim of the paper. Below L_ν denotes the uniform $1/\nu$ lattice on $[0, 1]$: $\mathrm{L}_\nu = \{0, 1/\nu, 2/\nu, \ldots, 1\}$, $\nu = 1, 2, \ldots$. By $f_{\ell,\nu}$ denote the mapping $\mathrm{L}_\nu \mapsto \mathrm{L}_\nu$ defined by $f_{\ell,\nu}(\xi) = [f_\ell(\xi)]_\nu$, $\xi \in \mathrm{L}_\nu$. The mapping $f_{\ell,\nu}$ is a *ν-discretization* [16] of f_ℓ. If $\nu = 2^N$ the ν-discretization is natural theoretical model for implementation of the mapping f_ℓ in the *fixed point format* with N binary digits and radix point in the first position, see [7], pages 98-100.

Hypothesis 1. *There exist positive constants $c_\Delta(\ell)$, $c_\kappa(\ell)$ such that for large ν statistical characteristics concerning "cyclic" events for an ensemble of discretizations of mappings from the family $f_\ell(x)$, $\ell > 2$, are similar to sample statistical characteristics of analogous events for corresponding ensemble of random mappings with single absorbing centre with parameters $\Delta(\nu, \ell) =$*

$c_\Delta(\ell)\nu^{1/\ell}$ and $\kappa(\nu, l) = [c_\kappa(\ell)\nu^{2/\ell}]$ where $[\cdot]$ is usual rounding operator for real number.

Although we have no rigorous justification of this Hypothesis, as will be seen further there is close agreement between theoretical conclusions drawn from it and simulations. This strongly suggests that some such mechanism is present when discretization occurs in computations. A physical and heuristic justification see in [8].

Surely, the Hypothesis, as it just have been formulated is rather vague. It should be explained which statistical characteristics were meant and what does it mean "analogous events". We will discuss this question in the next section.

2.2 Some asymptotics of the model

The role of the Hypothesis is in the fact that in contrast to statistics of discretizations, the statistics of the model admit straightforward theoretical analysis. The basic characteristics of the model is the length of transient process. We shall formulate the necessary facts about this characteristic in this subsection.

Consider the random mapping with a single absorbing centre $T_{\Delta,\kappa}$. For each realization $T^{\omega}_{\Delta,\kappa}$ of this mapping and for each $i \in X$ the corresponding random trajectory $\mathbf{y} = y_0(i,\omega), \ldots, y_n(i,\omega), \ldots$, defined by $y_0(i,\omega) = i$, $y_n(i,\omega) = T^{\omega}_{\Delta,\kappa}(y_{n-1}(i,\omega))$ is eventually periodic. Define the *first recurrence or absorption moment* $\mathcal{M}_{\Delta,\kappa}(i,\omega)$ for this trajectory by the formulas

$$\mathcal{M}_{\Delta,\kappa}(i,\omega) = \min\{n : (y(i,\omega) = 0) \vee (y(i,\omega)_n = y(i,\omega)_j) \text{ for some } j < n\}.$$

where $\vee$ is the logical "or". Informally speaking, $\mathcal{M}_{\Delta,\kappa}(i,\omega)$ is the first n such that the trajectory $\mathbf{y}(i,\omega)$ either reach the absorbing state 0 or repeats itself. It is the first moment after which the trajectory is uniquely determined. We choose it as the basic characteristic in our constructions because of simple recurrence for the probabilities $p(n,\Delta,\kappa,i)$ of the event $\mathcal{M}_{\Delta,\kappa}(i,\omega) \geq n$:

$$p(n+1,\Delta,\kappa) = \left(1 - \frac{\Delta + n}{\Delta + \kappa}\right) p(n,\Delta,\kappa), \qquad p(1,\Delta,\kappa) = 1, \tag{2}$$

which will be proved in the appendix. The characteristic $\mathcal{M}_{\Delta,\kappa}(i,\omega)$ is awkward from the point of view of dynamical systems theory. More natural is the *first recurrence moment* or *length of transient process* $\mathcal{Q}_{\Delta,\kappa}(i,\omega)$ which is defined by

$$\mathcal{Q}_{\Delta,\kappa}(i,\omega) = \min\{n : y(i,\omega)_n = y(i,\omega)_j, \text{ for some } j < n\}. \tag{3}$$

Clearly,

$$\mathcal{M}_{\Delta,\kappa}(i,\omega) \le \mathcal{Q}_{\Delta,\kappa}(i,\omega) \le \mathcal{M}_{\Delta,\kappa}(i,\omega) + 1. \tag{4}$$

Define also *length* of a corresponding cycle by the formula

$$\mathcal{C}_{\Delta,\kappa}(i,\omega) = \min\{p > 0 : \exists n \text{ with } y_n(i,\omega) = y_{n+p}(i,\omega)\}. \tag{5}$$

Random variables $\mathcal{M}_{\Delta,\kappa}(i,\omega)$, $\mathcal{Q}_{\Delta,\kappa}(i,\omega)$ so as $\mathcal{C}_{\Delta,\kappa}(i,\omega)$ are identically distributed for any $i = 1, 2, \ldots, \kappa$. Denote the corresponding distribution functions by $D_{\mathcal{Q}}(i;\Delta,\kappa)$ and by $D_{\mathcal{C}}(i;\Delta,\kappa)$, $i = 0, 1, \ldots, \kappa$. It is convenient to expand functions $D_{\mathcal{M}}(i;\Delta,\kappa)$, $D_{\mathcal{Q}}(i;\Delta,\kappa)$ and $D_{\mathcal{C}}(i;\Delta,\kappa)$ to step functions defined for all $x \in [0,\kappa]$ by equalities $D_{\mathcal{M}}(x;\Delta,\kappa) = D_{\mathcal{M}}(\mathrm{trunc}(x);\Delta,\kappa)$,, $D_{\mathcal{Q}}(x;\Delta,\kappa) = D_{\mathcal{Q}}(\mathrm{trunc}(x);\Delta,\kappa)$, and $D_{\mathcal{C}}(x;\Delta,\kappa) = D_{\mathcal{C}}(\mathrm{trunc}(x);\Delta,\kappa)$.

Introduce functions

$$\begin{aligned} d_1(x;c_\Delta,c_\kappa) &= 1 - e^{\frac{c_\Delta^2-(x+c_\Delta)^2}{2c_\kappa}}, & (6)\\ d_2(x;c_\Delta,c_\kappa) &= 1 - e^{\frac{c_\Delta^2-(x+c_\Delta)^2}{2c_\kappa}}\left[1-\sqrt{\pi}\frac{x+c_\Delta}{\sqrt{2c_\kappa}}\,\mathrm{ERFCX}\left(\frac{x+c_\Delta}{\sqrt{2c_\kappa}}\right)\right], & (7)\\ d_*(x;c_\Delta,c_\kappa) &= \mathrm{ERFC}\left(c_\Delta\sqrt{\frac{(1-x)}{2c_\kappa x}}\right) & (8) \end{aligned}$$

where $\mathrm{ERFC}(t)$ and $\mathrm{ERFCX}(t)$ are the complementary error function and the scaled complementary error function, respectively, ([13], p. 166.):

$$\mathrm{ERFC}(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-s^2}\,ds, \qquad \mathrm{ERFCX}(t) = e^{t^2}\frac{2}{\sqrt{\pi}}\int_t^\infty e^{-s^2}\,ds. \tag{9}$$

Proposition 1. *For $\tau \to \infty$ and positive $c_\Delta, c_\kappa > 0$ the asymptotics*

$$\begin{aligned} D_{\mathcal{M}}(\tau x;c_\Delta\tau,c_\kappa\tau^2) &\sim d_1(x;c_\Delta,c_\kappa), & (10)\\ D_{\mathcal{Q}}(\tau x;c_\Delta\tau,c_\kappa\tau^2) &\sim d_1(x;c_\Delta,c_\kappa), & (11)\\ D_{\mathcal{C}}(\tau x;c_\Delta\tau,c_\kappa\tau^2) &\sim d_2(x;c_\Delta,c_\kappa) & (12) \end{aligned}$$

hold.

Asymptotic (10) follows from recurrence (2); (11) follows from (2) and (4; the last asymptotic implies (12) in a usual way. Detailed proofs are relegated to Appendix. Note also a corollary of the proposition above. Let $E(v)$ denote the mathematical expectation of a random variable v.

Corollary 1. *For $\tau \to \infty$ and positive $c_\Delta, c_\kappa > 0$ the asymptotics*

$$E(\mathcal{M}(\tau x; c_\Delta \tau, c_\kappa \tau^2)) \sim \sqrt{\frac{\pi c_\kappa}{2}} e^{\frac{c_\Delta^2}{2c_k}} \mathrm{ERFC}\left(\sqrt{\frac{c_\kappa}{2}}\right), \tag{13}$$

$$E(\mathcal{Q}(\tau x; c_\Delta \tau, c_\kappa \tau^2)) \sim \sqrt{\frac{\pi c_\kappa}{2}} e^{\frac{c_\Delta^2}{2c_k}} \mathrm{ERFC}\left(\sqrt{\frac{c_\kappa}{2}}\right), \tag{14}$$

$$E(\mathcal{C}(\tau x; c_\Delta \tau, c_\kappa \tau^2)) \sim \frac{1}{2}\sqrt{\frac{\pi c_\kappa}{2}} e^{\frac{c_\Delta^2 - c_\kappa^2}{2c_k}} \left(\sqrt{\frac{2c_\kappa}{\pi}} + \mathrm{ERFCX}\left(\sqrt{\frac{c_\kappa}{2}}\right)\right) \tag{15}$$

hold.

Note also one more asymptotic, which was used previously in [8, 9]. Define the *collapsing component* $Z_{\Delta,\kappa}$ of the mapping $T_{\Delta,\kappa}$ as a random subset of $(0, 1, \ldots, \kappa)$,

$$Z_{\Delta,\kappa} = \{i \in E(\kappa) : T^n_{\Delta,\kappa} i = 0 \text{ for some } n\}.$$

Introduce the random variable

$$\mathcal{P}_{\Delta,\kappa} = \frac{\#(Z_{\Delta,\kappa})}{\kappa},$$

where $\#(Z_{\Delta,\kappa})$ denotes cardinality of the set $Z_{\Delta,\kappa}$. The value $\mathcal{P}_{\kappa,\Delta}$ is the proportion of elements of $(0, 1, \ldots, \kappa)$ belonging to the collapsing component of the mapping $T_{\Delta,\kappa}$. Denote by $D_\mathcal{P}(x, \Delta, \kappa)$ the distribution function of the random variable $\mathcal{P}_{\kappa,\Delta}$ expanded on the whole interval $[0, 1]$.

Proposition 2. *For $\tau \to \infty$ and positive $c_\Delta, c_\kappa > 0$ the asymptotics*

$$D_\mathcal{P}(x; c_\Delta \tau, c_\kappa \tau^2) \sim d_*(x; c_\Delta, c_\kappa) \tag{16}$$

is valid.

3 Interpretations and experiments

3.1 Dictionary

Hypothesis 1 should be supplemented by a "dictionary" for reformulating statements concerning cycles of discretizations into statement concerning cycles of random mappings with an absorbing centre and vice versa. Fortunately, this dictionary seems to be quite natural. Before to proceed farther, mention three items of this dictionary.

a. The first recurrence moment $\mathcal{Q}_{\Delta(\nu,\ell),\kappa(\nu,\ell)}(i,\omega)$ for a random trajectory $\mathbf{y}(i,\omega)$, $i > 0$ for the random mapping $T_{\Delta(\nu,\ell),\kappa(\nu,\ell)}$ corresponds to the first recurrence moment for a trajectory of discretization $f_{\nu,\ell}$ with random initial point $\xi_0 \in \mathrm{L}_\nu$.

b. Length of cycle which is generated by a trajectory of the discretization $f_{\nu,\ell}$ with a random initial point $\xi_0 \in \mathrm{L}_\nu$ corresponds to length of cycle in random mapping $T_{\Delta(\nu,\ell),\kappa(\nu,\ell)}$

c. Proportion $P(\nu,\ell)$ of points $\xi \in \mathrm{L}_\nu$ which are eventually zero for the discretization $f_{\nu,\ell}$ is an analog of the proportion $\mathcal{P}_{\Delta(\nu,\ell),\kappa(\nu,\ell)}$.

Some experiments concerning items **a** and **b** see in the next subsection. An analogy mentioned in the item **c** was discussed in detail in [8, 9].

Let $\mathbf{S}$ a finite set of non-negative real numbers from $[0,1]$. Define the *distribution function of the set* $\mathbf{S}$, $\mathcal{D}(\cdot\,;\mathbf{S}) : [0,1] \to [0,1]$, by

$$\mathcal{D}(x;\mathbf{S}) = \frac{\#(\{s \in \mathbf{S} : s \le x\})}{\#(\mathbf{S})}, \qquad 0 \le x \le 1$$

where $\#(S)$ denotes cardinality of finite set S.

A sequence of numbers $\mathbf{u}_\nu = u_1, u_2, \ldots, u_\nu, \ldots$, is said to have the *stable distribution property* with limit $D(x)$ if $\lim_{\nu\to\infty} \mathcal{D}(x;\{u_1,u_2,\ldots,u_\nu\}) = D(x)$. Let $\boldsymbol{\xi}(\omega) = \boldsymbol{\xi} = \xi_1, \xi_2, \ldots, \xi_\nu, \ldots,\ \xi_\nu \in \mathrm{L}_\nu$ be a random sequence. For each (random) element ξ_ν we can consider the corresponding trajectory $\boldsymbol{\eta}(\ell,\nu) = \eta_1, \eta_2, \ldots, \eta_1 = \xi_\nu$ of the discretization $f_{\ell,\nu}$.

Consider the first moment $\mathcal{Q}_\nu(\xi_\nu)$ in which the trajectory $\boldsymbol{\eta}(\ell,\nu)$ repeats itself; consider also length $\mathcal{C}_\nu(\xi_\nu)$ of cyclic part of the sequence $\boldsymbol{\xi}$. From Hypothesis 1 and Proposition 1 it follows

Proposition 3. *There exist positive constants $c_\Delta(\ell)$, $c_\kappa(\ell)$ with the following property. For almost each sequence $\boldsymbol{\xi}(\omega)$ the sequence $\nu^{-1/\ell}\mathcal{Q}_\nu(\xi_\nu)$ has stable distribution property with the limit $d_1(x; c_\Delta(\ell), c_\kappa(\ell))$ and the sequence $\nu^{-1/\ell}\mathcal{C}_\nu(\xi_\nu)$ has stable distribution property with the limit $d_2(x; c_\Delta(\ell), c_\kappa(\ell))$.*

3.2 Numerical experiments

Proposition 3 admits experimental testing. For instance, to testify the first part of this proposition we can choose a pair of large and quite different numbers ν_1, ν_2, say $\nu_1 = 10^5$, $\nu_2 = 10^9$. Then choose a positive integer n with $1 \ll n \ll \min\{\nu_1, \nu_2\}$, for instance, $n = 1000$. Consider two finite sequences of lattices $\mathbf{L}(\nu_1, n) = \mathrm{L}_{\nu_1}, \mathrm{L}_{\nu_1+1} \ldots \mathrm{L}_{\nu_1+n}$ and $\mathbf{L}(\nu_2, n) = \mathrm{L}_{\nu_2}, \mathrm{L}_{\nu_2+1} \ldots \mathrm{L}_{\nu_2+n}$. Choose after that in each lattice from the first family above a random element $\xi_1(n)$ and choose a random point $\xi_2(n)$ in each lattice from the second family. Consider corresponding numbers $Q_j(l, \nu_i) = \nu_i^{-1/\ell}\mathcal{Q}_{\nu_i}(\xi_i(j))$ and $C_j(l, \nu_i) = \nu_i^{-1/\ell}\mathcal{C}_{\nu_i}(\xi_i(j))$, $i = 1, 2$, $j = 1, \ldots, n$. Denote further for $i = 1, 2$

$$\mathbf{Q}(l, n, \nu_i) = \{Q_j(\ell, n, \nu_i),\ j = 1, \ldots, n\}, \tag{17}$$

$$\mathbf{C}(l, n, \nu_i) = \{C_j(\ell, n, \nu_i),\ j = 1, \ldots, n\}. \tag{18}$$

Proposition 1 means that the following should be valid

Proposition 4. *There exist positive constants $c_\Delta(\ell)$, $c_\kappa(\ell)$ with the following properties.*

i. *Distributions $\mathcal{D}(x; \mathbf{Q}(\ell, n, \nu_1))$, $\mathcal{D}(x; \mathbf{Q}(\ell, n, \nu_2))$ are close one to another and both are close to $d_1(x; c_\Delta(\ell), c_\kappa(\ell))$.*

ii. *Distributions $\mathcal{D}(x; \mathbf{C}(\ell, n, \nu_1))$, $\mathcal{D}(x; \mathbf{C}(\ell, n, \nu_2))$ are close one to another and both are close to $d_2(x; c_\Delta(\ell), c_\kappa(\ell))$.*

This conclusion is in an excellent agreement with experiments. Figure 1 graphs 6 different distributions. The 3 curves above represent experimental results

$$\mathcal{D}(x; \mathbf{C}(3, 10^3, 10^5), \quad \mathcal{D}(x; \mathbf{C}(3, 10^3, 10^9) \tag{19}$$

and the theoretical prediction $d_2(x; c_\Delta, c_\kappa, 3)$ for $c_\Delta = 2.5$, $c_\kappa = 6.25$. The 3 curves below represent experimental results

$$\mathcal{D}(x; \mathbf{Q}(3, 10^3, 10^5)), \qquad \mathcal{D}(x; \mathbf{Q}(3, 10^3, 10^9))$$

and the function $d_1(x; c_\Delta, c_\kappa)$ again for $c_\Delta = 2.5$, $c_\kappa = 6.25$. Note, that to ajust parameters c_Δ, c_κ it was convenient to use Corollary 1. An agreement between experiment and theory is very good. Even the qualitative behaviour of the experimental curves (19) is impossible to imitate by the distributions suggested in [11] on the base of the using of completely random mappings.

Item **c** from the dictionary above and Hypothesis 1 lead to the conclusion that the distribution of the set $\mathbf{P}(\ell, n, \nu) = \{P_j(\ell, \nu) : j = 1, \ldots, n\}$ should be similar to the function $d_*(x; c_\Delta(\ell), c_\kappa(\ell))$. Figure 2 shows the sample distribution of the set $\mathbf{P}(3, 500, 2^{27})$ as a step function, compared against the smooth curve, of the distribution function with the density $d_*(x; c_\Delta, c_\kappa)$ for the same numbers $c_\Delta = 2.5$, $c_\kappa = 6.25$. This picture again strongly supported the main hypothesis of this paper.

4 Appendix. Proof of Proposition 1

For $i \neq 0$ denote by $p(n, \Delta, \kappa; i)$ the probability of the event

$$E_n = \{\omega : \ \mathcal{M}_{\Delta,\kappa}(i, \omega) \geq n\}.$$

Clearly, $p(n, \Delta, \kappa; i)$ doesn't depend on i; so the notation $p(n, \Delta, \kappa)$ is correct. By definition $p(1, \Delta, \kappa) = 1$. The event E_{n+1} can be written as

$$E_{n+1} = E_n \bigcap F_n$$

where

$$F_n = \{\omega : \ y_n(i, \omega) \neq 0 \text{ and } y_n(i, \omega) \neq y_j(i, \omega), \text{ for all } j = 0, 1, \ldots, n-1\}.$$

Hence

$$p(n{+}1, \Delta, \kappa) = \mathcal{P}(E_n \bigcap F_n) = \mathcal{P}\left(E_n \setminus \left(\left(\bigcup_{j=0}^{n-1} (E_n \bigcap G_{n,j})\right) \bigcup (E_n \bigcap H_n)\right)\right)$$

where $\mathcal{P}(F)$ denotes probability of the event F and

$$H_n = \{\omega : \ y_n(i, \omega) = 0\}, \tag{20}$$

$$G_{n,j} = \{\omega : \ y_n(i, \omega) = y_j(i, \omega)\}, \qquad j = 0, 1, \ldots, n-1. \tag{21}$$

All events $E\bigcap H_n$ and $E_n\bigcap G_{n,j}$, $j = 0,1,\ldots,n-1$ are mutually disjoint; that is

$$p(n+1,\Delta,\kappa) = p(n,\Delta,\kappa) - \mathcal{P}(E_n\bigcap H_n) - \sum_{j=0}^{n-1}\mathcal{P}(E_n\bigcap G_{n,j}). \qquad (22)$$

By definition of random mapping with single absorbing centre the event E_n is independent of each of the events H_n, $G_{n,j}$, $j = 0,1,\ldots,n-1$. Therefore (20), (21) can be rewritten as

$$\mathcal{P}(E_n\bigcap H_n) = p(n,\Delta,\kappa)\frac{\Delta}{\Delta+\kappa} \qquad (23)$$

$$\mathcal{P}(E_n\bigcap G_{n,j}) = p(n,\Delta,\kappa)\frac{1}{\Delta+\kappa}, \qquad j = 0,1,\ldots,n-1. \qquad (24)$$

and, finally, by (22) and (23)–(24) the recurrence

$$p(n+1,\Delta,\kappa) = \left(1-\frac{\Delta+n}{\Delta+\kappa}\right)p(n,\Delta,\kappa), \qquad p(1,\Delta,\kappa) = 1 \qquad (25)$$

holds. In particular, $p(n,\Delta,\kappa) = 0$ for $n \geq k+1$.

It will be convenient for us to treat the function $p(\cdot,\Delta,\kappa) = 0$ as being defined by the first argument not only for positive integer values, but for all real values, which are equal to or greater to or equal to 1. So set

$$p(1+x,\Delta,\kappa) = xp(2,\Delta,\kappa) + (1-x)p(1,\Delta,\kappa) = 1 - x\frac{\kappa-1}{\Delta+\kappa} \qquad (26)$$

for $1 < x < 2$. Then prolongate it recurrently on the interval $[1,\kappa+1]$ by

$$p(x+1,\Delta,\kappa) = \left(1-\frac{\Delta+x}{\Delta+\kappa}\right)p(x,\Delta,\kappa). \qquad (27)$$

And, at last, set $p(x,\Delta,\kappa) = 0$ for $x \geq \kappa+1$. Clearly, the function $p(x,\Delta,\kappa)$ will satisfy (27) for all $x \geq 1$ and will be continuous for these values of x.

Denote $\tau = \nu^{1/\ell}$, then $\Delta = c_\Delta\tau$ and $\kappa = c_\kappa\tau^2$. Hence the function

$$y_\tau(x) = p(\tau x+1,\Delta,\kappa) = p(x\tau+1,c_\Delta\tau,c_\kappa\tau^2) \qquad (28)$$

satisfies $y_\tau(x) = 0$ for $x \geq c_\kappa\tau$, and

$$y_\tau(x+\tau^{-1}) = \left(1-\frac{c_\Delta+x+\tau^{-1}}{c_\Delta+\tau c_\kappa}\right)y_\tau(x), \qquad 0 \leq x \leq c_\kappa\tau. \qquad (29)$$

Introduce also the function

$$z_\tau(x) = e^{\frac{2\tau c_\Delta x + \tau x^2}{2(c_\Delta + \tau c_\kappa)}} y_\tau(x), \qquad x \geq 0. \tag{30}$$

Then in view of (29) we can write

$$z_\tau(x + \tau^{-1}) = \left(1 - \frac{c_\Delta + x + \tau^{-1}}{c_\Delta + \tau c_\kappa}\right) e^{\frac{c_\Delta + x + \tau^{-1}}{c_\Delta + \tau c_\kappa}} z_\tau(x), \qquad 0 \leq x \leq c_\kappa \tau. \tag{31}$$

Lemma 1. *For any $\tau > 0$ the estimate $z_\tau(x) \leq 1$ is valid for all $x \geq 0$.*

Proof: From the estimate $e^{-x} \geq 1 - x$ it follows that

$$\left(1 - \frac{c_\Delta + x + \tau^{-1}}{c_\Delta + \tau c_\kappa}\right) \leq e^{-\frac{c_\Delta + x + \tau^{-1}}{c_\Delta + \tau c_\kappa}}$$

Hence by (31)

$$z_\tau(x + \tau^{-1}) \leq z_\tau(x), \qquad 0 \leq x \leq c_\kappa \tau. \tag{32}$$

In view of (26), (28) and (30)

$$z_\tau(x) = e^{\frac{2\tau c_\Delta x + \tau x^2}{2(c_\Delta + \tau c_\kappa)}} \left(1 - x \frac{\tau c_\Delta + \tau^{-1}}{c_\Delta + \tau c_\kappa}\right), \qquad 0 \leq x \leq \tau^{-1}.$$

and so, by inequality $e^{-x} \geq 1 - x$,

$$z_\tau(x) = e^{\frac{(2\tau c_\Delta - 1)x + \tau x^2}{2(c_\Delta + \tau c_\kappa)} - \frac{\tau c_\Delta x}{c_\Delta + \tau c_\kappa}} = e^{\frac{\tau x^2 - x}{2(c_\Delta + \tau c_\kappa)}}, \qquad 0 \leq x \leq \tau^{-1}.$$

But $\tau x^2 - x \leq 0$ for $0 \leq x \leq \tau^{-1}$, thus

$$z_\tau(x) \leq 1, \qquad 0 \leq x \leq \tau^{-1}.$$

From here and from (32) the statement of Lemma follows.

We note also a corollary of the lemma which will be used in a proof of the asymptotic fo length of cycles.

Corollary 2. *For any $\tau \geq 1$ the estimate $e_\tau(x) \leq e^{-\frac{x^2}{2(c_\Delta + c_\kappa)}}$ is valid for all $x \geq 0$.*

Proof: From (30) and from Lemma 1 it follows that

$$y_\tau(x) \le e^{-\frac{2\tau c_\Delta x + \tau x^2}{2(c_\Delta + \tau c_\kappa)}}, \qquad x \ge 0.$$

Here for sufficiently large values of τ

$$\frac{2\tau c_\Delta x + \tau x^2}{2(c_\Delta + \tau c_\kappa)} \ge \frac{\tau x^2}{2(c_\Delta + \tau c_\kappa)} \ge \frac{x^2}{2(c_\Delta + c_\kappa)}$$

from which the statement of Lemma follows.

Lemma 2. *Given $x_* > 0$, then $z_\tau(x) \to 1$ as $\tau \to \infty$ uniformly for $x \in [0, x_*]$.*

Proof: It is easy to see, that $e^{-x-x^2} \le 1 - x$ for all small positive x. Then from (31) it follows that

$$z_\tau(x + \tau^{-1}) \ge e^{\frac{c_\Delta + x + \tau^{-1}}{c_\Delta + \tau c_\kappa} - \frac{c_\Delta + x + \tau^{-1}}{c_\Delta + \tau c_\kappa} - \left(\frac{c_\Delta + x + \tau^{-1}}{c_\Delta + \tau c_\kappa}\right)^2} z_\tau(x) = e^{-\left(\frac{c_\Delta + x + \tau^{-1}}{c_\Delta + \tau c_\kappa}\right)^2} z_\tau(x) \tag{33}$$

for $x \in [0, x_*]$ and sufficiently large τ.

Fix now arbitrary $x \in [0, x_*]$ and define $x_\tau = x - \tau^{-1}[x\tau]$, $0 \le x_\tau < \tau^{-1}$, where the sign $[\cdot]$ denotes the integer part of the corresponding number. Then from (33) it follows, that

$$z_\tau(x) \ge e^{-\left(\frac{c_\Delta + x + \tau^{-1}}{c_\Delta + \tau c_\kappa]}\right)^2 [x\tau]} z_\tau(x_\tau).$$

So, uniformly for $x \in [0, x_*]$,

$$\liminf_{\tau \to \infty} z_\tau(x) \ge 1$$

but on the other hand by Lemma 1 $z_\tau(x) \le 1$ and thus, uniformly for $x \in [0, x_*]$,

$$\lim_{\tau \to \infty} z_\tau(x) = 1.$$

Lemma 2 is proved.

Now, from the estimate

$$1 - y_\tau(x) \le D_{\mathcal{M}}(\tau x; c_\Delta \tau, c_\kappa \tau^2) \le 1 - y_\tau(\tau^{-1}[x\tau]),$$

from (28), (30) and from Lemma 2 it follows that

$$D_{\mathcal{M}}(\tau x; c_\Delta \tau, c_\kappa \tau^2) \to 1 - e^{-\frac{2c_\Delta x + x^2}{2c_\kappa}}, \qquad \text{as } \tau \to \infty.$$

The last relation coincides with (10), which proves the first statement of Proposition 1.

We are beginning now the proof of the second statement of Proposion 1, relation (12). Denote by $q(n, \Delta, \kappa; i)$ the probability for the random mapping with a single absorbing centre $T_{\Delta,\kappa}$ to have a cycle of length n. Clearly, $q(n, \Delta, \kappa; i)$ do not depend on i and we will use the notation $q(n, \Delta, \kappa)$. Consider first $q(1, \Delta, \kappa)$. We can get a cycle of the length 1 on the $n - th$ step of the process if $\mathcal{M}_{\Delta,\kappa}(i, \omega) > n - 1$ happened and either $y_n(i, \omega) = 0$ or $y_n(i, \omega) = y_{n-1}(i, \omega)$. That is

$$q(1, \Delta, \kappa) = \frac{\Delta + 1}{\Delta + \kappa} \sum_{i=1}^{\kappa} p(i, \Delta, \kappa).$$

Analogously, we can get a cycle of the length $p > 1$ on the $(n+1) - th$ step of the process if $\mathcal{M}_{\Delta,\kappa}(i, \omega)$ happened and either $y_{n+1}(i, \omega) = y_{n-p+1}(i, \omega)$. Therefor

$$q(n, \Delta, \kappa) = \frac{1}{\Delta + \kappa} \sum_{i=n}^{\kappa} p(i, \Delta, \kappa), \qquad 2 \leq n \leq \kappa.$$

Or, since $p(i, \Delta, \kappa) = 0$ for $i \geq \kappa + 1$, as

$$q(1, \Delta, \kappa) = \frac{\Delta + 1}{\Delta + \kappa} \sum_{i=1}^{\infty} p(i, \Delta, \kappa) \tag{34}$$

and

$$q(n, \Delta, \kappa) = \frac{1}{\Delta + \kappa} \sum_{i=n}^{\infty} p(i, \Delta, \kappa), \qquad 2 \leq n \leq \kappa. \tag{35}$$

Calculate first the value of $q(1, \Delta, \kappa)$. Due to (28) and (34) we can represent the value of $q(1, \Delta, \kappa)$ in the following form

$$q(1, \Delta, \kappa) = q(1, c_\Delta \tau, c_\kappa \tau^2) = \tau \frac{\Delta + 1}{\Delta + \kappa} \sum_{i=1}^{\infty} y_\tau\left(\frac{i-1}{\tau}\right) \frac{1}{\tau}.$$

Here

$$\tau\frac{\Delta+1}{\Delta+\kappa}\to\frac{c_\Delta}{c_\kappa},\qquad y_\tau(x)\to e^{-\frac{2c_\Delta x+x^2}{2c_\kappa}}\qquad\text{as}\quad\tau\to\infty$$

uniformly with respect to x from any bounded set, as was shown above. At last, by Corollary 2 functions $y_\tau(x)$ positive and uniformly bounded by a summable function. Hence,

$$q(1,\Delta,\kappa)=q(1,c_\Delta\tau,c_\kappa\tau^2)\to\frac{c_\Delta}{c_\kappa}\int\limits_0^\infty e^{-\frac{2c_\Delta s+s^2}{2c_\kappa}}\,ds. \tag{36}$$

Introduce the function

$$\widetilde{q}(n,\Delta,\kappa)=\sum_{j=2}^{n}q(j,\Delta,\kappa),\qquad n\ge 2.$$

then by (35)

$$\widetilde{q}(n,\Delta,\kappa)=\frac{1}{\Delta+\kappa}\sum_{j=2}^{n}\sum_{i=n}^{\infty}p(i,\Delta,\kappa),\qquad n\ge 2. \tag{37}$$

Prolongate now the range of definition of the function $\widetilde{q}(\cdot,\Delta,\kappa)$ on the set of real numbers $x\ge 2$ by

$$\widetilde{q}(x,\Delta,\kappa)=\widetilde{q}(\mathrm{trunc}(x),\Delta,\kappa) \tag{38}$$

Note, that by definition of distribution

$$D_{\mathcal{C}}(\tau x;c_\Delta\tau,c_\kappa\tau^2)=q(1,c_\Delta\tau,c_\kappa\tau^2)+\widetilde{q}(x\tau,c_\Delta\tau,c_\kappa\tau^2),$$

so in view of (36 it remained only to find the limit of the value $\widetilde{q}(x\tau,c_\Delta\tau,c_\kappa\tau^2)$. Denote $n_x=\mathrm{trunc}(x\tau)$ then by (28), (37) and (38)

$$\widetilde{q}(x\tau,c_\Delta\tau,c_\kappa\tau^2)=\frac{\tau^2}{\Delta+\kappa}\sum_{j=2}^{n_x}\left(\sum_{i=j}^{\infty}y_\tau(\frac{i-1}{\tau})\frac{1}{\tau}\right)\frac{1}{\tau}$$

Since

$$\frac{\tau^2}{\Delta+\kappa}\to\frac{1}{c_\kappa},\qquad\frac{n_x}{\tau}\to x\qquad\text{as}\quad\tau\to\infty$$

then by repetition the same reasoning as in the case with $q(1,\Delta,\kappa)$ we obtain

$$\widetilde{q}(x\tau,c_\Delta\tau,c_\kappa\tau^2)\to\frac{1}{c_\kappa}\int\limits_0^x\int\limits_t^\infty e^{-\frac{2c_\Delta s+s^2}{2c_\kappa}}\,ds\,dt. \tag{39}$$

So, from (36), (39) it follows that

$$D_{\mathcal{C}}(\tau x; c_\Delta \tau, c_\kappa \tau^2) \to \frac{c_\Delta}{c_\kappa} \int_0^\infty e^{-\frac{2c_\Delta s + s^2}{2c_\kappa}}\, ds + \frac{1}{c_\kappa} \int_0^x \int_t^\infty e^{-\frac{2c_\Delta s + s^2}{2c_\kappa}}\, ds\, dt \tag{40}$$

and we need only to calculate the integrals in the right hand part. By changing the order of integration in the second integral in (40), we obtain

$$\begin{aligned} \frac{1}{c_\kappa} \int_0^x \int_t^\infty e^{-\frac{2c_\Delta s + s^2}{2c_\kappa}}\, ds\, dt &= \frac{1}{c_\kappa} \int_0^x \left(\int_0^s dt \right) e^{-\frac{2c_\Delta s + s^2}{2c_\kappa}}\, ds + \\ &+ \frac{1}{c_\kappa} \int_x^\infty \left(\int_0^x dt \right) e^{-\frac{2c_\Delta s + s^2}{2c_\kappa}}\, ds \end{aligned}$$

or, that is the same,

$$\frac{1}{c_\kappa} \int_0^x \int_t^\infty e^{-\frac{2c_\Delta s + s^2}{2c_\kappa}}\, ds\, dt = \frac{1}{c_\kappa} \int_0^x s e^{-\frac{2c_\Delta s + s^2}{2c_\kappa}}\, ds + \frac{x}{c_\kappa} \int_x^\infty e^{-\frac{2c_\Delta s + s^2}{2c_\kappa}}\, ds.$$

Hence

$$D_{\mathcal{C}}(\tau x; c_\Delta \tau, c_\kappa \tau^2) \to \frac{c_\Delta + x}{c_\kappa} \int_x^\infty e^{-\frac{2c_\Delta s + s^2}{2c_\kappa}}\, ds + \frac{1}{c_\kappa} \int_0^x (c_\Delta + s) e^{-\frac{2c_\Delta s + s^2}{2c_\kappa}}\, ds.$$

Make now the substitution $\frac{c_\Delta + s}{\sqrt{2c_\kappa}} = u$ in the integrals above, then

$$\begin{aligned} D_{\mathcal{C}}(\tau x; c_\Delta \tau, c_\kappa \tau^2) &\to 2\frac{c_\Delta + x}{\sqrt{2c_\kappa}} e^{\frac{c_\Delta^2}{2c_\kappa}} \int_{\frac{c_\Delta + x}{\sqrt{2c_\kappa}}}^\infty e^{-u^2}\, du + 2e^{\frac{c_\Delta^2}{2c_\kappa}} \int_{\frac{c_\Delta}{\sqrt{2c_\kappa}}}^{\frac{c_\Delta + x}{\sqrt{2c_\kappa}}} u e^{-u^2}\, du \\ &= 1 - e^{\frac{c_\Delta^2 - (x + c_\Delta)^2}{2c_\kappa}} + 2\frac{c_\Delta + x}{\sqrt{2c_\kappa}} e^{\frac{c_\Delta^2}{2c_\kappa}} \int_{\frac{c_\Delta + x}{\sqrt{2c_\kappa}}}^\infty e^{-u^2}\, du. \end{aligned}$$

As is easy to see the latter coincide with the representation (7), (12) which completes the proof of Proposition 1.

References

[1] P.M. Binder. Limit cycles in a quadratic discrete iteration. *Phys.-D* **57** (1992), no. 1-2, 31–38.

[2] A. Boyarsky and P. Gora, Why computers like Lebesgue measure, *Comp. Math. Applns*, **16** (1988), 321–329.

[3] B. Bollobas, "*Random graphs*", Academic Press, London, 1985.

[4] Y.D. Burtin. On a simple formula for random mappings and is applications, *J. Appl. Prob.* **17** (1980), 403–414.

[5] P. Diamond, P. Kloeden and A. Pokrovskii, An invariant measure arising in computer simulation of a chaotic dynamical system, *Journal of Nonlinear Science*, **4**, pp. 59 - 68.

[6] P. Diamond, P. Kloeden and A. Pokrovskii, Weakly chain recurrent points and spatial discretizations of dynamical systems, *Random & Computational Dynamics*, **2**, No 1, pp. 97 - 110.

[7] J.J.F. Cavanagh, *"Digital Computer Arithmetic. Design and Implementation*, McGraw–Hill Book Company, 1984.

[8] P. Diamond P. Kloeden A. Pokrovskii and M. Suzuki.) Collapsing Effects in Numerical simulation of a class of chaotic dynamical Systems and Random Mappings with a Single Attracting Centre. Deakin University. School of Computing & Mathematics. Technical Reports. Mathematics Series, **TR M94/06**, pp. 1–25. Geelong, Vic. Australia.

[9] P. Diamond P. Kloeden A. Pokrovskii and M. Suzuki.) Collapsing effects in numerical simulation of Chaotic Dynamical Systems. *Proceedings of 94 Korea Automatic Control Conference. International Session*, 753–757. October 18-20,1994 Taejon, Korea.

[10] Grassberger P. and Procaccia I., On the characterization of strange attractors, *Phys. Rev. Lett.*, **50** (1983), pp. 346–349.

[11] C. Gregobi, E. Ott and J.A. Yorke, Roundoff–induced periodicity and the correlation dimension of chaotic attractors, *Phys. Rev. A*, **34** (1988), 3688–3692.

[12] Y.E. Levy, Some remarks about computer studies of dynamical systems, *Physics Letters*, **88A** (1982), No 1, 1–3ct.

[13] *"MATLAB. Reference Guide"*, The MathWorks, Inc., Natick, 1992.

[14] W. di Melo and S. van Strien, "*One-dimensional dynamics*", Springer-Verlag, Berlin, 1993.

[15] V.E. Stepanov. Random mappings with a single attracting centre, *Theory Prob. Appl.*, **16** (1971), 155–161.

[16] H.J. Stetter, "*Analysis of Discretization Methods for Ordinary Differential equations*", Springer–Verlag, Berlin, 1976.

[17] Takens F., On the numerical determination of the dimension of an attractor, In: *Lecture Notes in Mathematics*, **1125**, Springer-Verlag, Berlin, 1986, pp. 99–106.

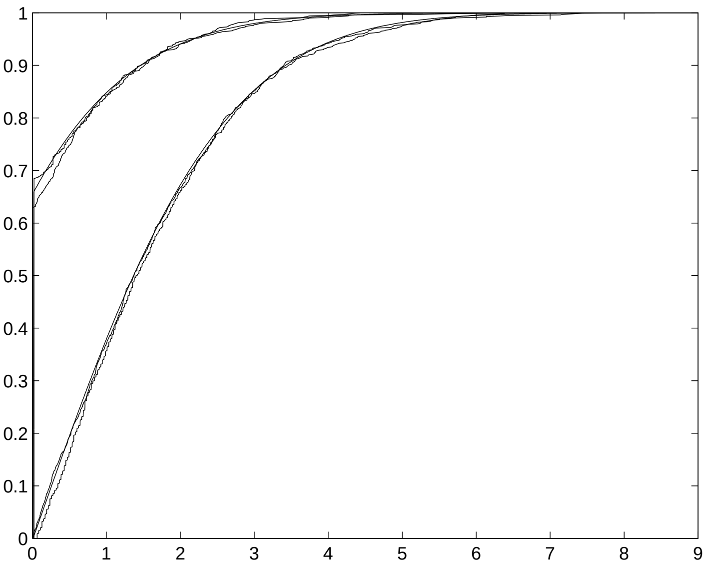

Figure 1: Three curves above represent distributions $\mathcal{D}(x; \mathbf{C}(3, 10^3, 10^5))$, $\mathcal{D}(x; \mathbf{C}(3, 10^3, 10^9))$ and $d_2(x; 2.5, 6.25, 3)$, whereas the three curves below represent $\mathcal{D}(x; \mathbf{Q}(3, 10^3, 10^5))$, $\mathcal{D}(x; \mathbf{Q}(3, 10^3, 10^9))$ and $d_1(x; 2.5, 6.25, 3)$.

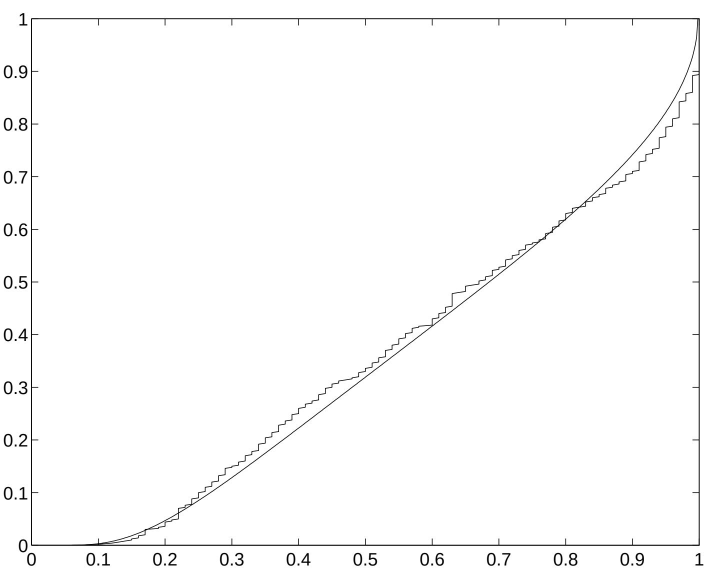

Figure 2: Sample distributions of the set $\mathbf{P}(3, 500, 2^{27})$ against $\delta_*(x; 2.5, 6.25)$.