A model for roundoff and collapse of chaotic dynamical systems *

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Abstract

Computer simulations of dynamical systems contain discretizations, *where finite machine arithmetic replaces continuum state space. For chaotic dynamical systems, the main features of this discretization are stochastically related to the parameters both of the underlying continuous system and of the computer arithmetic. A model of this process is required to to describe and analyze its statistical properties and this is carried out for the family of mappings $f_\ell(x) = 1 - |1 - 2x|^\ell$, $x \in [0, 1]$, $\ell > 2$. Computer modeling results are presented.*

1 Introduction

Chaotic mappings have trajectories which are exponentially sensitive to initial conditions and which behave apparently randomly. Interesting questions

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arise in analysis of computer simulations of chaotical systems. Many reasonable computer realizations of a dynamical system can be treated as a deterministic mapping of a finite subset L into itself and we will consider only realizations of such type. Such discretizations are also very sensitive to initial conditions and perturbations in the function, but each trajectory of a spatial discretization is eventually periodic, in contrast to chaotic orbits in a continuum. Consequently, the properties of a discretization are those of its cycles. Some such characteristics are:

1. The maximal length of cycles of a discretization [1, 11];
2. The proportion of initial points $\xi \in \mathrm{L}$ which collapse to a very short cycle [5, 6];
3. The length distribution of the transient, nonperiodic part, of a trajectory or its limit cycle, see [8] and references therein.

Whereas the Sinai-Ruelle-Bowen (SRB) invariant measures [12] describe the typical behaviour of trajectories of the original system for nearly all initial conditions, it is not yet clear what nontrivial statistical characteristics will suffice for discretizations. However, one more level of averaging can be done to obtain meaningful results. Rather than considering systems behaviour only with respect to a collection of randomly selected initial conditions, one can study an ensemble of discretizations on different lattices, or an ensemble of discretizations of different mappings for the same lattices, or both. Statistical properties of such ensembles are sufficiently robust and can be investigated in detail. See, for example, scaling of average length of the maximal cycle [1] and the analysis of average cycle length in [8]. It is important to develop models of the discretization process which will predict statistical properties of the characteristics enumerated above in terms of original mappings and demonstrate relationships between some of these different characteristics.

A canonical model of this type was suggested in [8] for mappings with a unique SRB measure μ. It can be roughly described as a completely random mapping [2] of the set $X(N) = \{0, 1, \ldots, N\}$ into itself when $N \sim 1/h^{\dim_c(\mu)}$, where h is the space step of the discretization and $\dim_c(\mu)$ is the correlation dimension of the measure μ [7]. This model was, in turn, a revision of an earlier model suggested in [11] where the Hausdorff dimension of the support $\mathrm{supp}(\mu)$ was used instead of the correlation dimension. This model works perfectly if the measure μ is spread fairly uniformly over its support, as happens if the correlation dimension coincides with the upper boxing dimension [7]. If this uniformity is not present, then the model is not quite adequate. For instance, it fails to explain collapsing effects [5] in computer simulations.

In this paper we discuss a rather different phenomenological model, which incorporates a special kind of random mapping. To get comparable results we will consider the application of this model to the classical family

$$f_\ell(x) = 1 - |2x-1|^\ell, \qquad 0 \le x \le 1 \tag{1}$$

where $\ell > 2$ is a parameter. Each mapping above has an absolutelly continuous SRB invariant measure μ_ℓ with a positive density and asymptotics $\mu_\ell[0,\gamma], \mu_\ell[1-\gamma,1] \sim \gamma^{1/\ell},\ \ \gamma \to 0+$. The correlation dimension of μ_ℓ is thus $2/\ell$, and differs from the upper box dimension of $\mathrm{supp}(\mu_\ell)$, which is 1.

1.1 Characteristics of discretizations.

Briefly, recall some definitions and notation concerning a dynamical system φ defined on finite set L. Let $\mathrm{Tr}(\xi_0,\varphi,\mathrm{L}) = \boldsymbol{\xi} = \xi_0,\xi_1,\ldots,\xi_n,\ldots,\ \xi_n = \varphi(\xi_{n-1}),\ n=1,2,\ldots,$ denote the trajectory of φ beginning at $\xi_0 \in \mathrm{L}$. For a positive integer m, the m-shift of the trajectory $\boldsymbol{\xi}$ is the sequence $\mathrm{S}^m(\boldsymbol{\xi},m) = \xi_m,\xi_{m+1},\ldots,$ which is also a trajectory of φ. A trajectory $\boldsymbol{\xi}$ is called a *cycle* if there exists a positive integer N with $\xi_N = \xi_0$. Then $\xi_i = \xi_{i+N}$ for each positive integer i. The minimal N satisfying $\xi_N = \xi_0$ is called the *period* of the cycle and is denoted by $C(\boldsymbol{\xi})$. Every trajectory of the system is eventually cyclic, that is, for some positive integer m the chopped trajectory $\mathrm{S}^m(\boldsymbol{\xi},m)$ is a cycle. The period of this cycle $\mathrm{S}^m(\boldsymbol{\xi})$ is denoted by $\mathcal{C}(\xi_0,\varphi,\mathrm{L})$. The minimal m with the property that $\mathrm{S}^m(\boldsymbol{\xi},m)$ is a cycle is the *length of the transient part* of a trajectory $\mathrm{Tr}(\xi_0,\varphi,\mathrm{L})$ and is denoted by $\mathcal{T}(\xi_0,\varphi,\mathrm{L})$. For an arbitrary $\xi_0 \in \mathrm{L}$ define the *first recurrence time* $\mathcal{Q}(\xi_0,\varphi,\mathrm{L})\min\{n : \xi_n = \xi_j,\ \text{for some } j < n\}$. $\mathcal{Q}(\xi_0,\varphi) = \mathcal{T}(\xi_0,\varphi) + \mathcal{C}(\xi_0,\varphi)$.

Let $\mathbf{S}$ be a finite set of non-negative real numbers from $[0,1]$. Define the *distribution function of the set* $\mathbf{S}$, $\mathcal{D}(\cdot\,;\mathbf{S}) : [0,1] \to [0,1]$, by

$$\mathcal{D}(x;\mathbf{S}) = \frac{\#(\{s \in \mathbf{S} : s \le x\})}{\#(\mathbf{S})}, \qquad 0 \le x \le 1$$

where $\#(S)$ denotes the cardinality of finite set S. The statistics of the discrete dynamical systems which will be studied are the functions

$$\mathcal{D}_{\mathcal{Q}}(x,\varphi) = \mathcal{D}(x,\{\mathcal{Q}(\xi):\xi\in\mathrm{L}\}), \tag{2}$$
$$\mathcal{D}_{\mathcal{C}}(x,\varphi) = \mathcal{D}(x,\{\mathcal{C}(\xi):\xi\in\mathrm{L}\}). \tag{3}$$

These are respectively the distribution of the first recurrence moment and the distribution of the period of the limit cycle.

Now consider the dynamical system induced by f_ℓ and realized on the lattice $\mathrm{L} = \mathrm{L}_\nu = \{0, 1/\nu, \ldots, (\nu-1)/\nu, 1\}$. The L_ν-discretization $f_{\ell,\nu}$ of a mapping f_ℓ is defined by $f_{\ell,\nu}(\xi) = [f_\ell(\xi)]_\nu$, $\xi \in \mathrm{L}_\nu$, where $[\alpha]_\nu$ is the roundoff operator: $[\alpha]_\nu = k/\nu$ if $(k-0.5)/\nu \leq \alpha < (k+0.5)/\nu$, for $k = 1, 2, \ldots, \nu-1$. If $\nu = 2^N$ then the L_ν-discretization is a natural theoretical model for implementation of a mapping f_ℓ in *fixed point arithmetic* with N binary digits and radix point in the first position (see [4], pages 98-100). We will examine asymptotic behaviour of the scaled distributions

$$\mathcal{D}_{\mathcal{Q}}(x, f_{\ell,1}), \mathcal{D}_{Q}(2^{1/\ell}x, f_{\ell,2}), \ldots, \mathcal{D}_{\mathcal{Q}}(\nu^{1/\ell}x, f_{\ell,\nu}), \ldots\ , \tag{4}$$
$$\mathcal{D}_{\mathcal{C}}(x, f_{\ell,1}), \mathcal{D}_{C}(2^{1/\ell}x, f_{\ell,2}), \ldots, \mathcal{D}_{\mathcal{C}}(\nu^{1/\ell}x, f_{\ell,\nu}), \ldots\ . \tag{5}$$

Recall one further statistical feature which was introduced in [5], the sequence

$$\mathcal{P}(f_{\ell,1}), \mathcal{P}(f_{\ell,2}), \ldots, \mathcal{P}(f_{\ell,\nu}), \ldots \tag{6}$$

where $\mathcal{P}(f_{\ell,\nu})$ is a proportion of initial values $\xi_0 \in \mathrm{L}_\nu$ such that the trajectory $\mathrm{Tr}(\xi_0, f_{\ell,\nu})$ collapses eventually to zero. Further details may be found in [5].

1.2 Principal results

Let $X(K) = \{0, 1, \ldots, K\}$ and let $\Delta > 0$. Define *a random mapping* $T_{\Delta,K}$*:*$X \to X$*, with a single absorbing centre* 0, by $T_{\Delta,K}(0) = 0$ and

$$P(T_{\Delta,K}(i) = j) = \begin{cases} \Delta/(K+\Delta) & \text{if} \quad i \neq 0, j = 0, \\ 1/(K+\Delta) & \text{if} \quad i, j \neq 0\ . \end{cases}$$

Here, $P(\cdot)$ denotes the probability of an event and the image of each element i, $i = 1, \ldots, K$, is chosen independently of the images of the others. It is important to note that a realization of the random mapping $T_{\Delta,K}$ is a deterministic dynamical system on $X(K)$. That is, images $T_{\Delta,K}(i)$ are chosen randomly as an event, but are thereafter fixed. Random mappings with a single absorbing centre are similar to, though differ from mappings with a single attracting centre [2, 3].

Define the random sequence $\boldsymbol{\xi} = \xi_1, \xi_2, \ldots, \xi_\nu, \ldots$, where the ξ_i are independent and uniformly distributed random variables on L_ν. Consider the random sequence of first recurrence times

$$\mathcal{Q}(\xi_1, f_{\ell,1}), \mathcal{Q}(\xi_2, f_{\ell,2}), \ldots \mathcal{Q}(\xi_\nu, f_{\ell,\nu}), \ldots \tag{7}$$

and of cycle periods

$$\mathcal{C}(\xi_1, f_{\ell,1}), \mathcal{C}(\xi_2, f_{\ell,2}), \ldots \mathcal{C}(\xi_\nu, f_{\ell,\nu}), \ldots \ . \tag{8}$$

Write $[x]$ for the integer part of x.

Hypothesis. *There exist positive constants $a = a\Delta(\ell)$, $b = b(\ell)$ such that the statistical properties of the sequences (7) and (8) are similar to those of sequences*

$$\mathcal{Q}(i_\nu; T_\nu) \quad \text{and} \quad \mathcal{C}(i_\nu; T_\nu), \qquad \nu = 1, 2, \ldots \tag{9}$$

where i_ν, $\nu = 1, 2, \ldots$, are independent random elements from the corresponding sets $X_\nu = \{1, 2, \ldots, [b\nu^{2/\ell}]\}$ and the T_ν are independent realizations of the random mappings $T_{\Delta(\nu),K(\nu)} = T_{a\nu^{1/\ell},\left[b\nu^{2/\ell}\right]}$. The statistical properties of the deterministic sequence $\{\mathcal{P}(f_{\ell,\nu})\}$ are similar to those of the random sequence $\{\mathcal{P}(T_\nu)\}$ where $\mathcal{P}(T)$ is the proportion of initial points in $X(K)$ whose $T-$trajectories are eventually absorbed by the absorbing centre 0.

As mentioned at in the introduction, $\dim_c(\mu_\ell) = 2/\ell$ for $\ell > 2$, where μ_ℓ is the SRB measure for f_ℓ. So the asymptotic estimate $\#(X_\nu) \sim b\nu^{2/\ell} = O(\nu^{\dim_c(\mu_\ell)})$ is in line with that suggested by the Grebogi–Ott–York model [8]. The asymptotic estimate $\lambda(\nu) \sim a\nu^{1/\ell}$ for the weight $\lambda(\nu) = \Delta(\nu)/(K(\nu) + \Delta(\nu))$ of the absorbing centre 0, mirrors that of $\mu_\ell([0, 1/(2\nu)]) = O(1/\nu^{1/\ell})$, which is the measure of the set in $[0, 1]$ which is rounded off to zero by the discretization $x \mapsto [x]_\nu$. A physical and heuristic justification of this construction was given in [5], adapting some ideas of [8, 11] and of some phenomenological models of hysteresis (see [10] and references therein). Although we have no rigorous justification of the Hypothesis, we will demonstrate that there is very close agreement between theoretical conclusions drawn from it and the results of computer experiments.

The Hypothesis plays an important role in that, in contrast to statistical studies of discretizations, the statistics of the model admit straightforward theoretical analysis. To show this, first introduce the functions

$$
\begin{aligned}
d_1(x;a,b) &= 1-e^{\frac{a^2-(x+a)^2}{2b}}, \\
d_2(x;a,b) &= 1-e^{\frac{a^2-(x+a)^2}{2b}}\left[1-\sqrt{\pi}\frac{x+a}{\sqrt{2b}}\operatorname{erfcx}\left(\frac{x+a}{\sqrt{2b}}\right)\right], \\
d_*(x;a,b) &= \operatorname{erfc}\left(a\sqrt{\frac{(1-x)}{bx}}\right),
\end{aligned}
$$

where

$$
\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-s^2}\,ds, \qquad \operatorname{erfcx}(t) = e^{t^2}\frac{2}{\sqrt{\pi}}\int_t^\infty e^{-s^2}\,ds,
$$

are respectively the complementary error function and the scaled complementary error function.

A sequence of numbers $\mathbf{u} = u_1, u_2, \ldots, u_\nu, \ldots$, is said to have the *stable distribution property* with the limit $D(x)$ if $\lim_{\nu\to\infty}\mathcal{D}(x;\{u_1,u_2,\ldots,u_\nu\}) = D(x)$. For a discussion of stable statistical properties, see [9].

Proposition 1 *Each of the random sequences*

$$
\nu^{1/\ell}\mathcal{Q}(i_\nu;T_\nu),\ \nu=1,2,\ldots \quad \text{and} \quad \nu^{1/\ell}\mathcal{C}(i_\nu;T_\nu),\ \nu=1,2,\ldots \tag{10}
$$

have a.e. the stable distribution property with respective limits $d_1(x;a,b)$, $d_2(x;a,b)$. *Furthermore, the random sequence of proportions of collapsing points* $\mathcal{P}(T_\nu)$ *has a.e. the stable distribution property with limit* $d_*(x;a,b)$.

A justification of the stable distribution property for the sequences (10) is in the appendix. That property for $\mathcal{P}(T_\nu)$ is a restatement of Proposition 3, [5].

Corollary 1 *Let* $\boldsymbol{\xi} = \xi_1, \xi_2, \ldots, \xi_\nu, \ldots$ *be a random sequence of independent and uniformly distributed elements from* L_ν. *Each of the sequences*

$$
\nu^{1/\ell}\mathcal{Q}(\xi_\nu, f_{\ell,\nu}), \qquad \nu^{1/\ell}\mathcal{C}(\xi_\nu, f_{\ell,\nu}), \tag{11}
$$

have a.e. the stable distribution property with continuous limits $d_{\mathcal{Q}}(x;\ell)$ *and* $d_{\mathcal{C}}(x;\ell)$. *Moreover, there exist constants* $a(\ell)$, $b(\ell)$ *such that*

$$
d_{\mathcal{Q}}(x;\ell) \approx d_1(x;a(\ell),b(\ell)), \qquad d_{\mathcal{C}}(x;\ell) \approx d_2(x;a(\ell),b(\ell)).
$$

The sequence $\mathcal{P}(f_{\ell,\nu})$ *has the stable distribution property with a continuous limit* $d_{\mathcal{P}}(x;\ell) \approx d_*(x;a(\ell),b(\ell))$.

This corollary is not completely satisfactory, neither from a theoretical point of view nor for numerical experiments, because it uses random notions to analyze a sequence of deterministic discretizations. To avoid this, we introduce one further definition. Let $U_\nu(\xi)$, $\nu = 1, 2, \ldots$ be a sequence of functions on on the sequence of finite sets $\{\mathcal{S}_\nu\}$. The sequence $\{U_\nu(\cdot)\}$ is said to be *Cesáro stable* with limit $D(x)$ if the sequence of distribution functions $w_\nu(x) = \mathcal{D}(x, \{U_\nu(\xi) : \xi \in \mathcal{S}_\nu\})$, $\nu = 1, 2, \ldots$, satisfies

$$\lim_{\nu\to\infty} \frac{1}{\nu} \sum_{n=1}^{\nu} u_n(x) = D(x).$$

This idea is closely related to the notion of the stable distribution property. To see this, consider the random sequence $\boldsymbol{\xi} = \xi_1, \xi_2, \ldots, \xi_\nu, \ldots$, where the ξ_i are independent and uniformly distributed on L_ν. *The sequence $\{U_\nu(\xi)\}$ is Cesáro stable with limit $D(x)$, provided that a.e. the sequence $\{U(\xi_\nu)\}$ has the stable distribution property with the same limit.* Now, Corollary 1 implies

Corollary 2 *The sequences of scaled distribution functions (4), (5) are Cesáro stable with continuous limits $d_{\mathcal{Q}}(x, \ell)$, $d_{\mathcal{C}}(x, \ell)$, while the sequence (6) has the stable distribution property with continuous limit $d_{\mathcal{P}}(x, \ell)$. There exist constants $a(\ell)$, $b(\ell)$ such that $d_{\mathcal{Q}}(x; \ell) \approx d_1(x; a(\ell), b(\ell))$, $\quad d_{\mathcal{C}}(x; \ell) \approx d_2(x; a(\ell), b(\ell))$, and $d_{\mathcal{P}}(x; \ell) \approx d_2(x; a(\ell), b(\ell))$.*

1.3 Numerical experiments

Corollary 2 admits of experimental verification. To test the result, choose a pair of large, distinct positive integers $\nu_1 \ll \nu_2$, say $\nu_1 = 10^5$, $\nu_2 = 10^7$. Then choose a positive integer n with $1 \ll n \ll \nu_1$, for instance, $n = 10^3$. Consider the two finite sequences of lattices $\mathrm{L}_{\nu_1}, \mathrm{L}_{\nu_1+1} \ldots \mathrm{L}_{\nu_1+n}$ and $\mathrm{L}_{\nu_2}, \mathrm{L}_{\nu_2+1} \ldots \mathrm{L}_{\nu_2+n}$. For $i = 1, 2$, define

$$\begin{aligned}
\mathbf{Q}(x; \ell, n, \nu_i) &= \frac{1}{n} \sum_{j=\nu_i+1}^{\nu_i+n} \mathcal{D}_Q(\nu^{1/\ell} x; f_{\ell,\nu}), \\
\mathbf{C}(x; \ell, n, \nu_i) &= \frac{1}{n} \sum_{j=\nu_i+1}^{\nu_i+n} \mathcal{D}_{\mathcal{C}}(\nu^{1/\ell} x; f_{\ell,\nu}), \\
\mathbf{P}(x; \ell, n, \nu_i) &= \mathcal{D}\left(x, \{\mathcal{P}(f_{\ell,\nu_i+1}), \ldots, \mathcal{P}(f_{\ell,\nu_i+n})\}\right).
\end{aligned}$$

Corollary 1 implies that there exist positive constants $a = a(\ell)$, $b = b(\ell)$ with the following properties.

(i) Functions $\mathbf{Q}(x;\ell,n,\nu_1)$, $\mathbf{Q}(x;\ell,n,\nu_2)$ are close to one another and both are close to $d_1(x;a(\ell),b(\ell))$.

(ii) Functions $\mathbf{C}(x;\ell,n,\nu_1)$, $\mathbf{C}(x;\ell,n,\nu_2)$ are close to one another and both are close to $d_2(x;a(\ell),b(\ell))$.

(iii) Functions $\mathbf{P}(x;\ell,n,\nu_1)$, $\mathbf{P}(x;\ell,n,\nu_2)$ are close to one another and both are close to $d_*(x;a(\ell),b(\ell))$.

This conclusion is in excellent agreement with experiments. Figure 1 graphs six different distributions. The three higher curves represent experimental results of $\mathbf{C}(x;3,10^3,10^5)$ and $\mathbf{C}(x;3,10^3,10^7)$ (jagged curves) compared with the theoretical prediction $d_2(x;a,b,3)$ for $a=2.5$, $b=6.5$ (smooth curve). The three lower curves are experimental results of $\mathbf{Q}(x;3,10^3,10^5)$ and $\mathbf{Q}(x;3,10^3,10^9)$ compared with the function $d_1(x;a,b)$, again for $a=2.5$, $b=6.5$. Item (iii) is also confirmed with reasonable precision by the numerical experiments for the same a,b (see Figures 1-3 in [6]). Note that it is not possible to imitate even the qualitative behaviour of the experimental curves with the distributions suggested in [8] using the model of completely random mappings as distinct from mappings with a single absorbing centre.

Other values of ℓ were studied in the same way. For $\ell \leq 4$ the close agreement of experimental computations with theory were very similar to those above. As ℓ increases further, with ν fixed, results still support the hypothesis, but the agreement is not quite as close. There is a simple explanation for this deterioration with increasing ℓ. The Hypothesis suggests that a reasonable model of the L_ν-discretization of the system f_ℓ is a random mapping with an absorbing centre defined on $\sim \nu^{2/\ell}$ points and the weight of the centre $O(\nu^{2/\ell})$. To be near the limit functions, this last quantity should be moderately large, say at least several hundred.So, if $\ell=5$, the discretization parameter ν has to be of the order of 10^8 for the empirical distributions to be near the limit distributions. Experiments readily confirm this last fact. Consequently, it becomes necessary to refine the grid L_ν as the exponent ℓ increases to retain the same degree of agreement of theory with experiment.

2 Appendix

Define the *first absorption time* $\mathcal{M}_{\Delta,K}(i,\omega)$ for the trajectory $\mathbf{y}(i,\omega)$ of a realization $T^{\omega}_{\Delta,K}$, with initial value at i, by

$$\mathcal{M}_{\Delta,K}(i,\omega) = \min\{n : (y(i,\omega)_n = 0) \vee (y(i,\omega)_n = y(i,\omega)_j) \text{ for some } j < n\},$$

where $\vee$ denotes the logical "or". That is, $\mathcal{M}_{\Delta,K}(i,\omega)$ is the first n such that the trajectory $\mathbf{y}(i,\omega)$ either falls into the absorbing state 0 or repeats itself and is thus absorbed by a cycle. It is the first time after which the trajectory is uniquely determined. Clearly, comparing with the first recurrence time,

$$\mathcal{M}_{\Delta,K}(i,\omega) \leq \mathcal{Q}_{\Delta,K}(i,\omega) \leq \mathcal{M}_{\Delta,K}(i,\omega) + 1. \tag{12}$$

It is chosen as the basic characteristic in our constructions because of a simple difference formula for the probability $p(n,\Delta,K,i)$ of the event $\mathcal{M}_{\Delta,K}(i,\omega) \geq n$:

$$p(n+1,\Delta,K) = \left(1 - \frac{\Delta+n}{\Delta+K}\right) p(n,\Delta,K), \qquad p(1,\Delta,K) = 1. \tag{13}$$

Clearly, all random variables $\mathcal{M}_{\Delta,K}(i,\omega)$, $i = 1,2,\ldots,K$, are identically distributed. Denote the joint distribution function of these random variables by $D_{\mathcal{M}}(x;\Delta,K)$. It is convenient to expand the functions $D_{\mathcal{M}}(x;\Delta,K)$ in terms of step functions defined for all $x \in [0,K]$ by the relation $D_{\mathcal{M}}(x;\Delta,K) = D_{\mathcal{M}}(\mathrm{trunc}(x);\Delta,K)$.

The recurrence relation (13) implies the asymptotic result

Proposition 2. *For $\tau \to \infty$ and $r,s > 0$*

$$D_{\mathcal{M}}(\tau x; r\tau, s\tau^2) \sim d_1(x;r,s).$$

The details of the calculation are cumbersome but straightforward. The assertion of Proposition 1 about the first sequence $\nu^{1/\ell}Q(x;T_{a\nu^{1/\ell},[b\nu^{1/\ell}]})$ follows from the proposition above and (12). The convergence of the second sequence $\nu^{1/\ell}C(x;T_{a\nu^{1/\ell},[b\nu^{1/\ell}]})$ can be derived in much the same way.

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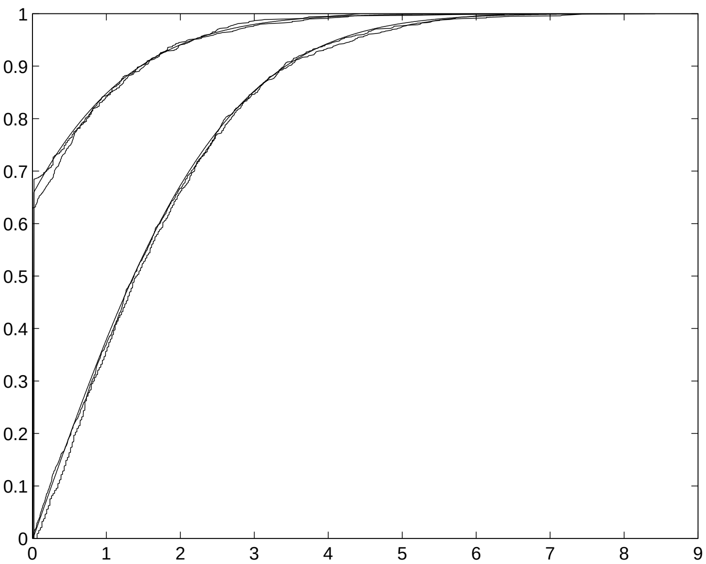

Figure 1: The three higher curves are empirical distributions (jagged curve) $\mathcal{D}(x; \mathbf{C}(3, 10^3, 10^5))$, $\mathcal{D}(x; \mathbf{C}(3, 10^3, 10^9))$ and the theoretical distribution (smooth) $d_2(x; 2.5, 6.5, 3)$. The three lower curves are $\mathcal{D}(x; \mathbf{Q}(3, 10^3, 10^5))$, $\mathcal{D}(x; \mathbf{Q}(3, 10^3, 10^9))$ and $d_1(x; 2.5, 6.5, 3)$.