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Dedicated to the memory of Leonid Volevich

Trajectory Attractors for Dissipative 2D Euler and Navier–Stokes Equations

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Abstract. A trajectory attractor $\mathfrak{A}$ is constructed for the 2D Euler system containing an additional dissipation term $-ru, r > 0$, with periodic boundary conditions. The corresponding dissipative 2D Navier–Stokes system with the same term $-ru$ and with viscosity $\nu > 0$ also has a trajectory attractor, $\mathfrak{A}_\nu$. Such systems model large-scale geophysical processes in atmosphere and ocean (see [1]). We prove that $\mathfrak{A}_\nu \to \mathfrak{A}$ as $\nu \to 0+$ in the corresponding metric space. Moreover, we establish the existence of the minimal limit $\mathfrak{A}_{\min}$ of the trajectory attractors $\mathfrak{A}_\nu$ as $\nu \to 0+$. We prove that $\mathfrak{A}_{\min}$ is a connected invariant subset of $\mathfrak{A}$. The connectedness problem for the trajectory attractor $\mathfrak{A}$ by itself remains open.

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INTRODUCTION

In the present paper, we construct a trajectory attractor $\mathfrak{A}$ for the 2D Euler system containing an additional dissipative term $-ru, r > 0$, and equipped with periodic boundary conditions. Here $u = (u^1(x,t), u^2(x,t))$ stands for the unknown periodic velocity vector field. Such systems come from geophysical models describing large-scale processes in atmosphere and ocean. The term $-ru$ parameterizes the main dissipation occurring in the planetary boundary layer (see, e.g., [1, Chap. 4]).

We also construct a trajectory attractor $\mathfrak{A}_\nu$ for the 2D Navier–Stokes system with the same dissipation term $-ru$ and with viscosity coefficient ν. In the above geophysical models, the viscosity term $\nu\Delta u$ is responsible for small-scale dissipation (note that $0 < \nu \ll r$ in physically relevant cases).

We prove that the Hausdorff deviation of the set $\mathfrak{A}_\nu$ from the set $\mathfrak{A}$ (in the corresponding metric space with metric $\rho(\cdot\,,\cdot)$) tends to zero as the viscosity ν vanishes,

$$\mathrm{dist}_\rho(\mathfrak{A}_\nu, \mathfrak{A}) \to 0 \quad \text{as} \quad \nu \to 0+.$$

We also study some important properties of the trajectory attractors $\mathfrak{A}$ and $\mathfrak{A}_\nu$ specified below.

Note that 2D Euler and Navier–Stokes systems with dissipation were considered in a number of papers (see, e.g., [2–4] for the 2D Euler system and [5–7] for the 2D Navier–Stokes system).

The method of trajectory attractors for evolution partial differential equations was developed in [8–11]. This approach is highly fruitful in the study of the long-time behavior of solutions to evolution equations for which the uniqueness theorem related to the corresponding initial-value problem is not proved yet (e.g., for the 3D Navier–Stokes system) or fails.

The paper is organized as follows. In Section 1, we study the dissipative 2D Euler system with periodic boundary conditions. Using the Galerkin method, we prove that the initial-value problem for this system has at least one weak distribution solution $u(x,t)$ such that $u(x,t) \in L_\infty(\mathbb{R}_+; H^1)$ and $\partial_t u(x,t) \in L_\infty(\mathbb{R}_+; H^{-1})$. Here H^1 stands for the space of periodic solenoidal vector fields with finite Sobolev H^1-norm and the space $H^{-1} = (H^1)^*$ for the dual to H^1. Moreover, the solution $u(x,t)$ thus constructed satisfies the corresponding energy inequality (see the next paragraph), which is of importance in our subsequent study. Note that the uniqueness theorem for weak solutions to the 2D Euler system in the class $L_\infty(\mathbb{R}_+; H^1)$ is not proved.

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In Section 2, we construct a trajectory attractor for the 2D dissipative Euler system (note that we failed to construct a trajectory attractor for the classical 2D Euler system without dissipation, for $r = 0$). We define spaces $\mathcal{F}_+^\infty$ and $\mathcal{F}_+^{\mathrm{loc}}$ ($\mathcal{F}_+^\infty \subset \mathcal{F}_+^{\mathrm{loc}}$), which contain the weak solutions $u(x,t)$ constructed in Section 2. We then introduce the space of trajectories (solutions) $\mathcal{K}_+(N) \subset \mathcal{F}_+^\infty$ depending on $N > 0$. The set $\mathcal{K}_+(N)$ consists of the weak solutions $u(x,t)$ of the system that satisfy the inequality

$$\operatorname{ess\,sup}\left\{\|u(\cdot, t+s)\|^2 \mid s \geqslant 0\right\} \leqslant Ne^{-rt} + r^{-1}\|g\|^2 \quad \forall t \in \mathbb{R}_+,$$

where $\|\cdot\| := \|\cdot\|_{H^1}$ stands for the norm in H^1 and g is the (known) external force for the dissipative Euler system. The space $\mathcal{F}_+^{\mathrm{loc}}$ is equipped with the weak topology Θ_+^{loc} generated by the weak convergence of sequences $\{v_n(x,t)\} \subset \mathcal{F}_+^{\mathrm{loc}}$. We prove that the trajectory space $\mathcal{K}_+(N)$ is bounded in $\mathcal{F}_+^\infty$ and closed in the topology Θ_+^{loc}. This theorem is very important in the subsequent investigations. Consider the translation semigroup $\{T(h), h \geqslant 0\}$ acting on a solution $u(x,t)$ by the formula $T(h)u(x,t) = u(x, h+t)$. It follows from the definition of the trajectory space that $\mathcal{K}_+(N)$ is invariant under $\{T(h)\} :\ T(h)\mathcal{K}_+(N) \subseteq \mathcal{K}_+(N)$ for all $h \geqslant 0$. Using these facts and applying the theory of trajectory attractors, we prove that the translation semigroup $\{T(h)\}$ acting on $\mathcal{K}_+(N)$ has a global attractor, $\mathfrak{A}(N)$, which we call the *trajectory attractor of the system*. Recall that $T(h)\mathfrak{A}(N) = \mathfrak{A}(N)$ for any $h \geqslant 0$. We then prove that the set $\mathfrak{A}(N)$ does not depend on N, $\mathfrak{A}(N) = \mathfrak{A}(0) =: \mathfrak{A}$, for any $N \geqslant 0$.

In Sections 3 and 4, we study the dissipative 2D Navier–Stokes system with periodic boundary conditions which contains an additional term $-ru$, $r > 0$. The corresponding initial-value problem is well-posed, and we construct a trajectory attractor $\mathfrak{A}_\nu$ for this system. We prove that $\mathrm{dist}_\rho(\mathfrak{A}_\nu, \mathfrak{A}) \to 0$ as $\nu \to 0+$.

In Section 5, we prove the existence of the minimal limit $\mathfrak{A}_{\min}$ of the trajectory attractors $\mathfrak{A}_\nu$ as $\nu \to 0+$, i.e., $\mathfrak{A}_{\min} \subseteq \mathfrak{A}, \mathfrak{A}_{\min}$ is closed in Θ_+^{loc}, $\mathrm{dist}_\rho(\mathfrak{A}_\nu, \mathfrak{A}_{\min}) \to 0$ as $\nu \to 0+$, and $\mathfrak{A}_{\min}$ is the minimal set satisfying these properties. We prove that the set $\mathfrak{A}_{\min}$ is connected in the topology Θ_+^{loc} and strictly invariant with respect to the translation semigroup. The question of whether or not the trajectory attractor $\mathfrak{A}$ by itself is a connected set in Θ_+^{loc} remains an open problem.

1. 2D EULER EQUATIONS WITH DISSIPATION

We consider the 2D Euler system with dissipation represented in the solenoidal form,

$$\begin{gathered}\partial_t u + B(u,u) + ru = g(x), \quad x = (x_1, x_2) \in \mathbb{T}^2, \\ (\nabla, u) := \partial_{x_1} u^1 + \partial_{x_2} u^2 = 0, \qquad u = (u^1(x,t), u^2(x,t)), \quad t \geqslant 0,\end{gathered} \tag{1.1}$$

where $\mathbb{T}^2 = (\mathbb{R} \bmod 2\pi)^2$ is the 2D torus and $B(u,v) = P\left(u^1\partial_{x_1} v + u^2 \partial_{x_2} v\right)$ is the standard bilinear term. Here and below, P stands for the orthogonal Leray projection from the space $L_2(\mathbb{T}^2)^2$ onto $H = \left[\left\{v \in C^\infty(\mathbb{T}^2)^2 \mid (\nabla, v) = 0\right\}\right]_{L_2(\mathbb{T}^2)^2}$ ($[X]_E$ stands for the closure of the set X in the topological space E). We similarly introduce the space $H^1 = \left[\left\{v \in C^\infty(\mathbb{T}^2)^2 \mid (\nabla, v) = 0\right\}\right]_{H^1(\mathbb{T}^2)^2} \Subset H$ and the standard scale of spaces $H^s, s \in \mathbb{R}$, where $H^0 = H$ and $H^{-s} = (H^s)^*$ is the dual space to $H^s, s \geqslant 0$. The norms in H and H^1 are denoted by $|\cdot|$ and $\|\cdot\|$, respectively. Recall that, for u satisfying $(\nabla, u) = \partial_{x_1} u^1 + \partial_{x_2} u^2 = 0$, we formally have[1]

$$B(u,v) = P\left(u^1\partial_{x_1} v + u^2\partial_{x_2} v\right) = P\left(\partial_{x_1}\left(u^1 v\right) + \partial_{x_2}\left(u^2 v\right)\right). \tag{1.2}$$

In (1.1), r is a positive dissipation coefficient. The pressure $p(x,t)$ was eliminated from the system by applying the operator P to both sides of the equations. We assume that $g(x) \in H^1$.

Note that the 2D Euler system with dissipation (1.1) was studied in a number of papers (see, e.g., [2–4]). System (1.1) describes large-scale geophysical processes in atmosphere and ocean for which the main dissipation occurs in the planetary boundary layer and is parameterized by the term $-ru$ (see, e.g., [1, Chap. 4]).

[1] Note that $B(u,v) \in H$ for $u, v \in H^2$ by the Gagliardo–Nirenberg inequality (see, e.g., [12]). Moreover, the trilinear form $(B(u,v), w)$ is continuous on $H^1 \times H^1 \times H^1$, and therefore $B(u,v) \in H^{-1}$ for $u, v \in H^1$.

The initial data are imposed at time $t = 0$,

$$u|_{t=0} = u_0(x), \qquad u_0(x) \in H^1. \tag{1.3}$$

The existence of a weak distribution solution $\{u(x,t), t \in \mathbb{R}_+\}$ to problem (1.1) and (1.3) can be proved, for example, by applying the Galerkin approximation method.

Consider the orthonormal basis (in H) $\{e_j(x) = \big(e_j^1(x), e_j^2(x)\big) \in H^2,\ j = 1, 2, \dots\}$ formed by eigenfunctions of the Stokes operator,

$$-P\Delta e_j(x) = \lambda_j e_j(x), \quad (\nabla, e_j) = 0, \quad x \in \mathbb{T}^2, \quad j = 0, 1, 2, \dots \tag{1.4}$$

We shall use the well known fact that the Stokes operator satisfies the relation $-P\Delta \equiv -\Delta$ in the spaces H^2 with periodic boundary conditions (see, e.g., [13]). Recall that $e_0(x) \equiv e_0$ is a constant vector and $0 = \lambda_0 < \lambda_1 < \lambda_2 \leqslant \dots \leqslant \lambda_j \to +\infty$ as $j \to \infty$.

The Galerkin approximations

$$u_m(x,t) = \sum_{j=0}^{m} c_{jm}(t) e_j(x), \quad m = 1, \dots,$$

satisfy the system

$$\partial_t u_m + \Pi_m B(u_m, u_m) + r u_m = \Pi_m g, \tag{1.5}$$

which is equivalent to the corresponding system of ordinary differential equations for the unknown real coefficients $c_{jm}(t)$, $j = 0, 1, \dots, m$. In (1.5), Π_m stands for the orthogonal projection in H onto the finite-dimensional subspace $[e_0(x), \dots, e_m(x)]$. At time $t = 0$, we consider the initial conditions

$$u_m|_{t=0} = u_m(0) = \Pi_m u_0, \tag{1.6}$$

where u_0 is the same as in (1.3).

Clearly, problem (1.5) and (1.6) has a unique solution $u_m(x,t) \in C^1([0, T_m]; H^2)$ for some $T_m > 0$. Taking the scalar product (in H) of equation (1.5) and the function $u_m(t) := u_m(\cdot, t)$, we obtain the following differential equation:

$$\frac{1}{2}\frac{d}{dt}\left|u_m(t)\right|^2 + r\left|u_m(t)\right|^2 = (g, u_m(t)) \quad \forall t \in [0, T_m]. \tag{1.7}$$

We have used here the well-known identity

$$(B(u,u), u) = 0, \quad u \in H^1. \tag{1.8}$$

Recall that $|\cdot| = \|\cdot\|_{L_2(\mathbb{T}^2)^2}$.

The differential relation (1.7) implies the inequality

$$\left|u_m(t)\right|^2 \leqslant \left|u_m(0)\right|^2 e^{-rt} + r^{-1}\left|g\right|^2 \leqslant \left|u_0\right|^2 e^{-rt} + r^{-1}\left|g\right|^2 \quad \forall t \in [0, T_m]. \tag{1.9}$$

Taking the inner product of equation (1.5) and $-P\Delta u_m = -\Delta u_m$ in the space H and using the standard identities $-(u_m, \Delta u_m) = \left|\nabla u_m\right|^2$ and $-(g, \Delta u_m) = (\nabla g, \nabla u_m)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left|\nabla u_m(t)\right|^2 - (B(u_m, u_m), \Delta u_m) + r\left|\nabla u_m(t)\right|^2 = (\nabla g, \nabla u_m(t)) \quad \forall t \in [0, T_m]. \tag{1.10}$$

As is known, for periodic boundary conditions ($x \in \mathbb{T}^2$), we have

$$(B(u,u), \Delta u) = 0 \quad \forall u \in H^2 \tag{1.11}$$

(see [13, Chap. VI, Lemma 3.1] and [4]). This identity plays a crucial role in our subsequent investigation.

It follows from (1.10) and (1.11) that

$$\frac{1}{2}\frac{d}{dt}\left|\nabla u_m(t)\right|^2 + r\left|\nabla u_m(t)\right|^2 = (\nabla g, \nabla u_m(t)) \quad \forall t \in [0, T_m]. \tag{1.12}$$

In turn, (1.12) yields

$$\left|\nabla u_m(t)\right|^2 \leqslant \left|\nabla u_m(0)\right|^2 e^{-rt} + r^{-1}\left|\nabla g\right|^2 \quad \forall t \in [0, T_m]. \tag{1.13}$$

Now, combining inequalities (1.9) and (1.13), we obtain the main estimate

$$\|u_m(t)\|^2 \leqslant \|u_m(0)\|^2 e^{-rt} + r^{-1}\|g\|^2 \leqslant \|u_0\|^2 e^{-rt} + r^{-1}\|g\|^2 \tag{1.14}$$

for all $t \in [0, T_m]$, where $\|v\|^2 = \|v\|_{H^1}^2 = |v|^2 + |\nabla v|^2$. It follows from (1.14) that a solution $u_m(x,t)$ to problem (1.5) and (1.6) can be extended to $\mathbb{R}_+$ (i.e., $T_m = +\infty$), $u_m(x,t) \in C_b^1(\mathbb{R}_+; H^1)$, and

$$\|u_m\|_{L_\infty(\mathbb{R}_+;H^1)}^2 := \operatorname{ess\,sup}\left\{\|u_m(t)\|^2 \mid t \geqslant 0\right\} \leqslant \|u_0\|^2 + r^{-1}\|g\|^2 \quad \forall m \in \mathbb{N}. \tag{1.15}$$

Since $u_0 \in H^1$, the initial data for system (1.5) satisfy the relation

$$u_m(0) = \Pi_m u_0 \to u_0 \quad (m \to \infty) \quad \text{strongly in} \quad H^1. \tag{1.16}$$

Inequality (1.15) implies the existence of a subsequence $\{m'\} \subset \{m\}$ such that

$$u_{m'}(\cdot,t) \rightharpoonup u(\cdot,t) \quad (m' \to \infty) \quad *\text{-weakly in} \quad L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H^1) \tag{1.17}$$

for some function $u(x,t) \in L_\infty(\mathbb{R}_+; H^1)$.

We claim that $u(x,t)$ is a weak solution to problem (1.1), (1.3) in the sense of distributions (for the space $D'(\mathbb{R}_+; H^{-1})$). Indeed, using the Galerkin system (1.5) and estimate (1.14), we can see that

$$\begin{aligned}\|\partial_t u_m\|_{H^{-1}} &\leqslant \|B(u_m,u_m)\|_{H^{-1}} + r\,\|u_m\|_{H^{-1}} + \|g\|_{H^{-1}} \leqslant C\left(\|u_m\|_{L_4(\mathbb{T}^2)^2}^2 + \|u_m\|_{H^1} + \|g\|_{H^1}\right)\\ &\leqslant C_1\left(\|u_0\|_{H^1}^2 + \|g\|_{H^1}^2 + 1\right), \quad t \geqslant 0.\end{aligned} \tag{1.18}$$

We have used here the inequality

$$\|B(u_m,u_m)\|_{H^{-1}} \leqslant \|u_m\|_{L_4(\mathbb{T}^2)^2}^2, \tag{1.19}$$

which follows from the representation (1.2) of the bilinear term B, and the inequality

$$\|u\|_{L_4(\mathbb{T}^2)^2}^2 \leqslant c\|u\|^2 \quad \forall u \in H^1, \tag{1.20}$$

which results from the embedding $H^1 \subset L_4(\mathbb{T}^2)^2$.

Relations (1.17) and (1.18) give

$$\partial_t u_{m'}(\cdot,t) \rightharpoonup \partial_t u(\cdot,t) \quad (m' \to \infty) \quad *\text{-weakly in } L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H^{-1}). \tag{1.21}$$

Using now (1.17), (1.21), and the Aubin compactness theorem (see [14, 15, 16]), we obtain

$$u_{m'}(\cdot,t) \rightharpoonup u(\cdot,t) \quad (m' \to \infty) \quad \text{strongly in} \quad L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H). \tag{1.22}$$

It follows from (1.21) and (1.22) (by using the routine argument similar to [12, 16, 17]) that

$$B(u_{m'},u_{m'}) \rightharpoonup B(u,u) \quad (m' \to \infty) \quad *\text{-weakly in } L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H^{-1}). \tag{1.23}$$

Now, with regard to relations (1.17), (1.21), and (1.23), we can pass to the limit as $m' \to \infty$ in equation (1.5) in the space of distributions $D'(\mathbb{R}_+; H^{-1})$ (see [16]), which shows that the function $u(x,t)$ is a weak distribution solution of equation (1.1) in the space $D'(\mathbb{R}_+; H^{-1})$, and it follows from (1.16) that $u(x,t)$ satisfies the initial condition (1.3). Finally, we see from inequality (1.14) that the limit function $u(x,t)$ satisfies the estimate

$$\operatorname{ess\,sup}\left\{\|u(t+s)\|^2 \mid s \geqslant 0\right\} \leqslant \|u(0)\|^2 e^{-rt} + r^{-1}\|g\|^2 \quad \forall t \in \mathbb{R}_+. \tag{1.24}$$

Proposition 1.1. *For every* $u_0(x) \in H^1$, *problem* (1.1), (1.3) *has a weak distribution solution* $u(x,t) \in L_\infty(\mathbb{R}_+; H^1)$ *satisfying* (1.24).

Remark 1.1. Any weak solution $u(x,t) \in L_\infty(\mathbb{R}_+; H^1)$ to problem (1.1), (1.3) satisfies the energy identity

$$\frac{1}{2}\frac{d}{dt}\,|u(t)|^2 + r\,|u(t)|^2 = (g, u(t)) \quad \forall t \geqslant 0,$$

where the function $|u(t)|^2$ is absolutely continuous (cf. (1.7)). However, a similar identity (see (1.12)) for the enstrophy function $|\nabla u(t)|^2$, $t \geqslant 0$ (or an inequality of the form (1.13) which follows from this identity), fails since the weak distribution solution is insufficiently smooth.

Remark 1.2. The uniqueness problem for a solution to (1.1),(1.3) in the weak class $L_\infty(\mathbb{R}_+; H^1)$ remains open. We face here a similar situation for the classical conservative 2D Euler system (1.1) with $r = 0$ for which existence and uniqueness theorems are proved in the class of functions $u(x,t)$ having the vortex $\nabla \times u := \partial_{x_1} u^2 - \partial_{x_2} u^1 \in L_\infty(\mathbb{R}_+; L_\infty(\mathbb{T}^2))$ if $\nabla \times u_0 \in L_\infty(\mathbb{T}^2)$ (see [18, 19]). These results can be extended to equations (1.1) with $r > 0$. However, in the next section, we show that no uniqueness theorem is needed in the study of trajectory attractors for the 2D Euler system with dissipation (1.1).

2. TRAJECTORY ATTRACTOR OF THE EULER EQUATIONS WITH DISSIPATION

Introduce the spaces $\mathcal{F}_+^\infty$ and $\mathcal{F}_+^{\mathrm{loc}}$,

$$\begin{aligned} \mathcal{F}_+^\infty &= \left\{ v(x,t),\ x \in \mathbb{T}^2, t \in \mathbb{R}_+ \mid v \in L_\infty(\mathbb{R}_+; H^1),\ \partial_t v \in L_\infty(\mathbb{R}_+; H^{-1}) \right\}, \\ \mathcal{F}_+^{\mathrm{loc}} &= \left\{ v(x,t),\ x \in \mathbb{T}^2, t \in \mathbb{R}_+ \mid v \in L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H^1),\ \partial_t v \in L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H^{-1}) \right\}. \end{aligned} \tag{2.1}$$

Recall that $z(t) \in L_\infty^{\mathrm{loc}}(\mathbb{R}_+; E)$ if and only if $z(t) \in L_\infty(0, M; E)$ for every $M > 0$. It is clear that $\mathcal{F}_+^\infty \subset \mathcal{F}_+{}^{\mathrm{loc}}$. The space $\mathcal{F}_+^{\mathrm{loc}}$ is equipped with the topology Θ_+^{loc} generated by the following weak convergence of sequences $\{v_n(x,t)\} \subset \mathcal{F}_+^{\mathrm{loc}}$: by definition, $v_n(\cdot) \rightharpoonup v(\cdot)$ $(n \to \infty)$ in the topology Θ_+^{loc} if, for every $M > 0$, we have $v_n(x,t) \rightharpoonup v(x,t)$ $(n \to \infty)$ $*$-weakly in $L_\infty(0, M; H^1)$ and $\partial_t v_n(x,t) \rightharpoonup \partial_t v(x,t)$ $(n \to \infty)$ $*$-weakly in $L_\infty(0, M; H^{-1})$.

Note that $\Theta_+^{\mathrm{loc}} \cap \mathcal{F}_+^{\mathrm{loc}}$ is a Fréchet–Uryhson Hausdorff space with a countable base. Moreover, every ball $\mathcal{B}(0,R) = \left\{ v \in \mathcal{F}_+^\infty \mid \|v\|_{\mathcal{F}_+^\infty} \leqslant R \right\}$ in $\mathcal{F}_+^\infty$ is a compact set in the weak topology Θ_+^{loc}. Therefore, the topology Θ_+^{loc} is metrizable on any ball $\mathcal{B}(0,R)$ (see, e.g., [20, 11]). The corresponding metric is denoted by $\rho(\cdot\,,\cdot)$. In fact, the metric $\rho(\cdot\,,\cdot) = \rho_R(\cdot\,,\cdot)$ depends on R. However, for any $R_1 > R$, the metric $\rho_{R_1}(\cdot\,,\cdot)$, regarded as a metric on $\mathcal{B}(0,R) \subset \mathcal{B}(0,R_1)$, is equivalent to the metric $\rho_R(\cdot\,,\cdot)$. Therefore, we can omit the index R in ρ. Note again that the topology Θ_+^{loc} is not metrizable on the whole space $\mathcal{F}_+^{\mathrm{loc}}$ or $\mathcal{F}_+^\infty$.

Now let us define the *trajectory space* $\mathcal{K}_+(N)$ for equation (1.1) in dependence on $N > 0$. By definition, a function $u(x,t) \in \mathcal{F}_+^\infty$ belongs to $\mathcal{K}_+(N)$ if (i) it is a distribution solution to equation (1.1) in the space $D'(\mathbb{R}_+; H^{-1})$ and (ii) $u(x,t)$ satisfies the inequality

$$\operatorname{ess\,sup} \left\{ \|u(t+s)\|^2 \mid s \geqslant 0 \right\} \leqslant N e^{-rt} + r^{-1} \|g\|^2 \quad \forall t \in \mathbb{R}_+. \tag{2.2}$$

The space $\mathcal{K}_+(N)$ is nonempty for any $N \geqslant 0$. Indeed, if $u_0 \in H^1$ and $\|u_0\|^2 \leqslant N$, then a (possibly nonunique) Galerkin weak solution $u(x,t)$ to (1.1) and (1.3) with specified initial data u_0 (see Proposition 1.1) is a distribution solution to (1.1) in $D'(\mathbb{R}_+; H^{-1})$ and satisfies inequality (2.2) (see (1.24) and note that $\|u_0\|^2 \leqslant N$). Therefore, $u(x,t)$ belongs to $\mathcal{K}_+(N)$.

Proposition 2.1. *For any fixed $N \geqslant 0$, the space $\mathcal{K}_+(N)$ is bounded in $\mathcal{F}_+^\infty$ and closed in the topology Θ_+^{loc}.*

Proof. The boundedness of $\mathcal{K}_+(N)$ in $\mathcal{F}_+^\infty$ follows from estimate (2.2) for $t = 0$ and for the following inequality similar to (1.18):

$$\begin{aligned} \|\partial_t u\|_{L_\infty(\mathbb{R}_+; H^{-1})} &\leqslant \|B(u,u)\|_{L_\infty(\mathbb{R}_+; H^{-1})} + r\,\|u\|_{L_\infty(\mathbb{R}_+; H^{-1})} + \|g\|_{H^{-1}} \\ &\leqslant C \left(\|u\|^2_{L_\infty(\mathbb{R}_+; L_4(\mathbb{T}^2)^2)} + \|u\|_{L_\infty(\mathbb{R}_+; H^1)} + \|g\| \right) \\ &\leqslant C_1 \left(\|u\|^2_{L_\infty(\mathbb{R}_+; H^1)} + \|g\|^2 + 1 \right) && (2.3) \\ &\leqslant C_1 \left(N + r^{-1}\|g\|^2 + \|g\|^2 + 1 \right) = C_1 N + R_1. && (2.4) \end{aligned}$$

Combining (2.4) and (2.2) for $t = 0$, we see that $\mathcal{K}_+(N)$ is bounded in $\mathcal{F}_+^\infty$.

We now claim that $\mathcal{K}_+(N)$ is closed in Θ_+^{loc}. Let $\{u_k(x,t)\}$ be a sequence in $\mathcal{K}_+(N)$ and let $u_k \to w$ $(k \to \infty)$ in Θ_+^{loc} for some $w \in \mathcal{F}_+^\infty$, i.e.,

$$u_k(\cdot,t) \rightharpoonup w(\cdot,t) \qquad (k \to \infty) \quad *\text{-weakly in} \quad L_\infty(0,M;H^1), \tag{2.5}$$

$$\partial_t u_k(\cdot,t) \rightharpoonup \partial_t w(\cdot,t) \quad (k \to \infty) \quad *\text{-weakly in} \quad L_\infty(0,M;H^{-1}) \quad \forall M > 0. \tag{2.6}$$

In particular, $\{u_k\}$ is bounded in $\mathcal{F}_+^\infty$. We claim that $w \in \mathcal{K}_+(N)$. The functions $u_k(x,t)$ satisfy the equations

$$\partial_t u_k + B(u_k,u_k) + ru_k = g(x), \quad (\nabla, u_k) = 0. \tag{2.7}$$

Let us show first that w is a weak distribution solution of system (1.1). Choose an arbitrary $M > 0$. Using (2.5) and (2.6), applying the Aubin compactness theorem (see [14–16]), and passing to a subsequence of $\{u_k\}$ (for which we preserve the notation $\{u_k\}$), we may assume that

$$u_k(t) \to w(t) \quad (k \to \infty) \quad \text{strongly in} \quad L_2(0,M;H). \tag{2.8}$$

Recall that $L_2(0,M;H) \subset L_2(\mathbb{T}^2 \times [0,M])^2$, and therefore we may also assume that

$$u_k(x,t) \to w(x,t) \quad (k \to \infty) \quad \text{for a.e.} \quad (x,t) \in \mathbb{T}^2 \times [0,M]. \tag{2.9}$$

Let us now study the behavior of the term $B(u_k,u_k)$ in (2.7). Identity (1.2) yields

$$B(u_k,u_k) = P\left[\partial_{x_1}\left(u_k^1 u_k\right) + \partial_{x_2}\left(u_k^2 u_k\right)\right]. \tag{2.10}$$

It follows from (2.9) that, for $j = 1, 2$,

$$u_k^j(x,t)u_k(x,t) \to w^j(x,t)w(x,t) \quad (k \to \infty) \quad \text{for a.e.} \quad (x,t) \in \mathbb{T}^2 \times [0,M]. \tag{2.11}$$

Recall that $\{u_n\}$ is bounded in $L_\infty(0,M;H^1)$. Hence, by inequality (1.20),

$$\{u_k^j u_k\} \quad \text{is bounded in} \quad L_\infty(0,M;H) \tag{2.12}$$

and in $L_2(\mathbb{T}^2 \times [0,M])^2$ as well. Applying the known lemma on the weak convergence (see [16]), we conclude from (2.11) and (2.12) that

$$u_k^j(t)u_k(t) \rightharpoonup w^j(t)w(t) \quad (k \to \infty)$$

weakly in $L_2(\mathbb{T}^2 \times [0,M])^2$ and $*$-weakly in $L_\infty(0,M;H)$ since (2.12) holds. Therefore, due to (2.10),

$$B(u_k(t),u_k(t)) \to B(w(t),w(t)) \quad (k \to \infty) \quad *\text{-weakly in} \quad L_\infty(0,M;H^{-1}). \tag{2.13}$$

We now see that, by (2.5), (2.6), and (2.13), we can pass to the limit as $k \to \infty$ in each term of equation (2.7) in the distribution space $D'(0,M;H^{-1})$ and find that the function $w(x,t)$ satisfies the equation

$$\partial_t w + B(w,w) + rw = g(x), \qquad (\nabla, w) = 0.$$

Since the number M was arbitrary, the function $w(x,t)$ is a weak distribution solution of (1.1) in the space $D'(\mathbb{R}_+;H^{-1})$.

Second, let us prove that $w(x,t)$ satisfies inequality (2.2). Recall that any $u_k(x,t)$ belongs to $\mathcal{K}_+(N)$, and thus $u_k(x,t)$ satisfies the inequality

$$\|u_k(t+\cdot)\|^2_{L_\infty(\mathbb{R}_+;H^1)} = \operatorname{ess\,sup}\left\{\|u_k(t+s)\|^2 \mid s \geqslant 0\right\} \leqslant Ne^{-rt} + r^{-1}\|g\|^2 \quad \forall t \in \mathbb{R}_+. \tag{2.14}$$

It follows from (2.5) that, for all $t \geqslant 0$,

$$\|w(t+\cdot)\|^2_{L_\infty(\mathbb{R}_+;H^1)} \leqslant \liminf_{k\to\infty} \|u_k(t+\cdot)\|^2_{L_\infty(\mathbb{R}_+;H^1)}$$

and hence (with regard to (2.14))

$$\operatorname{ess\,sup}\left\{\|w(t+s)\|^2 \mid s \geqslant 0\right\} \leqslant Ne^{-rt} + r^{-1}\|g\|^2 \quad \forall t \in \mathbb{R}_+.$$

We have thus proved that $w \in \mathcal{K}_+(N)$, and hence $\mathcal{K}_+(N)$ as well, is closed in Θ_+^{loc}. This proves Proposition 2.1.

As was noted above, the topology $\Theta_+^{\rm loc}$ is metrizable on any bounded set of the space $\mathcal{F}_+^\infty$. Therefore, by Proposition 2.2, the space $\mathcal{K}_+(N)$ equipped with the topology $\Theta_+^{\rm loc}$ is metrizable and compact.

Consider the translation semigroup $\{T(h)\} := \{T(h), h \geqslant 0\}$ acting on the spaces $\mathcal{F}_+^\infty$ and $\mathcal{F}_+^{\rm loc}$ by the formula $T(h)v(t) = v(t+h)$. The semigroup $\{T(h)\}$ clearly acts on the trajectory space $\mathcal{K}_+(N)$ of equation (1.1). Note that

$$T(h)\mathcal{K}_+(N) \subseteq \mathcal{K}_+(N) \quad \forall h \geqslant 0. \tag{2.15}$$

Indeed, if $u(t) \in \mathcal{K}_+(N)$, then the function $T(h)u(t) = u(t+h), t \geqslant 0$, is a weak solution of (1.1) as well because (1.1) is autonomous. Moreover, since $u(t)$ satisfies inequality (2.2), we see that, for all $h \geqslant 0$,

$$\operatorname{ess\,sup}\left\{\|u(t+h+s)\|^2 \mid s \geqslant 0\right\} \leqslant Ne^{-r(t+h)} + r^{-1}\|g\|^2 \leqslant Ne^{-rt} + r^{-1}\|g\|^2, \tag{2.16}$$

and hence $T(h)u(t)$ also satisfies (2.2), i.e., $T(h)u \in \mathcal{K}_+(N)$, and the proof of (2.15) is completed.

Proposition 2.1 and Eq. (2.15) imply that the translation semigroup $\{T(h)\}$ acts on the compact metric space $\mathcal{K}_+(N)$. It can readily be seen that the semigroup $\{T(h)\}$ is continuous on $\mathcal{F}_+^{\rm loc}$ (and hence on $\mathcal{K}_+(N)$) in the topology $\Theta_+^{\rm loc}$. These facts imply that the semigroup $\{T(h)\}|_{\mathcal{K}_+(N)}$ has a global attractor $\mathfrak{A}(N) \subseteq \mathcal{K}_+(N)$, which is referred to as the *trajectory attractor* of equation (1.1) (see [9, 11]). Recall that

$$\mathfrak{A}(N) = \bigcap_{\theta \geqslant 0}\left[\bigcup_{h \geqslant \theta} T(h)\mathcal{K}_+(N)\right]_{\Theta_+^{\rm loc}}. \tag{2.17}$$

The set $\mathfrak{A}(N)$ is strictly invariant with respect to $\{T(h)\}:\ T(h)\mathfrak{A}(N) = \mathfrak{A}(N)$ for all $h \geqslant 0$, and, for any trajectory set $B \subseteq \mathcal{K}_+(N)$, the Hausdorff deviation satisfies the relation

$$\operatorname{dist}_\rho(T(h)B, \mathfrak{A}(N)) \to 0 \quad (h \to +\infty) \tag{2.18}$$

(see, e.g., [21, 13, 11]). Recall that the Hausdorff deviation of a set X from a set Y in a Banach space E is the quantity

$$\operatorname{dist}_E(X,Y) := \sup_{x \in X} \operatorname{dist}_E(x,Y) = \sup_{x \in X}\inf_{y \in Y}\|x-y\|_E. \tag{2.19}$$

Proposition 2.2. *The trajectory attractor $\mathfrak{A}(N)$ does not depend on N, $\mathfrak{A}(N) = \mathfrak{A}$ for all $N \geqslant 0$, and*

$$\operatorname{dist}_\rho(T(h)B, \mathfrak{A}) \to 0 \quad (h \to +\infty) \quad \forall B \subseteq \mathcal{K}_+(N). \tag{2.20}$$

Moreover, $\mathfrak{A} \subseteq \mathcal{K}_+(0)$, i.e.,

$$\|u(\cdot)\|^2_{L_\infty(\mathbb{R}_+;H^1)} = \operatorname{ess\,sup}\left\{\|u(s)\|^2 \mid s \geqslant 0\right\} \leqslant r^{-1}\|g\|^2 \quad \forall u \in \mathfrak{A}. \tag{2.21}$$

Proof. It follows from the definition of $\mathcal{K}_+(N)$ that $\mathcal{K}_+(N) \subseteq \mathcal{K}_+(N_1)$ for any $N_1 \geqslant N$. Therefore, formula (2.17) implies that $\mathfrak{A}(N) \subseteq \mathfrak{A}(N_1)$ for $N_1 \geqslant N$. At the same time, we can see from inequality (2.2) that $T(h)\mathcal{K}_+(N_1) \subseteq \mathcal{K}_+(N)$ for any $h \geqslant r^{-1}\ln(N_1/N)$. Using (2.17) once again yields $\mathfrak{A}(N_1) \subseteq \mathfrak{A}(N)$ for any $N_1 \geqslant N$. We conclude that $\mathfrak{A}(N_1) = \mathfrak{A}(N)$ for $N_1 \geqslant N$. In particular, $\mathfrak{A}(N) = \mathfrak{A}(0) \subseteq \mathcal{K}_+(0)$, and (2.21) is also established.

Using (2.21) and (2.3), we obtain the following assertion.

Corollary 2.1. *For any $u \in \mathfrak{A}$, the following inequalities hold*:

$$\|u(\cdot)\|^2_{L_\infty(\mathbb{R}_+;H^1)} \leqslant r^{-1}\|g\|^2, \tag{2.22}$$

$$\|\partial_t u(\cdot)\|_{L_\infty(\mathbb{R}_+;H^{-1})} \leqslant C_1\left((1+r^{-1})\|g\|^2 + 1\right). \tag{2.23}$$

Now let us define the space $\mathcal{F}^\infty$ similarly to $\mathcal{F}_+^\infty$ (replacing $\mathbb{R}_+$ by $\mathbb{R}$ in (2.1)), and let Π_+ be the operator of restriction to the semiaxis $\mathbb{R}_+$.

The *kernel* $\mathcal{K}$ of system (1.1) consists of all functions $\{u(x,t),\ t \in \mathbb{R}\} \in \mathcal{F}^\infty$ which are bounded (in H^1) complete weak distribution solutions of (1.1) on the entire time axis $\mathbb{R}$ which satisfy the inequality

$$\|u(\cdot)\|^2_{L_\infty(\mathbb{R};H^1)} = \operatorname{ess\,sup}\left\{\|u(\cdot,s)\|^2 \mid s \in \mathbb{R}\right\} \leqslant r^{-1}\|g\|^2 \quad \forall u \in \mathcal{K}. \tag{2.24}$$

Proposition 2.3. *The attractor* $\mathfrak{A}$ *coincides with the set* $\Pi_+\mathcal{K}$,

$$\mathfrak{A} = \Pi_+\mathcal{K} \tag{2.25}$$

where $\mathcal{K}$ *is the kernel of* (1.1) *in the space* $\mathcal{F}^\infty$.

Proof. Let $u(\cdot) \in \mathcal{K}$. We claim that $\Pi_+u \in \mathfrak{A}$. Consider the set $B_u = \{\Pi_+u(h+\cdot) \mid h \in \mathbb{R}\}$. Each function $\Pi_+u(h+t), t \geqslant 0$, is clearly a weak solution of (1.1), and it follows from (2.24) that $B_u \in \mathcal{K}_+(0)$. Moreover, the set B is strictly invariant with respect to the semigroup $\{T(h)\}$: $T(h)B_u = B_u$ for all $h \geqslant 0$. Therefore, $B_u \subseteq \mathfrak{A}(0) = \mathfrak{A}$, and thus $\Pi_+\mathcal{K} \subseteq \mathfrak{A}$. Let us prove the converse inclusion. We must prove that any solution $u(t), t \geqslant 0$, belonging to $\mathfrak{A}$ can be extended to the negative semiaxis as a weak solution satisfying (2.24). Indeed, since $\mathfrak{A}$ is strictly invariant, there is a function $u_1(t), t \geqslant 0$, such that $u_1 \in \mathfrak{A} \subseteq \mathcal{K}_+(0)$ and $T(1)u_1(t) = u(t)$, i.e., $u_1(t+1) = u(t)$ for any $t \geqslant 0$. We now set $\tilde{u}(t) = u_1(t+1)$ for $t \geqslant -1$. Then $\tilde{u}(t)$ is a solution of (1.1), $\tilde{u}(t) = u(t)$ for $t \geqslant -1$, and

$$\operatorname{ess\,sup}\left\{\|\tilde{u}(t)\|^2 \mid t \geqslant -1\right\} \leqslant r^{-1}\|g\|^2.$$

Continuing this procedure for u_1 instead of u, we extend $u(t)$ as a solution to the sets $t \geqslant -2$, $t \geqslant -3$, etc. As a result, we obtain a complete weak solution $\tilde{u}(t), t \in \mathbb{R}$, of (1.1) satisfying the inequality

$$\operatorname{ess\,sup}\left\{\|\tilde{u}(t)\|^2 \mid t \in \mathbb{R}\right\} \leqslant r^{-1}\|g\|^2,$$

i.e., $\tilde{u} \in \mathcal{K}$, and $\Pi_+\tilde{u} = u$. Hence, $\mathfrak{A} \subseteq \Pi_+\mathcal{K}$, and identity (2.25) is proved.

Remark 2.1. The following embedding is continuous:

$$\Theta_+^{\mathrm{loc}} \cap \mathcal{F}_+^{\mathrm{loc}} \subseteq C^{\mathrm{loc}}(\mathbb{R}_+; H^\delta) \quad \forall \delta \in [0,1[$$

(see [16, 11]). The trajectory attractor $\mathfrak{A} = \Pi_+\mathcal{K}$ satisfies (2.20) and hence

$$\operatorname{dist}_{C([0,M];H^\delta)}(T(h)B, \mathfrak{A}) \to 0 \quad (h \to +\infty),$$

for any set $B \in \mathcal{K}_+(N)$, where M is an arbitrary positive number (see the definition of $\operatorname{dist}_E(\cdot\,,\cdot)$ in (2.19)).

Remark 2.2. In the case of $g = 0$, it follows from (2.25) and (2.24) that the trajectory attractor of (1.1) is trivial, $\mathfrak{A} = \{0\}$. Moreover, by (2.2), every weak solution to (1.1) tends to the zero solution exponentially.

3. DISSIPATIVE 2D NAVIER–STOKES SYSTEM AND LIMITS OF ITS SOLUTIONS AS VISCOSITY VANISHES

Consider the following 2D Navier–Stokes system with the dissipation term $-ru$ and with viscosity $\nu > 0$:

$$\partial_t u + B(u,u) - \nu\Delta u + ru = g(x), \quad (\nabla, u) = 0, \quad x = (x_1, x_2) \in \mathbb{T}^2,\ t \geqslant 0. \tag{3.1}$$

Here the notation is the same as that for the Euler system with dissipation (1.1) treated in Section 1. The pressure $p(x,t)$ is eliminated from the system by applying the Leray operator P.

System (3.1) also has a geophysical interpretation (see [1]). The essential dissipation occurs in the planetary boundary layer described by the term $-ru$, whereas the viscosity term $\nu\Delta u$ is responsible for small-scale dissipation (note that $0 < \nu \ll r$ in physically relevant cases).

Remark 3.1. Studying the classical Navier–Stokes system with viscosity $\nu > 0$ (at $r = 0$) and with periodic boundary conditions, one usually assumes that the functions u and g have zero means over the torus $\mathbb{T}^2$, in order to avoid any linear growth of the solutions. For $r > 0$, this assumption can be dropped since the term $-ru$ introduces an additional dissipation.

As is well known, the Cauchy problem (3.1), (1.3) is uniquely solvable. Moreover, for $u_0 \in H^1$, the corresponding solution $u(x,t)$ belongs to the class $C_b(\mathbb{R}_+; H^1) \cap L_2^b(\mathbb{R}_+; H^2)$, and we have $\partial_t u \in L_2^b(\mathbb{R}_+; H)$. Thus, the solution is strong (see [5–7]; the case $r = 0$ is considered, e.g., in [17, 13, 21, 11, 22, 9]). Note that the dissipative term $-ru$ does not influence in this result. Moreover, any solution $u(x,t)$ to problem (3.1), (1.3) satisfies the following identities:

$$\frac{1}{2}\frac{d}{dt}\left|u(t)\right|^2 + \nu\left|\nabla u(t)\right|^2 + r\left|u(t)\right|^2 = (g, u(t)), \tag{3.2}$$

$$\frac{1}{2}\frac{d}{dt}\left|\nabla u(t)\right|^2 + \nu\left|\Delta u(t)\right|^2 + r\left|\nabla u(t)\right|^2 = (\nabla g, \nabla u(t)), \tag{3.3}$$

which are analogous to the identities known for the standard 2D Navier–Stokes system with $r = 0$ (see, e.g., [13, 21, 11, 22]). Identities (1.8) and (1.11) are used in the proof of (3.2) and (3.3).

Identities (3.2) and (3.3) imply the following inequality:

$$\frac{d}{dt}\left\|u(t)\right\|^2 + r\left\|u(t)\right\|^2 + 2\nu\left|\nabla u(t)\right|^2 + 2\nu\left|\Delta u(t)\right|^2 \leqslant r^{-1}\|g\|^2. \tag{3.4}$$

Omitting the terms containing the coefficient ν, here we obtain the inequality

$$\left\|u(t)\right\|^2 \leqslant \left\|u_0\right\|^2 e^{-rt} + r^{-1}\|g\|^2, \tag{3.5}$$

similar to inequality (1.14). Recall that $\left\|v\right\|^2 = \left|\nabla v\right|^2 + \left|v\right|^2 = \left\|v\right\|_{H^1}^2$.

Integrating (3.4) over $[t, t+1]$ and using (3.5), we obtain

$$2\nu\int_t^{t+1}\left|\Delta u(s)\right|^2 ds \leqslant \left\|u_0\right\|^2 e^{-rt} + 2r^{-1}\|g\|^2. \tag{3.6}$$

It follows from (3.1) that

$$\begin{aligned}\left\|\partial_t u(t)\right\|_{H^{-1}} &\leqslant \left\|B(u(t), u(t))\right\|_{H^{-1}} + r\left\|u(t)\right\|_{H^{-1}} + \nu\left\|u(t)\right\|_{H^1} + \|g\|_{H^{-1}} \\ &\leqslant C\left(\left\|u(t)\right\|_{L_4(\mathbb{T}^2)^2}^2 + \left\|u(t)\right\|_{H^1} + \|g\|\right) \leqslant C_1\left(\left\|u(t)\right\|_{H^1}^2 + \|g\|^2 + 1\right),\end{aligned} \tag{3.7}$$

where the constant C_1 depends on r and does not depend on ν, $0 < \nu \leqslant 1$. The proof of (3.7) is similar to that of (1.18) and (2.4) and uses inequalities (1.19) and (1.20). Now, by (3.5), we can conclude that

$$\left\|\partial_t u(t)\right\|_{H^{-1}} \leqslant C_1\left(\left\|u_0\right\|^2 e^{-rt} + (1 + r^{-1})\|g\|^2 + 1\right). \tag{3.8}$$

Remark 3.2. Note that the constants on the right-hand sides of inequalities (3.5), (3.6), and (3.8) do not depend on ν, $0 < \nu \leqslant 1$. These estimates are similar to the inequalities established in Sections 1 and 2 for the weak solutions of the 2D Euler equation with dissipation.

Let us now study the behavior of solutions of system (3.1) as $\nu \to 0+$.

Let $\{u_\nu(x,t),\ 0 < \nu \leqslant 1\}$ be a family of solutions to the dissipative 2D Navier–Stokes system (3.1) such that

$$\left\|u_\nu(\cdot)\right\|_{L_\infty(\mathbb{R}_+; H^1)} \leqslant M, \qquad \left\|\partial_t u_\nu(\cdot)\right\|_{L_\infty(\mathbb{R}_+; H^{-1})} \leqslant M. \tag{3.9}$$

Using (3.5) and (3.8), we see that property (3.9) holds if the initial data u_0 for system (3.1) satisfy the inequality

$$\left\|u_\nu(0)\right\| \leqslant m \tag{3.10}$$

and $M = M(m)$ is sufficiently large.

Theorem 3.1. *Consider a sequence of solutions* $\{u_{\nu_k}(x,t),\ 0<\nu_k\leqslant 1\}$ *to* (3.1) *that satisfy* (3.9). *Assume that* $\nu_k\to 0+$ *as* $k\to\infty$. *Then there is a subsequence* $\{\nu_{k'}\}\subset\{\nu_k\}$ *such that*

$$u_{\nu_{k'}}(\cdot)\to w(\cdot)\quad as\quad k'\to\infty\quad in\quad \Theta_+^{\mathrm{loc}}, \tag{3.11}$$

where $w(x,t)$ *is a weak solution to the* 2D *Euler equation with dissipation* (1.1) *and* $w\in\mathcal{K}_+(M)$.

Proof. The function $u_{\nu_k}(x,t)$ satisfies the equation

$$\partial_t u_{\nu_k}+B(u_{\nu_k},u_{\nu_k})-\nu\Delta u_{\nu_k}+ru_{\nu_k}=g(x). \tag{3.12}$$

The sequence $\{u_{\nu_k}(x,t)\}$ is weakly compact in Θ_+^{loc} since it satisfies (3.9), and therefore contains a convergent subsequence $\{u_{\nu_{k'}}(x,t)\}$,

$$u_{\nu_{k'}}(\cdot)\to w(\cdot)\quad \text{as}\quad k'\to\infty\quad \text{in}\quad \Theta_+^{\mathrm{loc}}$$

for some $w(\cdot)\in\mathcal{F}_+^{\infty}$, i.e.,

$$u_{\nu_{k'}}(\cdot)\rightharpoonup w(\cdot)\qquad (k'\to\infty)\quad *\text{-weakly in}\quad L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H^1), \tag{3.13}$$

$$\partial_t u_{\nu_{k'}}(\cdot,t)\rightharpoonup \partial_t w(\cdot,t)\quad (k'\to\infty)\quad *\text{-weakly in}\quad L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H^{-1}). \tag{3.14}$$

Following the lines of reasoning in the proof of Proposition 2.1 (from formula (2.5) to formula (2.13)), we obtain

$$B(u_{\nu_{k'}}(t),u_{\nu_{k'}}(t))\to B(w(t),w(t))\quad (k'\to\infty)\quad *\text{-weakly in}\quad L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H^{-1}). \tag{3.15}$$

Consider the term $\nu\Delta u_{\nu_k}$ in equation (3.12). By estimate (3.9), we have

$$\|\nu_k\Delta u_{\nu_k}(\cdot)\|_{L_\infty(\mathbb{R}_+;H^{-1})}\leqslant \nu_k C\,\|u_{\nu_k}(\cdot)\|_{L_\infty(\mathbb{R}_+;H^1)}\leqslant \nu_k CM\to 0\quad (\nu_k\to 0+), \tag{3.16}$$

or, equivalently,

$$\nu\Delta u_{\nu_k}(\cdot)\to 0\quad \text{strongly in}\quad L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H^{-1}). \tag{3.17}$$

Combining relations (3.13), (3.14), (3.15), and (3.17), we can pass to the limit in equation (3.12) as $k'\to\infty$ in the distribution space $D'(\mathbb{R}_+;H^{-1})$ and see that the function $w(x,t)$ satisfies the equation

$$\partial_t w+B(w,w)+rw=g(\cdot)$$

in the distribution sense, i.e., w is a weak solution to (1.1).

It remains to show that $w(x,t)\in\mathcal{K}_+(M)$. Since $u_\nu(\cdot)\in C^b(\mathbb{R}_+;H^1)$, it follows from (3.9) that

$$\|u_{\nu_k}(0)\|\leqslant M.$$

Each function $u_{\nu_k}(x,t)$ satisfies inequality (3.5),

$$\operatorname{ess\,sup}\left\{\|u_{\nu_k}(t+s)\|^2\mid s\geqslant 0\right\}\leqslant \|u_{\nu_k}(0)\|^2 e^{-rt}+r^{-1}\|g\|^2\leqslant Me^{-rt}+r^{-1}\|g\|^2$$

(see also (3.9)). Since

$$\|w(t+\cdot)\|^2_{L_\infty(\mathbb{R}_+;H^1)}\leqslant \liminf_{k\to\infty}\|u_{\nu_k}(t+\cdot)\|^2_{L_\infty(\mathbb{R}_+;H^1)}$$

(see (3.13)), we have

$$\operatorname{ess\,sup}\left\{\|w(t+s)\|^2\mid s\geqslant 0\right\}\leqslant Me^{-rt}+r^{-1}\|g\|^2\quad \forall t\in\mathbb{R}_+,$$

and therefore $w\in\mathcal{K}_+(M)$.

4. CONVERGENCE OF THE TRAJECTORY ATTRACTORS OF THE DISSIPATIVE 2D NAVIER–STOKES SYSTEM AS $\nu \to 0+$

In this section, we study the relationship between the trajectory attractor $\mathfrak{A}_\nu$ of the dissipative 2D Navier–Stokes system (3.1) as $\nu \to 0+$ and the trajectory attractor $\mathfrak{A}$ of the 2D Euler equations (1.1) with dissipation.

Similarly to Section 2, we consider the trajectory space

$$\mathcal{K}_+^\nu \subset C_b(\mathbb{R}_+; H^1) \cap L_2^b(\mathbb{R}_+; H^2) \subset \mathcal{F}_+^\infty \tag{4.1}$$

of system (3.1) with a chosen viscosity coefficient $\nu > 0$. By definition, $\mathcal{K}_+^\nu$ consists of all solutions $u(x,t)$, $t \geqslant 0$, to this system with initial data $u_0 = u(0) \in H^1$. The spaces $\mathcal{F}_+^\infty, \mathcal{F}_+^{\mathrm{loc}}$, and Θ_+^{loc} were defined in Section 2. The trajectory space $\mathcal{K}_+^\nu$ is closed in Θ_+^{loc}.

The translation semigroup $\{T(h)\}$ acts on the trajectory space $\mathcal{K}_+^\nu$ by the usual formula $T(h)u(t) = u(t+h)$. It is clear that $T(h)\mathcal{K}_+^\nu \subseteq \mathcal{K}_+^\nu$ for any $h \geqslant 0$. We claim that the translation semigroup $\{T(h)\}$ acting on $\mathcal{K}_+^\nu$ has a trajectory attractor, $\mathfrak{A}_\nu \subset \mathcal{K}_+^\nu$, which attracts bounded (in $\mathcal{F}_+^\infty$) families of solutions to system (3.1) in the topology Θ_+^{loc}. (See a similar proof in [11, 22] in the case of $r = 0$ and Section 2 for $\nu = 0$.)

Recall that the set $\mathfrak{A}_\nu$ is strictly invariant with respect to $\{T(h)\}$,

$$T(h)\mathfrak{A}_\nu = \mathfrak{A}_\nu \quad \forall h \geqslant 0, \tag{4.2}$$

and, for any bounded (in $\mathcal{F}_+^\infty$) set $B_\nu \subseteq \mathcal{K}_+^\nu$, we have

$$\mathrm{dist}_\rho(T(h)B_\nu, \mathfrak{A}_\nu) \to 0 \quad (h \to +\infty), \tag{4.3}$$

where ρ is a metric generating the topology Θ_+^{loc} on a ball in $\mathcal{F}_+^\infty$ containing B_ν (see Section 2).

Using inequalities (3.5), (3.6), and (3.8), we obtain the following assertion.

Proposition 4.1. *The trajectory attractors $\mathfrak{A}_\nu$ are uniformly bounded for $0 < \nu \leqslant 1$ in the space $\mathcal{F}_+^\infty$ and, for any $u_\nu \in \mathfrak{A}_\nu$, we have*

$$\|u_\nu(\cdot)\|_{L_\infty(\mathbb{R}_+; H^1)} \leqslant r^{-1}\|g\|^2, \tag{4.4}$$

$$\|\partial_t u_\nu(\cdot)\|_{L_\infty(\mathbb{R}_+; H^{-1})} \leqslant C_1\left((1 + r^{-1})\|g\|^2 + 1\right), \tag{4.5}$$

$$\nu \int_t^{t+1} |\Delta u_\nu(s)|^2\, ds \leqslant r^{-1}\|g\|^2.$$

Comparing Corollary 2.1 and Proposition 4.1, we see that the trajectory attractor $\mathfrak{A}$ of the 2D Euler equations (1.1) with dissipation and the trajectory attractors $\mathfrak{A}_\nu$ of the dissipative 2D Navier–Stokes system (3.1) belong to the same ball $\mathcal{B}(0, R_0)$ in $\mathcal{F}_+^\infty$ of the radius R_0, where $R_0^2 = \max\left\{r^{-1}\|g\|^2, C_1\left((1 + r^{-1})\|g\|^2 + 1\right)\right\}$,

$$\mathfrak{A} \subset \mathcal{B}(0, R_0) \quad \text{and} \quad \mathfrak{A}_\nu \subset \mathcal{B}(0, R_0) \quad \forall \nu \in]0, 1]. \tag{4.6}$$

Now let us study the Hausdorff deviation of $\mathfrak{A}_\nu$ from $\mathfrak{A}$ as ν tends to zero in the topology Θ_+^{loc} generated by the metric ρ described in Section 2.

The main result of this section is the following theorem.

Theorem 4.1. *The trajectory attractors $\mathfrak{A}_\nu$ of system* (3.1) *converge in metric ρ as $\nu \to 0+$ to the trajectory attractor $\mathfrak{A}$ of system* (1.1),

$$\mathrm{dist}_\rho(\mathfrak{A}_\nu, \mathfrak{A}) \to 0+ \quad \textit{as} \quad \nu \to 0+. \tag{4.7}$$

Let B_ν be bounded (in $\mathcal{F}_+^\infty$) sets of trajectories of system (3.1),

$$\|B_\nu\|_{\mathcal{F}_+^\infty} \leqslant M \quad (0 < \nu \leqslant 1). \tag{4.8}$$

In this case,

$$\mathrm{dist}_\rho(T(h)B_\nu, \mathfrak{A}) \to 0 \quad \textit{as} \quad \nu \to 0+ \quad \textit{and} \quad h \to +\infty. \tag{4.9}$$

Proof. It suffices to prove (4.9), which implies (4.7) if one takes $B_\nu = \mathfrak{A}_\nu$ (by property (4.2)). Assume that relation (4.9) fails to hold. Then there is a neighborhood $\mathcal{O}(\mathfrak{A})$ in Θ_+^{loc} and two sequences ν_n and h_n, $\nu_n \to 0+$ and $h_n \to +\infty$ as $n \to \infty$, such that

$$T(h_n)B_{\nu_n} \not\subset \mathcal{O}(\mathfrak{A}). \tag{4.10}$$

Thus, there are solutions $w_{\nu_n} \in B_{\nu_n}$ such that the functions

$$W_{\nu_n}(t) := T(h_n)w_{\nu_n}(t) = w_{\nu_n}(h_n + t)$$

do not belong to $\mathcal{O}(\mathfrak{A})$,

$$W_{\nu_n}(t) \notin \mathcal{O}(\mathfrak{A}) \quad \forall n \in \mathbb{N}. \tag{4.11}$$

Note that $W_{\nu_n}(t)$ is a solution to equation (3.1) on the semiaxis $[-h_n, +\infty)$ with $\nu = \nu_n$ since $W_{\nu_n}(t)$ is a backward shift of $w_{\nu_n}(t)$ at time h_n. Recall that equation (3.1) is autonomous. Moreover, it follows from (4.8) that

$$\operatorname*{ess\,sup}_{s\geqslant -h_n}\|W_{\nu_n}(s)\|_{H^1} + \operatorname*{ess\,sup}_{s\geqslant -h_n}\|\partial_t W_{\nu_n}(s)\|_{H^{-1}} \leqslant M, \tag{4.12}$$

and inequality (3.5) yields

$$\operatorname*{ess\,sup}_{s\geqslant -h_n+\tau}\|W_{\nu_n}(s)\|_{H^1} = \operatorname*{ess\,sup}_{s\geqslant \tau}\|w_{\nu_n}(s)\|_{H^1} \leqslant \|w_{\nu_n}(0)\|e^{-r\tau} + r^{-1}\|g\|^2 \leqslant Me^{-r\tau} + r^{-1}\|g\|^2 \quad \forall \tau \geqslant 0. \tag{4.13}$$

Inequality (4.12) implies that the sequence $\{W_{\nu_n}(\cdot)\}$ is $*$-weakly compact in the space

$$\Theta_{-M,M} = L_{\infty,*w}(-M,M;H^1) \cap \{v \mid \partial_t v \in L_{\infty,*w}(-M,M;H^{-1})\}$$

for every M if we consider elements ν_n whose indices n satisfy the inequality $h_n \geqslant M$. Therefore, for any chosen $M > 0$, we can find a subsequence $\{\nu_{n'}\} \subset \{\nu_n\}$ such that $\{W_{\nu_{n'}}(\cdot)\}$ converges weakly in $\Theta_{-M,M}$. Therefore, using the standard Cantor diagonal procedure, we can construct a function $W(t), t \in \mathbb{R}$, and a subsequence $\{\nu_{n''}\} \subset \{\nu_n\}$ such that

$$W_{\nu_{n''}} \to W \quad *\text{-weakly in} \quad \Theta_{-M,M} \quad \text{as} \quad n'' \to \infty \quad \text{for any} \quad M > 0. \tag{4.14}$$

For the limit function $W(t), t \in \mathbb{R}$, it follows from (4.12) and (4.13) that

$$\|W(\cdot)\|_{L_\infty(\mathbb{R};H^1)} + \|\partial_t W(\cdot)\|_{L_\infty(\mathbb{R};H^{-1})} \leqslant M, \tag{4.15}$$

$$\|W(\cdot)\|_{L_\infty(\mathbb{R};H^1)} \leqslant Me^{-r\tau} + r^{-1}\|g\|^2 \quad \forall \tau \geqslant 0. \tag{4.16}$$

Passing to the limit in (4.16) as $\tau \to \infty$, we obtain

$$\|W(\cdot)\|_{L_\infty(\mathbb{R};H^1)} \leqslant r^{-1}\|g\|^2. \tag{4.17}$$

Let us now apply Theorem 3.1, where we can assume that all the functions are defined on the semiaxis $[-M, +\infty)$ instead of $[0, +\infty)$ (because the equations are autonomous). In this case, it follows from (4.14) that $W(x,t)$ is a weak solution of the 2D Euler system with dissipation (1.1) for any $t \in \mathbb{R}$ and, due to (4.17), the solution $W(x,t)$ satisfies estimate (2.24), i.e., $W \in \mathcal{K}$, where $\mathcal{K}$ is the kernel of equation (1.1). However, $\Pi_+\mathcal{K} = \mathfrak{A}$ (see Proposition 2.3). Thus, $\Pi_+W \in \mathfrak{A}$. At the same time, we have proved that

$$W_{\nu_{n''}} \to \Pi_+ W \quad \text{in} \quad \Theta_+^{\mathrm{loc}} \quad \text{as} \quad n \to \infty \tag{4.18}$$

(see (4.14)). In particular, for a large n'',

$$W_{\nu_{n''}} \in \mathcal{O}(\Pi_+ W) \subseteq \mathcal{O}(\mathfrak{A}). \tag{4.19}$$

This contradicts (4.11). Therefore, (4.9) is true. Finally, to prove (4.7), we apply (4.9) for $B_\nu = \mathfrak{A}_\nu$.

Remark 4.1. Recall that $\Theta_+^{\mathrm{loc}} \cap \mathcal{F}_+^{\mathrm{loc}} \subset C^{\mathrm{loc}}(\mathbb{R}_+; H^\delta), 0 \leqslant \delta < 1$, and the convergences (4.7) and (4.9) also hold in the uniform metric $C([0,M];H^\delta)$ for any $M > 0$ (see Remark 2.1),

$$\operatorname{dist}_{C([0,M];H^\delta)}(\mathfrak{A}_\nu, \mathfrak{A}) \to 0 \quad (\nu \to 0+), \tag{4.20}$$

$$\operatorname{dist}_{C([0,M];H^\delta)}(T(h)B_\nu, \mathfrak{A}) \to 0 \quad (\nu \to 0+,\ h \to +\infty). \tag{4.21}$$

In conclusion, we formulate two propositions which follow from the well-posedness of the Cauchy problem for the dissipative 2D Navier–Stokes system (see, e.g., [13, 21]). We use these assertions in the next section.

Proposition 4.2. *For any* $\nu > 0$, *the trajectory attractor* $\mathfrak{A}_\nu$ *of equation* (3.1) *is connected in the topological space* $\Theta_+^{\mathrm{loc}} \cap \mathcal{F}_+^{\mathrm{loc}}$.

Proposition 4.3. *The family of sets* $\{\mathfrak{A}_\nu, 0 < \nu \leqslant 1\}$ *is upper semicontinuous in* Θ_+^{loc}, *i.e., for every* ν, $0 < \nu \leqslant 1$, *and for any neighborhood* $\mathcal{O}(\mathfrak{A}_\nu)$, *there is a* $\delta = \delta(\alpha, \mathcal{O}) > 0$ *such that*

$$\mathfrak{A}_{\nu'} \subseteq \mathcal{O}(\mathfrak{A}_\nu) \quad \forall \nu' > 0, \quad |\nu' - \nu| < \delta. \tag{4.22}$$

5. MINIMAL LIMIT OF THE TRAJECTORY ATTRACTORS A_ν AS $\nu \to 0+$

Let $\mathfrak{A}_\nu$ be the trajectory attractor of the dissipative 2D Navier–Stokes system (3.1) for some $\nu > 0$, and let $\mathfrak{A}$ be the trajectory attractor of the Euler system (1.1) with dissipation. As was proved above, $\mathfrak{A}, \mathfrak{A}_\nu \subset \mathcal{B}_0 \quad \forall \nu \in \,]0,1]$, where $\mathcal{B}_0 = \mathcal{B}(0, R_0)$ is the ball in $\mathcal{F}_+^{\mathrm{b}}$ (see (4.6)) whose radius R_0 does not depend on ν,

$$\|\mathfrak{A}\|_{\mathcal{F}_+^{\mathrm{b}}} \leqslant R_0 \quad \text{and} \quad \|\mathfrak{A}_\nu\|_{\mathcal{F}_+^{\mathrm{b}}} \leqslant R_0 \quad \forall \nu, \quad 0 < \nu \leqslant 1. \tag{5.1}$$

Recall that the ball $\mathcal{B}_0$ is compact in the topology Θ_+^{loc}, and the Uryson compactness theorem implies that the subspace $\mathcal{B}_0 \cap \Theta_+^{\mathrm{loc}}$ equipped with the topology Θ_+^{loc} is metrizable (see [11] for more details). Denote the corresponding metric in $\mathcal{B}_0 \cap \Theta_+^{\mathrm{loc}}$ by $\rho(\cdot\,,\cdot)$ and by $\mathcal{B}_\rho$ the metric space by itself. This metric space is compact. As was proved in Theorem 4.1,

$$\operatorname{dist}_{\mathcal{B}_\rho}(\mathfrak{A}_\nu, \mathfrak{A}) \to 0 \quad \text{as} \quad \nu \to 0+, \tag{5.2}$$

where $\operatorname{dist}_{\mathcal{B}_\rho}(\cdot\,,\cdot)$ stands for the Hausdorff deviation of sets in $\mathcal{B}_\rho$ (see (2.19)). Note that, in fact, the limit relation in (5.2) is stronger than that in (4.20).

Recall that the set $\mathfrak{A} \subset \mathcal{B}_\rho$ is closed in $\mathcal{B}_\rho$. Let $\mathfrak{A}_{\min}$ be the minimal closed subset of $\mathfrak{A}$ which satisfies the attracting property (5.2), i.e.,

$$\lim_{\nu \to 0+} \operatorname{dist}_{\mathcal{B}_\rho}(\mathfrak{A}_\nu, \mathfrak{A}_{\min}) = 0,$$

and $\mathfrak{A}_{\min}$ belongs to every closed subset $\mathfrak{A}' \subseteq \mathfrak{A}$ for which

$$\lim_{\nu \to 0+} \operatorname{dist}_{\mathcal{B}_\rho}(\mathfrak{A}_\nu, \mathfrak{A}') = 0.$$

We refer to the set $\mathfrak{A}_{\min}$ as the *minimal limit of the trajectory attractors* $\mathfrak{A}_\nu$ *as* $\nu \to 0+$.

We claim that the set $\mathfrak{A}_{\min}$ exists and is unique. We have just to prove that

$$\mathfrak{A}_{\min} = \bigcap_{0<\delta\leqslant 1} \Big[\bigcup_{0<\nu\leqslant\delta} \mathfrak{A}_\nu \Big]_{\mathcal{B}_\rho}. \tag{5.3}$$

The set on the right-hand side of (5.3) is clearly nonempty. It is easy to prove that a point w belongs to the right-hand side of (5.3) if and only if there are $w_{\nu_n} \in \mathfrak{A}_{\nu_n}$, where $n = 1, 2, \ldots$ and $\nu_n \to 0+$ as $n \to \infty$, such that $\rho(w_{\nu_n}, w) \to 0$ as $n \to \infty$. By (5.2), such a limit point w always belongs to $\mathfrak{A}$ and, moreover, it belongs to *every* closed attracting set $\mathfrak{A}'$. The set (5.3) is attracting for $\mathfrak{A}_\nu$ as $\nu \to 0+$. Indeed, assuming the converse, we see that there is a sequence $w_{\nu_n} \in \mathfrak{A}_{\nu_n}$ such that $\nu_n \to 0+$ and

$$\operatorname{dist}_{\mathcal{B}_\rho}(w_{\nu_n}, \mathfrak{A}_{\min}) \geqslant \varepsilon \tag{5.4}$$

for some value $\varepsilon > 0$. Recall that $w_{\nu_n} \in \mathcal{B}_\rho$ and $\mathcal{B}_\rho$ is a compact metric space. Then, passing to a subsequence $\{w_{\nu_{n'}}\} \subset \{w_{\nu_n}\}$, we may assume that $\rho(w_{\nu_{n'}}, w') \to 0$ as $\nu_{n'} \to 0$ for some $w' \in \mathcal{B}_\rho$. Thus, by the above property, $w' \in \mathfrak{A}_{\min}$, which contradicts (5.4). We have proved that the set $\mathfrak{A}_{\min}$ defined in (5.3) is a minimal closed attracting subset of $\mathfrak{A}$.

Proposition 5.1. *The minimal limit* $\mathfrak{A}_{\min}$ *of trajectory attractors* $\mathfrak{A}_\nu$ *as* $\nu \to 0+$ *is a connected subset of* $\mathfrak{A}$ *in* $\mathcal{B}_\rho$.

Proof. Assume the converse. In this case, the set $\mathfrak{A}_{\min}$ is the union of two closed disjoint subsets $\mathfrak{A}^1_{\min}$ and $\mathfrak{A}^2_{\min}$, i.e.,

$$\mathfrak{A}_{\min} = \mathfrak{A}^1_{\min} \cup \mathfrak{A}^2_{\min} \quad \text{and} \quad \mathfrak{A}^1_{\min} \cap \mathfrak{A}^2_{\min} = \varnothing.$$

Since the metric space $\mathcal{B}_\rho$ is compact, there are two open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ in $\mathcal{B}_\rho$ such that $\mathfrak{A}^1_{\min} \subset \mathcal{O}_1$, $\mathfrak{A}^2_{\min} \subset \mathcal{O}_2$, and $\mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing$. Clearly, $\mathfrak{A}_{\min} \subset \mathcal{O}_1 \cup \mathcal{O}_2$. Therefore, by (5.2), there is a number $\nu_0 > 0$ for which

$$\mathfrak{A}_\nu \subset \mathcal{O}_1 \cup \mathcal{O}_2 \quad \forall \nu, \quad 0 < \nu \leqslant \nu_0. \tag{5.5}$$

Note that every set $\mathfrak{A}_\nu$ is connected (see Proposition 4.2), i.e., $\mathfrak{A}_\nu \subset \mathcal{O}_1$ or $\mathfrak{A}_\nu \subset \mathcal{O}_2$ for all $\nu < \nu_0$. At the same time, since $\mathfrak{A}_{\min}$ is the *minimal* limit of $\mathfrak{A}_\nu$, we can find ν_1 and ν_2 such that

$$\mathfrak{A}_{\nu_1} \subset \mathcal{O}_1 \quad \text{and} \quad \mathfrak{A}_{\nu_2} \subset \mathcal{O}_2 \tag{5.6}$$

(otherwise, we can diminish $\mathfrak{A}_{\min}$). To be definite, assume that $0 < \nu_2 < \nu_1 < \nu_0$. Write

$$\delta^* = \sup\{\delta : \mathfrak{A}_\nu \subset \mathcal{O}_2,\ \nu_2 \leqslant \nu < \nu_2 + \delta\}. \tag{5.7}$$

Note that $\nu_2 + \delta^* \leqslant \nu_1 < \nu_0$, (see (5.6)) and $\mathfrak{A}_{\nu_2+\delta^*} \subset \mathcal{O}_1 \cup \mathcal{O}_2$ since $\nu_2 + \delta^* < \nu_0$ (see (5.5)).

Now we claim that $\mathfrak{A}_{\nu_2+\delta^*}$ cannot belong to $\mathcal{O}_2$. Indeed, if $\mathfrak{A}_{\nu_2+\delta^*} \subset \mathcal{O}_2$, then, by Proposition 4.3, there is a small $\delta_2 > 0$ such that $\mathfrak{A}_{\nu_2+\delta^*+\delta_2} \subset \mathcal{O}_2$. This contradicts the definition of δ^* in (5.7). At the same time, $\mathfrak{A}_{\nu_2+\delta^*}$ cannot belong to $\mathcal{O}_1$ either. Indeed, if $\mathfrak{A}_{\nu_2+\delta^*} \subset \mathcal{O}_1$, then, by Proposition 4.3 again, there is a small $\delta_1 > 0$ such that $\mathfrak{A}_{\nu_2+\delta^*-\delta_1} \subset \mathcal{O}_1$, which contradicts the definition of δ^*. However, all this contradicts the relation $\mathfrak{A}_{\nu_2+\delta^*} \subset \mathcal{O}_1 \cup \mathcal{O}_2$. This completes the proof.

Recall that the set $\mathfrak{A}_{\min}$ is compact. Let us prove the following assertion.

Proposition 5.2. *The minimal limit* $\mathfrak{A}_{\min}$ *of trajectory attractors* $\mathfrak{A}_\nu$ *as* $\nu \to 0+$ *is strictly invariant with respect to the translation semigroup* $\{T(h)\}$, *i.e.,*

$$T(h)\mathfrak{A}_{\min} = \mathfrak{A}_{\min} \quad \forall h \geqslant 0. \tag{5.8}$$

Proof. Consider an arbitrary $w \in \mathfrak{A}_{\min}$. By definition, there is a sequence $w_{\nu_n} \in \mathfrak{A}_{\nu_n}$ such that $\rho(w_{\nu_n}, w) \to 0$ as $\nu_n \to 0+$. The translation semigroup $\{T(h)\}$ is continuous in Θ_+^{loc}, and therefore $\rho(T(h)w_{\nu_n}, T(h)w) \to 0$ as $\nu_n \to 0+$. Since every $\mathfrak{A}_{\nu_n}$ is strictly invariant, we obtain $T(h)w_{\nu_n} \in \mathfrak{A}_{\nu_n}$. Thus, $T(h)w \in \mathfrak{A}_{\min}$ and we have proved that

$$T(h)\mathfrak{A}_{\min} \subseteq \mathfrak{A}_{\min} \quad \forall h \geqslant 0.$$

Let us now prove the inverse inclusion. For any $h \geqslant 0$ and an arbitrary $w \in \mathfrak{A}_{\min}$ with corresponding $w_{\nu_n} \in \mathfrak{A}_{\nu_n}$ such that $\rho(w_{\nu_n}, w) \to 0$ ($\nu_n \to 0+$), we must find $W \in \mathfrak{A}_{\min}$ such that $T(h)W = w$. Since $\mathfrak{A}_{\nu_n}$ is strictly invariant, there is an element $W_{\nu_n} \in \mathfrak{A}_{\nu_n}$ for which $T(h)W_{\nu_n} = w_{\nu_n}$. The sequence $\{W_{\nu_n}\}$ belongs to the compact set $\mathcal{B}_\rho$. Passing to a subsequence $\{\nu_{n'}\}$, we see that $W_{\nu_{n'}} \to W$ ($n' \to \infty$) for some $W \in \mathcal{B}_\rho$. Then $W \in \mathfrak{A}_{\min}$. Since $\{T(h)\}$ is continuous, we obtain $T(h)W_{\nu_{n'}} \to T(h)W$ ($n' \to \infty$). However, $T(h)W_{\nu_{n'}} = w_{\nu_{n'}}$, and thus we have $w_{\nu_{n'}} \to T(h)W$ ($n' \to \infty$) and $w_{\nu_n} \to w$ ($n \to \infty$) simultaneously. Hence, $T(h)W = w$, and we have proved that

$$\mathfrak{A}_{\min} \subseteq T(h)\mathfrak{A}_{\min} \quad \forall h \geqslant 0.$$

We thus obtain (5.8).

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