Averaging of 2D Navier–Stokes equations with singularly oscillating forces

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Abstract

For $\rho \in [0, 1)$ and $\varepsilon > 0$, the nonautonomous 2D Navier–Stokes equations with singularly oscillating external force

$$\partial_t u - \nu \Delta u + (u \cdot \nabla) u = -\nabla p + g_0(t) + \varepsilon^{-\rho} g_1(t/\varepsilon),$$

$$\nabla \cdot u = 0$$

are considered, together with the averaged equations

$$\partial_t u - \nu \Delta u + (u \cdot \nabla) u = -\nabla p + g_0(t),$$

$$\nabla \cdot u = 0$$

formally corresponding to the limiting case $\varepsilon = 0$. Under suitable assumptions on the external force, the uniform boundedness of the related uniform global attractors $\mathcal{A}^\varepsilon$ is established, as well as the convergence of the attractors $\mathcal{A}^\varepsilon$ of the first system to the attractor $\mathcal{A}^0$ of the second one as $\varepsilon \to 0^+$. When the Grashof number of the averaged equations is small, the convergence rate of $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$ is controlled by $K\varepsilon^{1-\rho}$.

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1. Introduction

Let $\rho \in [0, 1)$ be a fixed parameter, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial \Omega$ of class C^1 (although this assumption is inessential). We consider the nonautonomous two-dimensional Navier–Stokes equations with the nonslip boundary condition

$$\begin{cases} \partial_t u - \nu \Delta u + u^1 \partial_{x_1} u + u^2 \partial_{x_2} u = -\nabla p + g_0(x, t) + \varepsilon^{-\rho} g_1(x, t/\varepsilon), \\ \partial_{x_1} u^1 + \partial_{x_2} u^2 = 0, \qquad u|_{\partial \Omega} = 0, \end{cases} \tag{1.1}$$

ruling the flow of a fluid which fills an infinite cylinder of cross section Ω, whose motion is parallel to the plane of Ω. Here, $x = (x_1, x_2) \in \Omega$,

$$u = u(x, t) = (u^1(x, t), u^2(x, t))$$

is the unknown velocity vector field and $p = p(x, t)$ is the unknown pressure. The Laplace operator $\Delta := \partial_{x_1}^2 + \partial_{x_2}^2$ acts in x-space. The parameter $\nu > 0$ stands for the kinematic viscosity, while the density of the fluid is assumed to be constant and equal to 1. Along with (1.1), we consider the *averaged* Navier–Stokes equations

$$\begin{cases} \partial_t u - \nu \Delta u + u^1 \partial_{x_1} u + u^2 \partial_{x_2} u = -\nabla p + g_0(x, t), \\ \partial_{x_1} u^1 + \partial_{x_2} u^2 = 0, \qquad u|_{\partial \Omega} = 0, \end{cases} \tag{1.2}$$

formally corresponding to the case $\varepsilon = 0$.

Remark 1.1. Somehow, the last assertion unveils our main result. Indeed, in the more challenging situation $\rho > 0$, the fact that (1.2) could be considered the (formal) limit as $\varepsilon \to 0^+$ of (1.1) is far from being clear: in principle, the averaging effect due to the term t/ε could be completely destroyed by the blow up of the oscillation amplitude.

The function

$$g^\varepsilon(x, t) := \begin{cases} g_0(x, t) + \varepsilon^{-\rho} g_1(x, t/\varepsilon) & \varepsilon > 0, \\ g_0(x, t) & \varepsilon = 0, \end{cases}$$

represents the external force of systems (1.1) and (1.2), respectively. The aim of this work is to study the asymptotic properties of the nonautonomous Navier–Stokes equations depending on the small parameter ε, which reflects the rate of fast time oscillations in the term $\varepsilon^{-\rho} g_1(x, t/\varepsilon)$ with amplitude of order $\varepsilon^{-\rho}$. Both $g_0(x, t)$ and $g_1(x, t)$ are supposed to be translation bounded in the space $L_2^{\mathrm{loc}}(\mathbb{R}; [L_2(\Omega)]^2)$.

Remark 1.2. The model of the 2D Navier–Stokes equations subject to an oscillating external force, with a growing amplitude depending on the oscillation rate, was formulated in 2003 by Victor I Yudovich in a private communication with Mark I Vishik, at the conference dedicated to the 100th anniversary of Kolmogorov. Yudovich motivated the relevance of this model in view of applications to problems arising in vibration hydrodynamics, a field to which he turned his mathematical interests during his last years (see [30–32]).

The longtime behaviour of autonomous and nonautonomous 2D Navier–Stokes equations is a widely investigated subject, which attracted the attention of a large number of authors (we refer the reader to the monographs [2, 7, 11, 15, 21, 24, 25] and references therein). Some problems related to the homogenization and the averaging of uniform global attractors for such equations have been analysed in [8, 9, 18, 20, 26]. Analogous issues for other relevant evolution equations of mathematical physics with rapidly oscillating coefficients and terms have been studied in [3, 5, 10, 12–14, 16, 18, 20, 27, 28, 33].

In this paper, working in the usual phase space H of the Navier–Stokes equations (namely, the closure in $[L^2(\Omega)]^2$ of divergence-free functions), we prove the following facts concerning the family $\{\mathcal{A}^\varepsilon\}$ of uniform global attractors of the dynamical processes generated by systems (1.1) and (1.2), respectively:

(i) The family $\{\mathcal{A}^\varepsilon\}$ is uniformly (w.r.t. ε) bounded in H:

$$\sup_{\varepsilon \in [0,1]} \|\mathcal{A}^\varepsilon\|_H < \infty.$$

(ii) The attractors $\mathcal{A}^\varepsilon$ converge to $\mathcal{A}^0$ as $\varepsilon \to 0^+$ in the standard Hausdorff semidistance in H:

$$\lim_{\varepsilon \to 0^+} \left\{ \mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \right\} = 0.$$

These conclusions are drawn under certain boundedness assumptions on the function

$$G_1(t, \tau) = \int_\tau^t g_1(s) \, \mathrm{d}s, \qquad t \geqslant \tau.$$

We emphasize that the parameter ρ is allowed to belong to the interval $[0, 1)$. When $\rho > 0$, we are dealing with *singular* oscillations.

Similar results in the literature can be found in [8, 18, 20], which establish the convergence of the attractors in the *nonsingular* situation $\rho = 0$, where the uniform boundedness of the family $\{\mathcal{A}^\varepsilon\}$ in H is straightforward. If the Grashof number of the averaged equation is small, the paper [8] shows that the attractor $\mathcal{A}^0$ is exponential, and provides the estimate (for $\rho = 0$)

$$\mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant K\sqrt{\varepsilon},$$

for some $K > 0$. On the other hand, for small Grashof numbers of the averaged equation, our conclusion (ii) (in the general case $0 \leqslant \rho < 1$) improves to

$$\mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant K\varepsilon^{1-\rho},$$

for some $K > 0$. In particular, when $\rho = 0$, we obtain the Lipschitz continuity of the family $\{\mathcal{A}^\varepsilon\}$ at $\varepsilon = 0$.

Analogous averaging results for uniform global attractors of dissipative wave equations with singularly time oscillating external forces have been found in [5], whereas the paper [9] deals with homogenization of uniform global attractors of the nonautonomous 2D Navier–Stokes equations having the external force of the form $g_0(x, t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$, $\rho \in [0, 1)$, hence, with singular oscillations in the space variable.

Plan of the paper. In the next section, we introduce some notation and the basic assumptions. In section 3, we recall some results on the existence of the uniform global attractors $\mathcal{A}^\varepsilon$ associated, for every given $\varepsilon \in [0, 1]$, to (1.1) or (1.2). Then, section 4 is devoted to the analysis of a linear evolution Stokes equation in the presence of an oscillating external force. In section 5, the uniform bound for the attractors is established, while section 6 deals with the convergence $\mathcal{A}^\varepsilon \to \mathcal{A}^0$ as $\varepsilon \to 0^+$. In section 7, we prove the Hölder continuity of $\{\mathcal{A}^\varepsilon\}$ at $\varepsilon = 0$ when the Grashof number of the averaged equation is small (and so $\mathcal{A}^0$ is exponential).

2. Notation and basic assumptions

For $\tau \in \mathbb{R}$, we set $\mathbb{R}_\tau = [\tau, +\infty)$. Throughout the paper, C will stand for a *generic* positive constant, depending on Ω and ν, but *independent* of ε, g_0, g_1 and of the choice of the initial time $\tau \in \mathbb{R}$. Whenever needed, the dependence on ρ approaching the critical value 1 will be highlighted. In the following, we agree to omit the dependence on the space variable x. Given a normed space X, we usually denote the norm in X by $\| \cdot \|_X$, and we indicate by

$$\mathrm{dist}_X(B_1, B_2) := \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} \|b_1 - b_2\|_X$$

the Hausdorff semidistance in X from a set B_1 to a set B_2.

We introduce the usual Hilbert spaces associated with Navier–Stokes system

$$H := \overline{\left\{ u \in [C_0^\infty(\Omega)]^2 \mid \partial_{x_1} u^1 + \partial_{x_2} u^2 = 0 \right\}}^{[L_2(\Omega)]^2}$$

and

$$V := \overline{\left\{ u \in [C_0^\infty(\Omega)]^2 \mid \partial_{x_1} u^1 + \partial_{x_2} u^2 = 0 \right\}}^{[H^1(\Omega)]^2},$$

and we denote by

$$P : [L_2(\Omega)]^2 \to H$$

the Leray–Helmholtz orthogonal projection from $[L_2(\Omega)]^2$ onto H. Consider the (strictly) positive selfadjoint operator

$$A := -P\Delta$$

acting on H, with domain

$$D(A) := [H^2(\Omega)]^2 \cap V.$$

We call $\lambda > 0$ the first eigenvalue of the Stokes operator A. We also define, for $\sigma \in \mathbb{R}$, the scale of Hilbert spaces

$$H^\sigma := D(A^{\sigma/2})$$

with inner products and norms

$$\langle u, v \rangle_\sigma := \langle A^{\sigma/2} u, A^{\sigma/2} v \rangle_{[L_2(\Omega)]^2}, \qquad \|u\|_\sigma := \|A^{\sigma/2} u\|_{[L_2(\Omega)]^2}$$

(we agree to omit the index σ whenever $\sigma = 0$). In particular,

$$H^{-1} = H^{-1}(\Omega), \qquad H^0 = H, \qquad H^1 = V, \qquad H^2 = D(A),$$

and we have the generalized Poincaré inequality

$$\|u\|_{\sigma+1} \geqslant \lambda^{1/2} \|u\|_\sigma, \qquad \forall u \in H^{\sigma+1}. \tag{2.1}$$

Then, we introduce the standard bilinear and trilinear forms

$$B(u, u) := P\left(u^1 \partial_{x_1} u + u^2 \partial_{x_2} u\right),$$

$$b(u, v, w) := \langle B(u, v), w \rangle.$$

The form b is continuous on $H^1 \times H^1 \times H^1$ and satisfies the identities

$$b(u, v, w) = -b(u, w, v), \tag{2.2}$$

$$b(u, w, w) = 0, \tag{2.3}$$

and the inequalities

$$|b(u, v, w)| \leqslant c \|u\|^{1/2} \|u\|_1^{1/2} \|v\|_1 \|w\|^{1/2} \|w\|_1^{1/2}, \tag{2.4}$$

$$|b(u, v, w)| \leqslant c \|u\|^{1/2} \|u\|_1^{1/2} \|v\|^{1/2} \|v\|_1^{1/2} \|w\|_1, \tag{2.5}$$

where $c > 0$ is an absolute constant independent of the domain Ω (see [11, 21, 24]). Note that (2.5) is an immediate consequence of (2.2) and (2.4).

Assumptions on the external force. The functions $g_0(t)$ and $g_1(t)$ are taken from the space $L_2^b(\mathbb{R}; H)$ of translation bounded functions in $L_2^{\mathrm{loc}}(\mathbb{R}; H)$; namely,

$$\|g_0\|_{L_2^b}^2 := \sup_{t \in \mathbb{R}} \int_t^{t+1} \|g_0(s)\|^2 \, \mathrm{d}s = M_0^2, \tag{2.6}$$

$$\|g_1\|_{L_2^b}^2 := \sup_{t \in \mathbb{R}} \int_t^{t+1} \|g_1(s)\|^2 \, \mathrm{d}s = M_1^2, \tag{2.7}$$

for some $M_0, M_1 \geqslant 0$. As a straightforward consequence of (2.7), we have (see, e.g. [5])

$$\|g^\varepsilon\|_{L_2^b} \leqslant Q_\varepsilon, \tag{2.8}$$

having put

$$Q_\varepsilon = \begin{cases} M_0 + \sqrt{2}M_1\varepsilon^{-\rho} & \varepsilon > 0, \\ M_0 & \varepsilon = 0. \end{cases} \tag{2.9}$$

Observe that Q_ε is of the order $\varepsilon^{-\rho}$ as $\varepsilon \to 0^+$.

We conclude the section recalling an inequality and a Gronwall-type lemma needed in the sequel.

Lemma 2.1. *For every $\tau \in \mathbb{R}$, every nonnegative locally summable function φ on $\mathbb{R}_\tau$ and every $\beta > 0$, we have*

$$\int_\tau^t \varphi(s)\mathrm{e}^{-\beta(t-s)}\,\mathrm{d}s \leqslant \frac{1}{1-\mathrm{e}^{-\beta}}\sup_{\theta\geqslant\tau}\int_\theta^{\theta+1}\varphi(s)\,\mathrm{d}s, \tag{2.10}$$

for all $t \geqslant \tau$.

Proof. Writing $t - \tau = N + \varpi$, for some nonnegative integer N and some $\varpi \in [0, 1)$, we have

$$\int_\tau^t \varphi(s)\mathrm{e}^{-\beta(t-s)}\,\mathrm{d}s \leqslant \sum_{n=0}^{N-1}\mathrm{e}^{-\beta n}\int_{t-n-1}^{t-n}\varphi(s)\,\mathrm{d}s + \mathrm{e}^{-\beta N}\int_\tau^{\tau+\varpi}\varphi(s)\,\mathrm{d}s,$$

where the sum vanishes if $N = 0$. Therefore,

$$\int_\tau^t \varphi(s)\mathrm{e}^{-\beta(t-s)}\,\mathrm{d}s \leqslant \sum_{n=0}^{N}\mathrm{e}^{-\beta n}\sup_{\theta\geqslant\tau}\int_\theta^{\theta+1}\varphi(s)\,\mathrm{d}s \leqslant \frac{1}{1-\mathrm{e}^{-\beta}}\sup_{\theta\geqslant\tau}\int_\theta^{\theta+1}\varphi(s)\,\mathrm{d}s,$$

as claimed. $\qquad\square$

Lemma 2.2. *Let $\zeta : \mathbb{R}_\tau \to \mathbb{R}_+$ fulfil, for almost every $t \geqslant \tau$, the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta(t) + \varphi_1(t)\zeta(t) \leqslant \varphi_2(t), \tag{2.11}$$

where, for every $t \geqslant \tau$, the scalar functions φ_1 and φ_2 satisfy

$$\int_\tau^t \varphi_1(s)\,\mathrm{d}s \geqslant \beta(t-\tau)-\gamma, \qquad \int_t^{t+1}\varphi_2(s)\,\mathrm{d}s \leqslant M,$$

for some $\beta > 0$, $\gamma \geqslant 0$ and $M \geqslant 0$. Then,

$$\zeta(t) \leqslant \mathrm{e}^\gamma\zeta(\tau)\mathrm{e}^{-\beta(t-\tau)} + \frac{M\mathrm{e}^\gamma}{1-\mathrm{e}^{-\beta}}, \qquad \forall t \geqslant \tau.$$

Proof. Fix $t > \tau$, and define, for $s \in [\tau, t]$,

$$\omega(s) := \int_s^t \varphi_1(y)\,\mathrm{d}y \geqslant \beta(t-s)-\gamma.$$

Multiplying (2.11) by $\exp\left[\int_\tau^t \varphi_1(s)\,\mathrm{d}s\right]$ and integrating in t, we obtain

$$\zeta(t) \leqslant \zeta(\tau)\mathrm{e}^{-\omega(\tau)} + \int_\tau^t \mathrm{e}^{-\omega(s)}\varphi_2(s)\mathrm{d}s \leqslant \mathrm{e}^\gamma\zeta(\tau)\mathrm{e}^{-\beta(t-\tau)} + \mathrm{e}^\gamma\int_\tau^t \mathrm{e}^{-\beta(t-s)}\varphi_2(s)\mathrm{d}s.$$

From (2.10), we see that

$$\int_\tau^t \mathrm{e}^{-\beta(t-s)}\varphi_2(s)\,\mathrm{d}s \leqslant \frac{M}{1-\mathrm{e}^{-\beta}},$$

which concludes the proof. $\qquad\square$

3. Attractors for nonautonomous Navier–Stokes equations

3.1. Well-posedness of the problem

We rewrite (1.1) and (1.2) in the unitary abstract form

$$\partial_t u + \nu A u + B(u, u) = g^\varepsilon(t), \tag{3.1}$$

where the pressure p has disappeared by force of the application of the Leray–Helmholtz projection P. For any fixed $\varepsilon \in [0, 1]$ and any $\tau \in \mathbb{R}$, the Cauchy problem for (3.1), with initial data

$$u|_{t=\tau} = u_\tau \in H, \tag{3.2}$$

has a unique weak solution [2, 7, 11, 21, 24, 25]

$$u \in C(\mathbb{R}_\tau; H) \cap L_2^{\text{loc}}(\mathbb{R}_\tau; H^1)$$

such that

$$\partial_t u \in L_2^{\text{loc}}(\mathbb{R}_\tau; H^{-1}).$$

For every $t \geqslant \tau$, this solution satisfies the energy identity

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^2 + \nu \|u(t)\|_1^2 = \langle u(t), g^\varepsilon(t) \rangle.$$

Hence, we deduce the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^2 + \nu \|u(t)\|_1^2 \leqslant (\nu\lambda)^{-1} \|g^\varepsilon(t)\|^2,$$

which, in light of (2.8) and (2.10) (note that $\beta/(1 - \mathrm{e}^{-\beta}) \leqslant 1 + \beta$), readily yields, for every $t \geqslant \tau$ and every $\tau \in \mathbb{R}$,

$$\|u(t)\|^2 \leqslant \|u(\tau)\|^2 \mathrm{e}^{-\nu\lambda(t-\tau)} + (\nu\lambda)^{-2}(1 + \nu\lambda)Q_\varepsilon^2 \tag{3.3}$$

and

$$\|u(t)\|^2 + \nu \int_\tau^t \|u(s)\|_1^2 \, \mathrm{d}s \leqslant \|u(\tau)\|^2 + (\nu\lambda)^{-1} Q_\varepsilon^2 (t - \tau + 1). \tag{3.4}$$

Besides,

$$(t - \tau)\|u(t)\|_1^2 \leqslant \mathcal{Q}\left(t - \tau, \|u(\tau)\|^2, Q_\varepsilon^2\right), \tag{3.5}$$

where $\mathcal{Q}(\cdot, \cdot, \cdot)$ is a positive function, increasing in each argument (see [2, 6, 7, 25]).

3.2. Dynamical processes and attractors

If the functions $g_0(t)$ and $g_1(t)$ are translation bounded, i.e. conditions (2.6) and (2.7) hold, equation (3.1) generates the *dynamical process*

$$\{U_\varepsilon(t, \tau), \ t \geqslant \tau, \ \tau \in \mathbb{R}\}$$

acting on H by the formula

$$U_\varepsilon(t, \tau)u_\tau = u(t), \qquad t \geqslant \tau,$$

where $u(t)$ is the solution to (3.1) with initial data (3.2). It follows from (3.3) that the process $\{U_\varepsilon(t, \tau)\}$ has a *uniformly* (w.r.t. $\tau \in \mathbb{R}$) *absorbing set*

$$B^\varepsilon := \{u \in H \mid \|u\| \leqslant C Q_\varepsilon\}, \tag{3.6}$$

bounded in H for any fixed ε. That is, for any bounded set $B \subset H$ of initial data, there is a time $T = T(B, \varepsilon)$ such that

$$U_\varepsilon(t, \tau) B \subseteq B^\varepsilon, \qquad \forall \tau \in \mathbb{R}, \qquad \forall t \geqslant \tau + T.$$

Estimate (3.5) implies that

$$B_1^\varepsilon := \bigcup_{\tau \in \mathbb{R}} U_\varepsilon(\tau + 1, \tau) B^\varepsilon$$

is also uniformly absorbing. Moreover, B_1^ε is bounded in H^1, and therefore compact in H. A process having a compact uniformly absorbing set is called *uniformly compact* (see [6, 7, 17]).

Definition 3.1. A closed set $\mathcal{A} \subset H$ is called the *uniform* (w.r.t. $\tau \in \mathbb{R}$) *global attractor* of the process $\{U(t, \tau)\}$ acting on H if $\mathcal{A}$ is a uniformly attracting set, that is, for any bounded set $B \subset H$,

$$\mathrm{dist}_H(U(t, \tau) B, \mathcal{A}) \to 0 \qquad \text{as } t - \tau \to +\infty,$$

and $\mathcal{A}$ satisfies the following *minimality property*: $\mathcal{A}$ belongs to any closed uniformly attracting set of the process $\{U(t, \tau)\}$ (for brevity, we sometimes call $\mathcal{A}$ merely the attractor).

Since the process $\{U_\varepsilon(t, \tau)\}$ is uniformly compact, it has the uniform global attractor

$$\mathcal{A}^\varepsilon = \omega(\widetilde{B}) := \bigcap_{h>0} \left[\overline{\bigcup_{t-\tau \geqslant h} U_\varepsilon(t, \tau) \widetilde{B}}^H \right],$$

where $\widetilde{B}$ is an arbitrary bounded uniformly absorbing set of the process $\{U_\varepsilon(t, \tau)\}$ (see [6, 7, 17]); for example, we can set $\widetilde{B} = B^\varepsilon$. From (3.3), it is readily seen that

$$\|\mathcal{A}^\varepsilon\| \leqslant C Q_\varepsilon, \qquad \forall \varepsilon \in [0, 1],$$

with Q_ε given by (2.9). On the other hand, $\mathcal{A}^\varepsilon$ is also bounded in H^1, for each fixed ε, since $\mathcal{A}^\varepsilon \subseteq B_1^\varepsilon$. Nonetheless, it is clear that the size of the attractor $\mathcal{A}^\varepsilon$ in H (and so in H^1) may approach infinity as $\varepsilon \to 0^+$.

3.3. Structure of attractors

A function $\psi(t)$ with values in a Banach space X is called *translation compact* in $L_2^{\mathrm{loc}}(\mathbb{R}; X)$, and we write $\psi \in L_2^{\mathrm{tc}}(\mathbb{R}; X)$, if the family of its time translations $\{\psi(t + \tau) \mid \tau \in \mathbb{R}\}$ is precompact in the space $L_2^{\mathrm{loc}}(\mathbb{R}; X)$, endowed with the local uniform convergence topology in the space $L_2(-T, T; X)$, for every $T > 0$. It is well known that $L_2^{\mathrm{tc}}(\mathbb{R}; X) \subset L_2^b(\mathbb{R}; X)$. The *hull* of ψ in $L_2^{\mathrm{loc}}(\mathbb{R}; X)$ is the set

$$\mathcal{H}(\psi) := \overline{\{\psi(t + \tau) \mid \tau \in \mathbb{R}\}}^{L_2^{\mathrm{loc}}(\mathbb{R}; X)},$$

and the inequality

$$\|\hat{\psi}\|_{L_2^b} \leqslant \|\psi\|_{L_2^b}$$

holds for every $\hat{\psi} \in \mathcal{H}(\psi)$ (cf [7]). Several translation compactness criteria for functions with values in various spaces can be found in [7]. We remark that almost periodic functions (cf [1]) with values in X are translation compact, both in $C_b(\mathbb{R}; X)$ and in $L_2^{\mathrm{loc}}(\mathbb{R}; X)$. However, the class of translation compact functions is significantly wider, and turns out to be very effective in the study of nonautonomous dynamical systems and their attractors.

Assuming $g_0, g_1 \in L_2^{\text{tc}}(\mathbb{R}; H)$, the external force $g^\varepsilon(t)$ appearing in equation (3.1) clearly belongs to $L_2^{\text{tc}}(\mathbb{R}; H)$ as well. Besides, if $\varepsilon > 0$ and $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, then

$$\hat{g}^\varepsilon(t) = \hat{g}_0(t) + \varepsilon^{-\rho}\hat{g}_1(t/\varepsilon),$$

for some $\hat{g}_0 \in \mathcal{H}(g_0)$ and $\hat{g}_1 \in \mathcal{H}(g_1)$. In which case, to describe the structure of the uniform global attractor $\mathcal{A}^\varepsilon$, we consider the family of equations

$$\partial_t \hat{u} + \nu A\hat{u} + B(\hat{u}, \hat{u}) = \hat{g}^\varepsilon(t), \qquad \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon). \tag{3.7}$$

For every external force $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, equation (3.7) generates the process $\{U_{\hat{g}^\varepsilon}(t, \tau)\}$ on H, which shares similar properties as those of the process $\{U_\varepsilon(t, \tau)\}$, corresponding to the original equation (3.1) with external force $g^\varepsilon(t)$. Moreover, the map

$$(u_\tau, \hat{g}^\varepsilon) \mapsto U_{\hat{g}^\varepsilon}(t, \tau)u_\tau$$

is $(H \times \mathcal{H}(g^\varepsilon), H)$-continuous (see [7]).

Definition 3.2. The *kernel* $\mathcal{K}_{\hat{g}^\varepsilon}$ of equation (3.7) is the family of all its complete solutions $\{\hat{u}(t), \ t \in \mathbb{R}\}$ which are uniformly bounded in H. The set

$$\mathcal{K}_{\hat{g}^\varepsilon}(\tau) = \{\hat{u}(\tau) \mid \hat{u} \in \mathcal{K}_{\hat{g}^\varepsilon}\} \subset H$$

is called the *kernel section* of $\mathcal{K}_{\hat{g}^\varepsilon}$ at time $t = \tau$.

For every $\varepsilon \in [0, 1]$, the following representation of the uniform global attractor $\mathcal{A}^\varepsilon$ of equation (3.1) holds [7]:

$$\mathcal{A}^\varepsilon = \bigcup_{\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)} \mathcal{K}_{\hat{g}^\varepsilon}(0). \tag{3.8}$$

Actually, $\mathcal{K}_{\hat{g}^\varepsilon}(0)$ can be replaced by $\mathcal{K}_{\hat{g}^\varepsilon}(\tau)$, for an arbitrary $\tau \in \mathbb{R}$.

Remark 3.3. In fact, we could as well assume the translation compactness of $g_0(t)$ and $g_1(t)$ in a weaker space (cf [5]). For instance, in $L_2^{\text{loc}}(\mathbb{R}; H^{-1})$. Indeed, appealing to the results of the recent paper [23], the representation (3.8) still holds if we require the compactness of the families of time translations of g_0 and g_1 in $L_2^{\text{loc}}(\mathbb{R}; H)$ with respect to whatever metrizable topology. In which case, it suffices to replace $\mathcal{H}(g^\varepsilon)$ in (3.8) with the set

$$\left\{\hat{g} \in L_2^{\text{loc}}(\mathbb{R}; H) \mid \exists\{\tau_n\} \subset \mathbb{R} \text{ such that } g^\varepsilon(t + \tau_n) \to \hat{g}(t)\right\},$$

where now the convergence takes place in the assigned metric.

4. Evolution Stokes equation with oscillating external force

In this section, we dwell on the evolution Stokes equation with time dependent external force and with null initial data given at an initial time $\tau \in \mathbb{R}$

$$\partial_t V + AV = K(t), \qquad V|_{t=\tau} = 0.$$

The following lemma is straightforward.

Lemma 4.1. *If $K \in L_2^{\text{loc}}(\mathbb{R}; H^1)$, then the above problem has a unique solution*

$$V \in C(\mathbb{R}_\tau; H^2) \cap L_2^{\text{loc}}(\mathbb{R}_\tau; H^3).$$

Moreover, the inequalities

$$\|V(t)\|_2^2 \leqslant C \int_\tau^t e^{-\beta(t-s)}\|K(s)\|_1^2 \, ds$$

and

$$\int_t^{t+1} \|V(s)\|_3^2 \, ds \leqslant \|V(t)\|_2^2 + \int_t^{t+1} \|K(s)\|_1^2 \, ds$$

hold for every $t \geqslant \tau$ and some $\beta > 0$, independent of the initial time $\tau \in \mathbb{R}$.

Proof. Multiply the equation by $A^2 V$, and apply standard arguments (cf [2, 25]). □

Setting

$$K(t, \tau) = \int_\tau^t k(s)\,\mathrm{d}s, \qquad t \geqslant \tau, \quad \tau \in \mathbb{R},$$

the main result of the section reads as follows.

Proposition 4.2. *Let $k \in L_2^{\mathrm{loc}}(\mathbb{R}; H^{-1})$. Assume that*

$$\sup_{t \geqslant \tau,\, \tau \in \mathbb{R}} \left\{ \|K(t, \tau)\|^2 + \int_t^{t+1} \|K(s, \tau)\|_1^2\,\mathrm{d}s \right\} \leqslant \ell^2, \tag{4.1}$$

for some $\ell \geqslant 0$. Then, the solution $v(t)$ to the problem

$$\partial_t v + Av = k\,(t/\varepsilon), \qquad v|_{t=\tau} = 0, \tag{4.2}$$

with $\varepsilon \in (0, 1]$, satisfies the inequality

$$\|v(t)\|^2 + \int_t^{t+1} \|v(s)\|_1^2\,\mathrm{d}s \leqslant C\ell^2\varepsilon^2, \qquad \forall t \geqslant \tau,$$

where C is independent of k.

Proof. Without loss of generality, we may assume $\tau = 0$. Denoting

$$V(t) = \int_0^t v(s)\,\mathrm{d}s,$$

we have, for any $t \geqslant 0$,

$$\partial_t V(t) = v(t) = \int_0^t \partial_t v(s)\,\mathrm{d}s,$$

as $v(0) = 0$. Integrating (4.2) in time, we see that the function $V(t)$ solves the problem

$$\partial_t V + AV = K_\varepsilon(t), \qquad V|_{t=0} = 0, \tag{4.3}$$

with external force

$$K_\varepsilon(t) = \int_0^t k\,(s/\varepsilon)\,\mathrm{d}s = \varepsilon \int_0^{t/\varepsilon} k(s)\,\mathrm{d}s = \varepsilon K\,(t/\varepsilon, 0)\,.$$

It follows from (4.1) that

$$\sup_{t \geqslant 0} \|K_\varepsilon(t)\| \leqslant \ell\varepsilon$$

and

$$\int_t^{t+1} \|K_\varepsilon(s)\|_1^2\,\mathrm{d}s = \varepsilon^3 \int_{t/\varepsilon}^{(t+1)/\varepsilon} \|K(s, 0)\|_1^2\,\mathrm{d}s \leqslant 2\varepsilon^2 \sup_{t \geqslant 0} \left\{ \int_t^{t+1} \|K(s, 0)\|_1^2\,\mathrm{d}s \right\} \leqslant 2\ell^2\varepsilon^2.$$

Accordingly, by (2.10),

$$\int_0^t \mathrm{e}^{-\beta(t-s)} \|K_\varepsilon(s)\|_1^2\,\mathrm{d}s \leqslant C\ell^2\varepsilon^2,$$

and applying lemma 4.1, we obtain

$$\|V(t)\|_2^2 + \int_t^{t+1} \|V(s)\|_3^2\,\mathrm{d}s \leqslant C\ell^2\varepsilon^2.$$

Hence, on account of (4.3) and the equalities

$$v(t) = \partial_t V(t), \qquad \|AV(t)\| = \|V(t)\|_2, \qquad \|AV(t)\|_1 = \|V(t)\|_3,$$

we have

$$\|v(t)\| \leqslant \|V(t)\|_2 + \|K_\varepsilon(t)\| \leqslant C\ell\varepsilon$$

and

$$\|v(t)\|_1^2 \leqslant 2\|V(t)\|_3^2 + 2\|K_\varepsilon(t)\|_1^2,$$

from which we derive the integral estimate

$$\int_t^{t+1} \|v(s)\|_1^2 \, ds \leqslant C\ell^2\varepsilon^2.$$

This finishes the proof.   □

5. Uniform boundedness of the attractors

We now prove the uniform boundedness of $\mathcal{A}^\varepsilon$ in H. To this end, setting

$$G_1(t, \tau) = \int_\tau^t g_1(s) \, ds, \qquad t \geqslant \tau,$$

we assume that

$$\sup_{t \geqslant \tau, \tau \in \mathbb{R}} \left\{ \|G_1(t, \tau)\|^2 + \int_t^{t+1} \|G_1(s, \tau)\|_1^2 \, ds \right\} \leqslant \ell^2, \tag{5.1}$$

for some $\ell \geqslant 0$.

Theorem 5.1. *Within (5.1), the attractors $\mathcal{A}^\varepsilon$ are uniformly (w.r.t. ε) bounded in H, namely,*

$$\sup_{\varepsilon \in [0,1]} \|\mathcal{A}^\varepsilon\| < \infty.$$

Remark 5.2. Note that $\mathcal{A}^\varepsilon$ might not be uniformly bounded in the space H^1, even if assumption (5.1) is satisfied.

Proof. Let $u(t) = U_\varepsilon(t, \tau)u_\tau$ be the solution to (3.1)–(3.2) with initial data $u_\tau \in H$. For $\varepsilon > 0$, we consider the auxiliary evolution Stokes problem

$$\partial_t v + \nu A v = \varepsilon^{-\rho} g_1(t/\varepsilon), \qquad v|_{t=\tau} = 0. \tag{5.2}$$

Proposition 4.2 provides the estimate

$$\|v(t)\|^2 + \int_t^{t+1} \|v(s)\|_1^2 \, ds \leqslant C\ell^2\varepsilon^{2(1-\rho)}, \qquad \forall t \geqslant \tau. \tag{5.3}$$

Then, we introduce the function

$$w(t) = u(t) - v(t),$$

which satisfies the problem

$$\partial_t w + \nu A w + B(w + v, w + v) = g_0, \qquad w|_{t=\tau} = u_\tau.$$

Taking the scalar product by w, we obtain

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 + \nu \|w\|_1^2 + b(w + v, v, w) = \langle g_0, w \rangle,$$

where we used the relation $b(w + v, w, w) = 0$ provided by (2.3). In light of (2.4) and (2.5), we observe that

$$|b(w, v, w)| \leqslant c\|w\|\|w\|_1\|v\|_1 \leqslant \frac{\nu}{8}\|w\|_1^2 + C\|w\|^2\|v\|_1^2,$$

$$|b(v, v, w)| \leqslant c\|v\|\|v\|_1\|w\|_1 \leqslant \frac{\nu}{8}\|w\|_1^2 + C\|v\|^2\|v\|_1^2,$$

so that

$$|b(w + v, v, w)| \leqslant \frac{\nu}{4}\|w\|_1^2 + C\|w\|^2\|v\|_1^2 + C\|v\|^2\|v\|_1^2.$$

Moreover,

$$\langle g_0, w \rangle \leqslant \frac{\nu}{4}\|w\|_1^2 + C\|g_0\|^2.$$

Therefore, using the inequality

$$\|v(t)\|^2 \leqslant C\ell^2, \qquad \forall t \geqslant \tau,$$

coming from (5.3) (as $\varepsilon \leqslant 1$) along with the control

$$\lambda\|w\|^2 \leqslant \|w\|_1^2,$$

we are led to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \lambda\nu\|w\|^2 \leqslant C\|w\|^2\|v\|_1^2 + C\ell^2\|v\|_1^2 + C\|g_0\|^2,$$

which, upon defining the functions

$$\varphi_1(t) = \lambda\nu - C\|v(t)\|_1^2,$$

$$\varphi_2(t) = C\ell^2\|v(t)\|_1^2 + C\|g_0(t)\|^2,$$

can be rewritten in the more convenient form

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \varphi_1\|w\|^2 \leqslant \varphi_2.$$

For every $t \geqslant \tau$, the integral estimate (5.3) together with (2.6) entail

$$\int_\tau^t \varphi_1(s)\,\mathrm{d}s = \lambda\nu(t - \tau) - C\int_\tau^t \|v(s)\|_1^2\,\mathrm{d}s \geqslant \lambda\nu(t - \tau) - C\ell^2\varepsilon^{2(1-\rho)}(t - \tau + 1)$$

and

$$\int_t^{t+1} \varphi_2(s)\,\mathrm{d}s \leqslant C\left(\ell^4 + M_0^2\right).$$

At this point, we put

$$\beta := \lambda\nu/2,$$

noting the trivial implication

$$\varepsilon \leqslant \varepsilon_0(\rho, \ell) := \left[\beta/(C\ell^2)\right]^{1/(2-2\rho)} \Rightarrow C\ell^2\varepsilon^{2(1-\rho)} \leqslant \beta.$$

Accordingly, if $\varepsilon \leqslant \varepsilon_0$, the integral control for φ_1 improves to

$$\int_\tau^t \varphi_1(s)\,\mathrm{d}s \geqslant \beta(t - \tau) - \beta,$$

and lemma 2.2 applies with $\zeta(t) = \|w(t)\|^2$, yielding

$$\|w(t)\|^2 \leqslant Ce^{-\beta(t-\tau)}\|u_\tau\|^2 + C\left(\ell^4 + M_0^2\right), \qquad \forall t \geqslant \tau.$$

Recalling that $u = w + v$, and using again (5.3), we end up with

$$\|u(t)\|^2 \leqslant Ce^{-\beta(t-\tau)}\|u_\tau\|^2 + C\left(\ell^2 + \ell^4 + M_0^2\right), \qquad \forall t \geqslant \tau.$$

Thus, for every $\varepsilon \leqslant \varepsilon_0$, the process $\{U_\varepsilon(t, \tau)\}$ has the absorbing set

$$B_0 := \left\{u \in H \mid \|u\|^2 \leqslant C\left(\ell^2 + \ell^4 + M_0^2\right)\right\}.$$

On the other hand, if $\varepsilon_0 < \varepsilon \leqslant 1$, the process $\{U_\varepsilon(t, \tau)\}$ possesses also the absorbing set (cf (3.3) and (3.6))

$$B^{\varepsilon_0} = \left\{u \in H \mid \|u\| \leqslant CQ_{\varepsilon_0}\right\}.$$

In conclusion, for every $\varepsilon \in [0, 1]$, the bounded set

$$B_\star := B_0 \cup B^{\varepsilon_0}$$

is an absorbing set for $\{U_\varepsilon(t, \tau)\}$ which is independent of ε. Since $\mathcal{A}^\varepsilon \subset B_\star$, the proof is completed. $\qquad\square$

In fact, we also have an integral uniform boundedness in H^1 for all trajectories constituting the attractor $\mathcal{A}^\varepsilon$.

Corollary 5.3. *For every $\varepsilon \in [0, 1]$, the estimate*

$$\sup_{t\in\mathbb{R}} \sup_{u\in\mathcal{K}_{g^\varepsilon}} \int_t^{t+1} \|u(s)\|_1^2 \, ds = C < \infty$$

holds, for some $C = C(\ell)$.

The proof is left to the reader.

Remark 5.4. Condition (5.1) takes place, for instance, when $g_1 \in L_\infty(\mathbb{R}; H) \cap L_2^{loc}(\mathbb{R}; H^1)$ is a time periodic function of period $T > 0$ with zero mean, that is,

$$\int_0^T g_1(s) \, ds = 0.$$

Other examples of quasiperiodic and almost periodic in time functions satisfying (5.1) can be found in [6, 7].

Remark 5.5. In light of (2.1), a sufficient condition in order for (5.1) to hold is to require that

$$\sup_{t\in\mathbb{R}} \|G_1(t)\|_1 \leqslant \frac{\ell}{2}\left(\frac{\lambda}{1+\lambda}\right)^{1/2},$$

where G_1 is the primitive of g_1 given by

$$G_1(t) = \int_0^t g_1(s) \, ds, \qquad t \in \mathbb{R}.$$

Remark 5.6. In the more challenging case $\rho = 1$, the uniform boundedness of the attractors $\mathcal{A}^\varepsilon$ in ε can still be established recasting the above proof, under the condition

$$(\lambda\nu)^{-1} \sup_{t\geqslant\tau,\tau\in\mathbb{R}} \left\{\int_t^{t+1} \|G_1(s, \tau)\|_1^2 \, ds\right\} < c_0,$$

where $c_0 > 0$ is some absolute constant. The uniform boundedness in the general case remains an open problem.

6. Convergence of the attractors

The main result of the paper reads as follows.

Theorem 6.1. *Let $g_0, g_1 \in L_2^{tc}(\mathbb{R}; H)$, and let (5.1) hold. Then, the uniform global attractors $\mathcal{A}^\varepsilon$ converge to $\mathcal{A}^0$ in the limit $\varepsilon \to 0^+$ in the following sense:*

$$\lim_{\varepsilon \to 0^+} \left\{ \mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \right\} = 0.$$

The proof of the theorem requires some steps. Firstly, we study the deviation of two solutions to (3.1) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively, sharing the same initial data. We denote

$$u^\varepsilon(t) := U_\varepsilon(t, \tau) u_\tau,$$

with u_τ belonging to the absorbing set $B_\star$ found in the previous section. In particular, for $\varepsilon = 0$, since $u_\tau \in B_\star$, (3.4) yields the bound

$$\|u^0(t)\|^2 + \int_t^{t+1} \|u^0(s)\|_1^2 \, \mathrm{d}s \leqslant R_0^2, \tag{6.1}$$

for some $R_0 = R_0(\rho)$, as the size of $B_\star$ depends on ρ.

Lemma 6.2. *For every $\varepsilon \in (0, 1]$, every $\tau \in \mathbb{R}$ and every vector $u_\tau \in B_\star$, the deviation*

$$w(t) = u^\varepsilon(t) - u^0(t),$$

with $u^\varepsilon(0) = u^0(0) = u_\tau$, fulfils the estimate

$$\|w(t)\| \leqslant D\varepsilon^{1-\rho} e^{R(t-\tau)}, \qquad \forall t \geqslant \tau, \tag{6.2}$$

for some positive constants $D = D(\rho, \ell)$ and $R = R(\rho, \ell)$, both independent of ε.

Proof. Since the deviation $w(t)$ solves

$$\partial_t w + \nu A w + B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0) = \varepsilon^{-\rho} g_1(t/\varepsilon), \qquad w|_{t=\tau} = 0,$$

the difference

$$q(t) = w(t) - v(t),$$

where $v(t)$ is the solution to (5.2), fulfils the Cauchy problem

$$\partial_t q + \nu A q + B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0) = 0, \qquad q|_{t=\tau} = 0. \tag{6.3}$$

At this point, we take the scalar product in H of equation (6.3) and q, so getting

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|^2 + \nu\|q\|_1^2 + \langle B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0), q \rangle = 0. \tag{6.4}$$

From the equality

$$B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0) = B(u^0, q+v) + B(q+v, u^0) + B(q+v, q+v),$$

by means of (2.3), we derive

$$\langle B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0), q \rangle$$
$$= b(u^0, v, q) + b(q, u^0, q) + b(v, u^0, q) + b(q, v, q) + b(v, v, q). \tag{6.5}$$

We now proceed to estimate the terms in the right-hand side. Exploiting (2.4), we find

$$|b(q, u^0, q)| \leqslant c\|q\|_1\|q\|\|u^0\|_1 \leqslant \frac{\nu}{4}\|q\|_1^2 + C\|q\|^2\|u^0\|_1^2, \tag{6.6}$$

$$|b(q, v, q)| \leqslant c\|q\|_1\|q\|\|v\|_1 \leqslant \frac{\nu}{4}\|q\|_1^2 + C\|q\|^2\|v\|_1^2, \tag{6.7}$$

$$|b(v, v, q)| \leqslant c\|q\|_1\|v\|\|v\|_1 \leqslant \frac{\nu}{4}\|q\|_1^2 + C\|v\|^2\|v\|_1^2 \tag{6.8}$$

and

$$|b(u^0, v, q)| + |b(v, u^0, q)| \leqslant 2c\|u^0\|^{1/2}\|u^0\|_1^{1/2}\|v\|^{1/2}\|v\|_1^{1/2}\|q\|_1 \qquad (6.9)$$

$$\leqslant \frac{\nu}{4}\|q\|_1^2 + C\|u^0\|\|u^0\|_1\|v\|\|v\|_1.$$

Therefore,

$$|\langle B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0), q\rangle|$$

$$\leqslant \nu\|q\|_1^2 + C\|q\|^2\left(\|u^0\|_1^2 + \|v\|_1^2\right) + C\|v\|^2\|v\|_1^2 + C\|u^0\|\|u^0\|_1\|v\|\|v\|_1,$$

which, as $\|v\|$ satisfies (5.3) and $\|u^0\|$ satisfies (6.1), can be rewritten as

$$|\langle B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0), q\rangle| \leqslant \nu\|q\|_1^2 + h\|q\|^2 + f,$$

where we set

$$h(t) = C\left(\|u^0(t)\|_1^2 + \|v(t)\|_1^2\right)$$

and

$$f(t) = C\ell^2\varepsilon^{2(1-\rho)}\|v\|_1^2 + CR_0\ell\varepsilon^{(1-\rho)}\|u^0(t)\|_1\|v(t)\|_1.$$

Then, (6.4) turns into

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|^2 \leqslant h\|q\|^2 + f.$$

Recalling that $\|q(\tau)\| = 0$, the Gronwall lemma entails

$$\|q(t)\|^2 \leqslant 2\int_\tau^t f(s)\exp\left(2\int_s^t h(y)\,\mathrm{d}y\right)\mathrm{d}s \leqslant 2\exp\left(2\int_\tau^t h(s)\,\mathrm{d}s\right)\int_\tau^t f(s)\,\mathrm{d}s.$$

On the other hand, from (5.3) and (6.1), we learn that

$$\int_\tau^t h(s)\,\mathrm{d}s \leqslant C(R_0^2 + \ell^2)(t - \tau + 1)$$

and

$$\int_\tau^t f(s)\,\mathrm{d}s \leqslant C\ell^4\varepsilon^{4(1-\rho)}(t - \tau + 1) + CR_0\ell\varepsilon^{(1-\rho)}\int_\tau^t \|u^0(s)\|_1\|v(s)\|_1\,\mathrm{d}s$$

$$\leqslant C\ell^4\varepsilon^{4(1-\rho)}(t - \tau + 1) + CR_0\ell\varepsilon^{(1-\rho)}\left(\int_\tau^t \|u^0(s)\|_1^2\,\mathrm{d}s\right)^{1/2}\left(\int_\tau^t \|v(s)\|_1^2\,\mathrm{d}s\right)^{1/2}$$

$$\leqslant C\ell^4\varepsilon^{4(1-\rho)}(t - \tau + 1) + CR_0^2\ell^2\varepsilon^{2(1-\rho)}(t - \tau + 1)$$

$$\leqslant C\varepsilon^{2(1-\rho)}(\ell^4 + R_0^2\ell^2)(t - \tau + 1).$$

Consequently,

$$\|q(t)\|^2 \leqslant C\varepsilon^{2(1-\rho)}(\ell^4 + R_0^2\ell^2)(t - \tau + 1)\mathrm{e}^{C(R_0^2+\ell^2)(t-\tau+1)} \leqslant D_1^2\varepsilon^{2(1-\rho)}\mathrm{e}^{2R_1(t-\tau)},$$

for some $D_1 = D_1(\rho, \ell)$ and $R_1 = R_1(\rho, \ell)$. Finally, as $w = q + v$, using (5.3) to control $\|v\|$, we obtain the desired conclusion (6.2). $\qquad \square$

In order to study the convergence of the uniform global attractors, we actually need a generalization of lemma 6.2, which applies to the whole family of equations (3.7), with external forces $\hat{g} = \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$. To this end, we observe that every function $\hat{g}_1 \in \mathcal{H}(g_1)$ fulfils the inequality (5.1) (see [5] for a proof). More precisely, defining

$$\hat{G}_1(t, \tau) = \int_\tau^t \hat{g}_1(s)\,\mathrm{d}s, \qquad t \geqslant \tau,$$

we have

$$\sup_{t \geqslant \tau, \tau \in \mathbb{R}} \left\{ \|\hat{G}_1(t, \tau)\|^2 + \int_t^{t+1} \|\hat{G}_1(s, \tau)\|_1^2 \right\} \leqslant \ell^2. \tag{6.10}$$

For any $\varepsilon \in [0, 1]$, let

$$\hat{u}^\varepsilon(t) = U_{\hat{g}^\varepsilon}(t, \tau) y_\tau$$

be the solution to (3.7) with external force $\hat{g}^\varepsilon = \hat{g}_0 + \varepsilon^{-\rho} \hat{g}_1(\cdot/\varepsilon) \in \mathcal{H}(g^\varepsilon)$ and $y_\tau \in B_\star$. For $\varepsilon > 0$, we consider the deviation

$$\hat{w}(t) = \hat{u}^\varepsilon(t) - \hat{u}^0(t).$$

Lemma 6.3. *The inequality*

$$\|\hat{w}(t)\| \leqslant D\varepsilon^{1-\rho} e^{R(t-\tau)}, \qquad \forall t \geqslant \tau,$$

holds, with D and R as in lemma 6.2.

Proof. We repeat the proof of lemma 6.2, with $\hat{u}^\varepsilon$, $\hat{g}_0$ and $\hat{g}_1$ in place of u^ε, g_0 and g_1, respectively, noting that (6.1) still holds for $\hat{u}^0$, as the family $\{U_{\hat{g}^\varepsilon}(t, \tau)\}$, $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, is $(H \times \mathcal{H}(g^\varepsilon), H)$-continuous, and using (6.10) in place of (5.1). □

We can now complete the proof of the theorem, using the following argument from [5], which we report in some detail for the reader's convenience.

Proof of theorem 6.1. For $\varepsilon > 0$, let $u^\varepsilon \in \mathcal{A}^\varepsilon$. Thus, in view of (3.8), there exists a complete bounded trajectory $\hat{u}^\varepsilon(t)$ of equation (3.7), with some external force

$$\hat{g}^\varepsilon = \hat{g}_0 + \varepsilon^{-\rho} \hat{g}_1(\cdot/\varepsilon) \in \mathcal{H}(g^\varepsilon),$$

such that $\hat{u}^\varepsilon(0) = u^\varepsilon$. For every $L \geqslant 0$,

$$\hat{u}^\varepsilon(-L) \in \mathcal{A}^\varepsilon \subset B_\star.$$

From the straightforward equality

$$u^\varepsilon = U_{\hat{g}^\varepsilon}(0, -L)\hat{u}^\varepsilon(-L),$$

by applying lemma 6.3 with $t = 0$ and $\tau = -L$, we have that

$$\|u^\varepsilon - U_{\hat{g}_0}(0, -L)\hat{u}^\varepsilon(-L)\| \leqslant D\varepsilon^{1-\rho} e^{RL}. \tag{6.11}$$

On the other hand (see [7]), the set $\mathcal{A}^0$ attracts $U_{\hat{g}_0}(t, -L)B_\star$, uniformly as $\hat{g}_0 \in \mathcal{H}(g^0)$. Then, for every $\delta > 0$, there is $T = T(\delta) \geqslant 0$, independent of L, such that

$$\text{dist}_H(U_{\hat{g}_0}(T - L, -L)\hat{u}^\varepsilon(-L), \mathcal{A}^0) \leqslant \delta. \tag{6.12}$$

Setting $L = T$, and collecting the two above inequalities, we readily get

$$\text{dist}_H(u^\varepsilon, \mathcal{A}^0) \leqslant D\varepsilon^{1-\rho} e^{RT} + \delta.$$

Since $u^\varepsilon \in \mathcal{A}^\varepsilon$ and $\delta > 0$ are arbitrary, taking the limit $\varepsilon \to 0^+$, the conclusion follows. □

7. Hölder continuity of $\mathcal{A}^\varepsilon$ at $\varepsilon = 0$

In this final section, we consider the Navier–Stokes equations under the assumption that the *Grashof number* G_0 of the averaged system (3.1) with $\varepsilon = 0$ satisfies the following inequality:

$$G_0 := \frac{\|g_0\|_{L_2^b}}{\lambda \nu^2} < \frac{1}{c_1}, \tag{7.1}$$

where c_1 is the smallest possible absolute constant such that

$$|b(v, w, v)| = |b(v, v, w)| \leqslant c_1 \|v\| \|v\|_1 \|w\|_1, \qquad \forall v, w \in H^1. \tag{7.2}$$

Clearly, with reference to (2.4) and (2.5), $c_1 \leqslant c$.

Remark 7.1. As shown in [4],

$$c_1 \leqslant c_L^2 / \sqrt{2},$$

where c_L is the constant from the celebrated scalar Ladyzhenskaya inequality

$$\|f\|_{L_4(\Omega)} \leqslant c_L \|f\|^{1/2} \|\nabla f\|^{1/2}, \qquad \forall f \in H_0^1(\Omega).$$

The bound

$$c_L \leqslant 2/(27\pi)^{1/4} = 0.6590\ldots$$

is demonstrated in [22], whereas the sharp value

$$c_L = (\pi \cdot 1.8622\ldots)^{-1/4} = 0.6429\ldots$$

is obtained numerically in [29]. Therefore, (7.1) holds provided that

$$G_0 < 3.4206\ldots = (2\pi \cdot 1.8622\ldots)^{1/2} = \sqrt{2}/c_L^2 \leqslant 1/c_1.$$

Since $\lambda \geqslant 2\pi/|\Omega|$ (see [19]), this is certainly true whenever

$$\frac{|\Omega| \, \|g_0\|_{L_2^b}}{\nu^2} < 21.4923\ldots = (8\pi^3 \cdot 1.8622\ldots)^{1/2}.$$

When (7.1) holds, the paper [8] proves that the averaged equation has a unique complete solution $\{\bar{u}(t), \ t \in \mathbb{R}\}$ such that

$$\sup_{t \in \mathbb{R}} \|\bar{u}(t)\| < \infty.$$

Moreover, this solution attracts any other solution $u_{g^0}(t), t \geqslant \tau$, of the averaged equation with exponential rate as $t - \tau \to +\infty$, namely,

$$\|u_{g^0}(t) - \bar{u}(t)\| \leqslant C \|u_{g^0}(\tau) - \bar{u}(\tau)\| \mathrm{e}^{-\varkappa(t-\tau)}, \qquad \forall t \geqslant \tau, \quad \forall \tau \in \mathbb{R}, \tag{7.3}$$

for some $C > 0$ and $\varkappa > 0$. Then, the attractor $\mathcal{A}^0$ has the form

$$\mathcal{A}^0 = \overline{\{\bar{u}(t) \mid t \in \mathbb{R}\}}^H.$$

Besides, it follows from (7.3) that the uniform global attractor $\mathcal{A}^0$ is exponential, that is, it attracts any bounded (in H) set of initial data with exponential rate $\varkappa$.

Theorem 7.2. *Let the assumptions of theorem 6.1 hold. If the Grashof number G_0 of the averaged system satisfies (7.1), then we have the Hölder continuity property at $\varepsilon = 0$*

$$\mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant K_\rho \, \varepsilon^{1-\rho}, \tag{7.4}$$

for some $K_\rho > 0$ depending (besides on ρ) on ν, $\|g_0\|_{L_2^b}$ and $\|g_1\|_{L_2^b}$.

Proof. The main point is to show that, within assumption (7.1), lemmas 6.2 and 6.3 hold with the exponent $R = 0$ if $\varepsilon \leqslant \varepsilon_1$, for some $\varepsilon_1 = \varepsilon_1(\rho) > 0$.

We preliminarily observe that, in light of (3.3) and (3.4), the function

$$u^0(t) = U_0(t, \tau)u_\tau, \qquad u_\tau \in B_\star,$$

satisfies, for all $t \geqslant \tau$, the inequalities

$$\|u^0(t)\|^2 \leqslant R_\star^2 + (\nu\lambda)^{-2}(1 + \nu\lambda)M_0^2, \tag{7.5}$$

$$\nu \int_\tau^t \|u^0(s)\|_1^2 \mathrm{d}s \leqslant R_\star^2 + (\nu\lambda)^{-1}M_0^2(t - \tau + 1), \tag{7.6}$$

with M_0 as in (2.6), where $R_\star = R_\star(\rho)$ is the radius of the absorbing set $B_\star$.

We now proceed as in the proof of lemma 6.2, and we obtain equalities (6.4) and (6.5). This time, to estimate the left-hand sides of (6.6)–(6.8), we use inequality (7.2) in place of (2.4). This leads to

$$|b(q, u^0, q)| \leqslant c_1 \|q\|_1 \|q\| \|u^0\|_1 \leqslant \frac{\nu}{2}\|q\|_1^2 + \frac{c_1^2}{2\nu}\|q\|^2\|u^0\|_1^2,$$

$$|b(q, v, q)| \leqslant c_1 \|q\|_1 \|q\| \|v\|_1 \leqslant \frac{\mu}{4}\|q\|_1^2 + \frac{c_1^2}{\mu}\|q\|^2\|v\|_1^2,$$

$$|b(v, v, q)| \leqslant c_1 \|q\|_1 \|v\| \|v\|_1 \leqslant \frac{\mu}{4}\|q\|_1^2 + \frac{c_1^2}{\mu}\|v\|^2\|v\|_1^2,$$

where the constant $\mu > 0$ will be specified later. Similarly to (6.9), we also have

$$|b(u^0, v, q)| + |b(v, u^0, q)| \leqslant 2c\|u^0\|^{1/2}\|u^0\|_1^{1/2}\|v\|^{1/2}\|v\|_1^{1/2}\|q\|_1$$

$$\leqslant \frac{\mu}{2}\|q\|_1^2 + \frac{2c^2}{\mu}\|u^0\|\|u^0\|_1\|v\|\|v\|_1.$$

Therefore, owing to (6.5),

$$|\langle B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0), q\rangle| \leqslant (\nu/2 + \mu)\|q\|_1^2 + (c_1^2\nu^{-1}/2)\|q\|^2\|u^0\|_1^2 + c_1^2\mu^{-1}\|q\|^2\|v\|_1^2$$

$$+ c_1^2\mu^{-1}\|v\|^2\|v\|_1^2 + 2c^2\mu^{-1}\|u^0\|\|u^0\|_1\|v\|\|v\|_1.$$

Then, it follows from (6.4) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|q\|^2 + (\nu - 2\mu)\|q\|_1^2 - \left(c_1^2\nu^{-1}\|u^0\|_1^2 + 2c_1^2\mu^{-1}\|v\|_1^2\right)\|q\|^2$$

$$\leqslant 2c_1^2\mu^{-1}\|v\|^2\|v\|_1^2 + 4c^2\mu^{-1}\|u^0\|\|u^0\|_1\|v\|\|v\|_1. \tag{7.7}$$

Consequently, using the inequality

$$\lambda(\nu - 2\mu)\|q\|^2 \leqslant (\nu - 2\mu)\|q\|_1^2,$$

for $0 < 2\mu < \nu$, we obtain from (7.7) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|q\|^2 + \varphi_1\|q\|^2 \leqslant \varphi_2, \tag{7.8}$$

having set

$$\varphi_1(t) = \lambda(\nu - 2\mu) - c_1^2\nu^{-1}\|u^0(t)\|_1^2 - 2c_1^2\mu^{-1}\|v(t)\|_1^2,$$

$$\varphi_2(t) = 2c_1^2\mu^{-1}\|v(t)\|^2\|v(t)\|_1^2 + 4c^2\mu^{-1}\|u^0(t)\|\|u^0(t)\|_1\|v(t)\|\|v(t)\|_1.$$

Since $\|v\|_1^2$ satisfies (5.3), we see that

$$\int_\tau^t \|v(s)\|_1^2 \,\mathrm{d}s \leqslant C\ell^2\varepsilon^{2(1-\rho)}(t - \tau + 1), \qquad \forall t \geqslant \tau.$$

Hence, defining

$$\beta_1 := \left[\lambda\nu - c_1^2\nu^{-2}(\nu\lambda)^{-1}M_0^2\right] - 2\lambda\mu - 2c_1^2\mu^{-1}C\ell^2\varepsilon^{2(1-\rho)},$$

$$\gamma_1 := c_1^2\nu^{-2}R_\star^2 + c_1^2\nu^{-2}(\nu\lambda)^{-1}M_0^2 + 2c_1^2\mu^{-1}C\ell^2\varepsilon^{2(1-\rho)},$$

and taking advantage of (7.6), we find the inequality

$$\int_\tau^t \varphi_1(s)\,\mathrm{d}s \geqslant \lambda(\nu - 2\mu)(t - \tau) - c_1^2\nu^{-2}\left[R_\star^2 + (\nu\lambda)^{-1}M_0^2(t - \tau + 1)\right]$$

$$-2c_1^2\mu^{-1}\left[C\ell^2\varepsilon^{2(1-\rho)}(t - \tau + 1)\right] = \beta_1(t - \tau) - \gamma_1.$$

Due to (7.1) (recall that $M_0 = \|g_0\|_{L_2^b}$),

$$\lambda\nu - c_1^2\nu^{-2}(\nu\lambda)^{-1}M_0^2 > 0.$$

Thus, fixing $\mu > 0$ sufficiently small,

$$\left[\lambda\nu - c_1^2\nu^{-2}(\nu\lambda)^{-1}M_0^2\right] - 2\lambda\mu > 0.$$

Accordingly (μ is now fixed),

$$\beta := \left[\lambda\nu - c_1^2\nu^{-2}(\nu\lambda)^{-1}M_0^2\right] - 2\lambda\mu - 2c_1^2\mu^{-1}C\ell^2\varepsilon_1^{2(1-\rho)} > 0,$$

for a sufficiently small $\varepsilon_1 = \varepsilon_1(\mu) > 0$. Thus,

$$\beta_1 = \left[\lambda\nu - c_1^2\nu^{-2}(\nu\lambda)^{-1}M_0^2\right] - 2\lambda\mu - 2c_1^2\mu^{-1}C\ell^2\varepsilon^{2(1-\rho)} \geqslant \beta, \qquad \forall\varepsilon \leqslant \varepsilon_1. \tag{7.9}$$

For this chosen μ, we also have that

$$\gamma_1 = c_1^2\nu^{-2}R_\star^2 + c_1^2\nu^{-2}(\nu\lambda)^{-1}M_0^2 + 2c_1^2\mu^{-1}C\ell^2\varepsilon^{2(1-\rho)} \leqslant \gamma, \tag{7.10}$$

where

$$\gamma := c_1^2\nu^{-2}R_\star^2 + c_1^2\nu^{-2}(\nu\lambda)^{-1}M_0^2 + 2c_1^2\mu^{-1}C\ell^2 > 0.$$

In conclusion, for $\varepsilon \leqslant \varepsilon_1$, we end up with the inequality

$$\int_\tau^t \varphi_1(s)\,\mathrm{d}s \geqslant \beta(t - \tau) - \gamma, \qquad \forall t \geqslant \tau. \tag{7.11}$$

To estimate $\varphi_2(t)$, we note that $\|v\|$ and $\|u^0\|$ satisfy (5.3) and (7.5), respectively. Hence,

$$\varphi_2 \leqslant 2c_1^2\mu^{-1}C\ell^2\varepsilon^{2(1-\rho)}\|v\|_1^2 + 4c^2\mu^{-1}C^{1/2}\ell\varepsilon^{(1-\rho)}\tilde{R}_\star\|u^0\|_1\|v\|_1, \tag{7.12}$$

with

$$\tilde{R}_\star^2 := R_\star^2 + (\nu\lambda)^{-2}(1 + \nu\lambda)M_0^2.$$

Observing that $\|v\|_1$ satisfies (5.3), exploiting (7.6), and using the Cauchy inequality

$$\int_t^{t+1} \|u^0(s)\|_1\|v(s)\|_1\,\mathrm{d}s \leqslant \left(\int_t^{t+1} \|u^0(s)\|_1^2\,\mathrm{d}s\right)^{1/2}\left(\int_t^{t+1} \|v(s)\|_1^2\,\mathrm{d}s\right)^{1/2},$$

we learn from (7.12) that

$$\int_t^{t+1} \varphi_2(s)\,\mathrm{d}s \leqslant C\ell^4\varepsilon^{4(1-\rho)} + C\ell^2\varepsilon^{2(1-\rho)}\tilde{R}_\star\left(\int_t^{t+1} \|u^0(s)\|_1^2\,\mathrm{d}s\right)^{1/2}$$

$$\leqslant C\ell^4\varepsilon^{4(1-\rho)} + C\ell^2\varepsilon^{2(1-\rho)}\tilde{R}_\star\hat{R}_\star,$$

with

$$\hat{R}_\star^2 := \nu^{-1}\left[\tilde{R}_\star^2 + 2(\nu\lambda)^{-1}M_0^2\right].$$

Consequently,

$$\int_t^{t+1} \varphi_2(s)\,\mathrm{d}s \leqslant M := C\varepsilon^{2(1-\rho)}(\ell^4 + \ell^2 \tilde{R}_\star \hat{R}_\star). \tag{7.13}$$

At this point, we apply lemma 2.2 to the function $\zeta(t) = \|q(t)\|^2$, which fulfils (7.8). On account of (7.11)–(7.13) and the fact that $\zeta(\tau) = 0$, this yields

$$\|q(t)\|^2 \leqslant \frac{M\mathrm{e}^\gamma}{1 - \mathrm{e}^{-\beta}} = D_2^2 \varepsilon^{2(1-\rho)}, \qquad \forall \varepsilon \leqslant \varepsilon_1,$$

for some $D_2 = D_2(\rho, \ell)$. Finally, for the function $w = q + v$, using (5.3), we obtain the inequality (6.2) with $R = 0$ for all $\varepsilon \leqslant \varepsilon_1$. This also implies lemma 6.3 with $R = 0$.

We now proceed to prove (7.4). If $\varepsilon \leqslant \varepsilon_1$, we repeat the proof of theorem 6.1, but replacing (6.11) with the estimate

$$\|u^\varepsilon - U_{\hat{g}_0}(0, -L)\hat{u}^\varepsilon(-L)\| \leqslant D\varepsilon^{1-\rho}.$$

For $L = T$, this inequality together with (6.12) entail

$$\mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant D\varepsilon^{1-\rho} + \delta, \qquad \forall \delta > 0.$$

From the arbitrariness of $\delta > 0$, we eventually obtain

$$\mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant D\varepsilon^{1-\rho}, \qquad \forall \varepsilon \leqslant \varepsilon_1.$$

In order to extend the inequality to the case $1 \geqslant \varepsilon \geqslant \varepsilon_1$, we merely increase the constant D accordingly, so to get (7.4) for all $\varepsilon \in (0, 1]$ and for some $K_\rho > 0$. $\qquad\square$

Remark 7.3. It can be shown that, under condition (7.1), there is $\varepsilon_2 > 0$ such that, for all $\varepsilon \leqslant \varepsilon_2$, equation (3.1) has also a unique bounded complete solution $\{\bar{u}^\varepsilon(t),\ t \in \mathbb{R}\}$ such that

$$\|U_{g^\varepsilon}(t, \tau)u_\tau - \bar{u}^\varepsilon(t)\| \leqslant C\|u_\tau - \bar{u}^\varepsilon(\tau)\|\mathrm{e}^{-\varkappa_1(t-\tau)}, \qquad \forall t \geqslant \tau, \forall \tau \in \mathbb{R},$$

where $\varkappa_1 > 0$ and $C > 0$ are independent of ε. Thus, the uniform global attractor $\mathcal{A}^\varepsilon$ is exponential with rate $\varkappa_1$. Moreover, similarly to (7.4) we have the inequality

$$\mathrm{dist}_H(\mathcal{A}^0, \mathcal{A}^\varepsilon) \leqslant K_\rho\,\varepsilon^{1-\rho}.$$

In which case, the uniform attractors $\mathcal{A}^\varepsilon$ converge to $\mathcal{A}^0$ as $\varepsilon \to 0^+$ w.r.t. the *symmetric Hausdorff distance*. The proofs of these assertions are left to the reader.

Acknowledgments

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