Computer Robustness of Semi–hyperbolic Mappings

Phil Diamond
Deptartment of Mathematics
University of Queensland,
Brisbane, Qld 4072,
Australia

Peter Kloeden
School of Comp.&Maths.
Deakin University,
Geelong, Vic 3217,
Australia

Victor Kozyakin
IPPI, Russian Academy
19 Ermolovoy str.,
Moscow, 101447,
Russia

Alexei Pokrovskii
Deptartment of Mathematics
University of Queensland,
Brisbane, Qld 4072,
Australia

Abstract

Study of computer modelling of systems might necessarily involve some different ideas from those used when considering the underlying theoretical system acting on a continuum. In this paper the idea of *computer robustness* of a mapping f is introduced. The concept is based on the idea that computer trajectories will bear some relation to what is expected of the true behaviour if there is an inverse shadowing property between f and its realization on a computer.

1. INTRODUCTION

Consider a mapping $f : \Omega \to \Omega \subseteq \Re^N$. Provided that f is sufficiently smooth and hyperbolicity is present, quite a lot can be said about the dynamical system induced by f. Some general results which spring to mind include the Hartman–Grobman Theorem, Stable Manifold Theorem, Shadowing Lemma and structural stability results (see, for example, [6], [9]). Many of these results state that a C^r dynamical system preserves some of its structural properties under a small *smooth perturbation*.

However, many smooth mappings induce very complicated behaviour and are very often investigated computationally. When this is done, f is replaced by a computer realization, or model. This may arise in a number of ways, including perhaps a computational scheme used to solve a system of differential equations, evaluation of the function by approximation, maybe from the discretization imposed by finite machine arithmetic, or some combination of factors. Discrepancies may then occur between the behaviour of such a computer realization of f and its theoretical counterpart:

1. Asymptotically stable fixed points may disappear and be replaced by more complicated structures such as periodic cycles [2].

2. Stable or unstable invariant manifolds may be replaced by clouds of points around a fixed point [2], [3], [4].

3. Trajectories of some mappings f cannot be realized by any discretization of f. For example, $f(x) = 2x \bmod 1$ in binary arithmetic behaves quite differently from the mapping on the continuum $[0, 1]$.

These difficulties suggest that the study of computer modelling of systems might necessarily involve some different ideas from those used when considering the underlying theoretical system acting on a continuum. In this paper we introduce the idea of *computer robustness* of a mapping f. The concept is based on the idea that computer trajectories will bear some relation to what we would expect of the true behaviour if there is "inverse shadowing" between f and its realization on a computer. Let us explain this in greater detail.

Let $\widetilde{f}$ be a computer realization of the theoretical mapping f. It is desirable that estimates of the distance between the trajectories of the two maps should not depend explicitly on the time interval over which the trajectories are taken, but are instead uniform over Ω. For example, in the shadowing lemma ([6], [9]), a pseudo–orbit of a hyperbolic f is close to some trajectory of f. This result is usually interpreted to mean that any computed orbit of the realization $\widetilde{f}$ is close to some true orbit of f as long as both the exact and computed trajectories remain in the domain of hyperbolicity. But it may also be taken as saying that each trajectory of a close *approximation* $\widetilde{f}$ is near to some trajectory of f provided that f is hyperbolic.

There is an *inverse problem* here. In computer modelling it is important that an exact orbit can be closely modelled during the computation. This will be guaranteed if we can specify a natural class of functions $\mathcal{K}$ such that if $\widetilde{f} \in \mathcal{K}$ is a sufficiently close approximation to f, then $\widetilde{f}$ has at least one orbit near to a given trajectory of f. The most commonly investigated class $\mathcal{K}$ is that of C^r approximations. Under natural conditions, such as hyperbolicity, it follows from structural stability that such $\widetilde{f}$ can be found. Beyn [1] investigated this problem for a class of numerical methods for differential equations in the neighbourhood of a hyperbolic stationary point. However, spatial discretization arising from the finiteness of state space on a computer reduces the appropriateness of C^r approximation. Too, there is perhaps a need for some analogue of hyperbolicity itself, so that it relates not only to invariant sets of f but to a larger class of subsets $\Omega \subset \Re^d$.

Let us consider the problem of spatial arithmetic discretization a little more closely. In the computer modelling of $f : \Re^d \to \Re^d$ we have a finite discretized space $\mathrm{L} \subset \Re^d$ and a corresponding discretized mapping $f_{\mathrm{L}} : \mathrm{L} \to \mathrm{L}$ which is close to the restriction of f to L (see for instance Stetter [10].) The lattice L is uniform when fixed point arithmetic is used, and semilogarithmic if floating point is used. In this sort of situation, it is not generally possible always to approximate orbits of f by those of f_{L} – see, for instance, Theorem 5 of [5].

One way around this problem is to introduce a class of *continuous approximations* $f_{\mathbf{I}}$. But then, since C^0 and not C^r approximation is involved, it is not possible to use either standard forms of the shadowing lemma, nor structural stability theorems for the analysis of these computer models. On the other hand, it does turn out that, for what we call semi–hyperbolic mappings, orbits of f can always be approximated by those of $f_{\mathbf{I}}$ which are close to f in the C^0 sense.

This paper consists of four Sections. In Section 2 the idea of a robust trajectory is introduced. Section 3 is devoted to a central notion of the paper: the concept of semi–hyperbolicity. In Section 4 the main theorem about the inverse shadowing property for semi–hyperbolic mappings is formulated and proved. The last section compares the semi–hyperbolicity property with hyperbolic structures and gives an example of a semi–hyperbolic system which is not hyperbolic.

2. ROBUST TRAJECTORIES

Let $f : \Re^d \to \Re^d$. A finite sequence

$$\mathbf{x} = x_0, x_1, \ldots, x_N \tag{1}$$

will be called a (finite) *trajectory* of the mapping f if $x_n = f(x_{n-1}),\ n = 1, 2, \ldots, N$. A distance between two mappings f and φ will be given by the supremum norm

$$||f - \varphi||_\infty = \sup_{x \in \Re^d} \|f(x) - \varphi(x)\|\ ,$$

where $\|\cdot\|$ is a norm in $\Re^d$.

Let α be a positive real number. The trajectory (1) will be called *α–robust* if there exists $\varepsilon_0 > 0$ such that any continuous mapping φ satisfying

$$||f - \varphi||_\infty \le \varepsilon_0 \tag{2}$$

has at least one trajectory $y_0, y_1, \ldots, y_N$ such that

$$\|y_n - x_n\| \le \alpha ||f - \varphi||_\infty, \quad n = 0, 1, \ldots N. \tag{3}$$

Clearly, any trajectory (1) is $(1 + L + \ldots + L^N)$–robust if the mapping f satisfies the Lipschitz condition $\|f(x) - f(y)\| \le L\|x - y\|$ in some neighborhood of the trajectory. But it is a much more significant matter if the robustness constant α is independent of N, and uniformly so throughout a region $\Omega \subseteq \Re^d$. This is possible when f has the semi–hyperbolic property, and this is presented in the next section.

3. SEMI–HYPERBOLICITY

Throughout, the mapping f is assumed smooth in a region containing a set $\Omega \subseteq \Re^d$. The derivative of the mapping f at the point $x \in \Re^d$ will be denoted by Df_x. The four–tuple of nonnegative values

$$\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u), \qquad \lambda_s \le \lambda_u, \tag{4}$$

will be called a *split* if the eigenvalues δ_1 and δ_2 of the matrix

$$\Delta = \begin{pmatrix} \lambda_s & \mu_s \\ \mu_u & \lambda_u \end{pmatrix}$$

are real and satisfy $|\delta_1| < 1 < |\delta_2|$. Clearly, the four–tuple $(\lambda_s, \lambda_u, \mu_s, \mu_u)$ is a split if and only if

$$\lambda_s < 1 < \lambda_u \tag{5}$$

and

$$(1 - \lambda_s)(\lambda_u - 1) > \mu_s \mu_u. \tag{6}$$

For any given λ_s, λ_u satisfying (5) the four–tuple (4) is a split if the product $\mu_s \mu_u$ is small enough.

Given some split $\mathbf{s}$ and a positive real number h, the map f is called $(\mathbf{s}, h)$*–hyperbolic on the set* Ω if for any $x \in \Omega$ there exists decompositions $T_x \Re^d = E_x^s \oplus E_x^u$ with relating projectors P_x^s and P_x^u which satisfy the following inequalities:

$$\|P^s_{f(x)} Df_x u\| \le \lambda_s \|u\|, \quad u \in E_x^s\ ; \tag{7}$$

$$\|P^s_{f(x)} Df_x v\| \le \mu_s \|v\|, \quad v \in E_x^u\ ; \tag{8}$$

$$\|P^u_{f(x)} Df_x v\| \ge \lambda_u \|v\|, \quad v \in E_x^u\ ; \tag{9}$$

$$\|P^u_{f(x)} Df_x u\| \le \mu_u \|u\|, \quad u \in E_x^s\ ; \tag{10}$$

$$\|P_x^s\|, \|P_x^u\| \le h\ . \tag{11}$$

This idea should be distinguished from exponential dichotomies of difference equations [7]: whereas the dichotomy inequalities are symmetric, this is not the case with those above. The reason is that the parameters of a split cannot be simply equated with the exponents of a dichotomy. The quantities μ_s, μ_u are respectively measures of the lack of invariance of the stable component E_x^s and of the unstable component E_x^u. However, the parameters λ_s and λ_u respectively quantify the stability of E_x^s and the instability of E_x^u. These last are more akin to a dichotomous structure. Note also that the "sets where f is hyperbolic" of Robinson [8] and which give an almost invariant splitting of Df_x, arise on neighbourhoods of an invariant set, do not have the particular structure of (6) and are used in a quite different way.

If the mapping f is $(\mathbf{s}, h)$–hyperbolic on a set Ω which is in fact a trajectory of the mapping f, then the corresponding trajectory will be called $(\mathbf{s}, h)$*–hyperbolic*. If the mapping f (or its trajectory) is $(\mathbf{s}, h)$–hyperbolic for at least one split $(\mathbf{s}, h)$, then the mapping f (respectively, its trajectory) will be called *semi–hyperbolic*.

Remark. Although we are here concerned with robustness, which may be thought of as a sort of inverse shadowing property, it is possible to develop a shadowing theory for semi–hyperbolic mappings. Moreover, when Ω is an invariant set, there is some relationship between hyperbolic splitting and the almost invariant splitting, see Section 5.

4. MAIN THEOREM

Theorem 1. *Let the mapping f be defined and semi–hyperbolic on the open set $\Omega \subseteq \Re^d$. Then there exists $\alpha > 0$ such that any finite trajectory of f, which is wholly contained in Ω, is α–robust.*

The proof of Theorem 1 and explicit estimates for the value of α will follow from the next theorem.

Theorem 2. *Each $(\mathbf{s}, h)$–hyperbolic trajectory of a smooth mapping f is α–robust for every*

$$\alpha > \alpha_*(\mathbf{s}, h) = \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s \mu_u} h \ . \tag{12}$$

Proof: Let $\mathbf{x} = x_0, x_1, \ldots, x_N$, be the given trajectory of the mapping f,

$$x_{n+1} = f(x_n), \qquad n = 0, 1, \ldots, N.$$

Denote by $\mathcal{B}$ the space of $N-$sequences

$$\mathbf{z} = z_0, z_1, \ldots, z_N, \quad z_n \in \Re^d, \tag{13}$$

satisfying

$$P^s_{x_0} z_0 = P^u_{x_N} z_N = 0. \tag{14}$$

The set $\mathcal{B}$ can be treated as a subspace of the Nd–dimensional vector space $\Re^d \times \ldots \times \Re^d$ (N times), with the norm

$$\|\mathbf{z}\| = \max_{0 \le n \le N} \|z_n\| \ .$$

Let $\varphi : \Re^d \to \Re^d$ be a given mapping. Define an operator $W_\varphi : \mathcal{B} \to \mathcal{B}$, which transforms every sequence (13) into a sequence $\mathbf{w} = w_0, w_1, \ldots, w_N$ defined by the initial conditions (14) and the relations

$$\begin{aligned} P^s_{x_n} w_n &= P^s_{x_n}(\varphi(x_{n-1} + z_{n-1}) - x_n) \ , \\ P^u_{x_{n-1}} w_{n-1} &= (U_n)^{-1}(P^u_{x_n} z_n - P^u_{x_n} Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1} + \\ &\quad P^u_{x_n}(-\varphi(x_{n-1} + z_{n-1}) + x_n + Df_{x_{n-1}} z_{n-1})) \ , \end{aligned}$$

where $U_n : E^u_{x_{n-1}} \to E^u_{x_n}$, defined by $U_n v = P^u_{x_n} Df_{x_{n-1}} v$, is surjective. Note that $(U_n)^{-1}$ is well–defined by virtue of the inequality (9). The following lemma is immediate.

Lemma 1. *W_φ is continuous. For any fixed point $\mathbf{z}^* = z_0^*, z_1^*, \dots, z_N^*$ of W_φ, the sequence*

$$\mathbf{y}^* = x_0 + z_0^*,\ x_1 + z_1^*,\ \dots,\ x_N + z_N^*$$

is a trajectory of the mapping φ.

We require a few more notations and definitions. For any $\beta > 0$, denote by $\delta_\beta(\varepsilon)$ the largest positive value δ such that, for any $\|z\| \le \delta$, the following inequality is valid:

$$\|x_n + Df_{x_{n-1}} z - f(x_{n-1} + z)\| \ \le\ \beta\,\varepsilon.$$

For each $\mathbf{z} \in \mathcal{B}$ define the pair of real numbers

$$V^s(\mathbf{z}) = \max_{0 \le n \le N} \|P^s_{x_n} z_n\| \ , \tag{15}$$

$$V^u(\mathbf{z}) = \max_{0 \le n \le N} \|P^u_{x_n} z_n\| \ , \tag{16}$$

and denote by $\mathbf{V}(\mathbf{z})$ the two–dimensional column vector with coordinates $V^s(\mathbf{z})$, $V^u(\mathbf{z})$. Define the matrix

$$M = \begin{pmatrix} \lambda_s & \mu_s \\ \mu_u/\lambda_u & 1/\lambda_u \end{pmatrix}, \tag{17}$$

and the column vector

$$\mathbf{h} = (h, h/\lambda_u)^T .$$

Lemma 2. *Let $\beta > 0$. Then for each continuous mapping φ and each $\mathbf{z}$ from the set $\mathcal{W}_{\varphi,\beta} = \{\mathbf{z} \in \mathcal{B} : \|\mathbf{z}\| \le \delta_\beta(||f - \varphi||_\infty)\}$,*

$$\mathbf{V}(W_\varphi(\mathbf{z})) \le M\,\mathbf{V}(\mathbf{z}) + (1+\beta)||f - \varphi||_\infty \mathbf{h} \ . \tag{18}$$

Proof: First, estimate the value of $V^s(W_\varphi(\mathbf{z}))$. By definition

$$V^s(W_\varphi(\mathbf{z})) = \max_{0 \le n \le N} \|v^s_n\| \ , \tag{19}$$

where

$$v^s_n = P^s_{x_n}(\varphi(x_{n-1} + z_{n-1}) - x_n) \ . \tag{20}$$

Rewrite (20) as

$$v^s_n = I_1 + I_2 + I_3 + I_4 \ , \tag{21}$$

where

$$
\begin{aligned}
I_1 &= P^s_{x_n} Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1} \ , \\
I_2 &= P^s_{x_n} Df_{x_{n-1}} P^u_{x_{n-1}} z_{n-1} \ , \\
I_3 &= P^s_{x_n} (\varphi(x_{n-1} + z_{n-1}) - f(x_{n-1} + z_{n-1})) \ , \\
I_4 &= P^s_{x_n} (f(x_{n-1} + z_{n-1}) - (f(x_{n-1}) + Df_{x_{n-1}} z_{n-1})) \ .
\end{aligned}
$$

From (7),

$$\|I_1\| \ \leq \ \lambda_s \|P^s_{x_{n-1}} z_{n-1}\| \ , \tag{22}$$

and from (8),

$$\|I_2\| \ \leq \ \mu_s \, \|P^u_{x_{n-1}} z_{n-1}\| \ . \tag{23}$$

The relations (11) imply that

$$\|I_3\| \ \leq \ h \, ||f - \varphi||_\infty \ . \tag{24}$$

Lastly, the relations (11) and the definition of $\delta_\beta(||f - \varphi||_\infty)$ imply that

$$\|J_4\| \ \leq \ h\beta \, ||f - \varphi||_\infty \ . \tag{25}$$

From (21) and (22)–(25) it follows that

$$\|v^s_n\| \ \leq \ \lambda_s \, \|P^s_{x_{n-1}} z_{n-1}\| + \mu_s \, \|P^u_{x_{n-1}} z_{n-1}\| + (1 + \beta) \, ||f - \varphi||_\infty h \ . \tag{26}$$

By (19) we can rewrite (26) as

$$V^s(W_\varphi(\mathbf{z})) \ \leq \ \lambda_s \, (V^s(\mathbf{z}) + \mu_s \, V^s(\mathbf{z}) + (1 + \beta) \, ||f - \varphi||_\infty \, h) \ . \tag{27}$$

Now estimate the value of $V^u(W_\varphi(\mathbf{z}))$. By definition,

$$V^u(W_\varphi(\mathbf{z})) = \max_{0 \leq n \leq N} \|v^u_n\| \ , \tag{28}$$

where

$$
\begin{aligned}
v^u_{n-1} \ = \ & (U_n)^{-1} (P^u_{x_n} z_n - P^u_{x_n} Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1} + \\
& P^u_{x_n} (-\varphi(x_{n-1} + z_{n-1}) + f(x_{n-1}) + Df_{x_{n-1}} z_{n-1})) \ .
\end{aligned}
$$

Rewrite this last equation as

$$v^u_{n-1} = (U_n)^{-1} J_1 + (U_n)^{-1} J_1 + (U_n)^{-1} J_2 + (U_n)^{-1} J_3 + (U_n)^{-1} J_4 \ , \tag{29}$$

with

$$
\begin{aligned}
J_1 &= P^u_{x_n} z_n \ , && (30)\\
J_2 &= -P^u_{x_n} Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1} \ , && (31)\\
J_3 &= P^u_{x_n}(-\varphi(x_{n-1}+z_{n-1}) + f(x_{n-1}+z_{n-1}) \ , && (32)\\
J_4 &= -f(x_{n-1}+z_{n-1}) + (f(x_{n-1}) + Df_{x_{n-1}} z_{n-1}) \ . && (33)
\end{aligned}
$$

The relations (9) and (30) imply that

$$\|(U_n)^{-1} J_1\| \ \le \ \lambda_u^{-1} \|P^u_{x_n} z_n\| \ , \tag{34}$$

while the relations (9), (10) and (31) imply that

$$\|(U_n)^{-1} J_2\| \ \le \ \lambda_u^{-1} \mu_u \|P^s_{x_{n-1}} z_{n-1}\| \ . \tag{35}$$

The relations (9), (11), (32) give

$$\|(U_n)^{-1} J_3\| \ \le \ \lambda_u^{-1} h \, ||f-\varphi||_\infty \ . \tag{36}$$

Finally, the relations (9), (11), (33) and the definition of $\delta_\beta(||f-\varphi||_\infty)$ imply

$$\|(U_n)^{-1} J_4\| \ \le \ \lambda_u^{-1} h\beta \, ||f-\varphi||_\infty \ . \tag{37}$$

From (29) and (34)–(37) it follows that

$$\|v^u_{n-1}\| \ \le \ \lambda_u^{-1} \left(\|P^u_{x_n} z_n\| + \mu_u \|P^s_{x_{n-1}} z_{n-1}\| + (1+\beta) \, ||f-\varphi||_\infty \, h\right) \ . \tag{38}$$

By (28) we can rewrite (38) as

$$V^u(W_\varphi(\mathbf{z})) \ \le \ \lambda_u^{-1} \left(V^u(\mathbf{z}) + \mu_u \, V^s(\mathbf{z}) + (1+\beta) \, ||f-\varphi||_\infty \, h\right) \ . \tag{39}$$

Inequalities (27) and (39) are equivalent to the assertion of the lemma.□

Let us return to and complete the proof of Theorem 2. The spectral radius $\sigma(M)$ of the matrix

$$M = \begin{pmatrix} \lambda_s & \mu_s \\ \dfrac{\mu_u}{\lambda_u} & \dfrac{1}{\lambda_u} \end{pmatrix}$$

is just

$$\sigma(M) = \frac{1}{2}\left(\left(\frac{1}{\lambda_u} + \lambda_s\right) + \sqrt{\left(\frac{1}{\lambda_u} - \lambda_s\right)^2 + \frac{4\mu_s\mu_u}{\lambda_u}}\right).$$

The entries of the matrix M are positive. Therefore by the Perron-Frobenius theorem the spectral radius $\sigma(M)$ is the maximal eigenvalue and the corresponding eigenvector has positive coordinates. Without loss of generality, assume that this eigenvector takes the form $(1,\gamma)^t$, where

$$\gamma = \frac{1}{2\mu_s}\left(\left(\frac{1}{\lambda_u} - \lambda_s\right) + \sqrt{\left(\frac{1}{\lambda_u} - \lambda_s\right)^2 + \frac{4\mu_s\mu_u}{\lambda_u}}\right).$$

It follows that

$$\begin{pmatrix} \lambda_s & \mu_s \\ \frac{\mu_u}{\lambda_u} & \frac{1}{\lambda_u} \end{pmatrix}\begin{pmatrix} 1 \\ \gamma \end{pmatrix} = \sigma(M)\begin{pmatrix} 1 \\ \gamma \end{pmatrix}.$$

In $\Re^2$ introduce the auxiliary norm $\|\cdot\|_*$ by $\|(y_1,y_2)^t\|_* = \max\{\gamma|y_1|, |y_2|\}$. Clearly, the corresponding norm $\|M\|_*$ of the linear operator with the matrix (17) coincides with the spectral radius of M. Therefore, $\|My\|_* \le \sigma(M)\|y\|_*$ for all $y \in \Re^2$. Hence, by Lemma 2, for any positive β we have

$$\|\mathbf{V}(W_\varphi(\mathbf{z}))\|_* \;\le\; \sigma(M)\,\|\mathbf{V}(\mathbf{z})\|_* + (1+\beta)\,||f-\varphi||_\infty\,\|\mathbf{h}\|_*\;,\;\; \mathbf{z} \in \mathcal{W}_{\varphi,\beta}\;. \quad (40)$$

Choose a fixed real number $\alpha > \alpha_*(\mathbf{s},h)$, where $\alpha_*(\mathbf{s},h)$ is defined by (12), and write $\beta = \alpha/\alpha_*(\mathbf{s},h) - 1$. Note that by (5) and (6)

$$\sigma(M) \;<\; 1\;. \quad (41)$$

Clearly there exists $\varepsilon_0 > 0$ such that for

$$||f-\varphi||_\infty \;\le\; \varepsilon_0 \quad (42)$$

we have the inclusion

$$\left\{\mathbf{z}:\; ||\mathbf{V}(\mathbf{z})\|_* \;\le\; \frac{1+\beta}{1-\sigma(M)}\,\|\mathbf{h}\|_*\,(||f-\varphi||_\infty)\right\} \;\subseteq\; \mathcal{W}_{\varphi,\beta}\;.$$

By (40) and (41), for any f satisfying (42), the set

$$\mathcal{V}_{f,\beta} = \left\{\mathbf{z}:\; ||\mathbf{V}(\mathbf{z})\|_* \;\le\; \frac{1+\beta}{1-\sigma(M)}\,\|\mathbf{h}\|_*\,(||f-\varphi||_\infty)\right\}$$

is invariant for the operator W_φ. Then, because of the continuity of W_φ (see Lemma 1), there exists a point $\mathbf{z}^*$ satisfying $W_\varphi \mathbf{z}^* = \mathbf{z}^*$, such that

$$\mathbf{z}^* \in \mathcal{W}_{\varphi,\beta} \ . \tag{43}$$

From (43) and (18) it follows that

$$\mathbf{V}(W_\varphi(\mathbf{z}^*)) \ \leq \ M\,\mathbf{V}(\mathbf{z}^*) + (1+\beta)\,||f-\varphi||_\infty\,\mathbf{h}\ ,$$

and moreover that

$$\mathbf{V}(\mathbf{z}^*) \ \leq \ \frac{1+\beta}{1-M}\,||f-\varphi||_\infty\,\mathbf{h}\ .$$

In particular, $V^s(\mathbf{z}^*) + V^u(\mathbf{z}^*) \leq \alpha||f-\varphi||_\infty$. Further,

$$\max_{0\leq n\leq N} \|\mathbf{z}_n^*\| \ \leq \ \alpha\,||f-\varphi||\ . \tag{44}$$

By (44) and Lemma 1, for any continuous mapping φ satisfying (2) the sequence $x_0 + z_0^*$, $x_1 + z_0^*$, $\ldots$, $x_N + z_N^*$ is a trajectory of φ and satisfies (3). That is, the trajectory (1) is α–robust and the theorem is proved. □

5. SEMI–HYPERBOLICITY AND HYPERBOLICITY

The notion of semi–hyperbolicity introduced in Section 3 bears some similarity to that of a hyperbolic structure of a mapping (see for example [6]), but nevertheless differs from the latter in some essential respects.

First, it must be stressed that the idea of a hyperbolic structure is specific to invariant sets of a mapping. This is sufficient when one is interested in such properties of a mapping as its structural stability. But this approach is far from adequate when one is concerned with investigating the phase portrait of a mapping in some region that is not an invariant set. The notion of the semi–hyperbolicity precisely covers this case.

Splitting the space $\Re^d$ into the direct sum of subspaces E^s_x and E^u_x in Section 3 is similar to imposing a hyperbolic structure $E^s_\Lambda \oplus E^u_\Lambda$ on the invariant set Λ of a given mapping. But in the semi–hyperbolic case there are no demands of continuous dependence of subspaces E^s_x and E^u_x from x while for hyperbolicity these subspaces must depend on x continuously. Besides, in the hyperbolic case both subspaces E^s_x and E^u_x are invariant with respect to the differential of the mapping, and to verify the existence of such subspaces can be difficult in practice. For semi–hyperbolicity these subspaces are only required to be nearly invariant subspaces of the differential of the mapping, which can be more easily verified in many instances.

Semi-hyperbolicity seems to be especially well suited for analysis of noninvertible mappings, when the straightforward analogues of standard hyperbolic splittings often do not exist at all. An example of this kind is discussed in more detail below. Recall that the decomposition

$$T_x\Re^d = E^s_x \oplus E^u_x, \qquad x \in \Omega \tag{45}$$

is said to be *hyperbolic for f* if $\sup_{x\in\Omega}\{\|P^s_x\|\} < \infty$, the decomposition is invariant with respect to Df and

$$\begin{aligned} \|(Df^n)_x u\| &\leq a\lambda^n \|u\|, \qquad u \in E^s_x, \quad n = 1, 2, \dots\ , \\ \|(Df)_x v\| &\geq a^{-1}\mu^n \|v\|_\rho, \qquad u \in E^u_x \quad n = 1, 2, \dots\ . \end{aligned}$$

for some $\lambda < 1 < \mu$ and $a > 0$.

Consider the space $\Re^6$. We shall interpret points as of triplets $\mathbf{z} = (z_1, z_2, z_3)$ of complex numbers. Define a norm in $\Re^6$ by

$$|\mathbf{z}| = \max\{|z_1|, |z_2|, |z_3|\}.$$

Let $\Omega = \{(z_1, z_2, z_3) : |z_1| = |z_2| = |z_3| = 1\}$ and $U = \{\mathbf{z} : z_k \neq 0, k = 1, 2, 3\}$. There is a natural projection

$$\mathrm{P}(\mathbf{z}) = (z_1/|z_1|, z_2/|z_2|, z_3/|z_3|), \qquad \mathbf{z} \in U.$$

of U onto Ω. For each $\mathbf{z} \in \Omega$ denote

$$e_1(\mathbf{z}) = (iz_1, z_2, z_3), \quad e_2(\mathbf{z}) = (z_1, iz_2, z_3), \quad e_3(\mathbf{z}) = (z_1, z_2, iz_3), \tag{46}$$

These vectors form a basis of the tangent subspace $E^t_{\mathbf{z}} \subset \Re^6$, $\mathbf{z} \in \Omega$; denote also by $E^n_{\mathbf{z}}$ the corresponding normal space. The coordinates of $\mathbf{z} \in \Omega$ can be represented as $\mathbf{z} = (e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3})$. So points $\mathbf{z} \in \Omega$ may be identified with real triples $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$, $0 \leq \varphi_i < 2\pi$, $i = 1, 2, 3$. That is, Ω is an immersion of the 3-torus $\mathbf{T}^3$ in $\Re^6$.

Let $A = (a_{ij})$ be 3×3 matrix with integer entries a_{jk}. The matrix A defines an algebraic automorphism of $\mathbf{T}^3$ by $\boldsymbol{\varphi} \mapsto A\boldsymbol{\varphi}$, which can be described by

$$F_A(\mathbf{z}) = (z_1^{a_{11}} z_2^{a_{12}} z_3^{a_{13}},\ z_1^{a_{21}} z_2^{a_{22}} z_3^{a_{23}},\ z_1^{a_{31}} z_2^{a_{32}} z_3^{a_{33}}), \quad z \in \Omega.$$

Extend F_A to the whole of U by defining $F_A(\mathbf{z}) = F_A(\mathrm{P}(\mathbf{z}))$ for $\mathbf{z} \in U$.

In particular, consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_{1,2} = 2 \pm \sqrt{3}$, $\lambda_3 = 2$ with corresponding eigenvectors $\mathbf{v}_{1,2} = (1, \pm 1/\sqrt{3}, 0)$ and $\mathbf{v}_3 = (0, 0, 1)$. Then on Ω the mapping F_A admits the hyperbolic decomposition

$$T_{\mathbf{z}} = E^s_{\mathbf{z}} \oplus E^u_{\mathbf{z}}, \qquad x \in \Omega \tag{47}$$

defined by

$$E^s_{\mathbf{z}} = L(e_1(\mathbf{z}) - (1/\sqrt{3})e_2(\mathbf{z})) \oplus E^n_{\mathbf{z}}$$

and

$$E^u_{\mathbf{z}} = L(e_1(\mathbf{z}) + (1/\sqrt{3})e_2(\mathbf{z})) \oplus L(e_3(\mathbf{z})).$$

Here $L(\mathbf{v}) = \{\alpha\mathbf{v} : -\infty < \alpha < \infty\}$, $\mathbf{v} \in \Re^6$. Denote F_ε: $\Omega \to \Omega$ a mapping which satisfies the following conditions.

C1. $\|F_\varepsilon - F_A\|_{C^1} < \varepsilon$.

C2. $F_\varepsilon(x) = F_A(x)$ in the union of the set $S_1 = \{\mathbf{z} \in \Omega : z_1 = z_2 = 1\}$ and the set $S_2 = \{\mathbf{z} \in \Omega : \inf_{\mathbf{y}\in S_1} |\mathbf{y} - \mathbf{z}| < \varepsilon,\ |z_3 + 1| > 1/4\}$.

C3. $M\mathbf{v}_1 \notin E^u_{\mathbf{1}}$ where M is derivative of F_ε at the point $(1, 1, -1)$, and $\mathbf{1}$ denotes the vector $(1, 1, 1)$.

Extend F_ε to the whole of U by defining $F_\varepsilon(\mathbf{z}) = F_\varepsilon(\mathrm{P}(\mathbf{z}))$.

Proposition 1. *For sufficiently small ε the mapping F_ε admits a semi-hyperbolic decomposition, but admits no hyperbolic decomposition.*

Proof: The fact that the decomposition (47) is *semi-hyperbolic* for sufficiently small ε follows from the definitions and condition C1. It remains to prove that there exists no any *hyperbolic* decomposition. Suppose the contrary. Let

$$T_{\mathbf{z}} \ = \ E^s_{\varepsilon,\mathbf{z}} \oplus E^u_{\varepsilon,\mathbf{z}} \ , \quad \mathbf{z} \in \Omega \tag{48}$$

be a hyperbolic decomposition. Because $\mathbf{z} = \mathbf{1}$ is a fixed point of F_A, the decomposition (48) coincides at this point with the decomposition (47) by condition C2. For any integer positive N consider the point $\mathbf{z}_N = (1, 1, e^{i\varphi_N}) \in S_1 \bigcup S_2 \subset \Omega$, where $\varphi_N = \pi 2^{-N+1}$. By condition C2

$$\mathbf{z}_{N,n} \ = \ F^n_\varepsilon(\mathbf{z}_N) \ = \ \begin{cases} \mathbf{z}_{N-n}, & \text{if} \quad n < N \ , \\ \mathbf{0}, & \text{if} \quad n \geq N \ . \end{cases}$$

The subspace $E^u_{\varepsilon,\mathbf{z}_N}$ is contained in $E^t_{\mathbf{z}_N}$ and is transversal to $E^s_{\varepsilon,\mathbf{z}_N}$ for sufficiently large N. Together with condition C2 this implies that for all sufficiently large N the subspace $E^u_{\varepsilon,\mathbf{z}_{N,N-1}}$ almost contains the vector $\mathbf{v}_1$, because it is an eigenvector of the matrix A with the maximal eigenvalue $\lambda_1 = 2 + \sqrt{3}$. But the image of the vector $\mathbf{v}_1$ with respect to the mapping $(DF_\varepsilon)(\mathbf{z}_{N,N-1}) = M$ doesn't belong to $E^u_{\varepsilon,\mathbf{1}} = E^u_{\mathbf{1}}$, because condition C3 holds. This contradicts invariance of the decomposition (48). ∎

This research has been supported by the Australian Research Council Grant A 89132609.

Permanent address of Pokrovskii: Institute of Information Transmission Problems, Russian Academy of Science, Ermolovoy 19, Moscow, Russia

References

[1] M.-J. Beyn, On the numerical approximation of phase portraits near stationary points, *SIAM J. Numer. Anal.*, **24** (1987), 1095–1113.

[2] P.M. Binder, Machine iteration of a linear function: Local behaviour, *Computers Math. Applic.*, **21** (1991), 133–140.

[3] M. Blank, Pathologies generated by round-off in dynamical systems, *Physica D* **78** (1994), 93-114.

[4] P. Diamond and P. Kloeden, Spatial discretization of mappings, *Computers Math. Applic.*, **25** (1993), 85–94.

[5] P. Diamond, P. Kloeden and A. Pokrovskii, Weakly chain recurrent points and spatial discretizations of dynamical systems, *Random & Computational Dynamics*, **2** (1994), 97-110.

[6] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields*, Springer-Verlag, New–York, 1983.

[7] K.J. Palmer, Exponential dichotomies, the shadowing lemma and transerval homoclinic points, *Dynamics Reported, Vol. I*, U. Kirchgraber and H.O. Walther (ed.), John Wiley & Sons and B.G. Teubner, New York, 1988.

[8] C. Robinson, Stability theorems and hyperbolicity in dynamical systems, *Rocky Mountain J. Math.*, **7** (1977), 425–437.

[9] M. Shub, *Global Analysis of Dynamical Systems*, Springer–Verlag, New York, 1987.

[10] H.J. Stetter, *Analysis of Discretization Methods for Ordinary Differential Equations*, Springer–Verlag, Berlin, 1973.