ISSN 1061-9208, Russian Journal of Mathematical Physics, Vol. 16, No. 2, 2009, pp. 208–227. © Pleiades Publishing, Ltd., 2009.

Dedicated to the memory of Vladimir Borovikov

Trajectory Attractor of a Reaction–Diffusion System with a Series of Zero Diffusion Coefficients

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Received February 16, 2009

Abstract. We study a reaction-diffusion system of N equations with k nonzero and $N-k$ zero diffusion coefficients. More exactly, the first k equations of the system contain the terms $a_i\Delta u_i - f_j(\mathbf{u},\mathbf{v})$, $i=\overline{1,k}$, with the diffusion coefficient $a_i>0$. The right-hand sides of the other $N-k$ equations contain only nonlinear interaction functions $-h_j(\mathbf{u},\mathbf{v})$, $j=\overline{k+1,N}$, with zero diffusion. Here $\mathbf{u}=(u_1,\dots,u_k)$ and $\mathbf{v}=(v_{k+1},\dots,v_N)$ are unknown concentration vectors. Under appropriate assumptions on the interaction functions $\mathbf{f}(\,\cdot\,)$ and $\mathbf{h}(\,\cdot\,)$, we construct the trajectory attractor $\mathfrak{A}^{\mathbf{0}}$ of this reaction-diffusion system. We also find the trajectory attractors $\mathfrak{A}^{\boldsymbol{\delta}}$, $\boldsymbol{\delta}=(\delta_1,\dots,\delta_k)$, for the analogous reaction-diffusion systems having the terms $\delta_j\Delta v_j - h_j(\mathbf{u},\mathbf{v})$, $j=\overline{k+1,N}$, with small diffusion coefficients $\delta_j\geqslant 0$ in the last $N-k$ equations. We prove that the trajectory attractors $\mathfrak{A}^{\boldsymbol{\delta}}$ converge to $\mathfrak{A}^{\mathbf{0}}$ (in an appropriate topology) as $\boldsymbol{\delta}\to 0+$.

DOI: 10.1134/S1061920809020058

INTRODUCTION

The global attractors for reaction-diffusion equations have been studied in a number of papers (see, e.g., [1–5] and the references therein). The major consideration has been given to reaction-diffusion systems for which the initial-value problem has a unique strong solution in the corresponding function space.

The present paper deals with reaction-diffusion systems for which it is possible to construct global (in time) weak solutions, but the uniqueness theorem for the corresponding Cauchy problem fails or is not proved yet. This situation is quite typical for model equations of chemical kinetics. Moreover, we assume that some diffusion coefficients of the system vanish. For such a system, in Secs. 1 and 2, we construct the trajectory attractor and study the main properties of this set. We note that the method of trajectory attractors is extremely useful in the study of equations without unique solvability of the Cauchy problem (see, e.g., [6–11]).

Consider the system

$$\partial_t\mathbf{u} = \mathbf{a}\Delta\mathbf{u} - \mathbf{f}(\mathbf{u},\mathbf{v}) + \mathbf{g}_1, \qquad \mathbf{f} := (f_1,\dots,f_k), \tag{0.1}$$

$$\partial_t\mathbf{v} = \qquad - \mathbf{h}(\mathbf{u},\mathbf{v}) + \mathbf{g}_2, \qquad \mathbf{h} := (h_{k+1},\dots,h_N), \tag{0.2}$$

where $\mathbf{u} := (u_1(x,t),\dots,u_k(x,t))$, $\mathbf{v} := (v_{k+1}(x,t),\dots,v_N(x,t))$, $x\in\Omega\Subset\mathbb{R}^n$, $t\geqslant 0$, $\mathbf{g}_1 := (g_{1,1}(x),\dots,g_{1,k}(x))$, $\mathbf{g}_2 := (g_{2,k+1}(x),\dots,g_{2,N}(x))$, and $\mathbf{a}$ is a diagonal $k\times k$ matrix with positive elements (the diffusion coefficients). On the boundary $\partial\Omega$, zero conditions are assumed for the unknown vector functions $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$. The nonlinear interaction vector functions $\mathbf{f}(\mathbf{u},\mathbf{v})$ and $\mathbf{h}(\mathbf{u},\mathbf{v})$ can have an arbitrary polynomial growth in $\mathbf{u}$ and $\mathbf{v}$; however, $\mathbf{f}(\mathbf{u},\mathbf{v})$ and $\mathbf{h}(\mathbf{u},\mathbf{v})$ must satisfy some inequalities ensuring the main *a priori* estimates for a solution of system (0.1)

Partially supported by the Russian Foundation for Basic Research (projects nos. 08-01-00784 and 07-01-00500).

and (0.2) (see Section 1). Moreover, since the Laplace operator is lacking in equations (0.2), we assume in addition that

$$\sum_{j=k+1}^{N}\sum_{l=k+1}^{N}\frac{\partial h_j}{\partial v_l}(\mathbf{u},\mathbf{v})\xi_j\xi_l \geqslant \sigma\sum_{j=k+1}^{N}\xi_j^2, \qquad \forall(\xi_{k+1},\dots,\xi_N)\in\mathbb{R}^{N-k},$$
$$|\partial h_j/\partial u_i(\mathbf{u},\mathbf{v})| \leqslant D, \quad i=\overline{1,k}, \quad j=\overline{k+1,N}, \quad \forall(\mathbf{u},\mathbf{v})\in\mathbb{R}^k\times\mathbb{R}^{N-k},$$

where $\sigma>0$ and $D>0$ are chosen numbers. Also assume that the functions $\mathbf{g}_1(x)$ and $\mathbf{g}_2(x)$ are known and that $\mathbf{g}_1(\,\cdot\,)\in[L_2(\Omega)]^k$ and $\mathbf{g}_2(\,\cdot\,)\in[H_0^1(\Omega)]^{N-k}$.

Let us now define weak solutions $(\mathbf{u}(x,t),\mathbf{v}(x,t))$ to system (0.1), (0.2) and, using the Galerkin method, find weak solutions satisfying the following estimate:

$$\|\mathbf{v}(t)\|^2_{[H_0^1]^{N-k}} \leqslant \|\mathbf{v}(0)\|^2_{[H_0^1]^{N-k}}e^{-\sigma t} + C\big(\|\mathbf{u}(0)\|^2_{[L_2]^k}+\|\mathbf{v}(0)\|^2_{[L_2]^{N-k}}\big)e^{-\sigma t}+R^2, \quad \forall t\geqslant 0, \quad (0.3)$$

for some positive values σ, C, and R. Here $H_0^1 := H_0^1(\Omega)$ and $L_2 := L_2(\Omega)$.

In Section 2, we construct the trajectory attractor $\mathfrak{A}^\mathbf{0}$ for system (0.1) and (0.2). For this purpose, we define the class $\mathcal{K}^+(S)$ of weak solutions $(\mathbf{u}(x,t),\mathbf{v}(x,t))$ to (0.1) and (0.2) on the semiaxis $0<t<+\infty$ such that the component $\mathbf{v}(x,t)$ satisfies the inequality $\|\mathbf{v}(t)\|^2_{[H_0^1]^{N-k}}\leqslant Se^{-\sigma t}+R^2$ for any $t\geqslant 0$. Here $S>0$ is arbitrarily chosen, whereas σ and R are taken from (0.3).

Consider the time *translation semigroup* $\{T(\tau)\}:=\{T(\tau),\tau\geqslant 0\}$ acting on the *trajectory space* $\mathcal{K}^+(S)$ by the formula $T(\tau)(\mathbf{u}(t),\mathbf{v}(t))=(\mathbf{u}(t+\tau),\mathbf{v}(t+\tau))$.

We claim that $\mathcal{K}^+(S)$ is closed in the weak topology Θ_+^{loc} and is invariant with respect to $\{T(\tau),\tau\geqslant 0\}$, i.e., $T(\tau)\mathcal{K}^+(S)\subseteq\mathcal{K}^+(S)$ for $\tau\geqslant 0$ (see Section 2).

Moreover, we claim that the semigroup $\{T(\tau)\}|_{\mathcal{K}^+(S)}$ has a compact (in the topology Θ_+^{loc}) absorbing set $\mathcal{P}$. Therefore, $\{T(\tau)\}|_{\mathcal{K}^+(S)}$ has the global attractor $\mathfrak{A}^\mathbf{0}(S)$. The set $\mathfrak{A}^\mathbf{0}(S)$ is compact in Θ_+^{loc} and strictly invariant with respect to $\{T(\tau)\}$, $T(\tau)\mathfrak{A}^\mathbf{0}(S)=\mathfrak{A}^\mathbf{0}(S)$, for all $\tau\geqslant 0$, and $\mathfrak{A}^\mathbf{0}(S)$ attracts (in the topology Θ_+^{loc}) the bounded sets of the trajectories in the space $\mathcal{K}^+(S)$ as the time τ tends to $+\infty$. We shall also prove that $\mathfrak{A}^\mathbf{0}(S)=:\mathfrak{A}^\mathbf{0}$ does not depend on S. The set $\mathfrak{A}^\mathbf{0}$ is referred to as the *trajectory attractor* of the reaction-diffusion system (0.1)–(0.2).

To describe the structure of the trajectory attractor $\mathfrak{A}^\mathbf{0}$, we define (in Section 2) the *kernel* $\mathcal{K}^\mathbf{0}$ for system (0.1)–(0.2) that consists of all its bounded weak solutions $(\mathbf{u}(x,t),\mathbf{v}(x,t))$, $-\infty<t<+\infty$. We prove that $\mathfrak{A}^\mathbf{0}=\Pi_+\mathcal{K}^\mathbf{0}$, where Π_+ is the restriction operator on the semiaxis $\mathbb{R}_+$.

In Section 3, we obtain similar results for the reaction-diffusion system

$$\partial_t\mathbf{u}=\mathbf{a}\Delta\mathbf{u}-\mathbf{f}(\mathbf{u},\mathbf{v})+\mathbf{g}_1(x), \qquad \mathbf{u}|_{\partial\Omega}=\mathbf{0}, \quad (0.4)$$
$$\partial_t\mathbf{v}=\boldsymbol{\delta}\Delta\mathbf{v}-\mathbf{h}(\mathbf{u},\mathbf{v})+\mathbf{g}_2(x), \qquad \mathbf{v}|_{\partial\Omega}=\mathbf{0}, \quad (0.5)$$

where $\boldsymbol{\delta}=(\delta_{k+1},\dots,\delta_N)$ is the diagonal matrix with elements $\delta_j\geqslant 0$, and δ_j are small for all $j=\overline{k+1,N}$. Similarly to system (0.1) and (0.2), we construct trajectory attractors $\mathfrak{A}^{\boldsymbol{\delta}}$ for system (0.4) and (0.5).

In Section 4, we prove that $\mathfrak{A}^{\boldsymbol{\delta}}\to\mathfrak{A}^\mathbf{0}$ in the weak topology Θ_+^{loc} as $\boldsymbol{\delta}\to\mathbf{0}+$.

In conclusion, we note that the global attractor of a particular system (0.1)–(0.2) of two scalar equations ($N=2$ and $k=1$), the so-called called *partly dissipative system*,

$$\partial_t u=\Delta u-f(u,v)+g_1(x), \quad (0.6)$$
$$\partial_t v= \qquad -h(u,v)+g_2(x), \quad (0.7)$$

has been constructed in [12] under some additional conditions which ensure the unique solvability of the Cauchy problem for this system. For the case in which this condition fails, the trajectory attractor for system (0.6)–(0.7) was constructed in [13]. It was also proved that $\mathfrak{A}^\delta\to\mathfrak{A}^0$ as $\delta\to 0+$, where $\mathfrak{A}^\delta$ is the trajectory attractor of the reaction-diffusion system which differs from the above system (0.6)–(0.7) by the presence of the small diffusion term $\delta\Delta v$ on the right-hand side of equation (0.7).

1. REACTION-DIFFUSION SYSTEM WITH A SERIES OF ZERO DIFFUSION COEFFICIENTS

Consider the following reaction-diffusion system:

$$\partial_t \mathbf{u} = \mathbf{a}\Delta\mathbf{u} - \mathbf{f}(\mathbf{u},\mathbf{v}) + \mathbf{g}_1(x), \tag{1.1}$$

$$\partial_t \mathbf{v} = \qquad - \mathbf{h}(\mathbf{u},\mathbf{v}) + \mathbf{g}_2(x). \tag{1.2}$$

Here $\mathbf{u} = (u_1(x,t),\dots,u_k(x,t))$ and $\mathbf{v} = (v_{k+1}(x,t),\dots,v_N(x,t))$ are unknown vector functions, $1 \leqslant k < N$, $N \geqslant 2$, $x \in \Omega \Subset \mathbb{R}^n$, $\partial\Omega \in C^1$, and $t \in \mathbb{R}_+$. The diagonal elements of the matrix $\mathbf{a} = \operatorname{diag}(a_1, a_2, \dots, a_k)$ are positive, $a_i > 0$ $(i = \overline{1,k})$. The nonlinear vector functions $\mathbf{f}$ and $\mathbf{h}$ are of the form $\mathbf{f}(\mathbf{u},\mathbf{v}) = (f_1(\mathbf{u},\mathbf{v}),\dots,f_k(\mathbf{u},\mathbf{v}))$, $\mathbf{h}(\mathbf{u},\mathbf{v}) = (h_{k+1}(\mathbf{u},\mathbf{v}),\dots,h_N(\mathbf{u},\mathbf{v}))$, and $(\mathbf{u},\mathbf{v}) \in \mathbb{R}^k \times \mathbb{R}^{N-k} = \mathbb{R}^N$. Assume that $\mathbf{f}(\mathbf{u},\mathbf{v})$ and $\mathbf{h}(\mathbf{u},\mathbf{v})$ are continuous with respect to $(\mathbf{u},\mathbf{v}) \in \mathbb{R}^N$ and satisfy the following growth conditions:

$$\sigma\Big(\sum_{i=1}^k |u_i|^{p_i} + \sum_{j=k+1}^N |v_j|^{p_j}\Big) - C \leqslant \sum_{i=1}^k f_i(\mathbf{u},\mathbf{v})u_i + \sum_{j=k+1}^N h_j(\mathbf{u},\mathbf{v})v_j \leqslant C_0\Big(\sum_{i=1}^k |u_i|^{p_i} + \sum_{j=k+1}^N |v_j|^{p_j} + 1\Big), \tag{1.3}$$

$$\sum_{i=1}^k |f_i(\mathbf{u},\mathbf{v})|^{q_i} + \sum_{j=k+1}^N |h_j(\mathbf{u},\mathbf{v})|^{q_j} \leqslant C_0\Big(\sum_{i=1}^k |u_i|^{p_i} + \sum_{j=k+1}^N |v_j|^{p_j} + 1\Big), \quad \forall (\mathbf{u},\mathbf{v}) \in \mathbb{R}^N, \tag{1.4}$$

where σ, C, and C_0 are some positive constants and $p_r \geqslant 2$ and $q_r = p_r/(p_r - 1)$, $r = \overline{1,N}$, are constant. Moreover, assume that

$$h_j \in C^1(\mathbb{R}^N), \qquad h_j(0,0) = 0, \quad j = \overline{k+1,N}, \tag{1.5}$$

and that the following inequalities hold:

$$\sum_{j=k+1}^N \sum_{l=k+1}^N \frac{\partial h_j}{\partial v_l}(\mathbf{u},\mathbf{v})\xi_j\xi_l \geqslant \sigma \sum_{j=k+1}^N \xi_j^2, \quad \forall (\xi_{k+1},\dots,\xi_N) \in \mathbb{R}^{N-k}, \ \sigma > 0, \tag{1.6}$$

$$\left|\frac{\partial h_j}{\partial u_i}(\mathbf{u},\mathbf{v})\right| \leqslant D, \ i = \overline{1,k}, \ j = \overline{k+1,N}, \quad \forall (\mathbf{u},\mathbf{v}) \in \mathbb{R}^N. \tag{1.7}$$

Here D stands for a positive constant depending on $\mathbf{f}$ and $\mathbf{h}$. Note that the constant σ in (1.3) and (1.6) is the same. (These inequalities can be achieved by choosing a sufficiently small σ.)

The vector functions $\mathbf{g}_1(x)$ and $\mathbf{g}_2(x)$ in (1.1) and (1.2) satisfy the conditions

$$\mathbf{g}_1 = (g_{1,1},\dots,g_{1,k}) \in [L_2(\Omega)]^k, \qquad \mathbf{g}_2 = (g_{2,k+1},\dots,g_{2,N}) \in [H_0^1(\Omega)]^{N-k}. \tag{1.8}$$

Throughout the paper,

$$H := L_2(\Omega), \quad V := H_0^1(\Omega).$$

For a given Banach space X, we usually denote the norm in X by $\|\cdot\|_X$, and, for brevity, denote the norms in H and V by $\|\cdot\|$ and $\|\cdot\|_1$, respectively. Recall that the Poincaré inequality implies that the norm of a function w in $V = H_0^1(\Omega)$ is equivalent to the norm

$$\|w\|_1 := \|\nabla w\| = \Big(\int_\Omega \sum_{i=1}^n |\partial_{x_i} w(x)|^2 dx\Big)^{1/2}.$$

For brevity, the norms of vector functions in H^k, H^{N-k} and V^k, V^{N-k} are also denoted by $\|\cdot\|$ and $\|\cdot\|_1$, respectively.

At the boundary $\partial\Omega$, we pose the Dirichlet conditions

$$u_i|_{\partial\Omega} = 0, \quad i = \overline{1,k}, \quad \text{and} \quad v_j|_{\partial\Omega} = 0, \quad j = \overline{k+1,N}, \tag{1.9}$$

and, at $t = 0$, consider the initial conditions

$$u_i|_{t=0} = u_{i,0}, \quad i = \overline{1,k}, \quad \text{and} \quad v_j|_{t=0} = v_{0,j}, \quad j = \overline{k+1,N}, \tag{1.10}$$

where it is assumed that

$$u_{i,0} \in H, i = \overline{1,k}, \quad \text{and} \quad v_{0,j} \in H,\ j = \overline{k+1,N}. \tag{1.11}$$

Note that, for $k = 1$, system (1.1)–(1.2) is of the form

$$\partial_t u_1 = a_1 \Delta u_1 - f_1(u_1, v_2, \dots, v_N) + g_{1,1}(x), \tag{1.12}$$

$$\partial_t v_j = \qquad - h_j(u_1, v_2, \dots, v_N) + g_{2,j}(x), \quad j = \overline{2,N}, \tag{1.13}$$

and, in this case, only the first equation is partial differential indeed. System (1.13) consists of ordinary differential equations depending on the function $u_1 = u_1(x,t)$, and the spatial variable $x \in \Omega$ can be regarded as a parameter. The boundary and initial conditions for the entire system (1.12)–(1.13) are (1.9) and (1.10). This system corresponds to the case in which only equation (1.12) has a nonzero diffusion coefficient $a_1 = 1$, whereas the other diffusion coefficients vanish, $a_j = 0,\ j = \overline{2,N}$. Note that, for $N = 2$, these systems were studied in [12], where they were referred to as "partly dissipative" reaction-diffusion systems. In particular, the Fitz–Hugh–Nagumo system is of this form (see [1, 14, 15]).

For a given $M > 0$ and for any functions $u_i(\,\cdot\,) \in L_{p_i}(0,M; L_{p_i}(\Omega))$, $i = \overline{1,k}$, and $v_j(\,\cdot\,) \in L_{p_j}(0,M; L_{p_j}(\Omega))$, $j = \overline{k+1,N}$, it follows from (1.4) that

$$f_i(\mathbf{u}(\,\cdot\,), \mathbf{v}(\,\cdot\,)) \in L_{q_i}(0,M; L_{q_i}(\Omega)), \qquad i = \overline{1,k}, \tag{1.14}$$

$$h_j(\mathbf{u}(\,\cdot\,), \mathbf{v}(\,\cdot\,)) \in L_{q_j}(0,M; L_{q_j}(\Omega)), \qquad j = \overline{k+1,N}, \tag{1.15}$$

$$\sum_{i=1}^{k} \|f_i(\mathbf{u},\mathbf{v})\|^{q_i}_{L_{q_i}(0,M;L_{q_i})} + \sum_{j=k+1}^{N} \|h_j(\mathbf{u},\mathbf{v})\|^{q_j}_{L_{q_j}(0,M;L_{q_j})} \leqslant C_1\biggl(\sum_{i=1}^{k} \|u_i\|^{p_i}_{L_{p_i}(0,M;L_{p_i})} + \sum_{j=k+1}^{N} \|v_j\|^{p_j}_{L_{p_j}(0,M;L_{p_j})} + 1\biggr). \tag{1.16}$$

(Here and below, the symbols C_i stand for positive constants depending on $\mathbf{f}, \mathbf{h}$, and Ω.)

If it is known in addition that $u_i(\,\cdot\,) \in L_2(0,M;V)$ for $i = \overline{1,k}$, then

$$a_i \Delta u_i(\,\cdot\,) + g_{1,i}(\,\cdot\,) \in L_2(0,M; H^{-1}(\Omega)). \tag{1.17}$$

The Sobolev embedding theorem implies that $H^s_0(\Omega) \subset L_p(\Omega)$ for $s \geqslant n(1/2 - 1/p)$, and hence, for the conjugate spaces

$$H^{-s}(\Omega) = [H^s_0(\Omega)]^* \quad \text{and} \quad L_q(\Omega) = [L_p(\Omega)]^* \ (q^{-1} + p^{-1} = 1),$$

we have the embedding $L_q(\Omega) \subset H^{-s}(\Omega)$. Therefore, if $s \geqslant \max\{1, n(1/2 - 1/p_i)\}$ for $i = \overline{1,k}$, then, by (1.14) and (1.17), the right-hand sides of equations (1.1) belong to the space $L_{q_i}(0,M; H^{-s}(\Omega))$.

It is clear that, for $j = \overline{k+1,N}$, the right-hand sides of equations (1.2) belong to the space $L_{q_j}(0,M; L_{q_j}(\Omega))$ since $q_j \leqslant 2$ (this follows from (1.15)). Write $r_i = \max\{1, n(1/2 - 1/p_i)\}$ for $i = \overline{1,k}$. We can now seek solutions $u_i(x,t)$ and $v_j(x,t)$ of equations (1.1) and (1.2) in the spaces of distributions $\mathcal{D}'(0,M; H^{-r_i}(\Omega))$ $(i = \overline{1,k})$ and $\mathcal{D}'(0,M; L_{q_j}(\Omega))$ $(j = \overline{k+1,N})$, respectively (see [16]), such that

$$\partial_t u_i(\,\cdot\,) \in L_{q_i}(0,M; H^{-r_i}(\Omega)) \quad \text{and} \quad \partial_t v_j(\,\cdot\,) \in L_{q_j}(0,M; L_{q_j}(\Omega)).$$

A couple of vector functions $(\mathbf{u}(x,t),\mathbf{v}(x,t)), (x,t)\in\Omega\times\mathbb{R}_+$, is said to be a *weak solution* to system (1.1)–(1.2) if, for every $M>0$,

$$u_i(\,\cdot\,)\in L_{p_i}(0,M;L_{p_i}(\Omega))\cap L_2(0,M;V),\quad i=\overline{1,k},$$
$$v_j(\,\cdot\,)\in L_{p_j}(0,M;L_{p_j}(\Omega)),\qquad j=\overline{k+1,N},$$

the functions $u_i(x,t)$ satisfy equations (1.1) in the distribution sense of the spaces $\mathcal{D}'(0,M;H^{-r_i}(\Omega))$ for $i=\overline{1,k}$, and the functions $v_j(x,t))$ satisfy equations (1.2) in the spaces of distributions $\mathcal{D}'(0,M;L_{q_j}(\Omega))$ for $j=\overline{k+1,N}$ (see [2, 3, 16]).

Since a weak solution $(\mathbf{u}(x,t),\mathbf{v}(x,t))$ satisfies (1.1)–(1.2), we find that

$$\mathbf{u}(\,\cdot\,)\in L_\infty(0,M;H^k)\quad\text{and}\quad\mathbf{v}(\,\cdot\,)\in L_\infty(0,M;H^{N-k}).$$

Then, using the well-known Lions–Magenes lemma (see [17]), we obtain

$$\mathbf{u}(\,\cdot\,)\in C_w([0,M];H^k)\quad\text{and}\quad\mathbf{v}(\,\cdot\,)\in C_w([0,M];H^{N-k}).$$

Consequently, for every $t\geqslant 0$, the values $u_i(\cdot,t)$ and $v_j(\cdot,t)$ make sense in the space H and, in particular, the initial conditions (1.10) are meaningful. We often omit the spatial variable x in arguments of the functions $\mathbf{u}(\,\cdot\,)$ and $\mathbf{v}(\,\cdot\,)$ for brevity.

The solvability of the problem (1.1), (1.2), (1.9), and (1.10) is established by using the Galerkin approximation method (see, e.g., [3]). The procedure relies on *a priori* estimates given below. The Galerkin method uses the basis of the eigenvectors of the Laplace operator with Dirichlet boundary conditions,

$$-\Delta w_l(x)=\lambda_l w_l(x),\quad w_l|_{\partial\Omega}=0,\quad w_l(\,\cdot\,)\in C^2(\bar\Omega),\quad l=1,2,\dots,$$
$$0<\lambda_1<\lambda_2\leqslant\lambda_3\leqslant\dots,\quad \lambda_l\to+\infty\quad(l\to\infty).$$

Let us outline the main steps of the method (see, e.g., [1–3, 16]). Let $(\mathbf{u}^m(x,t),\mathbf{v}^m(x,t))$ be the Galerkin approximation of order $m\in\mathbb{N}$. Recall that

$$u_i^m(x,t)=\sum_{l=1}^m\alpha_{i,l}(t)w_l(x),\quad i=\overline{1,k};\qquad v_j^m(x,t)=\sum_{l=1}^m\beta_{j,l}(t)w_l(x),\quad j=\overline{k+1,N},$$

where the real functions $\{\alpha_{i,l}(t)\}_{l=1}^m$ and $\{\beta_{j,l}(t)\}_{l=1}^m$ are the solutions of the Galerkin system

$$\frac{d}{dt}\mathbf{u}^m=\mathbf{a}P_m\Delta\mathbf{u}^m-P_m\mathbf{f}(\mathbf{u}^m,\mathbf{v}^m)+P_m\mathbf{g}_1,\qquad \mathbf{u}^m(0)=P_m\mathbf{u}_0,\tag{1.18}$$
$$\frac{d}{dt}\mathbf{v}^m=\qquad -Q_m\mathbf{h}(\mathbf{u}^m,\mathbf{v}^m)+Q_m\mathbf{g}_2,\qquad \mathbf{v}^m(0)=Q_m\mathbf{v}_0.\tag{1.19}$$

Here P_m and Q_m are orthogonal projections in H^k and H^{N-k} to the finite-dimensional subspaces

$$H_m^k=\sum_{l=1}^m w_l(x)\mathbb{R}^k\subset H^k\quad\text{and}\quad H_m^{N-k}=\sum_{l=1}^m w_l(x)\mathbb{R}^{N-k}\subset H^{N-k}.$$

It is clear that the Cauchy problem for the system of ordinary differential equations (1.18)–(1.19) has a solution $(\mathbf{u}^m(x,t),\mathbf{v}^m(x,t))$ such that, for some $t_m>0$,

$$u_i^m(\,\cdot\,)\in C^1([0,t_m);V\cap C^2(\bar\Omega)),\quad i=\overline{1,k},\tag{1.20}$$
$$v_j^m(\,\cdot\,)\in C^1([0,t_m);V\cap C^2(\bar\Omega)),\quad j=\overline{k+1,N}.\tag{1.21}$$

(Recall that the solutions are linear combinations of eigenfunctions of the Laplacian, which are smooth functions with respect to $x\in\Omega$.) Moreover, since the pair $(\mathbf{u}^m(x,t),\mathbf{v}^m(x,t))$ satisfies an *a priori* estimate discussed below, we can assume that $t_m=+\infty$.

To pass to the limit in the approximations ($\mathbf{u}^m(x,t)$ and $\mathbf{v}^m(x,t)$) as $m \to \infty$ and to obtain an exact weak solution $(\mathbf{u}(x,t), \mathbf{v}(x,t))$ of system (1.1)–(1.2), we need some *a priori* estimates. Let us begin with the first "energy" estimate.

We claim that any weak solution $(\mathbf{u}(\,\cdot\,), \mathbf{v}(\,\cdot\,))$ of system (1.1)–(1.2) has the following properties: $\mathbf{u}(\,\cdot\,) \in C(\mathbb{R}_+; H^k), \mathbf{v}(\,\cdot\,) \in C(\mathbb{R}_+; H^{N-k})$, and the real function $\|\mathbf{u}(t)\|^2_{H^k} + \|\mathbf{v}(t)\|^2_{H^{N-k}}$ is absolutely continuous for $t \geqslant 0$ and satisfies the following "energy" differential identity:

$$\frac{1}{2}\frac{d}{dt}\left\{ \sum_{i=1}^{k} \|u_i(t)\|^2 + \sum_{j=k+1}^{N} \|v_j(t)\|^2 \right\} + \sum_{i=1}^{k} a_i \|\nabla u_i(t)\|^2 + \int_\Omega \sum_{i=1}^{k} f_i(\mathbf{u},\mathbf{v}) u_i(x,t)\,dx$$
$$+ \int_\Omega \sum_{j=k+1}^{N} h_j(\mathbf{u},\mathbf{v}) v_j(x,t)\,dx = \sum_{i=1}^{k} \langle g_{1,i}, u_i(t)\rangle + \sum_{j=k+1}^{N} \langle g_{2,j}, v_j(t)\rangle\,. \qquad (1.22)$$

Here and below, the symbol $\langle\cdot,\cdot\rangle$ stands for the inner product in H and $\|\cdot\|$ for the norm in H.

Note that every Galerkin approximation $(\mathbf{u}^m(t), \mathbf{v}^m(t))$ of order m also satisfies identity (1.22), where $\mathbf{u}$ and $\mathbf{v}$ are replaced by $\mathbf{u}^m$ and $\mathbf{v}^m$. To verify this assertion, we merely take the inner product in H^k of (1.18) and $\mathbf{u}^m(t)$ and the inner product in H^{N-k} of (1.19) and $\mathbf{v}^m(t)$. Then we add the results, integrate by parts in the term with the Laplacian, and use the elementary formulas

$$\frac{d}{dt}\|u_i^m(t)\|^2 = 2\Big\langle \frac{d}{dt}u_i^m(t), u_i^m(t)\Big\rangle, \qquad \frac{d}{dt}\|v_j^m(t)\|^2 = 2\Big\langle \frac{d}{dt}v_j^m(t), v_j^m(t)\Big\rangle,$$

which hold since the functions $(\mathbf{u}^m(t), \mathbf{v}^m(t))$ are sufficiently smooth (see (1.20)–(1.21)). The proof of (1.22) for an arbitrary weak solution $(\mathbf{u}(t), \mathbf{v}(t))$ of (1.1)–(1.2) is a more delicate question, and it can be carried out by using the approach described in [3] for general reaction-diffusion systems.

Proposition 1.1. *For any weak solution* $(\mathbf{u}(t), \mathbf{v}(t))$ *of problem* (1.1), (1.2), (1.10), *the following inequalities hold*:

$$\|\mathbf{u}(t)\|^2 + \|\mathbf{v}(t)\|^2 + 2a\int_0^t \|\nabla \mathbf{u}(s)\|^2 e^{-\sigma(t-s)}\,ds \leqslant \left(\|\mathbf{u}_0\|^2 + \|\mathbf{v}_0\|^2\right)e^{-\sigma t} + R_1^2, \qquad (1.23)$$

$$2a\int_t^{t+1} \|\nabla \mathbf{u}(s)\|^2\,ds + \sigma\int_t^{t+1}\left(\sum_{i=1}^{k}\|u_i(s)\|^{p_i}_{L_{p_i}} + \sum_{j=k+1}^{N}\|v_j(s)\|^{p_j}_{L_{p_j}}\right)ds$$
$$\leqslant \left(\|\mathbf{u}_0\|^2 + \|\mathbf{v}_0\|^2\right)e^{-\sigma t} + R_2^2, \quad \forall t \geqslant 0, \qquad (1.24)$$

where $a := \min\{a_i, i = \overline{1,k}\}$ *and* R_1 *and* R_2 *are some positive values depending on* $\sigma, C, |\Omega|, \|\mathbf{g}_1\|,$ *and* $\|\mathbf{g}_2\|$ (*recall that* $\|\mathbf{u}\| := \|\mathbf{u}\|_{H^k}$, $\|\mathbf{v}\| := \|\mathbf{v}\|_{H^{N-k}}$, $\|\mathbf{g}_1\| := \|\mathbf{g}_1\|_{H^k}$, *and* $\|\mathbf{g}_2\| := \|\mathbf{g}_2\|_{H^{N-k}}$).

Proof. Using inequality (1.3), we obtain the following differential inequality from identity (1.22):

$$\frac{1}{2}\frac{d}{dt}\{\|\mathbf{u}(t)\|^2 + \|\mathbf{v}(t)\|^2\} + a\|\nabla\mathbf{u}(t)\|^2 + \sigma\int_\Omega \Big\{\sum_{i=1}^{k}|u_i(x,t)|^{p_i} + \sum_{j=k+1}^{N}|v_j(x,t)|^{p_j}\Big\}dx$$
$$\leqslant \|\mathbf{u}(t)\|\|\mathbf{g}_1\| + \|\mathbf{v}(t)\|\|\mathbf{g}_2\| + C|\Omega|, \quad \forall t \geqslant 0. \qquad (1.25)$$

Using the elementary inequality $|b|^p \geqslant |b|^2 - 1$, for $p \geqslant 2$, we see from (1.25) that

$$\frac{1}{2}\frac{d}{dt}\{\|\mathbf{u}(t)\|^2 + \|\mathbf{v}(t)\|^2\} + a\|\nabla\mathbf{u}(t)\|^2 + \sigma\{\|\mathbf{u}(t)\|^2 + \|\mathbf{v}(t)\|^2\}$$
$$\leqslant \|\mathbf{u}(t)\|\|\mathbf{g}_1\| + \|\mathbf{v}(t)\|\|\mathbf{g}_2\| + C|\Omega| + 2\sigma. \qquad (1.26)$$

Applying the Cauchy inequality to the right-hand side of (1.26), we have

$$\frac{d}{dt}\{\|\mathbf{u}(t)\|^2+\|\mathbf{v}(t)\|^2\}+2a\|\nabla\mathbf{u}(t)\|^2+\sigma\{\|\mathbf{u}(t)\|^2+\|\mathbf{v}(t)\|^2\}$$
$$\leqslant \sigma^{-1}\|\mathbf{g}_1\|^2+\sigma^{-1}\|\mathbf{g}_2\|^2+2C|\Omega|+4\sigma. \tag{1.27}$$

For brevity, write

$$\zeta(t)=\|\mathbf{u}(t)\|^2+\|\mathbf{v}(t)\|^2,\quad \phi(t)=2a\|\nabla\mathbf{u}(t)\|^2,\quad R_0^2=\sigma^{-1}\|\mathbf{g}_1\|^2+\sigma^{-1}\|\mathbf{g}_2\|^2+2C|\Omega|+4\sigma.$$

This yields

$$\frac{d}{dt}\zeta(t)+\phi(t)+\sigma\zeta(t)\leqslant R_0^2.$$

Multiplying this inequality by $e^{\sigma t}$ and making some elementary manipulations with the integration with respect to time, we obtain the inequalities

$$\frac{d}{dt}[\zeta(t)e^{\sigma t}]+\phi(t)e^{\sigma t}\leqslant R_0^2e^{\sigma t}$$

and

$$\zeta(t)e^{\sigma t}-\zeta(0)+\int_0^t\phi(s)e^{\sigma s}ds\leqslant R_0^2\sigma^{-1}\left[e^{\sigma t}-1\right]\leqslant R_0^2\sigma^{-1}e^{\sigma t}.$$

Replacing the expressions for ζ and ϕ, we have

$$\|\mathbf{u}(t)\|^2+\|\mathbf{v}(t)\|^2+2a\int_0^t\|\nabla\mathbf{u}(s)\|^2e^{-\sigma(t-s)}ds\leqslant\left(\|\mathbf{u}(0)\|^2+\|\mathbf{v}(0)\|^2\right)e^{-\sigma t}+R_1^2, \tag{1.28}$$

where

$$R_1^2:=R_0^2\sigma^{-1}=\sigma^{-2}\|\mathbf{g}_1\|^2+\sigma^{-2}\|\mathbf{g}_2\|^2+2\sigma^{-1}C|\Omega|+4. \tag{1.29}$$

Using identity (1.22) and the Young inequality, similarly to (1.25)–(1.27), we see that

$$\frac{d}{dt}\{\|\mathbf{u}(t)\|^2+\|\mathbf{v}(t)\|^2\}+2a\|\nabla\mathbf{u}(t)\|^2$$
$$+\sigma\int_\Omega\Big\{\sum_{i=1}^k|u(x,t)|^{p_i}+\sum_{j=k+1}^N|v_j(x,t)|^{p_j}\Big\}dx\leqslant C_2(\|\mathbf{g}_1\|^2+\|\mathbf{g}_2\|^2+1),\quad\forall t\geqslant 0, \tag{1.30}$$

where the constant C_2 depends on σ and is independent of $\mathbf{g}_1$ and $\mathbf{g}_2$. Integrating inequality (1.30) over the segment $[t,t+1]$, we obtain

$$(\|\mathbf{u}(t+1)\|^2+\|\mathbf{v}(t+1)\|^2)+2a\int_t^{t+1}\|\nabla\mathbf{u}(s)\|^2ds+\sigma\int_t^{t+1}\Big(\sum_{i=1}^k\|u_i(s)\|_{L_{p_i}}^{p_i}+\sum_{j=k+1}^N\|v_j(s)\|_{L_{p_j}}^{p_j}\Big)ds$$
$$\leqslant(\|\mathbf{u}(t)\|^2+\|\mathbf{v}(t)\|^2)+C_2(\|\mathbf{g}_1\|^2+\|\mathbf{g}_2\|^2+1),$$

which, owing to (1.28), implies that

$$2a\int_t^{t+1}\|\nabla\mathbf{u}(s)\|^2ds+\sigma\int_t^{t+1}\Big(\sum_{i=1}^k\|u_i(s)\|_{L_{p_i}}^{p_i}+\sum_{j=k+1}^N\|v_j(s)\|_{L_{p_j}}^{p_j}\Big)ds\leqslant(\|\mathbf{u}(0)\|^2+\|\mathbf{v}(0)\|^2)e^{-\sigma t}+R_2^2, \tag{1.31}$$

where

$$R_2^2:=R_1^2+C_2\left(\|\mathbf{g}_1\|^2+\|\mathbf{g}_2\|^2+1\right). \tag{1.32}$$

Remark 1.1. We note that every Galerkin approximation $(\mathbf{u}^m(t),\mathbf{v}^m(t))$ also satisfies (1.23) and (1.24) with the same constants R_1 and R_2 because the proof of these estimates uses identity (1.22), $(\mathbf{u}^m(t),\mathbf{v}^m(t))$ satisfies this identity, and we clearly have $\|\mathbf{u}_0^m\|\leqslant\|\mathbf{u}_0\|$ and $\|\mathbf{v}_0^m\|\leqslant\|\mathbf{v}_0\|$. In particular, for a chosen $\mathbf{u}_0$ and $\mathbf{v}_0$, the functions $u_i^m(t)$ are uniformly bounded (with respect to $m\in\mathbb{N}$) in the spaces

$$L_\infty(0,M;H)\cap L_{p_i}(0,M;L_{p_i}(\Omega))\cap L_2(0,M;V)\quad\text{for}\quad i=\overline{1,k}, \tag{1.33}$$

and $v_j^m(t)$ are uniformly bounded (with respect to $m \in \mathbb{N}$) in

$$L_\infty(0,M;H) \cap L_{p_j}(0,M;L_{p_j}(\Omega)) \quad \text{for} \quad j = \overline{k+1,N}. \tag{1.34}$$

Unfortunately, the estimates in the spaces (1.34) are insufficient to pass to the limit in the Galerkin approximation for system (1.18)–(1.19) and to obtain a weak solution of the original problem (1.1), (1.2), (1.9), (1.10). To carry out this passage to the limit, we need a stronger *a priori* estimate that guarantees the uniform boundedness (with respect to $m \in \mathbb{N}$) of the family $v_j^m(t)$ in the space $L_\infty(0,M;V)$.

Up to now, we do not take assumptions (1.6) and (1.7) into account. It is time to use these conditions in the second *a priori* estimate, which enables us to find a weak solution to system (1.1), (1.2).

From now on, instead of (1.11), assume that the initial data $\mathbf{v}_0$ satisfy the condition

$$v_{0,j} \in V, \quad j = \overline{k+1,N}. \tag{1.35}$$

Multiply equations (1.2) by $-\Delta v_j$, integrate the result in x over Ω, take the sum over j, and write out the identity

$$\frac{1}{2}\frac{d}{dt}\sum_{j=k+1}^{N}\|\nabla v_j(t)\|^2 - \int_\Omega \sum_{j=k+1}^{N} h_j(\mathbf{u}(x,t),\mathbf{v}(x,t))\Delta v_j(x,t)dx = \sum_{j=k+1}^{N}\langle \nabla g_{2,j}, \nabla v_j(t)\rangle. \tag{1.36}$$

Here we have used the formulas of integration by part,

$$-\langle \partial_t v_j, \Delta v_j\rangle = \frac{1}{2}\frac{d}{dt}\|\nabla v_j(t)\|^2, \qquad -\langle g_{2,j}, \Delta v_j\rangle = \langle \nabla g_{2,j}, \nabla v_j(t)\rangle,$$

which hold if $v_j(t)$ are sufficiently regular (see, e.g., [3]), say, if $v_j(\,\cdot\,) \in C^1([0,M];C^2(\bar\Omega))$ for $j = \overline{k+1,N}$. This property holds for the vector function $\mathbf{v}^m(t)$ taken from an arbitrary Galerkin approximation. Therefore, every derivative $\mathbf{v}^m(t)$ satisfies (1.36), and we can carry out our transformations under the assumption that $\mathbf{v}(t)$ is sufficiently regular or $\mathbf{v}(t) = \mathbf{v}^m(t)$ is taken from a Galerkin approximation. We stress that arbitrary weak solution of system (1.1)–(1.2) need not satisfy (1.36). (This remark is also related to inequality (1.37) below.)

Let us return to the second *a priori* estimate. Integrate by parts in the integral on the left-hand side of (1.36) which contains the functions h_j. Taking into account the condition $h_j(0,0) = 0$ (see (1.5)) and using the zero boundary conditions (1.9) for v_j, we obtain

$$\begin{aligned}
-\int_\Omega \sum_{j=k+1}^{N} h_j(\mathbf{u},\mathbf{v})\Delta v_j dx &= \int_\Omega \sum_{r=1}^{n}\sum_{j=k+1}^{N} \partial_{x_r} h_j(\mathbf{u},\mathbf{v})\partial_{x_r} v_j dx \\
&= \int_\Omega \sum_{r=1}^{n}\sum_{j=k+1}^{N}\sum_{l=k+1}^{N} \partial h_j/\partial v_l(\mathbf{u},\mathbf{v})\partial_{x_r} v_j \partial_{x_r} v_l dx + \int_\Omega \sum_{r=1}^{n}\sum_{j=k+1}^{N}\sum_{i=1}^{k} \partial h_j/\partial u_i(\mathbf{u},\mathbf{v})\partial_{x_r} v_j \partial_{x_r} u_i dx \\
&\geqslant \sigma \sum_{r=1}^{n}\sum_{j=k+1}^{N} \|\partial_{x_r} v_j\|_1^2 - D\sum_{j=k+1}^{N}\sum_{i=1}^{k}\sum_{r=1}^{n}\|\partial_{x_r} v_j\|\|\partial_{x_r} u_i\| \\
&\geqslant \sigma\|\mathbf{v}\|_1^2 - D\Big(\sum_{j=k+1}^{N}\|\nabla v_j\|_1\Big)\Big(\sum_{i=1}^{k}\|\nabla u_i\|_1\Big) \geqslant \sigma\|\mathbf{v}\|_1^2 - D_1\|\mathbf{v}\|_1\|\mathbf{u}\|_1,
\end{aligned} \tag{1.37}$$

where we set $D_1 = \sqrt{k(N-k)}D$. Here, we have used conditions (1.6)–(1.7) and applied the Cauchy–Schwartz inequality. Using elementary inequalities, we find that

$$D_1\|\mathbf{v}\|_1\|\mathbf{u}\|_1 \leqslant \frac{1}{4}\sigma\|\mathbf{v}\|_1^2 + \sigma^{-1}D_1^2\|\mathbf{u}\|_1^2, \tag{1.38}$$

$$\sum_{j=k+1}^{N}\langle \nabla g_{2,j}, \nabla v_j\rangle \leqslant \frac{1}{4}\sigma\|\mathbf{v}\|_1^2 + \sigma^{-1}\|\mathbf{g}_2\|_1^2. \tag{1.39}$$

Inequalities (1.36)–(1.39) yield

$$\frac{d}{dt}\left(\|\mathbf{v}(t)\|_1^2\right)+\sigma\|\mathbf{v}(t)\|_1^2 \leqslant C_3\|\mathbf{u}\|_1^2+C_4\|\mathbf{g}_2\|_1^2, \tag{1.40}$$

where $C_3 = 2\sigma^{-1}D_1^2$ and $C_4 = 2\sigma^{-1}$. It follows from (1.40) that

$$\frac{d}{dt}\left(\|\mathbf{v}(t)\|_1^2 e^{\sigma t}\right) \leqslant C_3\|\mathbf{u}\|_1^2 e^{\sigma t}+C_4\|\mathbf{g}_2\|_1^2 e^{\sigma t}.$$

Then

$$\|\mathbf{v}(t)\|_1^2 e^{\sigma t} \leqslant \|\mathbf{v}(0)\|_1^2 + C_3\int_0^t \|\mathbf{u}(s)\|_1^2 e^{\sigma s}ds + C_4\sigma^{-1}\|\mathbf{g}_2\|_1^2 e^{\sigma t}$$

after integration with respect to time, and therefore

$$\|\mathbf{v}(t)\|_1^2 \leqslant \|\mathbf{v}(0)\|_1^2 e^{-\sigma t} + 2\sigma^{-1}D^2\int_0^t \|\mathbf{u}(s)\|_1^2 e^{-\sigma(t-s)}ds + 2\sigma^{-2}\|\mathbf{g}_2\|_1^2. \tag{1.41}$$

Finally, we estimate the integral on the right-hand side of (1.41) by using inequality (1.23), which gives

$$\int_0^t \|\mathbf{u}(s)\|_1^2 e^{-\sigma(t-s)}ds \leqslant \frac{1}{2a}\left(\|\mathbf{u}(0)\|^2+\|\mathbf{v}(0)\|^2\right)e^{-\sigma t}+\frac{R_1^2}{2a}.$$

Therefore, by (1.41), we obtain the following estimate:

$$\|\mathbf{v}(t)\|_1^2 \leqslant \|\mathbf{v}(0)\|_1^2 e^{-\sigma t} + C_5\left(\|\mathbf{u}(0)\|^2+\|\mathbf{v}(0)\|^2\right)e^{-\sigma t}+R^2, \tag{1.42}$$

where

$$R^2 = \sigma^{-1}D_1^2R_1^2/a + 2\sigma^{-2}\|\mathbf{g}_2\|_1^2, \qquad C_5 = \sigma^{-1}D_1^2 \tag{1.43}$$

(the value R_1 is defined in (1.29)).

It follows from inequality (1.42) that the family of Galerkin approximations $\mathbf{v}^m(t)$ is uniformly bounded (with respect to $m \in \mathbb{N}$) in the space $L_\infty(0, M; V)$. Combining this result with the boundedness of the complete Galerkin approximations $(\mathbf{u}^m(t), \mathbf{v}^m(t))$ in the spaces (1.33) and (1.34), we finally pass to the limit in the Galerkin system (1.18)–(1.19) by using the standard scheme (see, e.g., [2, 3]) and obtain a weak solution $(\mathbf{u}(t), \mathbf{v}(t))$ of the original problem (1.1), (1.2), (1.9), (1.10). This weak solution satisfies (1.42).

Hence, we have proved the following assertion.

Proposition 1.2. *Under assumption* (1.35), *problem* (1.1), (1.2), (1.9), *and* (1.10) *has a weak solution* $(\mathbf{u}(t), \mathbf{v}(t))$, *such that* $\mathbf{v}(\,\cdot\,) \in L_\infty(\mathbb{R}_+; V^{N-k})$, *and the following inequality holds*:

$$\|\mathbf{v}(t)\|_1^2 \leqslant \|\mathbf{v}_0\|_1^2 e^{-\sigma t} + C_5\left(\|\mathbf{u}_0\|^2+\|\mathbf{v}_0\|^2\right)e^{-\sigma t}+R^2, \qquad \forall t \geqslant 0, \tag{1.44}$$

where the value R and the constant C_5 are defined in (1.43).

The Lions–Magenes lemma mentioned above implies that the weak solution thus constructed satisfies the relation $\mathbf{v}(\,\cdot\,) \in C_w(\mathbb{R}_+; V^{N-k})$, i.e., for every $t \geqslant 0$, the value $\mathbf{v}(t) \in V^{N-k}$ is well defined, and inequality (1.44) holds for all $t \geqslant 0$.

2. TRAJECTORY ATTRACTOR OF REACTION-DIFFUSION SYSTEM

Let us now define the spaces $\mathcal{F}_+^{\mathrm{loc}}, \mathcal{F}_+^{\mathrm{b}}$ and the topology Θ_+^{loc} in $\mathcal{F}_+^{\mathrm{loc}}$. Write

$$\mathcal{F}_+^{\mathrm{loc}} = \left\{\begin{array}{ll} \multicolumn{2}{c}{(\mathbf{y}(x,t),\mathbf{z}(x,t)) = (y_i(x,t),\ i=\overline{1,k},\quad z_j(x,t),\ j=\overline{k+1,N}),\ x\in\Omega,\ t\geqslant 0,} \\ \multicolumn{2}{c}{\text{such that}} \\ \mathbf{y}\in L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H^k)\cap L_2^{\mathrm{loc}}(\mathbb{R}_+;V^k), & y_i\in L_{p_i}^{\mathrm{loc}}(\mathbb{R}_+;L_{p_i}(\Omega)),\ i=\overline{1,k}; \\ \mathbf{z}\in L_\infty^{\mathrm{loc}}(\mathbb{R}_+;V^{N-k}), & z_j\in L_{p_j}^{\mathrm{loc}}(\mathbb{R}_+;L_{p_j}(\Omega)),\ j=\overline{k+1,N}; \\ \multicolumn{2}{c}{\text{and}} \\ \partial_t y_i\in L_{q_i}^{\mathrm{loc}}(\mathbb{R}_+;H^{-r_i}(\Omega)),\ i=\overline{1,k}; & \partial_t z_j\in L_{q_j}^{\mathrm{loc}}(\mathbb{R}_+;L_{q_j}(\Omega)),\ j=\overline{k+1,N}. \end{array}\right\}. \tag{2.1}$$

In the space $\mathcal{F}_+^{\rm loc}$, we define the following local weak convergence topology. By definition, a sequence $\{(\mathbf{y}^\mu(\,\cdot\,),\mathbf{z}^\mu(\,\cdot\,)),\ \mu\in\mathbb{N}\}\subset\mathcal{F}_+^{\rm loc}$ converges to $(\mathbf{y}(\,\cdot\,),\mathbf{z}(\,\cdot\,))\in\mathcal{F}_+^{\rm loc}$ in $\Theta_+^{\rm loc}$ as $\mu\to\infty$ if, for each $M>0$, the following limit relations hold:

$$\left\{\begin{array}{lll} \mathbf{y}^\mu(\,\cdot\,)\rightharpoondown\mathbf{y}(\,\cdot\,) & \text{as }\ \mu\to\infty & *\text{-weakly in } L_\infty(0,M;H^k),\ \text{weakly in } L_2(0,M;V^k),\\ y_i^\mu(\,\cdot\,)\rightharpoondown y_i(\,\cdot\,) & \text{as }\ \mu\to\infty & \text{weakly in } L_{p_i}(0,M;L_{p_i}(\Omega)),\ i=\overline{1,k};\\ \mathbf{z}^\mu(\,\cdot\,)\rightharpoondown\mathbf{z}(\,\cdot\,) & \text{as }\ \mu\to\infty & *\text{-weakly in } L_\infty(0,M;V^{N-k}),\\ z_j^\mu(\,\cdot\,)\rightharpoondown z_j(\,\cdot\,) & \text{as }\ \mu\to\infty & \text{weakly in } L_{p_j}(0,M;L_{p_j}(\Omega)),\ j=\overline{k+1,N};\\ & \text{and} & \\ \partial_t y_i^\mu(\,\cdot\,)\rightharpoondown\partial_t y_i(\,\cdot\,) & \text{as }\ \mu\to\infty & \text{weakly in } L_{q_i}(0,M;H^{-r_i}(\Omega)),\ i=\overline{1,k},\\ \partial_t z_j^\mu(\,\cdot\,)\rightharpoondown\partial_t z_j(\,\cdot\,) & \text{as }\ \mu\to\infty & \text{weakly in } L_{q_j}(0,M;L_{q_j}(\Omega)),\ j=\overline{k+1,N}. \end{array}\right\}$$

The space $\mathcal{F}_+^{\rm loc}$ equipped with the topology $\Theta_+^{\rm loc}$ is a Hausdorff Frechét–Urysohn topological vector space having a countable topology base (see, e.g., [3]). We consider a linear subspace $\mathcal{F}_+^{\rm b}\subset\mathcal{F}_+^{\rm loc}$ consisting of the vector functions $(\mathbf{y},\mathbf{z})\in\mathcal{F}_+^{\rm loc}$ with finite norm

$$\begin{aligned}\|(\mathbf{y},\mathbf{z})\|_{\mathcal{F}_+^{\rm b}} := \|\mathbf{y}\|_{L_\infty(\mathbb{R}_+;H^k)} + \|\mathbf{y}\|_{L_2^{\rm b}(\mathbb{R}_+;V^k)} + \sum_{i=1}^{k}\Big[\|y_i\|_{L_{p_i}^{\rm b}(\mathbb{R}_+;L_{p_i})} + \|\partial_t y_i\|_{L_{q_i}^{\rm b}(\mathbb{R}_+;H^{-r_i})}\Big]\\ + \|\mathbf{z}\|_{L_\infty(\mathbb{R}_+;V)} + \sum_{j=1}^{N-k}\Big[\|z_j\|_{L_{p_j}^{\rm b}(\mathbb{R}_+;L_{p_j})} + \|\partial_t z_j\|_{L_{q_j}^{\rm b}(\mathbb{R}_+;L_{q_j})}\Big].\end{aligned}\tag{2.2}$$

Recall that the norm of a function ϕ in the space $L_p^{\rm b}(\mathbb{R}_+;X)$, where X is a Banach space and $p\geqslant1$, is defined by the formula $\|\phi\|^p_{L_p^{\rm b}(\mathbb{R}_+;X)}:=\sup_{t\geqslant0}\int_t^{t+1}\|\phi(s)\|_X^p ds$. Obviously, $\mathcal{F}_+^{\rm b}$ with the norm (2.2) is a Banach space.

Remark 2.1. Any ball $B_r=\Big\{\|(\mathbf{y},\mathbf{z})\|_{\mathcal{F}_+^{\rm b}}\leqslant r\Big\}$ in the space $\mathcal{F}_+^{\rm b}$ is compact in the topology $\Theta_+^{\rm loc}$. Moreover, the corresponding topological subspace $B_r|_{\Theta_+^{\rm loc}}$ is metrizable (see, e.g., [18]). (Note that the space $\mathcal{F}_+^{\rm loc}|_{\Theta_+^{\rm loc}}$ is not metrizable.)

Let us now define the space $\mathcal{K}_+(S)$ of solutions (trajectories) for system (1.1)–(1.2) that depends on a parameter $S>0$.

Definition 2.1. The space $\mathcal{K}_+(S)$ consists of the functions $(\mathbf{u}(\,\cdot\,),\mathbf{v}(\,\cdot\,))\in\mathcal{F}_+^{\rm loc}$ such that

(i) the couple $(\mathbf{u}(t),\mathbf{v}(t)), t\geqslant0$, is a weak solution of system (1.1)–(1.2);

(ii) the vector function $\mathbf{v}(t)$ satisfies the inequality

$$\|\mathbf{v}(t)\|_1^2\ \leqslant Se^{-\sigma t}+R^2,\quad \forall t\geqslant0,\tag{2.3}$$

where the values σ and R are taken from the inequality (1.44).

We note that, by Proposition 1.2, the trajectory space $\mathcal{K}_+(S)$ is nonempty.

Consider the translation semigroup $\{T(\tau)\}:=\{T(\tau),\tau\geqslant0\}$ acting on $\mathcal{F}_+^{\rm loc}$ by the formula

$$T(\tau)(\mathbf{y}(t),\mathbf{z}(t))=(\mathbf{y}(t+\tau),\mathbf{z}(t+\tau)),\quad t\geqslant0.\tag{2.4}$$

Clearly, the semigroup $\{T(\tau)\}$ takes $\mathcal{K}_+(S)$ to itself,

$$T(\tau)\colon\mathcal{K}_+(S)\to\mathcal{K}_+(S),\qquad\forall\tau\geqslant0.\tag{2.5}$$

Proposition 2.1. *The space $\mathcal{K}_+(S)$ belongs to $\mathcal{F}_+^{\rm b}$, and the following inequality holds*:

$$\|T(\tau)(\mathbf{u},\mathbf{v})\|_{\mathcal{F}_+^{\rm b}}\leqslant C_6\left(\|\mathbf{u}(0)\|^2+S\right)e^{-\rho\tau}+R_3^2,\quad\forall\tau\geqslant0,\tag{2.6}$$

where $C_6\geqslant0$ and $\rho>0$ are independent of $\mathbf{g}_1$ and $\mathbf{g}_2$, whereas $R_3=R_3(\|\mathbf{g}_1\|,\|\mathbf{g}_2\|)$.

Proof. Inequalities (1.23), (1.24), and (2.3) yield

$$\|T(\tau)\mathbf{u}\|^2_{L_\infty(\mathbb{R}_+;H^k)} + a\|T(\tau)\mathbf{u}\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;V^k)} + \sum_{i=1}^{k} \|T(\tau)u_i\|^{p_i}_{L^{\mathrm{b}}_{p_i}(\mathbb{R}_+;L_{p_i})}$$
$$+ \|T(\tau)\mathbf{v}\|^2_{L_\infty(\mathbb{R}_+;V^{N-k})} + \sum_{j=1}^{N-k} \|T(\tau)v_j\|^{p_j}_{L^{\mathrm{b}}_{p_j}(\mathbb{R}_+;L_{p_j})} \leqslant C_7(\|\mathbf{u}(0)\|^2 + S)e^{-\rho_1\tau} + R_4^2, \tag{2.7}$$

for suitable ρ_1 and R_4. To estimate the norms $\|T(\tau)\partial_t u_i\|_{L^{\mathrm{b}}_{q_i}(\mathbb{R}_+;H^{-r_i})}$ and $\|T(\tau)\partial_t v_j\|_{L^{\mathrm{b}}_{q_j}(\mathbb{R}_+;L_{q_j})}$, we merely use equations (1.1)–(1.2). Thus, we obtain the following estimate for $\partial_t u_i$:

$$\left[\int_t^{t+1} \|\partial_t u_i(s)\|^{q_i}_{H^{-r_i}} ds\right]^{1/q_i}$$
$$\leqslant \left[\int_t^{t+1} \|a_i \Delta u_i(s)\|^{q_i}_{H^{-r_i}} ds\right]^{1/q_i} + \left[\int_t^{t+1} \|f_i(\mathbf{u}(s),\mathbf{v}(s))\|^{q_i}_{H^{-r_i}} ds\right]^{1/q_i} + \|g_{1,i}\|_{H^{-r_i}}$$
$$\leqslant C_8\left[\int_t^{t+1} \|u_i(s)\|_1^2 ds\right]^{1/2} + C_9\left[\int_t^{t+1} \left(\sum_{l=1}^{k} \|u_l(s)\|^{p_l}_{L_{p_l}} + \sum_{j=1}^{N-k} \|v_j(s)\|^{p_j}_{L_{p_j}}\right) ds + 1\right]^{1/q_j}$$
$$+ C_{10}\|\mathbf{g}_1\| \leqslant C_{11}(\|\mathbf{u}(0)\|^2 + S)e^{-\rho_2\tau} + R_5^2, \quad \forall t \geqslant \tau.$$

where we have used estimate (1.24) and the inequality $q_i \leqslant 2$. Consequently,

$$\|T(\tau)\partial_t u_i\|_{L^{\mathrm{b}}_{q_i}(\mathbb{R}_+;H^{-r_i})} \leqslant C_{12}(\|\mathbf{u}(0)\|^2 + S)e^{-\rho_3\tau} + R_6^2. \tag{2.8}$$

Similarly, it follows from equation (1.2) that

$$\|T(\tau)\partial_t v_j\|_{L^{\mathrm{b}}_{q_j}(\mathbb{R}_+;L_{q_j})} \leqslant C_{13}(\|\mathbf{u}(0)\|^2 + S)e^{-\rho_4\tau} + R_7^2. \tag{2.9}$$

Combining (2.7), (2.8), and (2.9), we obtain (2.6). In particular, $\mathcal{K}_+(S) \subset \mathcal{F}_+^{\mathrm{b}}$.

Proposition 2.2. *The space $\mathcal{K}_+(S)$ is closed in Θ_+^{loc} for every $S \geqslant 0$.*

Proof. Consider an arbitrary sequence $(\mathbf{u}^\mu(t),\mathbf{v}^\mu(t)) =: (\mathbf{u}^\mu,\mathbf{v}^\mu) \in \mathcal{K}_+(S)$, $\mu = 1,2,\dots$, which converges as $\mu \to \infty$ in Θ_+^{loc} to an element $(\mathbf{u}(t),\mathbf{v}(t)) =: (\mathbf{u},\mathbf{v}) \in \mathcal{F}_+^{\mathrm{loc}}$. We claim that $(\mathbf{u},\mathbf{v}) \in \mathcal{K}_+(S)$. By the definition of the topology Θ_+^{loc}, for every segment $[0,M]$, the following convergences hold as $\mu \to \infty$:

$$\left.\begin{array}{lll} \mathbf{u}^\mu(\,\cdot\,) \rightharpoonup \mathbf{u}(\,\cdot\,) & *\text{-weakly in } L_\infty(0,M;H^k) \\ \mathbf{u}^\mu(\,\cdot\,) \rightharpoonup \mathbf{u}(\,\cdot\,) & \text{weakly in } L_2(0,M;V^k) \\ \mathbf{v}^\mu(\,\cdot\,) \rightharpoonup \mathbf{v}(\,\cdot\,) & *\text{-weakly in } L_\infty(0,M;V^{N-k}) \end{array}\right\}, \tag{2.10}$$

$$\left.\begin{array}{ll} u_i^\mu(\,\cdot\,) \rightharpoonup u_i(\,\cdot\,) & \text{weakly in } L_{p_i}(0,M;L_{p_i}(\Omega)),\ i=\overline{1,k} \\ v_j^\mu(\,\cdot\,) \rightharpoonup v_j(\,\cdot\,) & \text{weakly in } L_{p_j}(0,M;L_{p_j}(\Omega)),\ j=\overline{k+1,N} \end{array}\right\}, \tag{2.11}$$

$$\left.\begin{array}{ll} \partial_t u_i^\mu(\,\cdot\,) \rightharpoonup \partial_t u_i(\,\cdot\,) & \text{weakly in } L_{q_i}(0,M;H^{-r_i}(\Omega)),\ i=\overline{1,k} \\ \partial_t v_j^\mu(\,\cdot\,) \rightharpoonup \partial_t v_j(\,\cdot\,) & \text{weakly in } L_{q_j}(0,M;L_{q_j}(\Omega)),\quad j=\overline{k+1,N} \end{array}\right\}. \tag{2.12}$$

In particular, the sequences $\{u_i^\mu\}$ are bounded in $L_\infty(0,M;H)$, $L_2(0,M;V^k)$, $L_{p_i}(0,M;L_{p_i}(\Omega))$ for $i=\overline{1,k}$, the sequences $\{v_j^\mu\}$ are bounded in $L_\infty(0,M;V)$ and in $L_{p_j}(0,M;L_{p_j}(\Omega))$ for $j=\overline{k+1,N}$, whereas the sequences $\{\partial_t u_i^\mu\}$ and $\{\partial_t v_j^\mu\}$ are bounded in the spaces $L_{q_i}(0,M;H^{-r_i}(\Omega))$ and $L_{q_j}(0,M;L_{q_j}(\Omega))$, respectively. Hence, due to inequalities (1.16), the sequences $\{f_i(\mathbf{u}^\mu,\mathbf{v}^\mu)\}$ and $\{h_j(\mathbf{u}^\mu,\mathbf{v}^\mu)\}$ are bounded in the spaces $L_{q_i}(0,M;L_{q_i}(\Omega))$ and in $L_{q_j}(0,M;L_{q_j}(\Omega))$, respectively.

Then, passing (if necessary) to a subsequence $\{\mu'\} \subset \{\mu\}$ and keeping the notation $\{\mu\}$, we can assume that

$$\left.\begin{array}{ll} f_i(\mathbf{u}^\mu, \mathbf{v}^\mu) \rightharpoonup \varphi_i(\,\cdot\,) \quad (\mu \to \infty) & \text{weakly in } L_{q_i}(0, M; L_{q_i}(\Omega)) \\ h_j(\mathbf{u}^\mu, \mathbf{v}^\mu) \rightharpoonup \chi_j(\,\cdot\,) \quad (\mu \to \infty) & \text{weakly in } L_{q_j}(0, M; L_{q_j}(\Omega)) \end{array}\right\}, \tag{2.13}$$

where $\varphi_i = \varphi_i(x,t)$ and $\chi_j = \chi_j(x,t)$ are some elements of the spaces $L_{q_i}(0, M; L_{q_i}(\Omega))$ and $L_{q_j}(0, M; L_{q_j}(\Omega))$, respectively.

Since $(\mathbf{u}^\mu(t), \mathbf{v}^\mu(t))$ is a weak solution of system (1.1)–(1.2), we have

$$\partial_t \mathbf{u}^\mu = \mathbf{a}\Delta\mathbf{u}^\mu - \mathbf{f}(\mathbf{u}^\mu, \mathbf{v}^\mu) + \mathbf{g}_1(x), \qquad \partial_t \mathbf{v}^\mu = -\mathbf{h}(\mathbf{u}^\mu, \mathbf{v}^\mu) + \mathbf{g}_2(x).$$

Using (2.10), (2.12), and (2.13), we conclude that the couple $(\mathbf{u}(t), \mathbf{v}(t))$ satisfies the differential equations

$$\partial_t \mathbf{u} = \mathbf{a}\Delta\mathbf{u} - \boldsymbol{\varphi}(x,t) + \mathbf{g}_1(x), \qquad \partial_t \mathbf{v} = -\boldsymbol{\chi}(x,t) + \mathbf{g}_2(x), \quad 0 \leqslant t \leqslant M,$$

in the distribution sense. Recall that the sequence $\{u_i^\mu(t)\}$ is bounded in the space $L_2(0, M; V)$, and $\{\partial_t u_i^\mu(t)\}$ is bounded in $L_{q_i}(0, M; H^{-r_i}(\Omega))$. Moreover, the embedding $V \Subset H \equiv L_2(\Omega)$ is compact. Therefore, by the Aubin theorem (see [19, 20]), the sequence $\{u_i^\mu(t)\}$ formes a precompact set in the space $L_2(0, M; L_2(\Omega))$. This means that $u_i^\mu(\,\cdot\,) \rightharpoonup u_i(\,\cdot\,)$ $(\mu \to \infty)$ strongly in $L_2(\Omega\times]0, M[)$ for $i = \overline{1,k}$. Passing to a subsequence gives $\mathbf{u}^\mu(x,t) \to \mathbf{u}(x,t)$ $(\mu \to \infty)$ for a.e. $(x,t) \in \Omega\times]0, M[$. Similarly, we obtain $\mathbf{v}^\mu(x,t) \to \mathbf{v}(x,t)$ $(\mu \to \infty)$ for a.e. $(x,t) \in \Omega\times]0, M[$. Using the continuity of the vector functions $\mathbf{f}$ and $\mathbf{h}$, we obtain

$$\mathbf{f}(\mathbf{u}^\mu(x,t), \mathbf{v}^\mu(x,t)) \to \mathbf{f}(\mathbf{u}(x,t), \mathbf{v}(x,t)), \qquad \mathbf{h}(\mathbf{u}^\mu(x,t), \mathbf{v}^\mu(x,t)) \to \mathbf{h}(\mathbf{u}(x,t), \mathbf{v}(x,t))$$

as $\mu \to \infty$ for a.e. $(x,t) \in \Omega\times]0, M[$. Recall that the sequences $\{f_i(\mathbf{u}^\mu, \mathbf{v}^\mu)\}$ and $\{h_j(\mathbf{u}^\mu, \mathbf{v}^\mu)\}$ are bounded in the spaces $L_{q_i}(0, M; L_{q_i}(\Omega))$ and $L_{q_j}(0, M; L_{q_j}(\Omega))$, respectively. Applying the known Lions lemma concerning the weak convergence (see [16, Ch.1, Lemma 1.3]), we have the following limit relations as $\mu \to \infty$:

$$f_i(\mathbf{u}^\mu, \mathbf{v}^\mu) \rightharpoonup f_i(\mathbf{u}, \mathbf{v}) \quad \text{weakly in} \quad L_{q_i}(0, M; L_{q_i}(\Omega)), \quad i = \overline{1,k},$$

$$h_j(\mathbf{u}^\mu, \mathbf{v}^\mu) \rightharpoonup h_j(\mathbf{u}, \mathbf{v}) \quad \text{weakly in} \quad L_{qi}(0, M; L_{q_j}(\Omega)), \quad j = \overline{k+1, N}.$$

Hence, due to (2.13), we conclude that $\boldsymbol{\varphi}(x,t) \equiv \mathbf{f}(\mathbf{u}(x,t), \mathbf{v}(x,t))$ and $\boldsymbol{\chi}(x,t) \equiv \mathbf{h}(\mathbf{u}(x,t), \mathbf{v}(x,t))$ for a.e. $\Omega\times]0, M[$. That is, the couple $(\mathbf{u}(x,t), \mathbf{v}(x,t))$ is a weak solution of system (1.1)–(1.2). It remains to prove inequality (2.3) for the function $\mathbf{v}(x,t)$ thus constructed. Indeed, the functions $\mathbf{v}^\mu(x,t)$ satisfy (2.3), and therefore the inequality

$$\operatorname{ess\,sup}\{\|\mathbf{v}^\mu(\theta)\|_1^2 \mid t \leqslant \theta \leqslant t+1\} \leqslant Se^{-\sigma t} + R^2$$

holds for all $\mu \in \mathbb{N}$ and for every $\theta \geqslant 0$. Recall that $\mathbf{v}^\mu(\,\cdot\,) \rightharpoonup \mathbf{v}(\,\cdot\,)$ *-weakly in $L_\infty(0, M; V^{N-k})$ for any $M > 0$. Hence, for a chosen $t \geqslant 0$,

$$\operatorname{ess\,sup}\left\{\|\mathbf{v}(\theta)\|_1^2 \mid t \leqslant \theta \leqslant t+1\right\} \leqslant \liminf_{\mu\to\infty}\left\{\|\mathbf{v}^\mu(\theta)\|_1^2 \mid t \leqslant \theta \leqslant t+1\right\} \leqslant Se^{-\sigma t} + R^2. \tag{2.14}$$

The Lions–Magenes lemma implies that $\mathbf{v}(\,\cdot\,) \in C_w(\mathbb{R}_+; V^{N-k})$ and, in particular, the real function $\|\mathbf{v}(t)\|_1$, $t \geqslant 0$, is lower semicontinuous, i.e.,

$$\|\mathbf{v}(t)\|_1 \leqslant \liminf_{\theta\to t^+} \|\mathbf{v}(\theta)\|_1$$

(see, e.g., [3]). Applying this relation, together with (2.14), we see that

$$\|\mathbf{v}(t)\|_1 \leqslant Se^{-\sigma t} + R^2 \quad \text{for any} \quad t \geqslant 0.$$

We have established inequality (2.3) for the vector function $\mathbf{v}(t)$, and therefore $(\mathbf{u}, \mathbf{v}) \in \mathcal{K}_+(S)$. We have proved that $\mathcal{K}_+(S)$ is closed in Θ_+^{loc}.

Let us now study the translation semigroup $\{T(\tau)\}$ acting on the trajectory space $\mathcal{K}_+(S)$, beginning with the main definitions.

Definition 2.2. A set $P \subseteq \mathcal{K}_+(S)$ is said to be *absorbing* for the semigroup $\{T(\tau)\}$ if, for every bounded set $B \subset \mathcal{K}_+(S)$ in $\mathcal{F}_+^{\text{b}}$, there is a $\tau_1 = \tau_1(B) \geqslant 0$ such that $T(\tau)B \subseteq P$ for all $\tau \geqslant \tau_1$.

Definition 2.3. A set $P \subseteq \mathcal{K}_+(S)$ is said to be *attracting* for the semigroup $\{T(\tau)\}$ if any neighborhood $\mathcal{O}(P)$ of the set P in the topology Θ_+^{loc} is an absorbing set for $\{T(\tau)\}$, i.e., for every bounded set $B \subset \mathcal{K}_+(S)$ in $\mathcal{F}_+^{\text{b}}$, there is a $\tau_1 = \tau_1(B, \mathcal{O}) \geqslant 0$ such that $T(\tau)B \subseteq \mathcal{O}(P)$ for all $\tau \geqslant \tau_1$.

Definition 2.4. A set $\mathfrak{A} \subset \mathcal{K}_+(S)$ is called a *trajectory attractor* for the semigroup $\{T(\tau)\}$ on $\mathcal{K}_+(S)$ if $\mathfrak{A}$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$, compact with respect to Θ_+^{loc}, strictly invariant with respect to $\{T(\tau)\}$, i.e.,

$$T(\tau)\mathfrak{A} = \mathfrak{A}, \qquad \forall \tau \geqslant 0, \tag{2.15}$$

and $\mathfrak{A}$ is an attracting set for $\{T(\tau)\}$.

Let us now construct a trajectory attractor for $\{T(\tau)\}$ on $\mathcal{K}_+(S)$. Inequality (2.6) implies that the set $P = \Big\{(\mathbf{u}, \mathbf{v}) \in \mathcal{K}_+(S) \mid \|(\mathbf{u}, \mathbf{v})\|_{\mathcal{F}_+^{\mathrm{b}}} \leqslant 2R_3^2\Big\}$ is absorbing for the semigroup $\{T(\tau)\}$ on $\mathcal{K}_+(S)$. This set is bounded in $\mathcal{F}_+^{\mathrm{b}}$. Therefore, the topological subspace $P|_{\Theta_+^{\mathrm{loc}}}$ is compact and metrizable (see Remark 2.1). Using (2.5) and the obvious inequality

$$\|T(\tau)(\mathbf{u}, \mathbf{v})\|_{\mathcal{F}_+^{\mathrm{b}}} \leqslant \|(\mathbf{u}, \mathbf{v})\|_{\mathcal{F}_+^{\mathrm{b}}} \quad \text{for all} \quad \tau \geqslant 0,$$

we see that the semigroup $\{T(\tau)\}$ takes P to itself, $T(\tau)P \subseteq P$ for all $\tau \geqslant 0$. The semigroup $\{T(\tau)\}$ is continuous on P in the topology Θ_+^{loc}. Hence, we have a continuous semigroup acting on a compact metric space. Applying the general theorem on the existence of a global attractor of a semigroup (see, e.g., [1, 2, 4]), we conclude that the set

$$\mathfrak{A}(S) = \bigcap_{\tau \geqslant 0} \Big[\bigcup_{\theta \geqslant \tau} T(\theta)P\Big]_{\Theta_+^{\mathrm{loc}}}$$

serves as the global attractor of $\{T(\tau)\}$. Consequently, the set $\mathfrak{A}(S) \subset P$ has the following properties: $\mathfrak{A}(S)$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$, compact with respect to Θ_+^{loc}, strictly invariant ($T(\tau)\mathfrak{A}(S) = \mathfrak{A}(S)$ for any $\tau \geqslant 0$), and, as a global attractor, the set $\mathfrak{A}(S)$ attracts any set $B \subseteq P$. However, P is an absorbing set for $\{T(\tau)\}$. Hence, $\mathfrak{A}(S)$ attracts any $\mathcal{F}_+^{\mathrm{b}}$-bounded set $B \subset \mathcal{K}_+(S)$. Therefore, $\mathfrak{A}(S)$ is a trajectory attractor of $\{T(\tau)\}$ on $\mathcal{K}_+(S)$ in the topology Θ_+^{loc}.

Proposition 2.3. *The trajectory attractor thus constructed does not depend on* S, $\mathfrak{A}(S) = \mathfrak{A}$. *In particular,* $\mathfrak{A} = \mathfrak{A}(0)$, *i.e.,*

$$\sup\big\{\|\mathbf{v}(t)\|_1^2 \mid t \geqslant 0\big\} \leqslant R^2, \quad \forall (\mathbf{u}, \mathbf{v}) \in \mathfrak{A}, \tag{2.16}$$

where R *is as in* (2.3).

Proof. Let $S > 0$. It follows from the definition of $\mathcal{K}_+(S)$ that $\mathcal{K}_+(S) \subseteq \mathcal{K}_+(S_1)$ for all $S_1 \geqslant S$. Hence, $\mathfrak{A}(S) \subseteq \mathfrak{A}(S_1)$ for $S_1 \geqslant S$. Having (2.3), we note that

$$T(\tau)\mathcal{K}_+(S_1) \subseteq \mathcal{K}_+(S) \quad \text{for} \quad \tau \geqslant \sigma^{-1}\log(S_1/S)$$

and, in particular,

$$T(\tau)\mathfrak{A}(S_1) \subseteq \mathcal{K}_+(S) \quad \text{for} \quad \tau \geqslant \sigma^{-1}\log(S_1/S)\,.$$

However, the set $\mathfrak{A}(S_1)$ is strictly invariant, $\mathfrak{A}(S_1) = T(\tau)\mathfrak{A}(S_1)$, i.e., $\mathfrak{A}(S_1) \subseteq \mathcal{K}_+(S)$, and the set $\mathfrak{A}(S)$, as the attractor in $\mathcal{K}_+(S)$, attracts $\mathfrak{A}(S_1)$. Therefore, using the strict invariance of $\mathfrak{A}(S_1)$ again, we conclude that $\mathfrak{A}(S_1) \subseteq \mathfrak{A}(S)$ for all $S_1 \geqslant S$. Hence, $\mathfrak{A}(S_1) = \mathfrak{A}(S)$ for all $S_1 \geqslant S$. We have proved that $\mathfrak{A} = \mathfrak{A}(S)$ does not depend on S for $S > 0$, i.e., the inequality

$$\sup\big\{\|\mathbf{v}(t)\|_1^2 \mid t \geqslant 0\big\} \leqslant S + R^2 \quad \text{for all} \quad S > 0,$$

holds for every $(\mathbf{u}, \mathbf{v}) \in \mathfrak{A}$, and we obtain (2.16).

In conclusion of this section, we describe the structure of the trajectory attractor $\mathfrak{A}$ by using the notion of kernel of system (1.1), (1.2), which consists of all weak solutions of the system that are defined on the entire time axis.

Define the spaces $\mathcal{F}^{\mathrm{loc}}, \mathcal{F}^{\mathrm{b}}$ and the topology Θ^{loc} similarly to $\mathcal{F}_+^{\mathrm{loc}}, \mathcal{F}_+^{\mathrm{b}}$ and Θ_+^{loc} by replacing the semiaxis $\mathbb{R}_+$ ($t \geqslant 0$) by the entire axis $\mathbb{R}$ ($-\infty < t < \infty$). For example, the norm in $\mathcal{F}^{\mathrm{b}}$ is defined by the formula (cf. (2.2))

$$\|(\mathbf{y}, \mathbf{z})\|_{\mathcal{F}^{\mathrm{b}}} := \|\mathbf{y}\|_{L_\infty(\mathbb{R}; H^k)} + \|\mathbf{y}\|_{L_2^{\mathrm{b}}(\mathbb{R}; V^k)} + \sum_{i=1}^{k} \Big[\|y_i\|_{L_{p_i}^{\mathrm{b}}(\mathbb{R}; L_{p_i})} + \|\partial_t y_i\|_{L_{q_i}^{\mathrm{b}}(\mathbb{R}; H^{-r_i})}\Big]$$
$$+ \|\mathbf{z}\|_{L_\infty(\mathbb{R}; V^{N-k})} + \sum_{j=1}^{N-k} \Big[\|z_j\|_{L_{p_j}^{\mathrm{b}}(\mathbb{R}; L_{p_j})} + \|\partial_t z_j\|_{L_{q_j}^{\mathrm{b}}(\mathbb{R}; L_{q_j})}\Big]. \tag{2.17}$$

Definition 2.5. The *kernel* $\mathcal{K}$ of system (1.1)–(1.2) in the space $\mathcal{F}^{\mathrm{b}}$ consists of all weak solutions $\{\mathbf{u}(t), \mathbf{v}(t)\}, t \in \mathbb{R}$ of this system from $\mathcal{F}^{\mathrm{loc}}$ belonging to $\mathcal{F}^{\mathrm{b}}$ (i.e., having the finite norm $\|(\mathbf{u}, \mathbf{v})\|_{\mathcal{F}^{\mathrm{b}}}$, defined by (2.17)) which satisfy the inequality $\sup\{\|\mathbf{v}(t)\|_1 \mid t \in \mathbb{R}\} \leqslant R$, where the value R is taken from inequality (2.3).

Let Π_+ be the operator of restriction to $\mathbb{R}_+$. This operator takes a function $\{\phi(t), t \in \mathbb{R}\}$ to the function $\{\Pi_+\phi(t), t \geqslant 0\}$, where $\Pi_+\phi(t) \equiv \phi(t)$ for all $t \geqslant 0$.

Theorem 2.1. *The kernel* $\mathcal{K}$ *of system* (1.1)–(1.2) *is bounded in the space* $\mathcal{F}^{\mathrm{b}}$ *and compact with respect to the topology* Θ^{loc}. *The trajectory attractor* $\mathfrak{A}$ *of* (1.1)–(1.2) *coincides with the restriction of* $\mathcal{K}$ *to the semiaxis* $\mathbb{R}_+ : \mathfrak{A} = \Pi_+\mathcal{K}$.

The proof is straightforward.

3. REACTION-DIFFUSION SYSTEMS WITH A SERIES OF SMALL DIFFUSION COEFFICIENTS

In this section, we study the reaction-diffusion system that differs from system (1.1)–(1.2) in the following way: the second set of equations for the vector components $v_j, j = \overline{k+1, N}$, can contain diffusion terms $\delta_j \Delta v_j$ with small (possibly nonzero!) diffusion coefficients δ_j. Thus, the system reads as follows:

$$\partial_t \mathbf{u} = \mathbf{a}\Delta\mathbf{u} - \mathbf{f}(\mathbf{u}, \mathbf{v}) + \mathbf{g}_1(x), \tag{3.1}$$

$$\partial_t \mathbf{v} = \boldsymbol{\delta}\Delta\mathbf{v} - \mathbf{h}(\mathbf{u}, \mathbf{v}) + \mathbf{g}_2(x), \tag{3.2}$$

where $\boldsymbol{\delta} = \mathrm{diag}(\delta_{k+1}, \delta_{k+2}, \ldots, \delta_N)$, $\delta_j \geqslant 0$ $(j = \overline{k+1, N})$. If $\boldsymbol{\delta} = \mathbf{0}$, we obtain the reaction-diffusion system treated in Section 1. As above, we supply the system with the Dirichlet boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \qquad \mathbf{v}|_{\partial\Omega} = 0. \tag{3.3}$$

Here the notation is the same as in Section 1. In particular, the vector functions $\mathbf{f}$ and $\mathbf{h}$ satisfy (1.3)–(1.7), and $\mathbf{g}_1$ and $\mathbf{g}_2$ satisfy (1.8). The couple of vector functions $(\mathbf{u}(x,t), \mathbf{v}(x,t)) := (u_1(x,t), \ldots, u_k(x,t), v_{k+1}(x,t), \ldots, v_N(x,t))$, $(x,t) \in \Omega \times [0, M]$, is referred to as a weak solution of system (3.1)–(3.2) if

$$u_i(\,\cdot\,) \in L_{p_i}(0, M; L_{p_i}(\Omega)) \cap L_2(0, M; V), \quad i = \overline{1, k}, \tag{3.4}$$

$$v_j(\,\cdot\,) \in L_{p_j}(0, M; L_{p_j}(\Omega)) \cap L_2(0, M; V), \quad j = \overline{k+1, N}, \tag{3.5}$$

and the functions $u_i(x,t)$ and $v_j(x,t)$ satisfy equations (3.1) and (3.2) in the spaces of distributions $\mathcal{D}'(0, M; H^{-r_i}(\Omega))$ and $\mathcal{D}'(0, M; H^{-r_j}(\Omega))$, respectively. Here

$$\begin{aligned} r_i &= \max\{1, n(1/2 - 1/p_i)\}, \quad i = \overline{1, k}; \\ r_j &= \max\{1, n(1/2 - 1/p_j)\}, \quad j = \overline{k+1, N}. \end{aligned} \tag{3.6}$$

Remark 3.1. Unlike the case $\boldsymbol{\delta} \equiv \mathbf{0}$, which was treated in Section 1, we now assume that the components $v_j(x,t)$, $j = \overline{k+1, N}$, belong to the space $L_{p_j}(0, M; L_{p_j}(\Omega)) \cap L_2(0, M; V)$. Note that, for $\delta_j > 0$, the right-hand sides of (3.2) clearly belong to

$$L_2(0, M; H^{-1}(\Omega)) + L_{q_j}(0, M; L_{q_j}(\Omega)) \subset L_{q_j}(0, M; H^{-r_j}(\Omega)), \quad j = \overline{k+1, N}$$

because $q_j < 2$ and $L_{q_j}(\Omega), H^{-1}(\Omega) \subset H^{-r_j}(\Omega)$ for r_j defined in (3.6). Hence, we can consider the derivatives $\partial_t v_j(t)$ in equations (3.2) as distributions in $\mathcal{D}'(0, M; H^{-r_j}(\Omega))$. For $\delta_j = 0$, we also consider the distribution space $\mathcal{D}'(0, M; H^{-r_j}(\Omega))$ since $L_{q_j}(\Omega) \subset H^{-r_j}(\Omega)$. Then the definition of a weak solution for $\delta_j > 0$ and $\delta_j = 0$ is the same.

For an arbitrary weak solution $(\mathbf{u}(\,\cdot\,), \mathbf{v}(\,\cdot\,))$ of (3.1)–(3.2) we have

$$\partial_t u_i \in L_{q_i}(0, M; H^{-r_i}(\Omega)), \quad i = \overline{1, k}, \qquad \partial_t v_j \in L_{q_j}(0, M; H^{-r_j}(\Omega)), \quad j = \overline{k+1, N}.$$

By the Lions–Magenes lemma, $\mathbf{u}(\,\cdot\,) \in C_w([0,M];H^k)$ and $\mathbf{v}(\,\cdot\,) \in C_w([0,M];H^{N-k})$. Therefore, the values $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are well defined for all $t \geqslant 0$, and the following initial data are meaningful for (3.1)–(3.2):

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \in H^k, \tag{3.7}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0 \in H^{N-k}. \tag{3.8}$$

The existence of a weak solution for problem (3.1)–(3.3), (3.7), (3.8) can be proved by using the Galerkin method outlined in Section 1 in the case of $\boldsymbol{\delta} = 0$. In fact, if all the diffusion coefficients are positive, $\delta_j > 0, j = \overline{k+1,N}$, then, to construct a weak solution, it is sufficient to have only the first *a priori* estimate. Moreover, assumptions (1.6) and (1.7) are not needed for the existence of weak solutions, and the space H^{N-k} for the initial data in (3.8) is quite sufficient. At the same time, if $\delta_j = 0$ for some j, then, to construct a weak solution of the system, we also need the second *a priori* estimate.

Recall that any weak solution of (3.1)–(3.2) satisfies the energy identity

$$\frac{1}{2}\frac{d}{dt}\left\{\sum_{i=1}^{k}\|u_i(t)\|^2 + \sum_{j=k+1}^{N}\|v_j(t)\|^2\right\} + \sum_{i=1}^{k} a_i\|\nabla u_i(t)\|^2 + \sum_{i=k+1}^{N}\delta_j\|\nabla v_j(t)\|^2$$
$$+ \int_\Omega \sum_{i=1}^{k} f_i(\mathbf{u},\mathbf{v})u_i(x,t)dx + \int_\Omega \sum_{j=k+1}^{N} h_j(\mathbf{u},\mathbf{v})v_j(x,t)dx = \sum_{i=1}^{k}\langle g_{1,i}, u_i(t)\rangle + \sum_{j=k+1}^{N}\langle g_{2,j}, v_j(t)\rangle$$

(cf. (1.22)). This identity implies the following assertion.

Proposition 3.1. *For every weak solution* $(\mathbf{u}(t),\mathbf{v}(t)), t \geqslant 0$, *of system* (3.1)–(3.8) *with initial data* (3.7) *and* (3.8), *the following inequalities hold*:

$$\|\mathbf{u}(t)\|^2 + \|\mathbf{v}(t)\|^2 + 2a\int_0^t \|\nabla\mathbf{u}(s)\|^2 e^{-\sigma(t-s)}ds + 2\int_0^t \sum_{j=k+1}^{N}\delta_j\|\nabla v_j(s)\|^2 e^{-\sigma(t-s)}ds$$
$$\leqslant \left(\|\mathbf{u}_0\|^2 + \|\mathbf{v}_0\|^2\right)e^{-\sigma t} + R_1^2, \tag{3.9}$$

$$2\int_t^{t+1} a\|\nabla\mathbf{u}(s)\|^2 ds + 2\int_t^{t+1}\sum_{j=k+1}^{N}\delta_j\|\nabla v_j(s)\|^2 ds + \sigma\int_t^{t+1}\left(\sum_{i=1}^{k}\|u_i(s)\|_{L_{p_i}}^{p_i} + \sum_{j=k+1}^{N}\|v_j(s)\|_{L_{p_j}}^{p_j}\right)ds$$
$$\leqslant \left(\|\mathbf{u}_0\|^2 + \|\mathbf{v}_0\|^2\right)e^{-\sigma t} + R_2^2, \quad \forall t \geqslant 0, \tag{3.10}$$

where R_1 *and* R_2 *are the same as in* (1.29) *and* (1.32).

The proof is similar to that of Proposition 1.1.

In Section 4, we study the limit behavior of the trajectory attractors of the system (3.1)–(3.3) as $\boldsymbol{\delta} \to \mathbf{0}^+$. This limit would exist if we could construct "stronger" weak solutions for system (3.1)–(3.3) that are uniformly bounded (with respect to $\boldsymbol{\delta}$) in the space $\mathcal{F}_+^{\mathrm{b}}$ introduced in Section 2. To obtain a solution of this kind, we need the second *a priori* estimate similar to (1.44), which can be proved in the case of $\boldsymbol{\delta} = \mathbf{0}$. We must also consider stronger initial conditions for the function $\mathbf{v}(\,\cdot\,)$ which satisfies (1.35). In this way, we apply the differential identity

$$\frac{1}{2}\frac{d}{dt}\sum_{j=k+1}^{N}\|\nabla v_j^m(t)\|^2 + \sum_{j=k+1}^{N}\delta_j\|\Delta v_j^m(t)\|^2 - \int_\Omega\sum_{j=k+1}^{N}h_j(\mathbf{u}^m(x,t),\mathbf{v}^m(x,t))\Delta v_j^m(x,t)dx$$
$$= \sum_{j=k+1}^{N}\langle\nabla g_{2,j}, \nabla v_j(t)\rangle\,.$$

which is similar to identity (1.36) and holds for an arbitrary Galerkin approximation $(\mathbf{u}^m(t),\mathbf{v}^m(t))$ for problem (3.1)–(3.3), (3.7), (3.8). Using assumptions (1.6)–(1.7), we can prove the following assertion, similarly to the proof of Proposition 1.2.

Proposition 3.2. *Assume that* $\mathbf{v}_0 \in V^{N-k}$. *Then problem* (3.1)–(3.3), (3.7), (3.8) *has a weak solution* $(\mathbf{u}(t), \mathbf{v}(t))$ *belonging to the classes* (3.4)–(3.5) *and such that* $\mathbf{v}(\,\cdot\,) \in L_\infty(\mathbb{R}_+; V^{N-k})$ *and the inequality*

$$\|\mathbf{v}(t)\|_1^2 \leqslant \|\mathbf{v}_0\|_1^2 e^{-\sigma t} + C_5 \left(\|\mathbf{u}_0\|^2 + \|\mathbf{v}_0\|^2\right) e^{-\sigma t} + R^2$$

holds for any $t \geqslant 0$, *where the value* R *and the constant* C_5 *do not depend on* $\boldsymbol{\delta} = \{\delta_{k+1}, \dots, \delta_N\}$, $\delta_j \geqslant 0$, $j = \overline{k+1, N}$, *and they are the same as in Proposition* 1.2 (*see* (1.43)).

Let us now construct the trajectory attractor for reaction-diffusion system (3.1)–(3.2).

Consider the spaces $\tilde{\mathcal{F}}_+^{\rm loc}$ and $\tilde{\mathcal{F}}_+^{\rm b}$ and the topology $\tilde{\Theta}_+^{\rm loc}$, which almost coincide with the spaces $\mathcal{F}_+^{\rm loc}$ and $\mathcal{F}_+^{\rm b}$ and the topology $\Theta_+^{\rm loc}$ introduced in Section 2, with the following modifications:

- in $\tilde{\mathcal{F}}_+^{\rm loc}$, the functions $\partial_t z_j$ belong to $L_{q_j}^{\rm loc}(\mathbb{R}_+; H^{-r_j}(\Omega))$, $j = \overline{k+1, N}$;
- in $\tilde{\mathcal{F}}_+^{\rm b}$, the function $\partial_t z_j$ belongs to $L_{q_j}^{\rm b}(\mathbb{R}_+; H^{-r_j}(\Omega))$, $j = \overline{k+1, N}$;
- in $\tilde{\Theta}_+^{\rm loc}$, $\partial_t z_j^\mu(\,\cdot\,) \rightharpoonup \partial_t z_j(\,\cdot\,)$ as $\mu \to \infty$ weakly in $L_{q_j}(0, M; H^{-r_j}(\Omega))$, $j = \overline{k+1, N}$.

Since $L_{q_j}(\Omega) \subset H^{-r_j}(\Omega)$, it is clear that $\mathcal{F}_+^{\rm loc} \subset \tilde{\mathcal{F}}_+^{\rm loc}$, $\mathcal{F}_+^{\rm b} \subset \tilde{\mathcal{F}}_+^{\rm b}$, and $\Theta_+^{\rm loc} \subset \tilde{\Theta}_+^{\rm loc}$, and the second and third embeddings are continuous.

The trajectory spaces $\mathcal{K}_+^{\boldsymbol{\delta}}(S)$ for system (3.1)–(3.2) are defined similarly to the spaces $\mathcal{K}_+(S)$ corresponding to system (1.1)–(1.2) (see Definition 2.1). Note that $\mathcal{K}_+^{\mathbf{0}}(S) \equiv \mathcal{K}_+(S)$ (this is a simple exercise).

Definition 3.1. The space $\mathcal{K}_+^{\boldsymbol{\delta}}(S)$ consists of the functions $(\mathbf{u}(\,\cdot\,), \mathbf{v}(\,\cdot\,)) \in \tilde{F}_+^{\rm loc}$ such that

(i) $(\mathbf{u}(t), \mathbf{v}(t)), t \geqslant 0$, is a weak solution of system (3.1)–(3.2);

(ii) the vector function $\mathbf{v}(t)$ satisfies the inequality

$$\|\mathbf{v}(t)\|_1^2 \leqslant S e^{-\sigma t} + R^2, \quad \forall t \geqslant 0, \tag{3.11}$$

where the values σ and R are taken from inequality (1.44).

Using Proposition 3.2, we prove that the trajectory spaces $\mathcal{K}_+^{\boldsymbol{\delta}}(S)$ are nonempty for any $S > 0$.

Propositions 2.1 and 2.2 remain valid for the spaces $\mathcal{K}_+^0(S)$. Indeed, we can repeat the proofs of these assertions, taking into account Propositions 3.1 and 3.2 and omitting the positive terms containing $\boldsymbol{\delta}$ in (3.9) and (3.10).

It follows from Proposition 3.1 that the set $P^{\boldsymbol{\delta}} = \left\{(\mathbf{u}, \mathbf{v}) \in \mathcal{K}_+^{\boldsymbol{\delta}}(S) \mid \|(\mathbf{u}, \mathbf{v})\|_{\tilde{\mathcal{F}}_+^{\rm b}} \leqslant 2R_3^2\right\}$ is absorbing for the semigroup $\{T(\tau)\}$ acting on $\mathcal{K}_+^{\boldsymbol{\delta}}(S)$ by the formula (2.4). The set $P^{\boldsymbol{\delta}}|_{\tilde{\Theta}_+^{\rm loc}}$ is a metric space and, moreover, $P^{\boldsymbol{\delta}}$ is bounded in the norm of $\tilde{\mathcal{F}}_+^{\rm b}$.

Now let us construct the trajectory attractor $\mathfrak{A}^{\boldsymbol{\delta}}$ for system (3.1)–(3.2) by the formula

$$\mathfrak{A}^{\boldsymbol{\delta}} = \bigcap_{\tau \geqslant 0} \Big[\bigcup_{\theta \geqslant \tau} T(\theta) P^{\boldsymbol{\delta}} \Big]_{\tilde{\Theta}_+^{\rm loc}}.$$

The set $\mathfrak{A}^{\boldsymbol{\delta}}$ does not depend on S, and

$$\sup \left\{ \|\mathbf{v}(t)\|_1^2 \mid t \geqslant 0 \right\} \leqslant R^2 \quad \text{for any} \quad (\mathbf{u}, \mathbf{v}) \in \mathfrak{A}^{\boldsymbol{\delta}}.$$

The proof of this property is similar to that of Proposition 2.3.

Note that, for the “limit” case $\boldsymbol{\delta} = 0$, the trajectory attractor $\mathfrak{A}^{\mathbf{0}}$ coincides with the attractor $\mathfrak{A}$ constructed in Section 2.

Finally, let $\mathcal{K}^{\boldsymbol{\delta}}$ be the kernel of system (3.1)–(3.2) (formed by the weak solutions $(\mathbf{u}(t), \mathbf{v}(t))$ of system (3.1)–(3.2) that are defined for all $t \in \mathbb{R}$ and bounded in $\tilde{\mathcal{F}}^{\rm b}$ and satisfy the inequality

$$\sup \left\{ \|\mathbf{v}(t)\|_1 \mid t \in \mathbb{R} \right\} \leqslant R,$$

where R is taken from inequality (2.3)). Similarly to Theorem 2.1, we can prove that

$$\mathfrak{A}^{\boldsymbol{\delta}} = \Pi_+ \mathcal{K}^{\boldsymbol{\delta}}. \tag{3.12}$$

This formula is used in the next section.

Note that the absorbing sets $P^{\boldsymbol{\delta}}$ are uniformly bounded (with respect to $\boldsymbol{\delta}$) in the space $\tilde{\mathcal{F}}_+^{\mathrm{b}}$. Thus, the following assertion holds.

Corollary 3.1. *The family of trajectory attractors* $\{\mathfrak{A}^{\boldsymbol{\delta}}\}$ *is uniformly bounded* (*with respect to* $\boldsymbol{\delta} = \{\delta_{k+1}, \dots, \delta_N\}$, $\delta_j \geqslant 0, \overline{k+1, N}$) *in the norm of the space* $\tilde{\mathcal{F}}_+^{\mathrm{b}}$, *and the family of kernels* $\{\mathcal{K}^{\boldsymbol{\delta}}\}$ *is uniformly bounded* (*with respect to* $\boldsymbol{\delta} = \{\delta_{k+1}, \dots, \delta_N\}$, $\delta_j \geqslant 0, \overline{k+1, N}$) *in* $\tilde{\mathcal{F}}^{\mathrm{b}}$.

4. CONVERGENCE OF THE TRAJECTORY ATTRACTORS $\mathfrak{A}^{\boldsymbol{\delta}}$ AS $\boldsymbol{\delta} \to 0^+$

To begin with, consider a sequence of weak solutions $\{(\mathbf{y}^\mu(t), \mathbf{z}^\mu(t)), t \geqslant 0\}_{\mu\in\mathbb{N}}$ of system (3.1)–(3.2) belonging to the spaces $\mathcal{K}_+^{\boldsymbol{\delta}^\mu}(0)$. Here the diffusion coefficients are $\boldsymbol{\delta}^\mu = \{\delta^\mu_{k+1}, \dots, \delta^\mu_N\}$, $\delta^\mu_j \geqslant 0$, $\overline{k+1, N}$.

Proposition 4.1. *If the sequence* $\{(\mathbf{y}^\mu(t), \mathbf{z}^\mu(t)), t \geqslant 0\}$ *is bounded in the space* $\tilde{\mathcal{F}}_+^{\mathrm{b}}$, $(\mathbf{y}^\mu, \mathbf{z}^\mu) \in \mathcal{K}_+^{\boldsymbol{\delta}^\mu}(0)$ *and* $|\boldsymbol{\delta}^\mu| \to 0^+$ *as* $\mu \to \infty$, *then there is a subsequence of indices* $\{\mu'\} \subset \{\mu\}$ *and a couple* $(\mathbf{y}, \mathbf{z}) \in \tilde{\mathcal{F}}_+^{\mathrm{b}}$ *such that* $(\mathbf{y}^{\mu'}, \mathbf{z}^{\mu'}) \to (\mathbf{y}, \mathbf{z})$ *as* $\mu' \to \infty$ *in* $\tilde{\Theta}_+^{\mathrm{loc}}$, $(\mathbf{y}(t), \mathbf{z}(t))$ $(t \geqslant 0)$ *is a weak solution of system* (1.1)–(1.2), *and*

$$(\mathbf{y}, \mathbf{z}) \in \mathcal{K}_+^{\mathbf{0}}(0). \tag{4.1}$$

Proof. The functions $(\mathbf{y}^\mu(t), \mathbf{z}^\mu(t))$ satisfy the system

$$\partial_t \mathbf{y}^\mu = \mathbf{a}\Delta\mathbf{y}^\mu - \mathbf{f}(\mathbf{y}^\mu, \mathbf{z}^\mu) + \mathbf{g}_1(x), \tag{4.2}$$

$$\partial_t \mathbf{z}^\mu = \boldsymbol{\delta}^\mu\Delta\mathbf{z}^\mu - \mathbf{h}(\mathbf{y}^\mu, \mathbf{z}^\mu) + \mathbf{g}_2(x). \tag{4.3}$$

The sequence $\{(\mathbf{y}^\mu(\cdot), \mathbf{z}^\mu(\cdot))\}$ is precompact in $\tilde{\Theta}_+^{\mathrm{loc}}$ since it is bounded in $\tilde{\mathcal{F}}_+^{\mathrm{b}}$ (see Remark 1.1). Hence, there is a subsequence of indices $\{\mu'\} \subset \{\mu\}$ such that $(\mathbf{y}^{\mu'}(\cdot), \mathbf{z}^{\mu'}(\cdot)) \to (\mathbf{y}(\cdot), \mathbf{z}(\cdot))$ as $\mu' \to \infty$ in $\tilde{\Theta}_+^{\mathrm{loc}}$ for some couple $(\mathbf{y}(\cdot), z(\cdot)) \in \tilde{\mathcal{F}}_+^{\mathrm{b}}$. Therefore, for any $M > 0$, the following convergences take place as $\mu \to \infty$:

$$\left.\begin{array}{lll} \mathbf{y}^\mu(\cdot) \rightharpoonup \mathbf{y}(\cdot) & *\text{-weakly in } L_\infty(0, M; H^k) \\ \mathbf{y}^\mu(\cdot) \rightharpoonup \mathbf{y}(\cdot) & \text{weakly in } L_2(0, M; V^k) \\ \mathbf{z}^\mu(\cdot) \rightharpoonup \mathbf{z}(\cdot) & *\text{-weakly in } L_\infty(0, M; V^{N-k}) \end{array}\right\}, \tag{4.4}$$

$$\left.\begin{array}{ll} y_i^\mu(\cdot) \rightharpoonup y_i(\cdot) & \text{weakly in } L_{p_i}(0, M; L_{p_i}(\Omega)),\ i = \overline{1, k} \\ z_j^\mu(\cdot) \rightharpoonup z_j(\cdot) & \text{weakly in } L_{p_j}(0, M; L_{p_j}(\Omega)),\ j = \overline{k+1, N} \end{array}\right\}, \tag{4.5}$$

$$\left.\begin{array}{ll} \partial_t y_i^\mu(\cdot) \rightharpoonup \partial_t y_i(\cdot) & \text{weakly in } L_{q_i}(0, M; H^{-r_i}(\Omega)),\ i = \overline{1, k} \\ \partial_t z_j^\mu(\cdot) \rightharpoonup \partial_t z_j(\cdot) & \text{weakly in } L_{q_j}(0, M; H^{-r_j}(\Omega)),\ j = \overline{k+1, N} \end{array}\right\}. \tag{4.6}$$

Here for brevity, we denote the indices μ' by μ.

Let us now choose an arbitrary $M > 0$ and apply the reasoning in the proof of Proposition 2.2. Passing to a subsequence $\{\mu'\}$ (if necessary), which we denote by $\{\mu\}$ again, we see that

$$f_i(\mathbf{y}^\mu, \mathbf{z}^\mu) \rightharpoonup f_i(\mathbf{y}, \mathbf{z}) \quad \text{weakly in } L_{q_i}(0, M; L_{q_i}(\Omega)), \quad i = \overline{1, k}, \tag{4.7}$$

$$h_j(\mathbf{y}^\mu, \mathbf{z}^\mu) \rightharpoonup h_j(\mathbf{y}, \mathbf{z}) \quad \text{weakly in } L_{q_j}(0, M; L_{q_j}(\Omega)), \quad j = \overline{k+1, N}, \quad \text{as } \mu \to \infty. \tag{4.8}$$

It follows from the second formula in (4.4) that $\Delta\mathbf{y}^\mu(\cdot) \rightharpoonup \Delta\mathbf{y}(\cdot)$ weakly in $L_2(0, M; [H^{-1}]^k)$ as $\mu \to \infty$. Recall that, by assumption, the sequence $\{\mathbf{z}^\mu(\cdot)\}$ is bounded in $L_\infty(\mathbb{R}_+; V^{N-k})$. Hence,

$$\begin{aligned} \|\boldsymbol{\delta}^\mu\Delta\mathbf{z}^\mu\|_{L_\infty(0,M;[H^{-1}]^{N-k})} &\leqslant \max\left\{\delta_j,\ j = \overline{k+1, N}\right\} C' \|\mathbf{z}^\mu\|_{L_\infty(0,M;V^{N-k})} \\ &\leqslant |\boldsymbol{\delta}^\mu| C'' K \to 0 \qquad (\boldsymbol{\delta}^\mu \to 0^+), \end{aligned}$$

and therefore

$$\boldsymbol{\delta}^\mu \Delta \mathbf{z}^\mu \to 0 \quad \text{strongly in } L_\infty(0,M;[H^{-1}]^{N-k}) \quad \text{as} \quad \mu \to \infty. \tag{4.9}$$

It is clear that the convergences (4.4)–(4.9) proved above are stronger than the convergence in the space of distributions $\mathcal{D}'\left(0,M;\Pi_{i=1}^k H^{-r_i}(\Omega) \times \Pi_{j=k+1}^N H^{-r_j}(\Omega)\right)$. Passing to the limit as $\mu \to \infty$ in the system (4.2)–(4.3), we see that the couple of vector functions $(\mathbf{y}(t),\mathbf{z}(t)), t \in [0,M]$ satisfies the equations

$$\partial_t \mathbf{y} = \mathbf{a}\Delta \mathbf{y} - \mathbf{f}(\mathbf{y},\mathbf{z}) + \mathbf{g}_1(x), \qquad \partial_t \mathbf{z} = -\mathbf{h}(\mathbf{y},\mathbf{z}) + \mathbf{g}_2(x)$$

for any $M > 0$, i.e., $(\mathbf{y}(t),\mathbf{z}(t))$, $t \in \mathbb{R}_+$, is a weak solution of system (1.1)–(1.2). It remains to verify relation (4.1). By assumption, $(\mathbf{y}^\mu,\mathbf{z}^\mu) \in \mathcal{K}_+^{\boldsymbol{\delta}^\mu}(0)$, i.e., the vector functions $\mathbf{z}^\mu(\,\cdot\,)$ satisfy the inequality

$$\sup\left\{\|\mathbf{z}^\mu(t)\|_1^2 \mid t \geqslant 0\right\} \leqslant R^2 \quad \text{for any} \quad \mu \in \mathbb{N}.$$

Moreover, $\mathbf{z}^\mu(\,\cdot\,) \rightharpoonup \mathbf{z}(\,\cdot\,)$ $*$-weakly in $L_\infty(0,M;V^{N-k})$, for any $M > 0$, and therefore

$$\operatorname{ess\,sup}\left\{\|\mathbf{z}(t)\|_1^2 \mid t \geqslant 0\right\} \leqslant \liminf_{\mu\to\infty} \operatorname{ess\,sup}\left\{\|\mathbf{z}^\mu(t)\|_1^2 \mid t \geqslant 0\right\} \leqslant R^2.$$

Recall that the real function $\|\mathbf{z}(t)\|_1^2$ is lower semicontinuous for $t \geqslant 0$, which implies the inequality

$$\sup\left\{\|\mathbf{z}(t)\|_1^2 \mid t \geqslant 0\right\} \leqslant R^2.$$

This proves (3.11) for the function $\mathbf{z}$ with $S = 0$, i.e., $(\mathbf{y},\mathbf{z}) \in \mathcal{K}_+^{\mathbf{0}}(0)$.

Now let us state and prove the main theorem on the convergence of trajectory attractors.

Recall that the trajectory attractors $\mathfrak{A}^{\boldsymbol{\delta}}$ are uniformly bounded (with respect to $\delta_j \geqslant 0$, $j = \overline{k+1,N}$) in $\tilde{\mathcal{F}}_+^{\mathrm{b}}$ (see Corollary 3.1). Consequently, all these trajectory attractors lie inside a ball $\mathcal{B}_r \subset \tilde{\mathcal{F}}_+^{\mathrm{b}}$ with sufficiently large radius r,

$$\mathfrak{A}^{\boldsymbol{\delta}} \subset \mathcal{B}_r. \tag{4.10}$$

Note that the topological subspace $\mathcal{B}_r|_{\tilde{\Theta}_+^{\mathrm{loc}}}$ is metrizable (see Remark 1.1).

Theorem 4.1. *The trajectory attractors $\mathfrak{A}^{\boldsymbol{\delta}}$ of system* (3.1), (3.2) *converge in the topology $\tilde{\Theta}_+^{\mathrm{loc}}$ as $\boldsymbol{\delta} \to \mathbf{0}^+$ to the trajectory attractor $\mathfrak{A}^{\mathbf{0}}$ of system* (1.1)–(1.2),

$$\mathfrak{A}^{\boldsymbol{\delta}} \to \mathfrak{A}^{\mathbf{0}}\ (\boldsymbol{\delta} \to \mathbf{0}^+) \quad in \quad \tilde{\Theta}_+^{\mathrm{loc}}. \tag{4.11}$$

Proof. We must show that, for an arbitrary ε-neighborhood $\mathcal{O}_\varepsilon(\mathfrak{A}^{\mathbf{0}})$ of the set $\mathfrak{A}^{\mathbf{0}}$ in the topology $\tilde{\Theta}_+^{\mathrm{loc}}$, there is a number $\delta_0 = \delta_0(\varepsilon)$ such that

$$\mathfrak{A}^{\boldsymbol{\delta}} \subset \mathcal{O}_\varepsilon(\mathfrak{A}^{\mathbf{0}}) \quad \forall\, \boldsymbol{\delta}, \quad 0 \leqslant \delta_j \leqslant \delta_0, \quad j = \overline{k+1,N},$$

or, equivalently,

$$\mathfrak{A}^{\boldsymbol{\delta}} \subset \mathcal{O}_\varepsilon(\Pi_+\mathcal{K}^{\mathbf{0}}) \quad \forall\, \boldsymbol{\delta}, \quad 0 \leqslant \delta_j \leqslant \delta_0, \quad j = \overline{k+1,N}, \tag{4.12}$$

where $\mathcal{K}^{\mathbf{0}}$ is the kernel of system (1.1)–(1.2) (see (3.12)).

Assume that (4.12) fails. Then there is a sequence $\boldsymbol{\delta}^\mu \to \mathbf{0}^+$ $(\mu \to \infty)$ such that

$$\mathfrak{A}^{\boldsymbol{\delta}^\mu} \not\subset \mathcal{O}_\varepsilon(\Pi_+\mathcal{K}^{\mathbf{0}}) \tag{4.13}$$

for some $\varepsilon > 0$. Choose now an arbitrary sequence $\tau_\mu > 0$ such that

$$\tau_\mu \to \infty \qquad (\text{as} \quad \mu \to \infty).$$

Note that

$$T(\tau_\mu)\mathfrak{A}^{\boldsymbol{\delta}^\mu} = \mathfrak{A}^{\boldsymbol{\delta}^\mu}, \quad \forall \mu \in \mathbb{N}, \tag{4.14}$$

because the trajectory attractor is strictly invariant (see (2.15)), and therefore (4.13) and (4.14) yield

$$T(\tau_\mu)\mathfrak{A}^{\boldsymbol{\delta}^\mu} \not\subset \mathcal{O}_\varepsilon(\Pi_+\mathcal{K}^{\mathbf{0}}).$$

Hence, there are couples (u,v) such that

$$\mathbf{w}^\mu(\,\cdot\,) = (\mathbf{u}^\mu(\cdot),\mathbf{v}^\mu(\,\cdot\,)) \in \mathfrak{A}^{\boldsymbol{\delta}^\mu}, \tag{4.15}$$

and the functions

$$\mathbf{W}^\mu(t) := T(\tau_\mu)\mathbf{w}^\mu(t) = (\mathbf{u}^\mu(t+\tau_\mu), \mathbf{v}^\mu(t+\tau_\mu)), \qquad t \geqslant 0,$$

do not belong to $\mathcal{O}_\varepsilon(\Pi_+\mathcal{K}^{\mathbf{0}})$,

$$\mathbf{W}^\mu(\,\cdot\,) \notin \mathcal{O}_\varepsilon(\Pi_+\mathcal{K}^{\mathbf{0}}), \qquad \forall \mu \in \mathbb{N}. \tag{4.16}$$

We claim that the couple $\mathbf{W}^\mu(t) = (\mathbf{U}^\mu(t), \mathbf{V}^\mu(t))$ is a solution of system (3.1), (3.2) with the diffusion coefficients $\boldsymbol{\delta} = \boldsymbol{\delta}^\mu$ for $t \geqslant -\tau_\mu$, since $(\mathbf{u}^\mu(t+\tau_\mu), \mathbf{v}^\mu(t+\tau_\mu))$ is a solution of the system for $t+\tau_\mu \geqslant 0$, and the system is autonomous. Moreover,

$$\sup\{\|\mathbf{V}^\mu(t)\|_1 \mid t \geqslant -\tau_\mu\} \leqslant R, \tag{4.17}$$

due to (4.15), because $\mathfrak{A}^{\boldsymbol{\delta}^\mu} \subset \mathcal{K}^{\boldsymbol{\delta}^\mu}(0)$ (see Proposition 2.3).

For a given $\ell \leqslant 0$, denote by $\tilde{\mathcal{F}}^{\mathrm{loc}}_\ell$ and $\tilde{\mathcal{F}}^{\mathrm{b}}_\ell$ the spaces similar to $\tilde{\mathcal{F}}^{\mathrm{loc}}_0 := \tilde{\mathcal{F}}^{\mathrm{loc}}_+$ and $\tilde{\mathcal{F}}^{\mathrm{b}}_0 := \tilde{\mathcal{F}}^{\mathrm{b}}_+$, respectively, (which consist of functions defined on the semiaxis $]\ell, +\infty[$, see (2.1)), and the norm in $\tilde{\mathcal{F}}^{\mathrm{b}}_\ell$ is defined by formula (2.2) in which the semiaxis $\mathbb{R}_+ =]0, +\infty[$ is replaced by $\mathbb{R}_\ell =]\ell, +\infty[$. Define the topology $\tilde{\Theta}^{\mathrm{loc}}_\ell$ in the spaces $\tilde{\mathcal{F}}^{\mathrm{loc}}_\ell$ and $\tilde{\mathcal{F}}^{\mathrm{b}}_\ell$ similarly to that in $\tilde{\Theta}^{\mathrm{loc}}_+$.

It follows from (4.10) that the function $\mathbf{W}^\mu(t), t \geqslant -\tau_\mu$, belongs to the ball with radius r in the space $\tilde{\mathcal{F}}^{\mathrm{b}}_{-\tau_m}$,

$$\|\mathbf{W}^\mu(\,\cdot\,)\|_{\tilde{\mathcal{F}}^{\mathrm{b}}_{-\tau_\mu}} \leqslant r, \qquad \forall \mu. \tag{4.18}$$

Owing to Remark 2.1, for any chosen $M > 0$, the sequence $\{\mathbf{W}^\mu(t),\ \tau_\mu \geqslant M\}$ is precompact in the topology $\tilde{\Theta}^{\mathrm{loc}}_{-M}$. In other words, for every $M > 0$, there is a subsequence $\mu' = \mu'(M)$ such that $\{\mathbf{W}^{\mu'}(\,\cdot\,)\}$ converges in the topology $\tilde{\Theta}^{\mathrm{loc}}_{-M}$. Using the well-known Cantor diagonal construction, we can find a subsequence of indices $\{\mu''\} \subset \{\mu\}$ and a function $\mathbf{W}(t), t \in \mathbb{R}$, such that

$$\mathbf{W}^{\mu''}(\,\cdot\,) \to \mathbf{W}(\,\cdot\,)\ (\mu'' \to \infty) \quad \text{in} \quad \tilde{\Theta}^{\mathrm{loc}}_{-\infty} = \tilde{\Theta}^{\mathrm{loc}}, \tag{4.19}$$

and, by (4.18),

$$\|\mathbf{W}(\,\cdot\,)\|_{\tilde{\mathcal{F}}^{\mathrm{b}}} \leqslant r, \tag{4.20}$$

i.e., $\mathbf{W} \in \tilde{\mathcal{F}}^{\mathrm{b}}$, and, according to (4.17),

$$\sup\{\|\mathbf{V}(t)\|_1 \mid t \in \mathbb{R}\} \leqslant R. \tag{4.21}$$

We claim that the function $\mathbf{W}(t), t \in \mathbb{R}$, belongs to the kernel $\mathcal{K}^{\mathbf{0}}$ of the limit system (1.1)–(1.2). Indeed, we apply Proposition 4.1 to each subsequence $\{\mathbf{W}^{\mu'}(\,\cdot\,)\}$ convergent in $\tilde{\Theta}^{\mathrm{loc}}_{-M}$, where we clearly may replace the semiaxis $[0, +\infty[$ by the semiaxis $[-M, +\infty[$, since the reaction-diffusion system under the consideration is autonomous. Consequently, the function $\mathbf{W}(t)$, $t \geqslant -M$, is a weak solution of (1.1)–(1.2) for every $M > 0$ and, using (4.20) and (4.21), we see that

$$\mathbf{W}(\,\cdot\,) \in \mathcal{K}^{\mathbf{0}}.$$

It also follows from (4.19) that

$$\Pi_+\mathbf{W}^{\mu''}(\,\cdot\,) \to \Pi_+\mathbf{W}(\,\cdot\,)\ (\mu'' \to \infty) \quad \text{in} \quad \tilde{\Theta}^{\mathrm{loc}}_0$$

and thus, if μ'' is sufficiently large, then

$$\Pi_+\mathbf{W}^{\mu''}(\,\cdot\,) \in \mathcal{O}_\varepsilon(\Pi_+\mathbf{W}) \subset \mathcal{O}_\varepsilon(\Pi_+\mathcal{K}^{\mathbf{0}}),$$

which contradicts (4.16). This proves property (4.11).

We have also proved the following corollary.

Corollary 4.1. *The kernels $\mathcal{K}^{\boldsymbol{\delta}}$ of system* (3.1)–(3.2) *converge as $\boldsymbol{\delta} \to \mathbf{0}^+$ to the kernel $\mathcal{K}^{\mathbf{0}}$ of system* (1.1)–(1.2) *in the topology $\tilde{\Theta}^{\mathrm{loc}}$,*

$$\mathcal{K}^{\boldsymbol{\delta}} \to \mathcal{K}^{\mathbf{0}}\ (\boldsymbol{\delta} \to \mathbf{0}^+) \quad \textit{in} \quad \tilde{\Theta}^{\mathrm{loc}}.$$

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