# Existence and Stability of a Unique Equilibrium in Continuous-Valued Discrete-Time Asynchronous Hopfield Neural Networks*

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**Abstract**

This paper investigates a continuous-valued discrete-time analog of the well-known continuous-valued continuous-time Hopfield neural network model, first proposed by Takeda and Goodman. It is shown that the assumption of D-stability of the interconnection matrix, together with the standard assumptions on the activation functions, guarantee a unique equilibrium under a synchronous mode of operation as well as a class of asynchronous modes. For the synchronous mode, these assumptions are also shown to imply local asymptotic stability of the equilibrium. For the asynchronous mode of operation, two results are derived. First, using results of Kleptsyn and coworkers, it is shown that symmetry and stability of the interconnection matrix guarantee local stability of the equilibrium under a class of asynchronous modes – this is referred to as local absolute stability. Second, using results of Bhaya and coworkers, it is shown that, under the standard assumptions, if the nonnegative matrix whose elements are the absolute values of the corresponding elements of the interconnection matrix is stable, then the equilibrium is globally absolutely asymptotically stable under a class of asynchronous modes. The results obtained are discussed both from the point of view of their robustness as well as their relationship to earlier results.

## 1 Introduction

Takeda and Goodman [1] introduced two different continuous-valued discrete-time models of the Hopfield neural network [2, 3]. In their paper, they also introduced the concept of

*This research was partially supported by grants from CNPq and RHAE/CNPq of the Brazilian National Council for Scientific and Technological Research, and by grant no. 93012884 of the Russian Foundation for Fundamental Research and represents the results of research conducted under the auspices of an exchange program between COPPE/UFRJ, Rio de Janeiro and IPPI, Moscow.

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asynchronous transition modes in which “one particular neuron $i$ need not wait for the last neuron $n$ for synchronization, and when it decides its new state, it can make use of information about new states of other neurons that have already renewed their states.” The motivation cited by Takeda and Goodman for the introduction of asynchronism was that their digital computer simulations showed that “asynchronous transition modes greatly reduced oscillatory or wandering behavior.” From the point of view of modeling, since no evidence has been found for the existence of a central synchronizing clock in biological neural nets (Sejnowski [4, p.383]), it is also of interest to consider asynchronism in neural net models.

In the case of discrete-valued networks (i.e. when the state of each neuron is two-valued and the neuron activation functions are signum functions), the convergence behavior of Hopfield neural networks has been studied in various papers such as Gotsman et al.[5], Bruck and Goodman [6], Amari [7], Bruck [8] and Shrivastava et al.[9]. For continuous-valued networks, the brain-state-in-a-box model with saturation type activation functions has been much studied: Hui and Żak [10] is a recent example and cites other relevant papers.

In this paper, the direct synchronous and asynchronous transition mode models of Takeda and Goodman are analysed rigorously from the point of view of existence, uniqueness and stability of equilibria. It is shown that the Takeda-Goodman synchronous models may be written as follows :

$$x(k+1) = TF(x(k)) + (I - B)x(k) + u, \tag{1}$$

where $T$ is called the interconnection matrix of the neural network (usually assumed to be symmetric), $F$ is a diagonal nonlinear function (usually assumed to be monotonic, often sigmoidal, i.e having finite limits at $\pm\infty$), $B$ is a diagonal matrix (usually zero or the identity), and $u$ is a vector of inputs, assumed to be constant. In this paper, it will be assumed that (i) $T$ belongs to a class of matrices known as D-stable matrices, which includes, but is not restricted to, the class of symmetric stable matrices; (ii)$F$, in addition to being diagonal and sigmoidal, is continuously differentiable, with slope-limited component functions; and (iii) $B = I$, so that only the so-called direct transition mode will be considered.

Under these assumptions, it is shown that equation (1) admits a unique locally asymptotically stable equilibrium, the stability result being derived by linearization. When the hypothesis on the interconnection matrix is strengthened to diagonal stability, global asymptotic stability of the unique equilibrium is shown using a diagonal quadratic Liapunov function.

This stability result is then extended to the asynchronous or desynchronized case, by the introduction of the concept of stability under the class of all desynchronizations that satisfy a mild regularity assumption – here such a type of stability, following to Russian literature [11, 12, 13, 14, 15], is referred to as absolute stability, thou in another terms and even in ”nameless” form analogous notions were used, e.g., in [16, 17]. It is observed that, since an asymptotically stable equilibrium under the class of regular desynchronizations (defined precisely below) must continue to be an asymptotically stable equilibrium under a synchronous mode of operation, therefore regular desynchronization does not introduce new absolutely asymptotically stable equilibria – it remains to determine the stability type of the original (synchronous) equilibrium under regular desynchronization. A linearization principle for asynchronous systems due to Kozyakin [14] (also see [15, Chap.6]) is invoked to prove that, for the neural network studied in this paper, the (unique) equilibrium is also locally absolutely stable under the same condition as in the synchronous case, namely that a certain matrix (derived from the interconnection matrix) be stable (i.e. with spectral radius less than unity). Finally, a different approach due to Bhaya and coworkers [18] based on Liapunov function and majorization techniques, is used to derive a condition that ensures global asymptotic stability under partial asynchronism. A similar condition was derived in Tseng et al. [19], using different methods and slightly different hypotheses.

For certain applications, such as optimization [20], neural networks that admit a unique equilibrium are of interest. The results of this paper provide a guideline for the design of such networks: thier interconnection matrices should belong to certain classes of stable matrices, depending on the mode of operation and the type of stability desired.

The robustness of the results is discussed using some recent results [14, 21]. Earlier results in the literature [19, 22] are also discussed in the context of the results of this paper.

For the class of networks investigated, the contributions of this paper may be summarized as follows: (i) conditions are found under which the neural network admits a unique equilibrium under synchronous and asynchronous modes of operation; (ii) even if the updating instants (firing times) of the neurons are asynchronously determined, the equilibrium continues to maintain some stability properties – either local or global, depending on the strength of the condition imposed on the interconnection matrix; and (iii) it is possible to quantify a bound on desymmetrizing perturbations (on the nominal symmetric interconnection matrix) that preserve local absolute stability. These issues are important from the point of view of: (a) modelling, since no evidence has been found for the existence of a central synchronizing clock in biological neural nets, so that it is reasonable to assume that they operate asynchronously, and (b) practical implementation of artificial neural nets, since perfectly symmetric interconnection matrices are impossible to realize.

### 1.1 Neural network models

The direct and differential synchronous transition mode models introduced by Takeda and Goodman [1] are given below, following a brief motivational introduction.

Each neuron $i$ receives inputs $t_{ij}y_j$ from the other neurons $j$ and a bias input $u_i$

associated with itself. Thus the net input $x_i(k)$ to neuron $i$ at instant $k$ is

$$x_i(k) = \sum_{j=1}^{n} t_{ij} y_j(k) + u_i, \qquad i = 1, \ldots, n, \tag{2}$$

where $n$ is the number of neurons in the network and the $t_{ij}$s are the elements of an interconnection matrix representing the strengths of connections between neurons. At discrete times, switches turn on and the inputs $x_i$ are fed back to the corresponding neurons which then determine whether or not to change their states according to a threshold rule determined by nonlinear functions $f_i$ as follows:

$$y_i(k+1) = f_i(x_i(k)). \tag{3}$$

The outputs $y_i$ are distributed through the interconnection network to regenerate new inputs. Equations (2) and (3) define the *direct synchronous transition mode* model of Takeda and Goodman.

In the continuous-time model, neurons change their states according to the following equations:

$$\begin{aligned} \frac{dx_i}{dt} &= \sum_{i=1}^{n} t_{ij} y_j + u_i, \\ y_i &= g(x_i), \end{aligned}$$

where $t$ is continuous time and $g(x)$ is a continuous monotonic nonlinear function having finite limits as $x \to \pm\infty$.

In the differential synchronous transition mode, the differential equations above are approximated by difference equations and transitions are assumed to occur synchronously (i.e. all at the same instant). In this case, the equations that describe the model are:

$$x_i(k) - x_i(k-1) = \sum_{j=1}^{n} t_{ij} y_j(k) + u_i, \tag{4}$$

$$y_i(k+1) = f(x_i(k)) \tag{5}$$

There are two approaches to the analysis of (2) and (3). In the first one, the variable of interest is taken to be $y$ and the two equations can be combined into the following equation.

$$y_i(k+1) = f\left(\sum_{j=1}^{n} t_{ij} y_j(k) + u_i\right), \quad i = 1, \ldots, n. \tag{6}$$

This is the approach taken by Tseng et al. [19], who also consider asynchronous neural nets, about which more will be said later.

The second approach, followed in this paper, is to consider $x$ as the variable of interest, in which case (2) and (3) can be combined to yield:

$$x_i(k+1) = \sum_{j=1}^{n} t_{ij} f(x_j(k)) + u_i, \quad i = 1, \ldots, n. \tag{7}$$

Although (6) and (7) are equivalent representations of the neural network, the choice of representation affects the assumptions needed for the analysis as well as the simplicity of the latter. An additional point of interest is that the equations (4) and (5) can also be easily combined to give a single equation in the variable $x$, whereas the additional assumption of invertibility of the function $f$ is required to combine them into a single equation in the variable $y$.

The above discussion is summarized in the following general model, written for brevity in vector notation.

$$x(k+1) = (I - B)x(k) + TF(x) + u, \tag{8}$$

where, $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, $T = (t_{ij}) \in \mathbb{R}^{n \times n}$, $u = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$, $B = \mathrm{diag}(b_1, \ldots, b_n) \in \mathbb{R}^{n \times n}$, $F(x) = (f_1(x_1), \ldots, f_n(x_n))^T$.

Note that when $B = I$ and $f_i = f$ for all $i$, equation (8) reduces to equation (7). Similarly, when $B = 0$ and $f_i = f$ for all $i$, equation (8) reduces to the combined version of (4) and (5) in the variable $x$. This paper will concentrate on the case $B = I$, but it will not be assumed that $f_i = f$ for all $i$.

# 2 Synchronous neural networks

Some definitions from matrix theory that will be needed immediately are given below.

DEFINITION 2.1 *An $n \times n$ real matrix $A$ is defined to be* stable *if all its eigenvalues are less than unity in absolute value (i.e. the spectral radius, $\rho(A)$, is less than unity).*

DEFINITION 2.2 *An $n \times n$ real stable matrix $A$ is said to be* diagonally stable *if and only if there exists a positive diagonal matrix $P$ such that $(A^T PA - P)$ is a negative definite matrix. In other words, the class of diagonally stable matrices is a class for which the equation $A^T PA - P = -Q$ (for some $Q > 0$) admits a positive diagonal solution; this class is also denoted by the letter $\mathcal{D}$.*

DEFINITION 2.3 [23] *A $n \times n$ real stable matrix $A$ is said to be* D-stable *if and only if $AD$ is stable for any diagonal matrix $D$ that has diagonal elements in the interval* $[-1, 1]$. *This class is also denoted by the letter* $\mathbb{D}$.

REMARK 2.4 Note that the set of diagonally stable matrices is known to be a strict subset of the set of D-stable matrices, i.e. $\mathcal{D} \subset \mathbb{D}$, [23]. Many important classes of stable matrices are known to be diagonally stable and hence D-stable as well: for example, symmetric, symmetrizable, triangular, M-, Z-, quasi diagonally dominant etc. [23]. In particular, the case of symmetric matrices is interesting, since interconnection matrices are often assumed to be symmetric and additional results on asynchronism are available.

In this section the direct synchronous transition mode model introduced above in Section 1 will be studied. Rewriting equations (2) and (3) in vector form, this model is expressed as:

$$x(k+1) = TF(x(k)) + u. \tag{9}$$

The following assumptions will be made.

($\mathcal{T}$) *the interconnection matrix $T$ is D-stable.*

($\mathcal{U}$) *the input vector $u$ is a constant.*

($\mathcal{F}$) *$F$ is a slope-limited diagonal map of class $C^1$, i.e.*
$F : \mathbb{R}^n \to \mathbb{R}^n : (x_1, \ldots, x_n)^T \mapsto (f_1(x_1), \ldots, f_n(x_n))^T$*; where*

- *(i) For all $i$, the function $f_i(x_i)$ is continuously differentiable in $x_i$ and tends to finite limits as $x_i$ tends to $\pm\infty$; and,*
- *(ii) for all $i$, for all $x_i \in \mathbb{R}$, $0 < \frac{df_i(x_i)}{dx_i} \leq 1$.*

REMARK 2.5 Assumptions ($\mathcal{F}$) and ($\mathcal{U}$) are standard in the neural network literature, while assumption ($\mathcal{T}$) is generally replaced by the stronger assumption that $T$ is stable and symmetric.

The following lemma is reproduced from Ortega and Rheinboldt [24, Result 5.3.9,p.137] in the interests of making the paper self-contained.

LEMMA 2.6 *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable on all of $\mathbb{R}^n$, and that $F'(x)$ is nonsingular for all $x$ in $\mathbb{R}^n$. Then $F$ is a homeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$ if and only if $\lim_{\|x\|\to\infty} \|F(x)\| = \infty$.* ■

The following theorem is now obtained.

THEOREM 2.7 *Under the assumptions ($\mathcal{T}$), ($\mathcal{F}$) and ($\mathcal{U}$) above:*

- *(a) the neural network represented by (9) admits an equilibrium ($x(k) = x^e$, for all $k$) that is uniquely determined by the constant $u \in \mathbb{R}^n$ and depends continuously on $u$.*

- *(b) Furthermore, this unique equilibrium is locally asymptotically stable.*

*Proof.* The equilibrium or fixed-point equation corresponding to (9) is:

$$x = TF(x) + u. \tag{10}$$

Let $\phi(x)$ be defined as $x - TF(x)$. By assumption $(\mathcal{F})$(i), there exists a constant $\alpha$ such that

$$\|F(x)\| \leq \alpha, \qquad x \in \mathbb{R}^n.$$

Hence

$$\|\phi(x)\| \geq \|x\| - \alpha\|T\|, \qquad x \in \mathbb{R}^n$$

from which it follows that $\lim_{\|x\|\to\infty} \|\phi(x)\| = \infty$. Therefore, by Lemma 2.6, $\phi$ is a homeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$, which complete the proof of item (a) of the theorem.

To prove item (b), let the equilibrium solution be denoted $x^e$. Linearizing equation (9) about $x^e$ gives:

$$x(k+1) = TF'(x^e)x(k), \tag{11}$$

where $F'(x^e) = \text{diag}\left(f_1'(x_1^e), \ldots, f_n'(x_n^e)\right)$ and by assumption $(\mathcal{F})$(ii), $0 < f_i'(x_i^e) \leq 1$ for all $i$, so that, by assumption $(\mathcal{T})$, (11) describes an asymptotically stable linear system, as required to complete the proof of item (b). ■

The following theorem (a simplified version of a theorem in [25]) is needed for an extension of Theorem 2.7.

Theorem 2.8 *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, $\Phi : \mathbb{R}^n \to \mathbb{R}^n : (x_1, \ldots, x_n)^T \mapsto (\phi_1(x_1), \ldots, \phi_n(x_n))^T$ where, for all $i$, the functions $\phi_i$ satisfy $|\phi_i(x_i)| \leq |x_i|$. Then, for all functions $\Phi$ satisfying these conditions, the zero solution of the difference equation*

$$x(k+1) = A\Phi(x(k)) \tag{12}$$

*is globally asymptotically stable if the matrix $A$ is diagonally stable.*

*Proof.* Since $A$ is diagonally stable, there exists a positive diagonal matrix $P$ such that $A^TPA - P$ is negative definite. It can be shown that $V(x) = x^TPx$ is a global quadratic Liapunov function [25]. ■

With the help of this theorem and under a strengthening of assumption $(\mathcal{T})$ on the interconnection matrix $T$, the following global stability result can be obtained:

Theorem 2.9 *Let assumptions $(\mathcal{F})$ and $(\mathcal{U})$ hold and let the interconnection matrix $T$ be diagonally stable. Under these assumptions, the neural network represented by (9) admits an equilibrium ($x(k) = x^e$, for all $k$) that is uniquely determined by the constant $u \in \mathbb{R}^n$, depends continuously on $u$ and is globally asymptotically stable.*

*Proof.* The assertion on uniqueness and continuous dependence follows directly from Theorem 2.7, since $T$ diagonally stable implies that $T$ is D-stable. It remains to show that the dynamical equation (9) can be written in the form (12) and that hypotheses of Theorem 2.8 are satisfied. The equilibrium equation is

$$x^e = TF(x^e) + u.$$

Subtracting this equation from (9) leads to

$$x(k+1) - x^e = T(F(x(k)) - F(x^e)).$$

Defining $y(k)$ as $x(k+1) - x^e$ leads to

$$y(k+1) = T(F(y(k) + x^e) - F(x^e)).$$

Finally, defining $\Phi(y(k))$ as $F(y(k) + x^e) - F(x^e)$ gives

$$y(k+1) = T\Phi(y(k)). \tag{13}$$

Clearly $\Phi(\cdot)$ satisfies assumption $(\mathcal{F})$ and it is clear from the mean value theorem of calculus that its components $\phi_i$ satisfy $|\phi_i(y_i(k))| \leq |y_i(k)|$. It follows from Theorem 2.8 that the zero solution of (13) is globally asymptotically stable; equivalently, $x^e(k)$ is the globally asymptotically stable equilibrium of (9). ■

**Some remarks on this theorem**

Remark 2.10 For applications in which a unique equilibrium is of interest, such as optimization, the above theorem provides a design guideline as to the choice of interconnection matrix.

Remark 2.11 An interesting feature of the above theorem is that the global asymptotic stability is *robust* to perturbations in the interconnection matrix (since the set of diagonally stable matrices is known to be an open set) as well as to perturbations in the interconnection functions (since all that is required of the latter is that they belong to the so-called $[0, 1]$ sector, as is clear in the statement of Theorem 2.8).

Remark 2.12 If assumption $(\mathcal{T})$ is strengthened to $T$ symmetric and stable, then assumption $(\mathcal{F})$ can be weakened somewhat, but only at the price of many more technicalities in the proof.

Remark 2.13 The result of Theorem 2.9 can also be shown to hold for the model $z(k+1) = F(Tz(k))$, under the same assumptions: this is consistent with the discussion in section 1.1 above.

Remark 2.14 The asynchronous model to be introduced in Section 3 below includes the synchronous model as a special case and a global stability result will be derived for it, using a nonquadratic but diagonal Liapunov function, under a further strengthening of assumption $(\mathcal{T})$.

# 3 Asynchronous neural networks

The main objective of this section is to study the behavior of solutions of equation (9), under asynchronous updating laws, first locally by means of the linearization technique [14, 15], and then globally, using a result that depends on a weighted infinity norm type Liapunov function and a majorization technique [18, 26]. In order to do this, the next subsection introduces some preliminaries: terminology and basic results for asynchronous systems.

## 3.1 Preliminaries

Consider a system $W$ consisting of interacting subsystems $W_1, \ldots, W_n$ with the following general properties:

- The state of each subsystem $W_i, i = 1, \ldots, n$ is described by a finite-dimensional vector $x_i$, which is updated only at a discrete set of times $\{T_i^k\}$ and the updating procedure is instantaneous.
- For each subsystem $W_i, i = 1, \ldots, n$, only a finite number of updating instants $\{T_i^k\}$ can occur within any time interval of finite length.
- Each subsystem $W_i$ is updated infinitely often.

For the neural network models in this paper, each subsystem $W_i$ represents an individual neuron, is one-dimensional and composed of a summation node, a unit delayer and a nonlinear activation function.

The notion of a linear asynchronous system, also called a linear desynchronized system in the Russian literature [11, 12, 13, 14, 15] is expressed mathematically as follows. Let the $i^{\text{th}}$ component of the state, $x_i$, which describes subsystem $W_i$, be updated according to the law:

$$x_{i_{new}} = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + u_i \tag{14}$$

where $a_{ij}$ are the elements of the $i^{\text{th}}$ row of a given matrix $A = (a_{ij})$; $x_1, x_2, \ldots, x_n$ are the states of the subsystems of system $W$ at time instants immediately preceding the 'firing' of the component $W_i$; $x_{i_{new}}$ is the new state of the component $W_i$; and $u_i$ is the vector of external inputs of subsystem $W_i$. If all the subsystems change their state simultaneously, the system is called *synchronized* or *synchronous*. In general, the components do not change their states simultaneously, and the system is then called *desynchronized* or *asynchronous*.

Let $\omega$ be the set of indices of the subsystems that undergo updates at some time $T$. Denote by $A_\omega$ the matrix that is obtained from $A$ by replacing the rows $i \notin \omega$ with the corresponding rows of the identity matrix $I$. Let $X$ denote the state-space of the system $W$, $\mathbb{R}^n$ in this paper; and let $X_\omega$ denote the subspace of vectors $x = (x_1, \ldots, x_n)^T \in X$ for which $x_i = 0$ for $i \notin \omega$. Then the change in system $W$ is described by:

$$x_{new} = A_\omega x + u_\omega, \tag{15}$$

where $u_\omega = (u_1, \dots, u_n)^T \in X_\omega$. Now let $T_0 < T_1 < \cdots < T_n < \cdots$ be the time instants when the state of the system $W$ undergoes change. Let $x(k)$ denote the system state vector at time $T_k$ and $\omega(k)$ the set of indices of components that change at that instant. The equation of dynamics of the system $W$ can then be written as:

$$x(k+1) = A_{\omega(k)}x(k) + u(k), \quad u(k) \in X_{\omega(k)} \tag{16}$$

The dynamics of $W$ is thus described by a linear difference equation. Desynchronization or an asynchronous mode of operation of the system $W$ has the following effects: first, the time-varying matrix $A_{\omega(k)}$ has a specific form and, second, the 'input' or 'free' terms $u(k)$ are required to belong to the subspaces $X_{\omega(k)}$ compatible with the matrices $A_{\omega(k)}$ (this is an 'unnatural' requirement in the general theory of linear difference equations, but is justifiable in the present context on both modelling grounds as well as mathematical grounds [15]).

If the system $W$ is not subjected to external inputs, then its dynamics is described by the homogeneous linear equation:

$$y(k+1) = A_{\omega(k)}y(k). \tag{17}$$

The sequence of nonempty sets $\omega(k) \subseteq \{1, 2, \dots, n\}$ is called *regular* if the inclusion $i \in \omega(k)$ is satisfied for infinitely many $k$. The corresponding update law is referred to as *regular* or *totally asynchronous*.

*Partial asynchronism* [17], also referred to as (uniformly) *bounded-delay asynchronism* [27], is defined as follows: there exists $B$ such that for all $k$, $\omega(k) \cup \cdots \cup \omega(k+B-1) = \{1, \dots, n\}$.

The system $W$ is called *absolutely asymptotically stable (in the class of all desynchronizations)* if for any regular (respectively, partially asynchronous) sequence $\{\omega(k)\}$ each solution of the corresponding equation (17) tends to zero as $k \to \infty$.

## 3.2 General stability theorems for asynchronous systems

The following results due to Chazan and Miranker [16], Kleptsyn et al. [11, 12], and Bhaya et al. [18, 26] are fundamental in the theory of stability of asynchronous systems and will be used below.

THEOREM 3.1 [16] *Let $A = (a_{ij})$ and $S = (|a_{ij}|)$. The zero solution of equation (17) is absolutely asymptotically stable under the class of regular asynchronisms if the spectral radius of $S$, $\rho(S)$, is less than unity. If, in addition, $a_{ij} \geq 0$ and zero solution of equation (17) is absolutely asymptotically stable under the class of regular asynchronisms, then $\rho(A) = \rho(S) < 1$.* □

THEOREM 3.2 [12] *Let $A = (a_{ij})$ be a symmetric matrix. Then the zero solution of equation (17) is absolutely asymptotically stable under the class of regular asynchronisms if and only if $\rho(A) < 1$.* □

THEOREM 3.3 [26] *Let* $G(x) = (g_1(x), \ldots, g_n(x))^T$ *and let each* $g_i$ *satisfy the following 'block-Lipschitz' condition:*

$$\forall x, y \in \mathrm{Dom}(g_i), \quad \|g_i(x) - g_i(y)\| \le \sum_{j=1}^{n} l_{ij} \|x_j - y_j\|. \tag{18}$$

*Assume uniqueness of the fixed-point* $x^*$ *(in the domain of* $G$*), of the system below:*

$$\forall i \in \{1, \ldots, n\}, \quad x_{i_{new}} = g_i(x^i_{old}), \tag{19}$$

*where* $x_{i_{new}} \in \mathbb{R}$ *is the new value of the* $i^{\text{th}}$ *component after updating,* $x^i_{old}$ *denotes the vector* $x \in \mathbb{R}^n$ *available for the update of the* $i^{\text{th}}$ *component at the instant just before the update. The equilibrium* $x^*$ *is absolutely asymptotically stable under the class of partial asynchronisms if* $\rho(L) < 1$*, where* $L = (l_{ij})$*.* □

REMARK 3.4 Theorems 3.1 and 3.3 can be proved using a Liapunov function of the weighted infinity norm type and standard majorization techniques; whereas $x^T(I - A)x$ is a quadratic Liapunov function which can be used to prove Theorem 3.2.

Furthermore, Kozyakin [14] proved the following robust generalization of Theorem 3.2.

THEOREM 3.5 *Let* $A = M + N$*, where* $M$ *is a symmetric matrix and* $N$ *is antisymmetric. Then the zero solution of equation (17) is absolutely asymptotically stable, if* $\rho(M) < 1$ *and* $\rho(N) < \gamma(M)$*, where,*

$$\gamma(M) := \rho(M)\sqrt{\frac{1 - \rho(M)}{1 + \rho(M)}} \left( \frac{1}{\sqrt{1 - (1 - \rho(M)^2)^n}} - 1 \right). \tag{20}$$

□

REMARK 3.6 Theorem 3.5 can be stated in the following weaker form: assume that the linear system with an arbitrary matrix (not necessarily symmetric) is absolutely asymptotically stable in the class of all desynchronizations; then any close (in the sense of distance between matrices) system with an arbitrary matrix is also absolutely asymptotically stable. In this form, the assertion of theorem 3.5 can be generalized to systems with vector states of subsystems and arbitrary matrices [14, Thm.2].

Consider a system $W$ in the presence of external perturbations. Then its dynamics is described by equation (16). The system $W$ is called *absolutely stable in the presence of persistent perturbations (in the class of all desynchronizations)* if there exists a constant $\beta < \infty$ such that for all sequences of sets $\omega(k) \subseteq \{1, 2, \ldots, n\}$ and vectors $u(k) \in X_{\omega(k)}$, $\|u(k)\| \le 1$, the solution $x(k)$ of equation (16) satisfying the zero initial condition $x(0) = 0$ has the bound $\|x(k)\| \le \beta$ for $k \ge 1$. Stability in the presence of persistent perturbations is sometimes called the Perron property [28]. Note that this definition differs from the traditional Perron property for difference equations in that the vector $u(k)$ is required to be an element of the subspace $X_{\omega(k)}$ compatible with the matrices $A_{\omega(k)}$. Equation (17) does not possess the Perron property in the usual sense if $\omega(k) \neq \{1, 2, \ldots, n\}$ for infinitely many $k$. The main result for Perron stability is as follows.

THEOREM 3.7 [14, 15] *A linear system $W$ is absolutely stable in the presence of persistent perturbations in the class of all desynchronizations if and only if it is absolutely asymptotically stable in the class of all desynchronizations.*

REMARK 3.8 Standard linearization techniques (see, e.g. [28, 29]) can be applied to deduce theorems on stability of desynchronized systems in the first approximation by reference to Theorem 3.7.

## 3.3 Equilibrium and stability results

The asynchronous version of equation (9) is written in the following manner. Let $\omega(k)$ be a regular sequence, as defined above. Then, the asynchronous version of (9) is as follows:

$$x_{i_{new}} = \sum_{j}^{n} t_{ij} f_j(x_j) + u_i, \quad i \in \omega(k) \tag{21}$$

$$x_{i_{new}} = x_i, \quad i \notin \omega(k) \tag{22}$$

The following simple observation is fundamental.

LEMMA 3.9 *Under the assumptions of Theorem 2.7, the asynchronous equation (21) has at most one absolutely asymptotically stable equilibrium.*

*Proof.* By the definition of absolute asymptotic stability, any equilibrium with this property is required to be asymptotically stable under the class of all regular desynchronizations. Since this class includes the synchronized case, under which $\omega(k) = \{1, \ldots, n\}$, for all $k$, so that the asynchronous version (21) reduces to the synchronous version (1), the lemma follows. ■

To prove a local stability result, the following two lemmas are needed.

LEMMA 3.10 *Let $A \in \mathbb{R}^{n\times n}$ be symmetric and $D \in \mathbb{R}^{n\times n}$ be a diagonal matrix with positive diagonal elements. Then the diagonal similarity transformations $K = D^{-\frac{1}{2}}$ and its inverse symmetrize both products $DA$ and $AD$ respectively, and the result, in both cases, is the symmetric matrix $A_s = D^{\frac{1}{2}} A D^{\frac{1}{2}}$ .*

*Proof.* By calculation. ■

Denote by $(\mathcal{T}_{sym})$ the assumption that $T$ is symmetric and stable.

Lemma 3.11 *Under the assumptions ($\mathcal{F}$) and ($\mathcal{T}_{sym}$), the linearization of equation (9) about its unique equilibrium, $x^e$, can be written in an appropriate coordinates as:*

$$z(k+1) = T_L z(k), \tag{23}$$

*where $T_L$ is a symmetric matrix given by $T_L = [F'(x^e)]^{\frac{1}{2}} T [F'(x^e)]^{\frac{1}{2}}$.*

*Proof.* The linearization of (9) about the equilibrium $x^e$ is:

$$y(k+1) = TF'(x^e)y(k), \tag{24}$$

where $F'(x^e)$ is the positive diagonal Jacobian matrix (by assumptions ($\mathcal{F}$) and ($\mathcal{T}_{sym}$)) of the diagonal function $F$. Lemma 3.10 may now be used to assert that equation (24) can be transformed to equation (23) as announced above. ∎

A local stability result may now be obtained.

Theorem 3.12 *Let assumptions ($\mathcal{F}$) and ($\mathcal{T}_{sym}$) hold. Then, the equilibrium $x^e$ of (21) is locally absolutely asymptotically stable.*

*Proof.* Under the given assumptions, it follows from theorem 3.2 that the equation

$$z(k+1) = (T_L)_{\omega(k)} z(k), \tag{25}$$

where $T_L = [F'(x^e)]^{\frac{1}{2}} T [F'(x^e)]^{\frac{1}{2}}$, is absolutely asymptotically stable in the class of all regular desynchronizations. Under an asynchronous mode of operation specified by the sequence $\omega(k)$, assume that $i \in \omega(k)$ and that, without loss of generality, $x^e = 0$ and $f(x^e) = 0$. A Taylor expansion of the nonlinear terms on the right-hand side of equation (21) about $x^e$ yields:

$$x_{i_{new}} = \sum_{j}^{n} t_{ij} f_j'(x_j^e) x_j + u_i + \text{h.o.t.}, \tag{26}$$

where h.o.t. denotes higher order terms. From here and from [15, Thm. 6.7.2 ,p.289] the statement of the theorem follows. ∎

Finally, using Theorem 3.3, a global stability condition under partial asynchronism can be deduced immediately. The following assumptions are needed:

($\mathcal{F}_L$) $F(x) = (f_1(x_1), \ldots, f_n(x_n))^T$ and the nonlinear functions $f_i(x_i)$ are Lipschitz-continuous, with constants $\ell_{f_i}$, i.e.

$$|f_i(x) - f_i(y)| < \ell_{f_i} |x - y|; \tag{27}$$

($\mathcal{P}$) the asynchronism is partial.

Remark 3.13 Note that if assumption ($\mathcal{F}$) holds, then assumption ($\mathcal{F}_L$) holds with all the $\ell_{f_i}$ equal to 1, but the converse is not true.

A global asymptotic stability result may now be proved.

Theorem 3.14 *Under the assumptions ($\mathcal{F}_L$) and ($\mathcal{P}$), the unique equilibrium of (21), (22) is globally absolutely asymptotically stable under the class of partial asynchronisms, if $\rho(C) < 1$, where $C = (c_{ij})$ and $\forall i, j, \quad c_{ij} := |t_{ij}|\ell_{f_j}$.*

*Proof.* Immediate from Theorem 3.3 by calculation of the constants $l_{ij}$. ■

Remark 3.15 This theorem provides a guideline on the choice of an interconnection matrix for a neural network operating in the asynchronous mode and for which it is desired to have a unique globally asymptotically stable equilibrium.

Remark 3.16 If assumption ($\mathcal{F}$) holds, then all the $\ell_{f_j}$ may be taken equal to unity, so that the above condition reduces to $\rho(|T|) < 1$, thus generalizing Theorem 3.1. Since $\rho(|T|) < 1$ implies that $\rho(T) < 1$, it is also clear that this global stability result imposes a more restrictive condition than that of: (a) the local stability result Theorem 3.12, which, however, is only valid for symmetric matrices $T$, or small perturbations thereof; and (b) the global stability result of Theorem 2.9 for the synchronous case.

Remark 3.17 For the equivalent fixed-point iteration (see discussion in sec.1.1):

$$x_{i_{new}} = f_i \left( \sum_j t_{ij} x_j + u_i \right), \tag{28}$$

Tseng et al. [19] proved an asynchronous convergence result, under similar assumptions on the functions $f_i$ (basically ($\mathcal{F}$) and range($f_i$) $= [-1, 1]$) and an indecomposability assumption on $T$, using the Brouwer fixed-point theorem and allowing 'total' asynchronism (i.e. regular desynchronization). Essentially, their result is similar to that of theorem 3.14, since it requires $\rho(|T|) < 1$, where $T = (t_{ij})$ and, once again, this is consistent with the discussion in section 1.1.

# 4 Discussion of the results: Robustness issues

Since the widely-studied class of Hopfield-Tank type networks assumes symmetry of the interconnection matrix (the matrix $T$ in the above), it is clear that a study of the robustness of different properties of the network is very important, from the point of view of practical implementations.

Earlier results in this area, for continuous-time models, include [21] which proves structural stability of a class of Hopfield-Tank networks with diagonally stable interconnection matrices and [22] which carries out a robustness and perturbation analysis of Hopfield type networks.

In this context, the present paper makes two contributions in the discrete-time case. First, theorem 3.12 shows that stability properties of the system are maintained (at least locally), even if the system operates in a fairly general asynchronous mode. As already pointed out above, this is important from a modelling point of view: citing [19], which, in turn, cites [4], "... asynchronous neural networks are quite natural since biological neural connections may experience long propagation delays." Second, theorem 3.5 shows that small (where 'small' is quantifiable via equation (20)) desymmetrizing perturbations do not affect the stability result of Theorem 3.12.

Robustness of the condition of Theorem 3.14 may be seen from the following two facts. First, the stability of a nonnegative matrix is equivalent to its diagonal stability [27]. Second, it is known that the set of diagonally stable matrices is open; hence small perturbations do not affect the property of diagonal stability. Now, since Theorem 3.14 requires diagonal stability of the nonnegative (modulus of elements) matrix derived from the interconnection matrix, it is clear that this condition is robust to small perturbations in the elements of the interconnection matrix $T$. This is analogous to the result in [21] where the condition for structural stability was shown to be robust in a similar manner.

In general terms, this paper has carried out the study of existence and uniqueness conditions of equilibria of a class of continuous-valued, discrete-time Hopfield-Tank networks. Conditions have been given for the existence of a unique equilibrium in terms of the matrix theory concept of D-stability. An analysis has been made of the stability of this equilibrium as well as its robustness in two senses. First, the equilibrium is shown to be robust in the conventional sense, i.e., under perturbations of the interconnection matrix and of the activation functions of the network. Second, it is shown that, under certain conditions, asynchronous modes of operation are also permissible, since they do not change the global asymptotic stability property of the equilibrium – thus the network is robust to desynchronization as well, which is a satisfactory result from the point of view of modelling biological neural networks, which are believed to operate asynchronously.

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