Perturbation of linear asynchronous systems

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Over the past few years systems whose parts behave to some extent autonomously have attracted meticulous attention of investigators. Examples are the control systems with independently functioning pulse elements, computer networks, multiprocessor computer systems, and so on. Distinctive feature of such systems is the possibility for their parts to operate asynchronously. To synchronize work of different parts of a system, sometimes special measures are undertaken [1]. If such compulsory synchronization is impossible (see examples in [2–5]) then one is forced to deal with the so-called desynchronized [5–7] or asynchronous [2–4] systems. If additionally the law of 'desynchronization' is not known in advance then there naturally arise the problem on absolute stability of a system (see [8, 9]) in the class of 'all possible desynchronizations'. Below we present an answer to the principal question about conditions which enable preservation of the property of absolute stability under all possible 'small' perturbations of linear desynchronized (asynchronous) systems.

1. Consider a system W consisting of components (parts, elements) $W_1, W_2, \ldots, W_N$. Let the state of the component W_i is described by a vector $x_i \in R^{n_i}$, $n_i \geq 1$, and is changed (updated) at some discrete instants of time in accordance with the law

$$(1)\qquad x_{i,\text{new}} = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{iN}x_n + f_i,$$

where a_{ij} are matrices of appropriate dimensions and f_i is the vector of external perturbations of the component W_i.

In general, the state of several components of the system W can be changed simultaneously; let ω be the set of their indices. Denote by A_ω the block matrix obtained from the matrix $A = (a_{ij})$ by replacing its strings with the indices $i \notin \omega$ by the corresponding strings of the identity matrix I. Let X be the state space of the system W, i.e., the set of vectors $x = \{x_1, x_2, \ldots, x_n\} \in X$, where $x_i \in R^{n_i}$, and X_ω be the subspace of vectors $x = \{x_1, x_2, \ldots, x_n\} \in X$ for which $x_i = 0$ when $i \notin \omega$. Then change (updating) of the state of the system W is described by the vector equality

$$x_{\text{new}} = A_\omega x + F_\omega, \quad \text{where} \quad F_\omega = \{f_1, f_2, \ldots, f_N\} \in X_\omega.$$

Let $\ldots < T_0 < T_1 < \ldots < T_n < \ldots$ be all the updating instants for the system W. By denoting the state vector of the system at the moment T_n by $x(n)$ and the set of indices of all the components updated at this moment by $\omega(n)$, we arrive to the equation of dynamics of the system W (cf. [2, 5, 6]):

$$(2)\qquad x(n+1) = A_{\omega(n)}x(n) + F(n), \qquad F(n) \in X_{\omega(n)}.$$

If $\omega(n) \equiv \{1, 2, \ldots, N\}$ then the system W is called [3–5] *synchronous* or *synchronized* while in the opposite case it is called *asynchronous* or *desynchronized.*

2. The system W will be called *absolutely stable by Perron* (in the class of all possible desynchronizations) if there exists a $\beta < \infty$ such that, for any sequences of the sets $\omega(n) \in \{1, 2, \ldots, N\}$ and the vectors $F(n) \in X_{\omega(n)}$, $\|F(n)\| \leq 1$, for the solutions $x(n)$ of the corresponding equations (2) satisfying the initial condition $x(0) = 0$ the estimates $\|x(n)\| \leq \beta$ are hold for all $n \geq 0$ (cf. [9, 10]). This definition differs from the traditional definition of the 'Perron property' for the difference equations [10] by the requirement that the perturbation vectors $F(n)$ must belong to the subspaces $X_{\omega(n)}$ matched with the matrices $A_{\omega(n)}$. Equations (2) do not possess the Perron property in the usual meaning [10].

T h e o r e m 1. *A system W with the matrix A is absolutely stable by Perron if and only if for some $\varepsilon > 0$ there exists a norm $\|\cdot\|_\varepsilon$ on X such that $\|A_\omega x + F_\omega\|_\varepsilon \le \|x\|_\varepsilon$ for any $\omega \in \{1, 2, \dots, N\}$, $x \in X$ and $F_\omega \in X_\omega$ satisfying $\|F_\omega\|_\varepsilon \le \varepsilon\|x\|_\varepsilon$.*

Theorem 1 allows to establish some properties of systems absolutely stable by Perron. Confine ourselves to one example. Let A and B are some matrices. Then $A_\omega x - B_\omega x \in X_\omega$ for any vector $x \in X$ and index set $\omega \in \{1, 2, \dots, N\}$. From here and from Theorem 1 then follows

T h e o r e m 2. *If a system W with the matrix A is absolutely stable by Perron then any system with the matrix B close to the matrix A is also absolutely stable by Perron.*

Remark that as a rule to answer the question about absolute stability by Perron of a specific system, by direct using of the definition, is not an easy task.

3. Let the system W be not affected by external perturbations. Then its dynamics is described by the equation

$$(3) \quad x(n+1) = A_{\omega(n)}x(n).$$

The sequence $\{\omega(n)\}$ of non-empty subsets of the set $\{1, 2, \dots, N\}$ will be called *regular* if for each $i = 1, 2, \dots, N$ the inclusions $i \in \omega(n)$ take place for infinitely many values of n. We shell say [5–7] that the system W is *absolutely asymptotically stable* (in the class of all desynchronizations) if for any regular sequence $\{\omega(n)\}$ each solution of the corresponding equation (3) tends to zero as $n \to \infty$.

Examples of two classes of absolutely asymptotically stable asynchronous systems are given in [6, 7]; both of them consist of systems with the scalar states of the components. The first class constitute the systems with the matrices $A = (a_{ij})$ for which the spectral radius of the auxiliary matrix $S = (|a_{ij}|)$ is less than 1. The second class constitute the systems with the symmetric matrices $A = (a_{ij})$ whose spectral radius is less than 1. Simple modification of arguments from [6] leads to conclusion about absolute stability by Perron of the systems from the first class. The question about absolute stability by Perron of the systems from the second class is more difficult; the answer to it follows from the next Theorem 3.

T h e o r e m 3. *A linear system is absolutely stable by Perron if and only if it is absolutely asymptotically stable.*

One of the most important corollaries from Theorem 3 is the following

T h e o r e m 4. *If a system W with the matrix A is absolutely asymptotically stable then any system with the matrix close to the matrix A is also absolutely asymptotically stable.*

4. Subordinate systems. Let α be a non-empty subset of the set $\{1, 2, \dots, N\}$. Let us 'remove' from the system W all the components with the indices $i \notin \alpha$ and the arising 'loose' inputs of the remaining components we feed by some signals. The obtained system will be referred to as *the system subordinate to* W, and will be denoted by W^α. To describe the law of changing the state of the component W_i ($i \in \alpha$) treating as a component of the system W^α the summands $a_{ij}x_j$ with the indices $j \notin \alpha$ in equation (1) should be replaced by the external perturbations. Let us identify the state space of the subordinate system W^α with X_α. Denote by A^α the block matrix obtained from the matrix $A = (a_{ij})$ by 'clearing' all the elements a_{ij} for which $i \notin \alpha$ or $j \notin \alpha$. Then the equation of dynamics of the subordinate system W^α will take the form similar to (2):

$$x(n+1) = (A^\alpha)_{\omega(n)}x(n) + F(n), \qquad F(n) \in X_{\omega(n)},$$

where $x(n) \in X_\alpha$, $\omega(n) \subseteq \alpha$.

T h e o r e m 5. *If a linear asynchronous system W is absolutely asymptotically stable then any subordinate system W^α is also absolutely asymptotically stable.*

Theorem 5 has no analogs in the case of synchronous systems.

5. Outline the scheme of proving Theorem 3. Let the system W with the matrix A be absolutely stable by Perron. Then, in the norm $\|\cdot\|_\varepsilon$ defined by Theorem 1, for some $q<1$ there hold the inequalities

$$(4)\qquad \|A_\omega\|_\varepsilon \le 1, \qquad \|A_{\omega_1}A_{\omega_2}\cdots A_{\omega_k}\|_\varepsilon \le q,$$

if $\omega, \omega_1, \dots, \omega_k \subseteq \{1,2,\dots,N\}$ and $\omega_1 \cup \omega_2 \cup \dots \cup \omega_k = \{1,2,\dots,N\}$. Inequalities (4) imply absolute asymptotic stability of the system W.

Converse assertion of Theorem 3 is less trivial. First, it is proved that absolute asymptotic stability is equivalent to existence of a norm in which for the matrices A_ω inequalities (4) hold. From here, it is derived existence of a constant $\gamma<\infty$ such that any solution $x(n)$ of equation (3) satisfies the inequality

$$\sum_{n=0}^{\infty}\|x(n+1)-x(n)\| \le \gamma\|x(0)\|.$$

Then on X it is defined the function

$$\nu(x) = \sup_{\{\omega(n)\}\in\Omega, x(0)=x, n=0} \sum_{n=0}^{\infty}\|x(n+1)-x(n)\|,$$

where Ω is the family of all regular sequences of subsets of the set $\{1,2,\dots,N\}$. The function $\nu(x)$ is a norm, and moreover

$$(5)\qquad \nu(A_\omega x) + \|(A_\omega - I)x\| \le \nu(x), \qquad \omega \in \{1,2,\dots,N\}.$$

By Theorem 5 for each set $\omega \in \{1,2,\dots,N\}$ there exists a limit $Q_\omega = \lim_{n\to\infty} A_\omega^n$ which is a projector onto the subspace of solutions of the equation $A_\omega x = x$. Let us define now the norm $\|\cdot\|_*$ as follows

$$\|x\|_* = \max_{\omega\in\{1,2,\dots,N\}} \mu(\omega)\nu(Q_\omega x).$$

Inequalities (5) allows to choose the quantities $\mu(\omega)$ in such a way that for some $\varepsilon>0$ the norm $\|\cdot\|_*$ will satisfy the conditions of Theorem 1. Therefore the system W is absolutely stable by Perron.

6. The idea of proof of Theorem 3 helps to establish a new criterion of absolute stability of asynchronous systems. Let the system W have the scalar states of components. If the matrix A of the system W is symmetric and its eigenvalues lie in an interval $[-\rho,\rho]$ with $\rho<1$ then the system W, as is known [3–6], is absolutely asymptotically stable. Now, let $A = B + C$, where the matrix B is symmetric with eigenvalues lying in an interval $[-\rho,\rho]$ with $\rho<1$, and the matrix C is skew-symmetric. Denote by r the spectral radius of the matrix C.

Theorem 6. *If*

$$(6)\qquad r < \rho\sqrt{\frac{1-\rho}{1+\rho}}\left(\frac{1}{\sqrt{1-(1-\rho^2)^N}}-1\right)$$

then the system W with the matrix $A = B + C$ is absolutely asymptotically stable.

When $\rho = 0$ the condition (6) takes the form $r < N^{-1/2}$; already in the case $N = 2$ this condition is rather rough.

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