Bi-shadowing in infinite dimensional systems and delay equations*

A. Al-Nayef[†] P. Diamond[‡] P. Kloeden[†]
V. Kozyakin[§] A. Pokrovskii[†¶]

Introduction

The concept of bi-shadowing comes in two parts: direct shadowing in which there is some true trajectory near a given pseudo–trajectory; and indirect or inverse shadowing in which there is some pseudo–trajectory near a given true trajectory. Shadowing results (cf. [9, 12]) typically establish only direct shadowing and involve rather stringent assumptions such as that the dynamical system is generated by a hyperbolic diffeomorphism. Many useful properties of hyperbolic diffeomorphisms are retained by the semi–hyperbolicity mappings that were introduced in [4] for local diffeomorphisms (see also Anosov [2] where related concepts are discussed) and extended to Lipschitz mappings in [5, 6, 7].

Here it will be shown that semi–hyperbolicity of a Lipschitz mapping on a given set implies bi-shadowing for a wide class of dynamical systems in infinite dimensional Banach spaces. Definitions of shadowing and bi–shadowing are given in the next section and that of semi-hyperbolicity for

*P. Diamond, P. Kloeden and A. Pokrovskii were supported by the Australian Research Council Grant A 8913 2609, V. Kozyakin was partially supported by the Russian Foundation for Fundamental Research Grant 93-01-00884.

†School of Computing and Mathematics, Deakin University, Geelong, Victoria 3217, Australia.,

‡Department of Mathematics, University of Queensland, Brisbane, Qld 4072, Australia.

§Institute of Information Transmission Problems, Russian Academy of Sciences, 19 Ermolovoy str., Moscow 101447, Russia.

¶Permanent address: Institute of Information Transmission Problems, Russian Academy of Sciences.

Lipschitz mappings in Section 2. The main results of the paper are stated in Section 3, an example of its application to delay equation is introduced in Section 4. Note that infinite dimensional perturbations of finite-dimensional semi-hyperbolic mappings were also considered in [3].

1 Bi-shadowing With Respect To Completely Continuous Perturbations

Let E be a Banach space. Consider a mapping $f : \mathrm{X} \mapsto \mathrm{X}$ where X is a subset of E.

A *trajectory* of a discrete-time dynamical system on the state space X generated by the mapping f is a sequence $\mathbf{x} = \{x_n\} \subset \mathrm{X}$ satisfying

$$x_{n+1} = f(x_n) \tag{1}$$

for $n = 0, 1, 2, \ldots, N_1$ or $n = -N_0, \ldots, -1, 0, 1, \ldots, N_1$ where $N_0, N_1 \leq \infty$, whereas a sequence $\mathbf{y} = \{y_n\} \subset \mathrm{X}$ with

$$\|y_{n+1} - f(y_n)\|_E \leq \gamma, \qquad \gamma > 0, \tag{2}$$

for such n is called a γ *pseudo–trajectory* of the dynamical system. In both cases the qualifier finite may be appended when $N_0, N_1 < \infty$ and infinite otherwise.

Pseudo–trajectories arise naturally due to the presence of roundoff error in computer calculations of trajectories, though accumulated roundoff error can rapidly destroy any meaningful connection between a computed pseudo-trajectory and an original trajectory. The concept of shadowing provides an alternative, more practical form of comparison of trajectories and pseudo–trajectories. A trajectory $\mathbf{x} = \{x_n\}$ is said to ϵ–*shadow* a γ pseudo–trajectory $\mathbf{y} = \{y_n\}$ if

$$\|x_n - y_n\|_E \leq \epsilon \tag{3}$$

for all n belonging to some contiguous set (which is usually of finite length depending on the trajectories and the parameters).

The gist of a *Shadowing Lemma* (cf. [12]) is that, under certain assumptions on f such as hyperbolicity, for every $\epsilon > 0$ there exists a $\gamma > 0$ such that each γ pseudo–trajectory is ϵ–shadowed by a true trajectory. From this it is often concluded that the behaviour of a computed system reflects that of the original system, at least over finite time intervals, in the sense that there will always be some true trajectory near any observed, pseudo–trajectory.

The inverse question as to whether every true trajectory can be approximated by some pseudo–trajectory is of no less practical importance. While any γ pseudo-trajectory is possible in principle, only those belonging to some particular class $\mathcal{T}$ occur in practice. These might be generated by a particular discretization method being applied or arise from specific processes associated with computer arithmetic. Typically, only general characteristics of such pseudo–trajectories will be known rather than a complete definition of $\mathcal{T}$ itself. The problem of *inverse shadowing* with respect to such a class $\mathcal{T}$ is to determine whether every true trajectory of a given system f can be approximated by some pseudo–trajectories from $\mathcal{T}$. A discussion of direct and indirect shadowing can be found in [10], Appendix C.

The class of pseudo–trajectories $\mathcal{T}$ plays a somewhat different role in the two forms of shadowing. In the classical shadowing lemma, $\mathcal{T}$ consists of all conceivable pseudo–trajectories of f and is thus as large as possible. On the other hand, inverse shadowing should be compatible with more restricted classes such as T_φ containing the trajectories of a completely continuous mapping φ: $X \to X$ that is sufficiently C^0 close to f. Recall, that a continuous mapping φ is said to be *completely continuous* if the image $\varphi(X_0)$ of any bounded set $X_0 \subseteq X$ is relatively compact in E. It will be shown that the two forms of shadowing with respect to such classes are usually both present (see also [4, 5, 6, 7]).

Let $\mathbf{Tr}(f, K, \gamma)$ denote the totality of finite or infinite γ pseudo–trajectories (2) belonging entirely to a subset $K \subseteq \mathrm{X}$. Since a true trajectory can be regarded as a $\gamma = 0$ pseudo–trajectory, the set of all finite or infinite trajectories which belong entirely to K will be denoted by $\mathbf{Tr}(f, K, 0)$. Since a trajectory is also a γ pseudo–trajectory for any $\gamma > 0$, $\mathbf{Tr}(f, K, 0) \subset \mathbf{Tr}(f, K, \gamma)$. The inclusion is strict because not every pseudo–trajectory is a trajectory. Set

$$\|\varphi - f\|_\infty = \sup_{x\in \mathrm{X}} \|\varphi(x) - f(x)\|_E.$$

A dynamical system generated by a mapping $f : \mathrm{X} \mapsto \mathrm{X}$ is said to be *bi–shadowing with positive parameters α and β on a subset K of* X if for any given finite pseudo-trajectory $\mathbf{y} = \{y_n\} \in \mathbf{Tr}(f, K, \gamma)$ with $0 \le \gamma \le \beta$ and any completely continuous mapping $\varphi : \mathrm{X} \mapsto \mathrm{X}$ satisfying

$$\gamma + \|\varphi - f\|_\infty \le \beta \tag{4}$$

there exists a trajectory $\mathbf{x} = \{x_n\} \in \mathbf{Tr}(\varphi, \mathrm{X}, 0)$ such that

$$\|x_n - y_n\|_E \leq \alpha(\gamma + \|\varphi - f\|_\infty) \tag{5}$$

for all n for which $\mathbf{y}$ is defined.

Bi-shadowing conceptualizes the robust relationship between observed dynamical behaviour of a dynamical system and its computer simulations, and can also be interpreted as a form of dynamical structural stability when restricted to specific classes of mappings, such as continuous mappings. It implies both the direct shadowing and inverse shadowing properties discussed above: taking $\varphi \equiv f$ in (4) and (5) gives $\alpha\gamma$–shadowing of any γ pseudo–trajectory $\mathbf{y} \in \mathbf{Tr}(f, K, \gamma)$ by a true trajectory $\mathbf{x} \in \mathbf{Tr}(f, K, 0)$. Inverse shadowing follows by taking $\gamma = 0$, because if $\mathbf{y} \in \mathbf{Tr}(f, K, 0)$ is a true trajectory of f, hence a γ pseudo-trajectory with $\gamma = 0$, there exists a trajectory $\mathbf{z}$ of φ, $\mathbf{z} \in \mathbf{Tr}(\varphi, \mathrm{X}, 0)$ which $\alpha\beta$–shadows $\mathbf{y}$ by (4) and (5).

Cyclic behaviour is often of particular interest in dynamical systems, with a trajectory $\mathbf{x} = \{x_n\}_{n=0}^{N} \in \mathbf{Tr}(f, K, 0)$ being called a *cycle of period* N if $x_N = x_0$. Analogously, a pseudo-trajectory $\mathbf{y} = \{y_n\}_{n=0}^{N} \in \mathbf{Tr}(f, K, \gamma)$ will be called a γ *pseudo-cycle of period* N if $\|y_N - y_0\|_E \leq \gamma$. Let $\mathcal{C}(f, K, \gamma) \subset \mathbf{Tr}(f, K, \gamma)$ denote the totality of γ pseudo–cycles of any period belonging entirely to the subset K of X, with $\mathcal{C}(f, K, 0) \subset \mathbf{Tr}(f, K, 0)$ denoting the totality of proper cycles of any period which are contained entirely in K. Obviously $\mathcal{C}(f, K, 0) \subset \mathcal{C}(f, K, \gamma)$ for every $\gamma > 0$. A counterpart of bi–shadowing for cycles and pseudo–cycles is also useful: a dynamical system generated by a mapping $f : \mathrm{X} \mapsto \mathrm{X}$ is said to be *cyclically bi–shadowing with positive parameters* α *and* β *on a subset* K *of* X if for any given pseudo–cycle $\mathbf{y} \in \mathcal{C}(f, K, \gamma)$ with $0 \leq \gamma \leq \beta$ and any mapping completely continuous $\varphi : \mathrm{X} \mapsto \mathrm{X}$ satisfying (4) there exists a proper cycle $\mathbf{x} \in \mathcal{C}(\varphi, \mathrm{X}, 0)$ of period N equal to that of $\mathbf{y}$ such that (5) holds for $n = 0, 1, \ldots, N$. Note that the cycle $\mathbf{x}$ here is required only to be in X rather than in the subset K.

2 Semi-hyperbolic Mappings in Banach Spaces

A four-tuple $\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ of nonnegative real numbers is called a *split* if

$$\lambda_s < 1 < \lambda_u \tag{6}$$

and

$$(1 - \lambda_s)(\lambda_u - 1) > \mu_s \mu_u. \tag{7}$$

Clearly, for any given λ_s, λ_u satisfying (6) the four-tuple $\mathbf{s}$ is a split if the product $\mu_s\mu_u$ is small enough.

Let $\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ be a split and K a subset of X. A Lipschitz mapping $f : \mathrm{X} \mapsto \mathrm{X}$ is said to be $\mathbf{s}$–*semi-hyperbolic on the set* K if there exist positive real numbers k, δ such that for each $x \in K$ there exists a splitting (decomposition)

$$E = E_x^s \oplus E_x^u \tag{8}$$

with corresponding projectors P_x^s and P_x^u satisfying the following four properties:

SH0. The space E_x^u is finite dimensional for all x and $\dim(E_x^u) = \dim(E_{f(x)}^u)$ if $x, f(x) \in K$.

SH1. $\sup_{x\in K}\{\|P_x^s\|_E, \|P_x^u\|_E\} \leq k$.

SH2. The inclusion

$$x + u + v \in \mathrm{X} \tag{9}$$

and the inequalities

$$\begin{aligned}
\|P_{f(x)}^s\left(f(x+u+v) - f(x+\widetilde{u}+v)\right)\|_E &\leq \lambda_s\|u-\widetilde{u}\|_E, & (10)\\
\|P_{f(x)}^s\left(f(x+u+v) - f(x+u+\widetilde{v})\right)\|_E &\leq \mu_s\|v-\widetilde{v}\|_E, & (11)\\
\|P_{f(x)}^u\left(f(x+u+v) - f(x+\widetilde{u}+v)\right)\|_E &\leq \mu_u\|u-\widetilde{u}\|_E, & (12)\\
\|P_{f(x)}^u\left(f(x+u+v) - f(x+u+\widetilde{v})\right)\|_E &\geq \lambda_u\|v-\widetilde{v}\|_E & (13)
\end{aligned}$$

hold for all $x \in K$ with $f(x) \in K$ and all $u, \widetilde{u} \in E_x^s, v, \widetilde{v} \in E_x^u$ such that $\|u\|_E, \|\widetilde{u}\|_E, \|v\|_E, \|\widetilde{v}\|_E \leq \delta$.

Note that continuity in x of the splitting subspaces E_x^s, E_x^u or of the projectors P_x^s, P_x^u is not assumed here, nor is invariance of the splitting subspaces, as is the case in the definition of hyperbolicity of a diffeomorphism.

3 Main Results

The main result of this paper is that semi–hyperbolicity is sufficient to ensure bi–shadowing of a dynamical system generated by a Lipschitz mapping with respect to perturbed systems generated by completely continuous mappings.

Theorem 1. *Let $f : \mathrm{X} \mapsto \mathrm{X}$ be a Lipschitz mapping which is $\mathbf{s}$–semi-hyperbolic on a subset K of X with constants k, δ. Then it is bi–shadowing on K with parameters*

$$\alpha(\mathbf{s}, k) = k \, \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1-\lambda_s)\,(\lambda_u - 1) - \mu_s \mu_u} \tag{14}$$

and

$$\beta(\mathbf{s}, k, \delta) = \delta k^{-1} \, \frac{(1-\lambda_s)(\lambda_u - 1) - \mu_s \mu_u}{\max\{\lambda_u - 1 + \mu_s, 1 - \lambda_s + \mu_u\}}. \tag{15}$$

This result not only generalizes existing variants of the Shadowing Lemma to a far broader class of dynamical systems, but also includes inverse as well as direct shadowing.

The proof of the theorem is not dissimilar to that of a finite-dimensional analogue ([7]). Nevertheless, to make this paper selfsufficient and to highlight some differences from the finite-dimensional case, a complete proof is given in Appendix.

Cyclic bi–shadowing is also a consequence of semi-hyperbolicity.

Theorem 2. *Let $f : \mathrm{X} \mapsto \mathrm{X}$ be a Lipschitz mapping which is $\mathbf{s}$–semi-hyperbolic on a subset K of X with constants k, δ. Then it is cyclically bi–shadowing on K with parameters $\alpha(\mathbf{s}, k)$ and $\beta(\mathbf{s}, k)$ given by (14) and (15), with respect to completely continuous mappings $\varphi : \mathrm{X} \mapsto \mathrm{X}$.*

4 Application to delay equation

Consider the linear delay equation

$$x'(t) = Ax(t) + Bx(t-h). \tag{16}$$

Here $x(t) \in \mathbb{R}^d$, A and B are real d-matrices and h is a positive constant. We shall call this equation *hyperbolic* if

$$\det(wI - A - e^{wB}) = 0 \tag{17}$$

does not have a purely imaginative solution $w = ip$.

To each solution w of this equation there corresponds a solution of the delay equation (16) of the form

$$e^{wt}a \qquad -\infty < t < \infty \tag{18}$$

where a is an eigenvector of the matrix $wI - A - e^{wB}$ with the eigenvalue w.

We will also consider nonlinear delay equations of the form

$$y'(t) = Ay(t) + By(t-h) + F(y(t), y(t-h)). \tag{19}$$

Here $F(y,v)$ is a continuous $\mathbb{R}^d$-valued function, which is locally Lipschitz in y and A,B are as before. Denote by $L(F)$ the set of all continuous function $y(t)$, $t \geq -h$ satisfying the equation (19) for $t > 0$. In particular, $L(0)$ denotes the set of all continuous function $x(t)$, $t \geq -h$ satisfying the equation (16) for $t > 0$.

Theorem 3. *Let the equation* (16) *be hyperbolic. Then there exists a constant* $\gamma > 0$ *with the following properties.*

(a) *For each* $x(t) \in L(0)$ *and for each uniformly bounded* $F(x,u)$ *there exists a continuous function* $y(t) \in L(F)$, *satisfying the inequality*

$$|y(t) - x(t)| < \gamma \sup_{y,v} |F(y,v)|, \qquad t \geq -h. \tag{20}$$

(b) *Let* $F(y,v)$ *be a uniformly bounded and* $y(t) \in L(F)$. *Then there exists a function* $x(t) \in L(0)$ *satisfying* (20).

This demonstrates robustness of solutions of a hyperbolic delay equation with respect to arbitrary continuous perturbations of small amplitude. In particular, any nonlinear perturbation (19) of a linear equation (16) has bounded at $t \to \infty$ solutions which shadow a given bounded at $t \to \infty$ solution of the linear equation. Let us consisely describe the main steps in the proof of this theorem

Step 1. For each continuous function $\xi(s)$, $s \in [-h, 0]$ the equation (19) has a unique solution $y(t; \xi, F)$, $t \geq -h$ which is continuous and satisfies $y(s; \xi, F) = \xi(s)$, $s \in [-h, 0]$, because $F(y,v)$ is supposed continuous, uniformly bounded and satisfy local Lipschitz condition in y. Introduce the shift operator S_F for equation (19) by

$$(S_F \xi)(\tau) = y(h - \tau; \xi, F), \qquad -h \leq \tau \leq 0.$$

The operator S_F, is completely continuous as an operator in the space $C = C([-h, 0], \mathbb{R}^d)$.

In particular, denote by S the shift operator for the linear equation (16). The operator S is a *linear* completely continuous operator in C.

Step 2. Let $\xi(\tau)$, $\tau \in [-h,0]$ be an eigenfunction of the complexification of the operator S with a complex eigenvalue w. Then by the definition $\xi(s)$ satisfies the equation

$$w\xi'(\tau) = wA\xi(\tau) + B\xi(\tau), \qquad -h \le \tau \le 0.$$

Thus the set of nonzero eigenvalues of the linear operator S coincides with the set of complex number $z = e^{hw}$ where w is a solution of the equation (17) (The corresponding complex eigenfunction are restrictions of functions (18) on $[-h,0]$.)

Sincw S is completely continuous, the spectrum of S consists of zero and all complex numbers e^{wh} where w is a solution of the equation (17).

Step 3. By the previous step and the hyperbolicity of the linear equation (16) the spectrum $\sigma(S)$ of the linear operator S consists of two disjoint parts $\sigma(S) = \sigma^s(S)\bigcup\sigma^u(S)$, such that σ^s is located strictly inside the unit disc of a the complex plane and σ^u is located strictly outside the unit disc. By the decomposition theorem ([11], p.421), it means that the space C can be decomposed into a direct sum

$$C = E^s \oplus E^u \tag{21}$$

so that both E^s and E^u are invariant for S, the spectrum of the restriction $\sigma(S|_{E^s}) = \sigma^s$ of S onto E^s and the spectrum of the restriction $\sigma(S|_{E^u}) = \sigma^u$ Further, since S is completely continuous, the subspace E^u is finite-dimensional. Note that the parallel projection P^s of C onto E^s in the direction of E^u can be written in an explicit form as

$$P^s = -\frac{1}{2\pi i}\int_{|z|=1}(S - zI)^{-1}\,dz. \tag{22}$$

Step 4. Introduce an auxiliary norm $\|\cdot\|_s$ onto the subspace E^s by

$$\|x\|_s = \sum_{n=0}^{\infty}\|S^n x\|_C.$$

Clearly this norm is equivalent to the norm $\|\cdot\|$ and the restriction of the operator S onto E^s contracts in this norm with some constant $\lambda_s < 1$. Analogously, introduce an auxiliary norm $\|\cdot\|_u$ onto the subspace E^u by

$$\|x\|_u = \sum_{n=0}^{\infty} \|S^{-n}x\|_C.$$

This norm is also equivalent to the C-norm and the restriction of the operator S onto E^u expands in this norm with some constant $\lambda_u > 1$. Introduce in C an auxiliary norm $\|\cdot\|_*$ by $\|\xi\|_* = \max\{\|P^s\xi\|_s, \|P^u\xi\|_u\}$ where P^s is defined by (22) and $P^u = I - P^s$. Denote by $\mathbf{s}$ the split $(\lambda_s, \lambda_u, 0, 0)$. By construction, the linear operator S is $\mathbf{s}$-semihyperbolic with constants k, δ where $k = \max\{\|P^s\|_*, \|P^u\|_*\}$ and δ is an arbitrary positive number.

Step 5. In Step 1 the shift operator S_F of the nonlinear equation (19) is completely continuous. This operator also satisfies the estimate

$$\|S_F\xi - S\xi\|_* < \gamma_1 \sup_{y,v} |F(y,v)|, \qquad t \geq -h.$$

for some positive γ_1. Thus Theorem 1 is applicable and, taking into account the equivalence of norms $\|\cdot\|_C$ and $\|\cdot\|_*$, as a corollary to that theorem it follows that:

Corollary 1. *There exist a constant $\gamma > 0$ with the following properties.*

(a) *For each trajectory*

$$\boldsymbol{\eta} = \eta_0, \eta_1, \ldots \tag{23}$$

of the shift operator S there exists a trajectory

$$\boldsymbol{\eta}^F = \eta^F_{(0)}, \eta^F_1, \ldots \tag{24}$$

of the operator S_F with

$$\|\eta_n - \eta^F_n\|_C \leq \gamma \sup_{y,v} |F(y,v)|, \tag{25}$$

(b) *For each trajectory* (24) *of the shift operator S_F there exists a trajectory* (23) *of the operator S satisfying* (25)*.*

Theorem 3 then follows. □

Note mention that the construction of the last section can be carried out also for some systems described by parabolic equations. Also note that some hysteresis perturbations, like Prandtl, Besseling and Ishlinskii models in plasticity or Preisach, Giltay and Madelung models in magnetizm ([8]) can be taken into account both in analysis of delay and parabolic equations.

5 Appendix

For a given split $\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ and semi-hyperbolicity constant k define the matrix

$$M(\mathbf{s}) = \begin{pmatrix} \lambda_s & \mu_s \\ \mu_u/\lambda_u & 1/\lambda_u \end{pmatrix} \tag{26}$$

and a two-dimensional vector $\mathbf{a} = (a, b)^T$ by formula

$$\mathbf{a} = (I - M(\mathbf{s}))^{-1}\mathbf{k}, \qquad \text{where} \qquad \mathbf{k} = k(1, 1/\lambda_u)^T. \tag{27}$$

Then the bi–shadowing constants (14) and (15) satisfy

$$\alpha(\mathbf{s}, k) = a + b, \qquad \beta(\mathbf{s}, k, \delta) = \delta \min\left\{a^{-1}, b^{-1}\right\}. \tag{28}$$

First observe the following: denote by $B_x(r)$ the closed ball centred at x of the radius r in the linear space E^u_x. For each $x \in K$ and each $z \in \mathbb{R}^d$ satisfying $\|P^s_x z\|_E \le \delta$ introduce the finite-dimensional mapping $F_{x,z} : B_x(\delta) \mapsto E^u_{f(x)}$ by $F_{x,z}(v) = P^u_{f(x}(f(x + P^s_x z + v) - f(x + P^s_x z))$.

Lemma 1. *Let $0 \le r \le \delta$. Then*

$$F_{x,z}(B_x(r)) \supseteq B_{f(x)}(\lambda_u r). \tag{29}$$

Proof: Consider only the case $r > 0$. Denote by $\partial B_x(r)$ and $B^o_x(r)$ the boundary and the internity of $B_x(r)$. Clearly,

$$F_{x,z}(0) = P^u_{f(x)}(f(x + P^s_x z) - f(x + P^s_x z)) = 0 \in B^o_{f(x)}(\lambda_u r); \tag{30}$$

on the other hand, by the inequality (12)

$$F_{x,z}(\partial B_x(r)) \bigcap B^o_{f(x)}(\lambda_u r) = \emptyset. \tag{31}$$

By Property SH0, and the principle of domain invariance (see, e.g., [1], p.396) (31) implies $\partial F_{x,z}(B_x(r)) \bigcap B^o_{f(x)}(\lambda_u r) = \emptyset$. The last equality together with (30) imply (29) and the lemma is proved. □

From this lemma and from inequality (12) it follows immediately that

Lemma 2. *The operator $Q_{x,z} = F^{-1}_{x,z}$ is defined and continuous on $B_{f(x)}(\lambda_u \delta)$ and satisfies the estimate $\|Q_{x,z}(v)\|_E \le \lambda_u^{-1} \|v\|_E$.*

Denote by $\mathcal{Z}_N$ the space of N-tuples $\mathbf{z} = (z_0, z_1, \dots, z_N) \in \mathbb{R}^d$. The set $\mathcal{Z}_N$ can be treated as the Banach space $E \times \ldots \times E$ (N times), with the norm

$$\|\mathbf{z}\|_{E^N} = \max_{0 \le n \le N} \|z_n\|_E.$$

Let $\mathbf{x} = \{x_0, x_1, \dots, x_N\}$ be a given γ pseudo-trajectory of the system f. Let φ be a given completely continuous mapping. Within this proof, suppose that the value

$$\beta = \gamma + \|f - \varphi\|_\infty \tag{32}$$

satisfies

$$\beta \le \beta(\mathbf{s}, k, \delta). \tag{33}$$

Introduce an operator $H : \mathcal{Z}_N \mapsto \mathcal{Z}_N$, which transforms $\mathbf{z} = (z_0,\ z_1,\ \dots,\ z_N) \in \mathcal{Z}_N$ into $H(\mathbf{z}) = \mathbf{w} = (w_0,\ w_1,\ \dots,\ w_N) \in \mathcal{Z}_N$ defined by the relations

$$P^s_{x_0} w_0 = 0 \tag{34}$$

and

$$P^s_{x_n} w_n = P^s_{x_n}(\varphi(x_{n-1} + z_{n-1}) - x_n) \tag{35}$$

for $n = 1, 2, \dots, N$, and the relations

$$P^u_{x_N} w_N = 0 \tag{36}$$

and

$$\begin{aligned} P^u_{x_{n-1}} w_{n-1} \quad = \quad & Q_{x_{n-1}, z_{n-1}}(P^u_{x_n}(-\varphi(x_{n-1} + z_{n-1}) + f(x_{n-1} + z_{n-1}) \\ & + x_n - f(x_{n-1} + P^s_{x_{n-1}} z_{n-1}) + z_n)) \end{aligned} \tag{37}$$

for $n = 0, 1, \dots, N-1$.

Consider the set

$$S(\beta) = \{\mathbf{z} \in \mathcal{Z}_N : \ \|P^s_{x_n} z_n\|_E \le a\beta \text{ and } \|P^u_{x_n} z_n\|_E \le b\beta,\ n = 0, 1, \dots, N\ \}. \tag{38}$$

By (28) and (33)

$$a\beta \le a\beta(\mathbf{s}, k, \delta) \le \delta, \qquad b\beta \le b\beta(\mathbf{s}, k, \delta) \le \delta$$

and so by (9) trajectories from $\mathcal{S}(\beta)$ belong to X.

Lemma 3.

a. *The operator H is defined and completely continuous for* $\mathbf{z}$ *belonging to the set* $S(\beta)$.

b. *For any fixed point* $\mathbf{z} = (z_0, z_1, \ldots, z_N) \in S(\beta)$ *of* H*, the sequence*

$$\mathbf{y} = \{x_0 + z_0,\ x_1 + z_1,\ \ldots,\ x_N + z_N\}$$

is a trajectory of the system φ.

Proof: **a**. Clearly, by (33) the right hand side of (35) is defined and completely continuously depends on $\mathbf{z} \in S(\beta)$. So we need only prove that for any $n = 1, 2, \ldots, N$ the right hand side of the finite-dimensional equality (37) is defined and continuous for $\mathbf{z} \in S(\beta)$. By Lemma 2 it is sufficient to establish the inequality

$$\|P^u_{x_n}(-\varphi(x_{n-1}+z_{n-1})+f(x_{n-1}+z_{n-1})+x_n-f(x_{n-1}+P^s_{x_{n-1}}z_{n-1})+z_n)\|_E \le \lambda_u\delta\ .$$

Rewrite the last inequality in the form

$$\|J_1 + J_2 + J_3\|_E \le \lambda_u\delta,$$

where

$$\begin{aligned} J_1 &= P^u_{x_n}(-\varphi(x_{n-1}+z_{n-1}) + f(x_{n-1}+z_{n-1}) + x_n - f(x_{n-1}))\ , \\ J_2 &= P^u_{x_n}(f(x_{n-1}) - f(x_{n-1}+P^s_{x_{n-1}}z_{n-1}))\ , \\ J_3 &= P^u_{x_n}z_n\ . \end{aligned} \tag{39}$$

Estimate $\|J_1\|_E, \|J_2\|_E, \|J_3\|_E$. To estimate $\|J_1\|_E$, note that by (32) $\|\varphi - f\|_\infty \le \beta - \gamma$ and also $\|x_n - f(x_{n-1})\|_E \le \gamma$, so by the property SH1

$$\|J_1\|_E \le \beta k. \tag{40}$$

From the inequality (12),

$$\|J_2\|_E \le \mu_u\|P^s_{x_{n-1}}z_{n-1}\|_E. \tag{41}$$

Clearly,

$$\|J_3\|_E = \|P^u_{x_n}z_n\|_E. \tag{42}$$

On the other hand, $\mathbf{z} \in S(\beta)$ implies that

$$\|P^s_{x_{n-1}}z_{n-1}\|_E \le \beta a, \qquad \|P^u_{x_n}z_n\|_E \le \beta b. \tag{43}$$

From by (41) – (43) $\|J_1+J_2+J_3\|_E \leq \|J_1\|_E+\|J_2\|_E+\|J_3\|_E \leq \beta(k+a\mu_u+b)$ and it remains to establish the inequality

$$\beta(k + a\mu_u + b) \leq \lambda_u \delta. \tag{44}$$

From (26), (27) it is seen that

$$\begin{aligned} a &= k\frac{\lambda_u - 1 + \mu_s}{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u}, \\ b &= k\frac{1 - \lambda_s + \mu_u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u}. \end{aligned}$$

Put $k + a\mu_u + b = \lambda_u b$, in rewrite (44) as

$$\beta\lambda_u b \leq \lambda_u \delta \tag{45}$$

But then (45) follows from (28) and assertion **a** is proved.

b. It is sufficient to establish that

$$x_n + z_n = \varphi(x_{n-1} + z_{n-1}), \qquad n = 1, 2, \ldots, N. \tag{46}$$

Because **z** is a fixed point of H, equations (35) and (37) can be rewritten as

$$P^s_{x_n} z_n = P^s_{x_n}(\varphi(x_{n-1} + z_{n-1}) - x_n), \tag{47}$$

and

$$\begin{aligned} P^u_{x_{n-1}} z_{n-1} \;\; = \;\; & Q_{x_{n-1},z_{n-1}}(P^u_{x_n}(-\varphi(x_{n-1} + z_{n-1}) + f(x_{n-1} + z_{n-1}) + \\ & x_n - f(x_{n-1} + P^s_{x_{n-1}} z_n) + z_n)) \;. \end{aligned} \tag{48}$$

From (47) it follows that

$$P^s_{x_n}(x_n + z_n) = P^s_{x_n}\varphi(x_{n-1} + z_{n-1}). \tag{49}$$

Applying the nonlinear, finite-dimensional operator $F_{x_{n-1},z_{n-1}} = Q^{-1}_{x_{n-1},z_{n-1}}$ to both sides of (48), obtain

$$\begin{gathered} P^u_{x_n}\Big(f(x_{n-1} + z_{n-1}) - f(x_{n-1} + P^s_{x_{n-1}} z_{n-1})\Big) = \\ P^u_{x_n}\left((x_n + z_n - \varphi(x_{n-1} + z_{n-1})) + (f(x_{n-1} + z_{n-1}) - f(x_{n-1} + P^s x_{n-1} z_{n-1}))\right). \end{gathered}$$

and simplifying

$$0 = P^u_{x_n}(z_n - \varphi(x_{n-1} + z_{n-1}) + x_n).$$

That is,

$$P^u_{x_n}(x_n + z_n) = P^u_{x_n}\varphi(x_{n-1} + z_{n-1}) \tag{50}$$

and (46) follows from (49) and (50). The lemma is proved. □

Lemma 4. *The set $S(\beta)$ is invariant for H.*

Proof: First, rewrite (35) in the form

$$P^s_{x_n} w_n = (I_1 + I_2),$$

where

$$\begin{aligned} I_1 &= P^s_{x_n}\left((\varphi(x_{n-1}+z_{n-1}) - f(x_{n-1}+z_{n-1})) + (f(x_{n-1}) - x_n)\right) \ , \\ I_2 &= P^s_{x_n}\left(f(x_{n-1}+z_{n-1}) - f(x_{n-1})\right) \ . \end{aligned}$$

Similarly, rewrite (37) in the form

$$P^u_{x_{n-1}} w_{n-1} = Q_{x_{n-1},z_{n-1}}(J_1 + J_2 + J_3)$$

where J_1, J_2, J_3 are defined in (39).

Estimate $\|I_1\|_E$ and $\|I_2\|_E$. To estimate $\|J_1\|_E$ remark that by (32) $\|\varphi - f\|_\infty \le \beta - \gamma$ and also $\|x_n - f(x_{n-1})\|_E \le \gamma$, so by the property SH1

$$\|I_1\|_E \le k\beta. \tag{51}$$

By (10), (11)

$$\|I_2\|_E \le \lambda_s \|P^s_{x_{n-1}} z_{n-1}\|_E + \mu_s \|P^u_{x_{n-1}} z_{n-1}\|_E. \tag{52}$$

Now for each $\mathbf{z} \in \mathcal{Z}_N$ define the pair of real nonnegative numbers

$$m^s(\mathbf{z}) = \max_{0\le n\le N} \|P^s_{x_n} z_n\|_E, \qquad m^u(\mathbf{z}) = \max_{0\le n\le N} \|P^u_{x_n} z_n\|_E$$

and denote by $\mathbf{m}(\mathbf{z})$ the two-dimensional column vector with coordinates $m^s(\mathbf{z}), m^u(\mathbf{z})$. From the estimates (51), (52) and definition (38) of $S(\beta)$, it follows that

$$m^s(H\mathbf{z}) = m^s(\mathbf{w}) \le \beta k + \beta\lambda_s a + \beta\mu_s b. \tag{53}$$

Analogously, from (40) – (42), the definition (38) of $S(\beta)$ and Lemma 2, it follows that

$$m^u(H\mathbf{z}) = m^u(\mathbf{w}) \le \lambda_u^{-1}(\beta\mu_u a + \beta b + \beta k). \tag{54}$$

Inequalities (53, (54) are equivalent to the coordinate-wise estimate

$$\mathbf{m}(H z) = \mathbf{m}(\mathbf{w}) \le \beta M(\mathbf{s})\mathbf{a} + \beta\mathbf{k}, \qquad \mathbf{z} \in H. \tag{55}$$

In view of (27) we have

$$\begin{aligned}\beta M(\mathbf{s})\mathbf{a} + \beta\mathbf{k} &= \beta(M(\mathbf{s})(I - M(\mathbf{s}))^{-1} + I)\mathbf{k} \\ &= \beta(I - M(\mathbf{s}))^{-1}\mathbf{k} = \beta\mathbf{a}.\end{aligned}$$

Henceforth, (55) is equivalent to

$$\mathbf{m}(H\mathbf{z}) = \mathbf{m}(\mathbf{w}) \leq \beta\mathbf{a}, \qquad \mathbf{z} \in H,$$

which means that the set $S(\beta)$ is invariant for H. □

Let us finish the proof of Theorem 1. In view of Assertion **a** of Lemma 3, the operator H is completely continuous on the convex set $S(\beta)$ and in view of Lemma 4 $H(\mathcal{S}(\beta)) \subseteq S(\beta)$. So by the Schauder fixed point theorem, H has a fixed point $\mathbf{z} = (z_0, z_1, \ldots, z_N) \in S(\beta)$ and hence by Assertion **b** of Lemma 3 the sequence

$$\mathbf{x}^* = \{x_0 + z_0,\ x_1 + z_1,\ \ldots,\ x_N + z_N\}$$

is a trajectory of the system φ. By definition (38) of the set $S(\beta)$,

$$\|z_n\|_E \leq \|P^s_{x_n} z_n\|_E + \|P^u_{x_n} z_n\|_E \leq (a+b)\beta, \qquad n = 0, 1, \ldots, N,$$

and thus by (28), (32),

$$\|z_n\|_E \leq \alpha(\mathbf{s}, k)(\gamma + \|\varphi - f\|_\infty), \qquad n = 0, 1, \ldots, N,$$

or

$$\|x_n - x^*_n\|_E \leq \alpha(\mathbf{s}, k)(\gamma + \|\varphi - f\|_\infty), \qquad n = 0, 1, \ldots, N.$$

Theorem 1 is proved. □

References

[1] Alexandroff P. and Hopf H., *Topologie*, Springer–Verlag, Berlin, 1974.

[2] Anosov D., Geodesic flows on closed Riemannian manifolds with negative curvature, *Proc. Steklov Inst. Math.* **90** (1967), 1–235.

[3] Diamond P., Kloeden P., Kozyakin V., Krasnosel'skii M. and Pokrovskii A., Robustness of trajectories of dynamical systems with respect to a class of hysteresis perturbations, *Doklady Akademii Nauk*, **343** (1995), No 1, 1 – 3. (English translation to appear in *Physics Doklady*).

[4] Diamond P., Kloeden P., Kozyakin V. and Pokrovskii A., Semi-hyperbolic Mappings, *Journal of Nonlinear Science*, to appear.

[5] Diamond P., Kloeden P., Kozyakin V. and Pokrovskii A., Computer Robustness of Semi-hyperbolic Mappings, *Random & Computational Dynamics*, to appear.

[6] Diamond P., Kloeden P., Kozyakin V. and Pokrovskii A., Expansivity of Semi-hyperbolic Lipschitz Mappings, *Bull. Austral. Math. Soc.*, **51** (1995), 301–308.

[7] Diamond P., Kloeden P., Kozyakin V. and Pokrovskii A., Robustness of observed behaviour of semi-hyperbolic dynamical systems. *Automation and Remote Control*, (Special issue dedicated to 75 years of M. Krasnosel'skii), to appear.

[8] Krasnosel'skii M. and Pokrovskii A., *Systems with Hysteresis*, Springer-Verlag, New York, 1989.

[9] Palis J. and de Melo W., *Geometric Theory of Dynamical Systems. An Introduction*, Springer-Verlag, Heidelberg, 1982.

[10] Pilyugin S. Yu., *The Space of Dynamical Systems with the C^0–Topology,* Lecture Notes in Mathematics Vol. 1571, Springer–Verlag, Heidelberg, 1994.

[11] Riesz F. and Nagy B.Sz., *"Functional Analysis"*, Frederick Ungar Publ. Co, New York, 1978.

[12] Ruelle D., *Elements of Differentiable Dynamics and Bifurcations Theory*, Academic Press, Inc, Boston, 1989.