Robustness of Dynamical Systems to a Class of Nonsmooth Perturbations*

P.Diamond[†] P.Kloeden[‡] V.Kozyakin[§]
M.Krasnosel'skii[§] A.Pokrovskii[¶]

1 Introduction

Consider a smooth mapping $f : \Re^d \to \Re^d$. Throughout, this will be referred to as the system f. The dynamical system f generated by a difference equation of the form

$$x_n = f(x_{n-1}), \quad n = 1, 2, \dots, \tag{1}$$

is often used in technical, physical or mechanical applications, where f usually occurs via a Poincaré section. Realistically, a system (1) can describe the actual underlying system only approximately. Thus an important mathematical problem is the robustness of the system to perturbations. Classical results in this direction state that a C^r dynamical system preserves some of its structural properties under a small *smooth* perturbation [4, 8, 9]. However, there are some kinds of *nonsmooth* perturbations which are very important.

*P. Diamond, P. Kloeden and A. Pokrovskii were supported by the Australian Research Council Grant A 8913 2609, V.Kozyakin and M.Krasnosel'skii were partially supported by the the Russian Foundation for Fundamental Research Grant 93012884

[†]Department of Mathematics, University of Queensland, Brisbane, Qld 4072, Australia.

[‡]Department of Computing and Mathematics, Deakin University, Geelong, Victoria 3217, Australia.

[§]Institute of Information Transmission Problems, Russian Academy of Sciences, 19 Ermolovoy str., Moscow 101447, Russia.

[¶]Department of Mathematics, University of Queensland, Brisbane, Qld 4072, Australia. Permanent address: Institute of Information Transmission Problems, Russian Academy of Sciences.

In this paper we apply a recently proposed technique for the analysis of numerical solutions of chaotic systems [2] to analyze a specific class of perturbations which arise in systems with weak hysteresis nonlinearities. An important feature of such models is that hysteresis nonlinearities are treated as continuous but nonsmooth dynamical systems W, often with an infinite dimensional set of internal states. This includes such nonlinearities as play, stop, the Besseling–Ishlinskii and Preisach–Giltay models and so on. Further details may be found in [6].

In such situations the natural description of state space of a perturbed system (1) is $\mathcal{Q} = \Re^d \times \Omega$. So it is more realistic to describe the dynamics of the perturbed system W by relations of the form

$$(x_n, \omega_n) = W(x_{n-1}, \omega_{n-1}) = (\varphi(x_{n-1}, \omega_{n-1}), \psi(x_{n-1}, \omega_{n-1})). \tag{2}$$

Here $\varphi : \Re^d \times \Omega \to \Re^d$ and $\psi : \Re^d \times \Omega \to \Omega$ are continuous mappings. Some concrete examples of systems which arise in the theory of hysteresis nonlinearities are given in Sections 3 and 4.

We are concerned with the relationship between the trajectories of a smooth system f and those of systems W which are close to f in some sense. An appropriate measure of the distance between the two types of system is described in Section 2. It is important to note that, without extra assumptions, the system (1) is not structurally stable in general. See [3, 7] for a discussion of this for systems with asymptotically stable equilibria or periodic orbits.

A natural, additional assumption is hyperbolicity of f, perhaps in a neighbourhood of a hyperbolic fixed point. In these circumstances, estimates of the distance between trajectories of f and its perturbation W should not depend explicitly upon the time interval over which the trajectories are considered. Instead, it is preferable that any estimate should be uniform so long as the trajectories remain in the region in question. This is the principal question that we address in this paper.

In Section 2, the principal results are stated. Informally speaking, any given trajectory of the system generated by f, lying in the hyperbolic region, can be C^0-approximated by the first component of some trajectory of any sufficiently close perturbation W. It must be emphasised that this is *not an analog of the shadowing property*, but rather an inverse of it. Sections 3 and 4 contain examples of applications of these general results in the analysis of differential equations with specific hysteresis nonlinearities.

2 Main Theorem

We first introduce a few notations and definitions.

It will be convenient to restrict attention to a fixed open set $\mathcal{X} \subseteq \Re^d$ and to characterize the distance between the systems generated by W and f on $\mathcal{X}$ by

$$\rho(W, f) = \sup_{x \in \mathcal{X}, \omega \in \Omega} |\varphi(x, \omega) - f(x)|.$$

A finite sequence $\boldsymbol{x} = x_0, x_1, \ldots, x_N$ is called a *finite trajectory* of the system f if $x_n = f(x_{n-1})$, $n = 1, 2, \ldots, N$. We shall just refer to trajectories, it being henceforth understood that they are all finite. For any compact set $\mathcal{M} \subset \mathcal{X}$ denote by $\mathcal{T}(f, \mathcal{M})$ the totality of trajectories of the system (1) belonging to $\mathcal{M}$. Analogously, for $\omega \in \Omega$ denote by $\mathcal{T}(W, \omega)$ the totality of finite trajectories

$$(x_0, \omega_0), (x_1, \omega_1), \ldots, (x_N, \omega_N) \tag{3}$$

of the system W satisfying $\omega_0 = \omega$.

Let α be a positive real number and let $\mathcal{M} \subseteq \mathcal{X}$ be a compact set. The system (1) is called *α-robust in $\mathcal{M}$ with respect to continuous perturbations* if there exists $\varepsilon > 0$ such that for any given trajectory $\boldsymbol{x}^* = x_0^*, x_1^*, \ldots, x_N^* \in \mathcal{T}(f, \mathcal{M})$, any continuous system (2) satisfying $\rho(W, f) < \varepsilon$, and any $\omega \in \Omega$ there exists a trajectory (3) from $\mathcal{T}(W, \omega)$ such that $\|x_n - x_n^*\| \leq \alpha\rho(W, f)$, $n = 0, 1, \ldots N$. This last definition should be distinguished from that of the shadowing property [4] and is in fact the inverse of it. It means that *any given* trajectory of the *semi-hyperbolic* system f is C^0-approximated by the first component of *some* trajectory of perturbed systems W. A similar idea was used in a quite different situation in [1].

The derivative of the mapping f at the point $x \in \Re^d$ will be denoted by Df_x. The four-tuple of nonnegative values $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$, will be called a *split* if

$$\lambda_s < 1 < \lambda_u \tag{4}$$

and

$$(1 - \lambda_s)(\lambda_u - 1) > \mu_s \mu_u. \tag{5}$$

For any given λ_s, λ_u satisfying (4) the four-tuple $\boldsymbol{s}$ is a split if the product $\mu_s \mu_u$ is small enough. Given some split $\boldsymbol{s}$ and a positive real number k, the system f is called *$(\boldsymbol{s}, k)$-hyperbolic on the set $\mathcal{X}$* if for each $x \in \mathcal{X}$ there exist a decomposition $T_x\Re^d = E_x^s \oplus E_x^u$ with corresponding projectors P_x^s and P_x^u

which satisfy the following inequalities:

$$
\begin{aligned}
\|P^s_{f(x)} Df_x u\| &\leq \lambda_s \|u\|, \quad u \in E^s_x, && (6)\\
\|P^s_{f(x)} Df_x v\| &\leq \mu_s \|v\|, \quad v \in E^u_x, && (7)\\
\|P^u_{f(x)} Df_x v\| &\geq \lambda_u \|v\|, \quad v \in E^u_x, && (8)\\
\|P^u_{f(x)} Df_x u\| &\leq \mu_u \|u\|, \quad u \in E^s_x, && (9)\\
\|P^s_x\|, \|P^u_x\| &\leq k. && (10)
\end{aligned}
$$

The system f will be called *semi-hyperbolic on the set* $\mathcal{X}$ if they are $(\boldsymbol{s}, k)$-hyperbolic on the set $\mathcal{X}$ for some split $\boldsymbol{s}$ and a positive real number k.

Theorem 1 *Let f be $(\boldsymbol{s}, k)$-hyperbolic in $\mathcal{M}$. Then it is α-robust in $\mathcal{M}$ with respect to continuous perturbations for every*

$$\alpha > \alpha_*(\boldsymbol{s}, k) = \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} k. \tag{11}$$

3 First Example

Recall that the nonlinearity *stop with threshold value h* or transducer stop ([6], p. 23–24) is a system $\mathbf{U}_h$ with the state space $[-h, h]$, scalar inputs $u(t)$ and outputs $\omega(t)$. For a smooth input $u(t)$, $t \geq 0$, and initial state $\omega_0 \in [-h, h]$ the corresponding output $\omega(t) = (U_h[\omega_0]u)(t)$, $t \geq 0$, is defined as a unique absolutely continuous solution of the problem

$$\omega' = q(\omega, u'(t)), \qquad \omega(0) = \omega_0$$

where

$$q(\omega, u) = \begin{cases} \min\{u, 0\} & \text{if} \quad \omega \geq h, \\ u & \text{if} \quad |\omega| < h, \\ \max\{u, 0\} & \text{if} \quad \omega \leq -h. \end{cases}$$

Consider the system described by the equations

$$x' = G(x, \omega), \qquad \omega(t) = (U_h[\omega_0](c, x))(t). \tag{12}$$

Here $x \in \Re^d$; $h > 0$ and $\omega \in [-h, h]$ are parameters, c is a fixed vector from $\Re^d$ and U_h is the stop nonlinearity with threshold value h. Equations of such type arise as description of mechanical systems with elastic-plastic Prager elements, technical systems with plays or stops and many control systems.

Suppose that the function G satisfies a global Lipschitz condition. Then the equation (12) has a unique solution for any initial condition $x(0) = x_0$ and each initial state ω_0 of the hysteresis nonlinearity U_h. Let the shift operator $S_h(x_0, \omega_0)$ denote the image of the initial value (x_0, ω_0) after unit time along the trajectories of the system (12). Suppose that $F(x) = G(x, 0)$ is a smooth function, satisfying $F(0) = 0$ and the matrix DF_0 does not have eigenvalues with zero real part. Similarly, let $S_0(x_0)$ be the image of the initial value x_0 after unit time along the trajectories of equation

$$x' = F(x). \tag{13}$$

The mappings $W(x, \omega) = S_h(x, \omega)$ and $f(x) = S_0(x)$ generate dynamical systems W and f respectively, where the state space of the system W is the product $\Re^d \times [-h, h]$

Clearly, the system f is semi-hyperbolic in some open ball $\mathcal{B}$ centered at the origin. From Theorem 1 it follows immediately

Theorem 2 *There exist $\alpha > 0$ and $h_0 > 0$ with the following property: for any trajectory $x(t) \in \mathcal{B}$, $0 \leq t < t_* \leq \infty$, of the equation* (13) *and any $h \leq h_0$ there exists a trajectory $(x_h(t), \omega_h(t))$, $0 \leq t < t_*$, of* (12) *satisfying*

$$|x(t) - x_h(t)| \leq \alpha h, \quad 0 \leq t < t_*.$$

Corollary 1 *There exist $\alpha > 0$ and $h_0 > 0$ with the following property: for any $x_0 \in \mathcal{B}$ belonging to the stable manifold of the equation* (13) *there exists a trajectory $(x_h(t), \omega_h(t))$, $t \geq 0$, of* (12) *satisfying*

$$|x_0(t) - x_h(t)| \leq \alpha h, \quad t > 0.$$

This result can be treated as a kind of "the stable manifold robustness theorem" with respect to hysteresis perturbations of a system.

Analogues of Theorem 2 are valid for equations with such nonlinearities as play or generalized play, with multi-dimensional plays and stops, with Mizes and Treska models [6], and so on.

4 Second Example

Let U_h be the stop nonlinearity with threshold h, as in Example 1. Consider h as a parameter, $0 \leq h \leq \infty$, and let μ be a Borel measure on $[0, \infty]$

satisfying

$$\int_0^\infty h\, d\mu(h) < \infty.$$

Denote by $\mathcal{H}$ the totality of continuous functions $z(h)$, $h \geq 0$, satisfying $|z(h)| \leq h$. Now introduce a system W_μ, with scalar inputs and outputs and with state space $\mathcal{H}$, as follows. For a given smooth input $u(t)$, $t \geq 0$, and an initial state $z_0 \in \mathcal{H}$, the corresponding output $z(t) = (W_\mu[z_0]u)(t)$, $t \geq 0$, is defined as

$$z(t) = \int_0^\infty (U[z_0(h)]u)(t)\, d\mu(h).$$

A model of this type includes fundamental mechanical models such as the Ishlinskii and Besseling systems ([6], p. 342–346). It might be thought of as describing a continuum of linked transducers.

Suppose that the function G is globally Lipschitz, as in previous section. Consider the system described by equations

$$x' = G(x, z), \qquad z(t) = (W_\mu[z_0](c, x))(t). \tag{14}$$

This extends the system (12). Again, (14) has a unique solution $x(t)$, $t \geq 0$ for each initial condition $x(0) = x_0$. Define the corresponding shift operator $S_\mu(x_0)$. From Theorem 1 it follows that

Theorem 3 *There exist $\alpha > 0$ and $\varepsilon_0 > 0$ with the following property: for any trajectory $x(t)$, $0 \leq t < t_* \leq \infty$, of the equation* (13)*, for any measure μ satisfying*

$$r(\mu) = \int_0^\infty h\, d\mu(h) \leq \varepsilon_0$$

and any $z(h) \in \mathcal{H}$, there exists a trajectory $(x_\mu(t), z_\mu(t))$, $0 \leq t < t_*$, of (14) *satisfying*

$$|x(t) - x_\mu(t)| \leq \alpha r(\mu), \quad 0 \leq t < t_*.$$

Analogues of Theorem 3 are valid for models such as the multi-dimensional Ishlinskii system, the Preisach–Giltay model [6] and its multi-dimensional analogue [5].

5 Proof of Theorem 1

Fix some trajectory $\boldsymbol{x} = x_0, x_1, \ldots, x_N$ of the mapping f,

$$x_{n+1} = f(x_n), \qquad n = 0, 1, \ldots, N.$$

Denote by $\mathcal{Z}$ the space of $N-$sequences $\boldsymbol{z} = z_0, z_1, \ldots, z_N$, $z_n \in \Re^d$, satisfying

$$P^s_{x_0} z_0 = P^u_{x_N} z_N = 0. \tag{15}$$

The set $\mathcal{Z}$ can be treated as a subspace of the Nd-dimensional vector space $\Re^d \times \ldots \times \Re^d$ (N times), with the norm

$$\|\boldsymbol{z}\| = \max_{0 \le n \le N} \|z_n\|.$$

Let W be a given continuous system (2) and let $\omega_0 \in \Omega$ be some parameter value. Introduce for the given W and $\boldsymbol{x}$ an operator $H : \mathcal{Z} \to \mathcal{Z}$, which transforms every sequence $\boldsymbol{z} \in \mathcal{Z}$ into a sequence $\boldsymbol{w} = w_0, w_1, \ldots, w_N$ defined by the boundary conditions (15) and the relations

$$\begin{aligned} P^s_{x_n} w_n &= P^s_{x_n}(\varphi(x_{n-1} + z_{n-1}, \omega_{n-1}) - x_n), && (16)\\ P^u_{x_{n-1}} w_{n-1} &= Q_n^{-1} P^u_{x_n}(z_n - Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1}) + \\ & \quad Q_n^{-1} P^u_{x_n}(-\varphi(x_{n-1} + z_{n-1}, \omega_{n-1}) + x_n) + \\ & \quad Q_n^{-1} P^u_{x_n} Df_{x_{n-1}} z_{n-1}, && (17) \end{aligned}$$

where $Q_n : E^u_{x_{n-1}} \to E^u_{x_n}$, defined by $Q_n v = P^u_{x_n} Df_{x_{n-1}} v$, is surjective and

$$\omega_n = \psi(x_{n-1} + z_{n-1}, \omega_{n-1}), \quad n = 1, 2, \ldots N; \qquad \omega_0 = \omega_0. \tag{18}$$

Note that Q_n^{-1} is well-defined by virtue of the inequality (8).

Lemma 1 *Operator H is continuous. For any fixed point $\boldsymbol{z} = z_0, z_1, \ldots, z_N$ of H, the sequence*

$$\boldsymbol{q} = (x_0 + z_0, \omega_0),\ (x_1 + z_1, \omega_1),\ \ldots,\ (x_N + z_N, \omega_N),$$

where

$$\omega_n = \psi(x_{n-1} + z_{n-1}, \omega_{n-1}), \quad n = 0, 1, \ldots, N, \tag{19}$$

is a trajectory of the system W.

Proof. Continuity of H follows straightforwardly from the continuity of φ and ψ, smoothness of f and relations (14), (21), (22) and (23). Hence, it is sufficient to establish that

$$x_n + z_n = \varphi(x_{n-1} + z_{n-1}, \omega_{n-1}). \tag{20}$$

Because $\boldsymbol{z}$ is a fixed point of H, equations (16) and (17) can be rewritten as

$$\begin{aligned} P^s_{x_n} z_n &= P^s_{x_n}(\varphi(x_{n-1}+z_{n-1},\omega_{n-1})-x_n), \qquad (21)\\ P^u_{x_{n-1}} z_{n-1} &= Q_n^{-1}P^u_{x_n}(z_n - Df_{x_{n-1}}P^s_{x_{n-1}}z_{n-1}) + \\ &\quad Q_n^{-1}P^u_{x_n}(-\varphi(x_{n-1}+z_{n-1},\omega_{n-1})+x_n) + \\ &\quad Q_n^{-1}P^u_{x_n}Df_{x_{n-1}}z_{n-1}, \qquad (22) \end{aligned}$$

where ω_n, $n=1,2,\dots N$, are satisfying to (19). From (21) it follows that

$$P^s_{x_n}(x_n+z_n) = P^s_{x_n}\varphi(x_{n-1}+z_{n-1},\omega_{n-1}). \qquad (23)$$

Rewrite (22) as

$$\begin{aligned} P^u_{x_{n-1}} z_{n-1} &= Q_n^{-1}P^u_{x_n}(z_n - \varphi(x_{n-1}+z_{n-1},\omega_{n-1})+x_n) + \\ &\quad Q_n^{-1}P^u_{x_n}Df_{x_{n-1}}P^u_{x_{n-1}}z_{n-1}, \qquad (24) \end{aligned}$$

or, what is equivalent,

$$0 = Q_n^{-1}P^u_{x_n}(z_n - \varphi(x_{n-1}+z_{n-1},\omega_{n-1})+x_n).$$

That is,

$$P^u_{x_n}(x_n+z_n) = P^u_{x_n}\varphi(x_{n-1}+z_{n-1},\omega_{n-1}). \qquad (25)$$

From (23) and (25) it follows (20). Lemma 1 is proved. ∎

We require a few more notations to continue the proof of Theorem 1. For the given trajectory $\boldsymbol{x}\in\mathcal{T}(f,\mathcal{M})$ and each $\boldsymbol{z}\in\mathcal{Z}$, define the pair of real nonnegative numbers

$$m^s(\boldsymbol{z}) = \max_{0\le n\le N}\|P^s_{x_n}z_n\|, \qquad m^u(\boldsymbol{z}) = \max_{0\le n\le N}\|P^u_{x_n}z_n\|,$$

and denote by $\boldsymbol{m}(\boldsymbol{z})$ the two-dimensional column vector with coordinates $m^s(\boldsymbol{z})$, $m^u(\boldsymbol{z})$. Define the matrix

$$M = \begin{pmatrix} \lambda_s & \mu_s \\ \mu_u/\lambda_u & 1/\lambda_u \end{pmatrix}, \qquad (26)$$

and the column vector

$$\boldsymbol{k} = (k, k/\lambda_u)^T. \qquad (27)$$

Given $\varepsilon > 0$, denote by $\delta(\varepsilon)$ the largest positive value δ such that, for any $x \in \mathcal{M}$ and any $\|z\| \le \delta$, the following inequality holds:

$$\|f(x) + Df_x z - f(x+z)\| \le \varepsilon, \quad x + z \in \mathcal{X}.$$

Given $\varepsilon > 0$, introduce the set

$$\mathcal{W}(\varepsilon) = \{\boldsymbol{z} \in \mathcal{Z} : \|\boldsymbol{z}\| \le \delta(\varepsilon)\}, \tag{28}$$

Lemma 2 *Let $\beta > 0$. Then for each trajectory $\boldsymbol{x} \in \mathcal{T}(f, \mathcal{M})$, each continuous system W and each $\boldsymbol{z}$ from the set $\mathcal{W}(\beta\rho(W, f))$ the following inequality holds:*

$$\boldsymbol{m}(H(\boldsymbol{z})) \le M\,\boldsymbol{m}(\boldsymbol{z}) + (1+\beta)\rho(W, f)\boldsymbol{k}. \tag{29}$$

Proof. First, estimate the value of $m^s(H(\boldsymbol{z}))$. By definition

$$m^s(H(\boldsymbol{z})) = \max_{0 \le n \le N} \|v_n^s\|, \tag{30}$$

where

$$v_n^s = P_{x_n}^s(\varphi(x_{n-1} + z_{n-1}, \omega_{n-1}) - x_n). \tag{31}$$

In (31) and below ω_n is defined by (18). Rewrite (31) as

$$v_n^s = I_1 + I_2 + I_3 + I_4, \tag{32}$$

where

$$\begin{aligned}
I_1 &= P_{x_n}^s Df_{x_{n-1}} P_{x_{n-1}}^s z_{n-1}, \\
I_2 &= P_{x_n}^s Df_{x_{n-1}} P_{x_{n-1}}^u z_{n-1}, \\
I_3 &= P_{x_n}^s(\varphi(x_{n-1} + z_{n-1}, \omega_{n-1}) - f(x_{n-1} + z_{n-1})), \\
I_4 &= P_{x_n}^s(f(x_{n-1} + z_{n-1}) - (f(x_{n-1}) + Df_{x_{n-1}} z_{n-1})).
\end{aligned}$$

From (6),

$$\|I_1\| \le \lambda_s \|P_{x_{n-1}}^s z_{n-1}\|,$$

and from (7),

$$\|I_2\| \le \mu_s\,\|P_{x_{n-1}}^u z_{n-1}\|.$$

The relations (10) imply that

$$\|I_3\| \le k\,\rho(W, f).$$

Lastly, the relations (10) and the definition of $\delta(\beta||f-\varphi||_\infty)$ imply that

$$\|I_4\| \le k\beta\, \rho(W,f).$$

From (32) and from obtained estimates of the norms $\|I_1\|$, $\|I_2\|$, $\|I_3\|$ and $\|I_4\|$ it follows that

$$\|v_n^s\| \le \lambda_s \, \|P^s_{x_{n-1}} z_{n-1}\| + \mu_s \, \|P^u_{x_{n-1}} z_{n-1}\| + (1+\beta)\,\rho(W,f)k. \tag{33}$$

By (30), we can rewrite (33) as

$$m^s(H(\boldsymbol{z})) \le \lambda_s \, m^s(\boldsymbol{z}) + \mu_s \, m^u(\boldsymbol{z}) + (1+\beta)\,\rho(W,f)\,\gamma. \tag{34}$$

Now estimate the value of $m^u(H(\boldsymbol{z}))$. By definition,

$$m^u(H(\boldsymbol{z})) = \max_{0\le n\le N} \|v_n^u\|, \tag{35}$$

where

$$\begin{aligned} v^u_{n-1} \quad &= \quad Q_n^{-1} P^u_{x_n}(z_n - Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1}) + \\ &\qquad Q_n^{-1} P^u_{x_n}(-\varphi(x_{n-1}+z_{n-1},\omega_{n-1}) + f(x_{n-1}) + Df_{x_{n-1}} z_{n-1}). \end{aligned}$$

Rewrite this last equation as

$$v^u_{n-1} = Q_n^{-1} J_1 + Q_n^{-1} J_1 + Q_n^{-1} J_2 + Q_n^{-1} J_3 + Q_n^{-1} J_4, \tag{36}$$

with

$$\begin{aligned} J_1 &= P^u_{x_n} z_n, \\ J_2 &= -P^u_{x_n} Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1}, \\ J_3 &= P^u_{x_n}(-\varphi(x_{n-1}+z_{n-1},\omega_{n-1}) + f(x_{n-1}+z_{n-1})), \\ J_4 &= P^u_{x_n}(-f(x_{n-1}+z_{n-1}) + f(x_{n-1}) + Df_{x_{n-1}} z_{n-1}). \end{aligned}$$

The relations (8) imply that

$$\|Q_n^{-1} J_1\| \le \lambda_u^{-1} \, \|P^u_{x_n} z_n\|,$$

while the relations (8) and (9) imply that

$$\|Q_n^{-1} J_2\| \le \lambda_u^{-1} \mu_u \, \|P^s_{x_{n-1}} z_{n-1}\|.$$

The relations (8) and (10) give

$$\|Q_n^{-1} J_3\| \leq \lambda_u^{-1} k \, \rho(W, f).$$

Finally, the relations (8) and (10) and the definition of $\delta(\cdot)$ imply that

$$\|Q_n^{-1} J_4\| \leq \lambda_u^{-1} h\beta \, \rho(W, f).$$

From (36) and from obtained estimates of the norms $\|Q_n^{-1} J_1\|$, $\|Q_n^{-1} J_2\|$, $\|Q_n^{-1} J_3\|$ and $\|Q_n^{-1} J_4\|$ it follows that

$$\|v_{n-1}^u\| \leq \lambda_u^{-1} \left(\|P_{x_n}^u z_n\| + \mu_u \, \|P_{x_{n-1}}^s z_{n-1}\| + (1+\beta)\, \rho(W, f)\, k\right). \tag{37}$$

By (35) we can rewrite (37) as

$$m^u(H(\boldsymbol{z})) \leq \lambda_u^{-1} \left(m^u(\boldsymbol{z}) + \mu_u \, m^s(\boldsymbol{z}) + (1+\beta)\, \rho(W, f)\, k\right). \tag{38}$$

Inequalities (34) and (38) are equivalent to the assertion of Lemma 2. ∎

Let us return to and finish the proof of Theorem 1. Choose a real number $\alpha > \alpha_*(\boldsymbol{s}, k)$, where $\alpha_*(\boldsymbol{s}, k)$ is defined by (11). It is now sufficient to prove that

Lemma 3 *Let $\alpha > \alpha_*(\boldsymbol{s}, k)$. Then there exists $\varepsilon > 0$ such that for any given trajectory $\boldsymbol{x} = x_0, x_1, \ldots, x_N \in \mathcal{T}(f, \mathcal{M})$, any continuous system W satisfying*

$$\rho(W, f) < \varepsilon, \tag{39}$$

and any $\omega \in \Omega$, there exists a trajectory

$$\boldsymbol{q} = (y_0, \omega_0), (y_1, \omega_1) \ldots, (y_N, \omega_N)$$

from $\mathcal{T}(W, \omega)$ such that

$$\|y_n - x_n\| \leq \alpha \rho(W, f), \qquad n = 0, 1, \ldots N. \tag{40}$$

Proof. The spectral radius $\sigma(M)$ of the matrix (26) is just

$$\sigma(M) = \frac{1}{2\mu_s} \left(\left(\frac{1}{\lambda_u} + \lambda_s \right) + \sqrt{\left(\frac{1}{\lambda_u} - \lambda_s \right)^2 + \frac{4\mu_s \mu_u}{\lambda_u}} \right).$$

The entries of the matrix M are positive. Therefore by the Perron-Frobenius theorem the spectral radius $\sigma(M)$ is the maximal eigenvalue and the corresponding eigenvector has positive coordinates. Without loss of generality, assume that this eigenvector takes the form $(1,\gamma)^T$, where

$$\gamma = \frac{1}{2\mu_s}\left(\left(\frac{1}{\lambda_u} - \lambda_s\right) + \sqrt{\left(\frac{1}{\lambda_u} - \lambda_s\right)^2 + \frac{4\mu_s\mu_u}{\lambda_u}}\right).$$

It follows that

$$\begin{pmatrix} \lambda_s & \mu_s \\ \dfrac{\mu_u}{\lambda_u} & \dfrac{1}{\lambda_u} \end{pmatrix} \begin{pmatrix} 1 \\ \gamma \end{pmatrix} = \sigma(M) \begin{pmatrix} 1 \\ \gamma \end{pmatrix}.$$

In $\Re^2$ introduce the auxiliary norm $\|\cdot\|_*$ by

$$\|(y_1,y_2)^T\|_* = \max\{\gamma|y_1|, |y_2|\}.$$

Clearly, the corresponding norm $\|M\|_*$ of the linear operator with the matrix (26) coincides with the spectral radius of M, $||M\boldsymbol{u}||_* \le \sigma(M)||\boldsymbol{u}||_*$ for all $\boldsymbol{u} \in \Re^2$. By (4) and (5)

$$\sigma(M) = \|M\|_* < 1. \tag{41}$$

Let ε be any positive number such that the set

$$\mathcal{P}_\varepsilon = \left\{\boldsymbol{z} :\ \|\boldsymbol{m}(\boldsymbol{z})\|_* \le \frac{1+\beta}{1-\sigma(M)}\,\varepsilon\,\|\boldsymbol{k}\|_*\right\}$$

satisfies the inclusion $\mathcal{P}_\varepsilon \subseteq \mathcal{W}(\beta\varepsilon)$, where $\mathcal{W}(\cdot)$ is defined by (28), and

$$\beta = \frac{\alpha}{\alpha_*(\boldsymbol{s},k)} - 1. \tag{42}$$

Clearly such an $\varepsilon > 0$ exists.

Now, for the given trajectory $\boldsymbol{x} \in \mathcal{T}(f,\mathcal{M})$, the system W satisfying (39) and $\omega \in \Omega$, it remains to construct a trajectory $\boldsymbol{q} \in \mathcal{T}(W,\omega)$ satisfying (40). Consider the set

$$\mathcal{P} = \left\{\boldsymbol{z} :\ \|\boldsymbol{m}(\boldsymbol{z})\|_* \le \frac{1+\beta}{1-\sigma(M)}\,\rho(W,f)\,\|\boldsymbol{k}\|_*\right\}.$$

Clearly $\mathcal{P} \subseteq \mathcal{P}_\varepsilon \subseteq \mathcal{W}(\beta\varepsilon)$. Further,

$$\mathcal{P} \subseteq \mathcal{W}(\beta\rho(W,f)), \tag{43}$$

where $\mathcal{W}(\cdot)$ is defined by (28). By (43) and Lemma 2,

$$\|\boldsymbol{m}(H(\boldsymbol{z}))\|_* \le \sigma(M)\,\|\boldsymbol{m}(\boldsymbol{z})\|_* + (1+\beta)\,\rho(W,f)\,\|\boldsymbol{k}\|_*, \boldsymbol{z} \in \mathcal{P}.$$

That is, $\mathcal{P}$ *is invariant for the operator* H. The set $\mathcal{P}$ is clearly convex and closed, and is nonempty by (41). Then, because of the continuity of H proved in Lemma 1, there exists a point $\boldsymbol{z}$ satisfying $H(\boldsymbol{z}) = \boldsymbol{z}$ such that

$$\boldsymbol{z} \in \mathcal{W}(\beta\rho(W,f)). \tag{44}$$

From (44) and (29) it follows that

$$\boldsymbol{m}(H(\boldsymbol{z})) \le M\,\boldsymbol{m}(\boldsymbol{z}) + (1+\beta)\,\rho(W,f)\,\boldsymbol{k},$$

and, moreover, that

$$\boldsymbol{m}(\boldsymbol{z}) \le \Big((1+\beta)\,(1-M)^{-1}\,\boldsymbol{k}\Big)\;\rho(W,f). \tag{45}$$

Obviously,

$$(1-M)^{-1} = \frac{\lambda_u}{(1-\lambda_s)(\lambda_u-1)} \begin{pmatrix} 1-\frac{1}{\lambda_u} & -\frac{\mu_u}{\lambda_u} \\ -\mu_s & 1-\lambda_s \end{pmatrix}. \tag{46}$$

From (45), (46), (42) and (27) it follows that $m^s(\boldsymbol{z}) + m^u(\boldsymbol{z}) \le \alpha\,\rho(W,f)$. Furthermore,

$$\max_{0\le n\le N} \|\boldsymbol{z}_n\| \le \alpha\,\rho(W,f). \tag{47}$$

Set $y_n = x_n + z_n$ and put

$$\omega_n = \psi(x_{n-1} + z_{n-1}, \omega_{n-1}), \quad n = 0,1,\ldots,N.$$

By (47) and Lemma 1, the sequence

$$\boldsymbol{q} = (y_0,\omega_0),\ (y_1,\omega_1),\ \ldots,\ (y_N,\omega_N)$$

is a trajectory of W and satisfies (40). That is, Lemma 3 and Theorem 1 are proved. ∎

References

[1] M.-J. Beyn, On the numerical approximation of phase portraits near stationary points, *SIAM J. Numer. Anal.*, **24**, No 5, 1987, pp. 1095–1113.

[2] P. Diamond, P. Kloeden, V. Kozyakin and A. Pokrovskii, Computer Robustness of Semi-hyperbolic Mappings *submitted for publication*.

[3] I. Fomenko, M. Krasnosel'skii and A. Pokrovskii, *Self Induced Oscillations in Systems with Hysteresis. Soviet. Phys. Dokl.*, **34**, No 12, pp. 1066 – 1068.

[4] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields*, Springer-Verlag, New-York, 1983.

[5] M. Krasnosel'skij, I. Mejergojz, A. Pokrovskii and D. Rachinskiji, State Spaces for Vector Continual Systems of Relays. *Soviet Phys. Dokl.*, 1992.

[6] M. Krasnosel'skii and A. Pokrovskii, "*Systems with Hysteresis*" Nauka, Moscow 1983; Springer-Verlag, Berlin, 1992.

[7] M. Krasnosel'skii and A. Pokrovskii, The Averaging Principle for Hysteresis In: "*Asymptotic Methods in Mathematical Physic*", "Naukova Dumka", Kiev, pp. 126 – 133, 301. (Russian).

[8] Palis S., Melo W. de, Geometric Theory of Dynamical Systems: an Introduction, Springer-Verlag, New York, 1982.

[9] M. Shub, *Global Analysis of Dynamical Systems*, Springer-Verlag, New York, 1987.