Attractors for non-autonomous Navier–Stokes system and other PDEs

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Introduction

One of the major mathematical problems in the study of evolution equations arising in different branches of mechanics and physics is the study of the final behaviour of solutions of these equations when time is large or tends to infinity. The related important question concerns the stability of solutions as time $t \to +\infty$ or the nature of instability if a solution is unstable in some sense. In the last decades, considerable progress in this areas has been achieved in the study of autonomous partial differential equations. For a number of basic autonomous evolution equations of mathematical physics, it was shown that the long time behaviour of their solutions is characterized by finite dimensional global attractors (see, e.g., the books [T88, L91, BV89, H88, CoF89, SY02] and the references cited therein).

Non-autonomous evolution PDEs and their global attractors are less studied. However in the last decade, a notable advance has been made in this perspective area of mathematical researches. In particular, the global attractor has been constructed and studied for the non-autonomous 2D Navier–Stokes system with external force depending on time t. We note that a process $\{U(t,\tau)\} := \{U(t,\tau) \mid t \geq \tau;\ t,\tau \in \mathbb{R}\}$ corresponds to this system which maps every solution $u := u(t)$ at time τ into the value of this solution u at time $t \geq \tau$: $u(\tau) \longmapsto U(t,\tau)u(\tau) := u(t)$. The process $\{U(t,\tau)\}$ is a two-parameter family of mappings acting in the phase space of the evolution equation. Therefore, the study of the behaviour of solutions $u(t)$ as $t \to +\infty$ of the considering non-autonomous evolution equation is equivalent to the investigation of the corresponding process $\{U(t,\tau)\}$ as $t \to +\infty$. Thus, in the study of solution $u(t)$ of non-autonomous equations, processes $\{U(t,\tau)\}$ play the same role as semigroups $\{S(t), t \geq 0\}$ do in the study of solutions $u(t)$ of autonomous equations as time $t \to +\infty$.

In the present survey paper, we mostly study non-autonomous partial differential equations and the corresponding processes $\{U(t,\tau)\}$. Particular emphasis is placed to the study of the global attractor of the non-autonomous 2D Navier–Stokes system.

In Chapter 1, we sketch out the general theory of global attractors of semigroups and some basic autonomous equations of mathematical physics. Besides, we consider questions related to the dimension and the ε-entropy of invariant sets and we present upper estimates for the fractal dimension and for the ε-entropy of global attractors of autonomous equations. We derive such estimates with reasonable details for the 2D Navier–Stokes system, for the dissipative wave equation, and for the complex Ginzburg–Landau equation.

In Chapter 2, we study the uniform global attractors of general processes and non-autonomous equations. We note that, studying global attractors of such an equation, there is a good reason to introduce a notion of its time symbol $\sigma(t)$. The time symbol of a non-autonomous equation is the collection of all time-dependent terms of this equation. Along with solutions dynamics, we study the symbols dynamics as $t \to +\infty$. In Chapter 2, we formulate theorems on the existence of the uniform global attractor $\mathcal{A}$ of the process $\{U_\sigma(t,\tau)\}$ corresponding to a non-autonomous equation with translation compact symbol $\sigma(t)$. Besides, we present a theorem on the structure of the set $\mathcal{A}$. Then we study the uniform global attractor $\mathcal{A}$ of the 2D Navier–Stokes system with time-dependent external force that is the symbol of this system. We study in great details the case, when this system has a unique bounded complete solution $\{z(t), t \in \mathbb{R}\}$ that attracts all other solutions $\{u(t), t \geq \tau\}$ of this 2D Navier–Stokes system as $t \to +\infty$ with exponential rate. We also consider similar problems for the non-autonomous dissipative wave equation and for the non-autonomous Ginzburg–Landau equation.

We note, that a number of important questions related to the global attractors of non-autonomous equations and the corresponding processes were considered, e.g., in the books [Ha91, H88, CV02a, SY02], references therein, and in many papers cited in Bibliography of this report.

It is well known that the fractal dimension of the global attractor of a general non-autonomous PDE can be infinite (see, e.g., the example in the end of Chapter 2). However, the ε-entropy of the global attractor is always finite since the attractor is a compact set. In Chapter 3, we present estimates for the ε-entropy of global attractors of non-autonomous equations with translation compact symbols. We also consider applications of these general results to the non-autonomous 2D Navier–Stokes system and to some other equations of mathematical physics. Particular attention is devoted to the case, where, for example, the external force of the 2D Navier–Stokes system is a quasiperiodic function in time with k rationally independent frequencies. In this case, the global attractor has the finite fractal dimension and the upper estimate for its dimension has a summand k. This means that the fractal dimension can grow with no limit as k tends to infinity. The corresponding examples are constructed in the paper.

In Chapter 4, we study the global attractor $\mathcal{A}_\varepsilon$ of the 2D Navier–Stokes system with singularly oscillating external force of the form $g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t),\ 0 \leq \rho \leq 1,\ 0 < \varepsilon \leq 1$. The behaviour of $\mathcal{A}_\varepsilon$ as $\varepsilon \to 0+$ is under discussion. The analogous problem is studied in Chapter 5 for the non-autonomous complex Ginzburg–Landau equation.

For the readers convenience, each chapter is supplied with a short exposition of its contents.

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Chapter 1

Attractors of autonomous equations

In this chapter we present some fundamental results concerning the global attractors of semigroups corresponding to autonomous evolution equations. The detailed materials can be found in many books on infinite-dimensional dynamical systems and attractors. See, for example, D.Henry [He81], R.Temam [T88], J.K.Hale [H88], A.V.Babin and M.I.Vishik [BV89], O.A.Ladyzhenskaya [L91], M.I.Visik [V92], A.Eden, C.Foias, B.Nicolaenco, and R.Temam [E–T95], I.D.Chueshov [Ch99], C.Foias,O.Manley,R.Rosa, and R.Temam [F–T01], J.C.Robinson [R01], G.Sell and Y.You [SY02], V.V.Chepyzhov and M.I.Vishik [CV02a]. Below, we present a short survey of the known methods and results concerning global attractors of autonomous evolution equations.

1.1 Semigroups and their global attractors

In this section, we consider a general (nonlinear) semigroup $\{S(t)\}$ acting on a set E. Usually E is a complete metric space or a Banach space. In particular, E can be a closed subset of a Banach space.

Definition 1.1.1 A family of mappings $S(t) : E \to E$ depending on a real parameter $t \geq 0$ (time) is called a *semigroup* acting on E and is denoted by $\{S(t)\}$ if it satisfies the *semigroup identity*

$$S(t_1)S(t_2) = S(t_1 + t_2), \quad \forall t_1, t_2 \geq 0, \tag{1.1}$$

and

$$S(0) = \mathrm{Id}. \tag{1.2}$$

Here and below, we denote by Id the identity operator. In the case where $S(t)$ is defined for any real t and identity (1.1) holds for any t_1 and t_2 from $\mathbb{R}$ we shall call $\{S(t)\}$ a *group.*

We now introduce some notations which will be used to describe properties of semigroups. We assume that a semigroup $\{S(t)\}$ acts in a complete metric or a Banach space E. Let $\mathcal{B}(E)$ be the collection of all bounded sets in E with respect to the metric in E.

The semigroup $\{S(t)\}$ is called (E,E)-*bounded* if $S(t)B \in \mathcal{B}(E)$ for every $B \in \mathcal{B}(E)$ and for all $t \geq 0$. The semigroup $\{S(t)\}$ is called *uniformly* (E,E)-*bounded* if for every $B \in \mathcal{B}(E)$ there exists $B_1 \in \mathcal{B}(E)$ such that $S(t)B \subset B_1$ for all $t \geq 0$.

The dynamical system we are going to study are *dissipative.* In application to general semigroups, the dissipation means the existence of bounded or compact absorbing or attracting sets.

A set $B_0 \subset E$ is called *absorbing* for a semigroup $\{S(t)\}$ if for every $B \in \mathcal{B}(E)$ there exists $T = T(B) > 0$ such that $S(t)B \subset B_0$ for all $t \geq T$. A set $P \subset E$ is called *attracting* for $\{S(t)\}$ if for any $B \in \mathcal{B}(E)$

$$\operatorname{dist}_E(S(t)B, P) \to 0 \quad \text{as } t \to +\infty.$$

Here

$$\operatorname{dist}_E(X,Y) = \sup_{x\in X} \inf_{y \in Y} \|y - x\|_E\,;\ X, Y \subseteq E. \tag{1.3}$$

This value is called the *Hausdorff (non-symmetric) distance* from the set X to the set Y. Clearly, any absorbing set is attracting as well.

The semigroup $\{S(t)\}$ is said to be a *compact semigroup* if there exists a compact absorbing set P, $P \Subset E$, for $\{S(t)\}$. The semigroup $\{S(t)\}$ is said to be an *asymptotically compact semigroup* if there exists a compact attracting set K, $K \Subset E$. This two notions generally reflect the dissipativity of dynamical systems under the consideration.

We are going to study continuous semigroups. The semigroup $\{S(t)\}$ is called (E,E)-*continuous* if each mapping $S(t)$ for $t \geq 0$ is continuous from E into E.

We are going to study the behavior of semigroups $\{S(t)\}$ as time $t \to +\infty$. This limit behavior can be described in terms of global attractors.

Definition 1.1.2 A set $\mathcal{A} \in \mathcal{B}(E)$ is called a *global attractor* for $\{S(t)\}$, if it has the following properties:

1. $\mathcal{A}$ is compact in E ($\mathcal{A} \Subset E$);

2. $\mathcal{A}$ is an attracting set for $\{S(t)\}$, that is, for every $B \in \mathcal{B}(E)$,

$$\operatorname{dist}_E(S(t)B, \mathcal{A}) \to 0 \textit{ as } t \to +\infty;$$

3. $\mathcal{A}$ is strictly invariant with respect to $\{S(t)\}$, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$.

As it was shown in [BV89] that the global attractor $\mathcal{A}$ for $\{S(t)\}$ is the maximal bounded invariant set for $\{S(t)\}$ (see also [L75, L82, L91]) This means the following: if $Y \in \mathcal{B}(E)$ and $S(t)Y = Y$ for all $t \geq 0$, then $Y \subset \mathcal{A}$. This implies, in particular, that the global attractor for $\{S(t)\}$ is unique.

Definition 1.1.3 For a bounded set $B \in \mathcal{B}(E)$, the set

$$\omega(B) = \bigcap_{h \geq 0} \left[\bigcup_{t \geq h} S(t)B \right]_E \tag{1.4}$$

is said to be an ω-*limit set* for B. Here $[\,\cdot\,]_E$ denotes the closure in E.

We now formulate the classical attractor existence theorem.

Theorem 1.1.1 *Let $\{S(t)\}$ be a continuous semigroup in a complete metric space E having a compact attracting set K, $K \Subset E$. Then the semigroup $\{S(t)\}$ has a global attractor $\mathcal{A}$ ($\mathcal{A} \subseteq K$). The attractor $\mathcal{A}$ coincides with $\omega(K)$: $\mathcal{A} = \omega(K)$. (If E is a Banach space, then the set $\mathcal{A}$ is connected).*

The proof is given, for example, in [BV89, T88].

We need one more notion to describe the general structure of a global attractor. A curve $u(s)$, $s \in \mathbb{R}$, is called a *complete trajectory* of the semigroup $\{S(t)\}$ if

$$S(t)u(s) = u(t+s) \quad \forall s \in \mathbb{R},\, t \in \mathbb{R}_+. \tag{1.5}$$

Definition 1.1.4 The *kernel* $\mathcal{K}$ of the semigroup $\{S(t)\}$ consists of all its bounded complete trajectories:

$$\mathcal{K} = \{\; u(\cdot) \mid u(s) \text{ satisfies (1.5) and } \|u(s)\|_E \le C_u \text{ for } s \in \mathbb{R}\}\,.$$

Definition 1.1.5 The *kernel section* at a time $s \in \mathbb{R}$ is the following set from E :

$$\mathcal{K}(s) = \{u(s) \mid u \in \mathcal{K}\}\,.$$

Remark 1.1.1 Speaking informally, the kernel $\mathcal{K}$ of the semigroup $\{S(t)\}$ corresponding to autonomous equation (see Section 1.2) consists of all its solutions $u(t)$ determined on the whole time axis $\{t \in \mathbb{R}\}$ that are bounded in E. The kernel includes equilibrium points, periodic, quasiperiodic, and almost periodic orbits. Heteroclinic and homoclinic orbits belong to $\mathcal{K}$ as well and in general, the structure of $\mathcal{K}$ can be extremely complex even with chaotic behaviour of its elements, i.e., bounded complete trajectories.

Theorem 1.1.2 *Under the assumptions of Theorem 1.1.1 the global attractor $\mathcal{A}$ of the semigroup $\{S(t)\}$ coincides with the kernel section $\mathcal{K}(0)$,*

$$\mathcal{A} = \mathcal{K}(0). \tag{1.6}$$

One can replace here 0 by an arbitrary s, $s \in \mathbb{R}$.

The proof is given, e.g., in [BV89].

In the next sections, we apply Theorems 1.1.1 and 1.1.2 to various semigroups $\{S(t)\}$ corresponding to partial differential equations arising in mathematical physics.

1.2 Cauchy problem and corresponding semigroup

For simplicity, below we suppose that E is a Banach space. (Nevertheless, E can be a complete metric space.) Let $\{S(t)\}$ act on the whole Banach space E. Such semigroups are usually generated by evolution equations of the form

$$\partial_t u = A(u), \tag{1.7}$$

where A is a (nonlinear) operator defined on a Banach space E_1 and A maps E_1 into another Banach space E_0. We suppose that $E_1 \subseteq E \subseteq E_0$, where all embeddings are dense. We now construct a semigroup $\{S(t)\}$ acting on E that corresponds to equation (1.7).

We assume that, for an arbitrary element $v_0 \in E$, equation (1.7) with initial data

$$u|_{t=0} = u_0 \tag{1.8}$$

has a unique solution $u(t), t \geq 0$, such that $u(t) \in E$ for all $t \geq 0$. The meaning of the expression "$u(t)$ is a solution of the Cauchy problem (1.7) and (1.8)" should be clarified in each particular case. Usually for every fixed $T > 0$, solutions $u(t)$, $0 \leq t \leq T$, of (1.7) are taken from the class $\mathcal{F}_T$ of functions satisfying the conditions $u(\cdot) \in L_\infty(0, T; E)$ and $u(\cdot) \in L_p(0, T; E_1)$, where E_1 is a Banach space on which the operator A is defined and $1 < p \leq \infty$. Moreover, $A(u(\cdot)) \in L_q(0, T; E_0)$ for some q, $1 < q < \infty$, and $\partial_t u(\cdot) \in L_q(0, T; E_0)$ (the derivative is taken in the distribution sense). Equation (1.7) in this case is understood as an equality in $L_q(0, T; E_0)$. Thus $u(t)$ satisfies (1.7) in the distribution sense in $\mathcal{D}'(]0, T[; E_0)$ (see [Lio69, BV89] for the details). Using various embedding theorems, (see, e.g., [LioM68, T79]) usually it can be shown that $u(t) \in C_w([0, T]; E)$ and even $u(t) \in C([0, T]; E)$ and (1.8) makes sense: $u(t) \to u_0$ weakly or strongly in the space E as $t \to 0+$. Moreover, $u(t) \in E$ for every $t \in [0, T]$. In particular examples, it is convenient to take the space E_0 sufficiently large, since the extension of E does not cause any difficulties and makes the verification of the conditions $A(u) \in E_0$, $\partial_t u \in E_0$ more easy.

Operators $S(t) : E \to E$ generated by equation (1.7) are usually defined as follows. For an arbitrary element $u_0 \in E$, we consider the corresponding solution $u(t)$, $t \geq 0$, of problem (1.7), (1.8). For all $\tau \geq 0$, the element $u(\tau)$ of the space E is uniquely defined. Therefore, the formula

$$S(\tau) : u_0 = u|_{t=0} \mapsto u|_{t=\tau} \tag{1.9}$$

defines the family of mappings $\{S(\tau),\ \tau \geq 0\}$, $S(\tau) : E \to E$.

We state that these mappings form a semigroup. Indeed, let $v_0 \in E$, $v_1 = S(t_1)v_0$, $t_1 > 0$, and $v_2 = S(t_2 + t_1)v_0, t_2 > 0$. Obviously, v_0, v_1 and v_2 are the values of the solution $u(\cdot) \in \mathcal{F}_{t_2+t_1}$ at $t = 0$, $t = t_1$, and $t = t_2 + t_1$, respectively. Consider now the function $u_1(t) = u(t + t_1)$, $t \in [0, t_2]$. Since $u(\cdot) \in \mathcal{F}_{t_2+t_1}$, it follows that $u_1(\cdot) \in \mathcal{F}_{t_2}$. It is also clear that $u_1(t)$ is a solution of (1.7). Obviously also, $u_1|_{t=0} = v_1$, $u_1|_{t=t_2} = v_2$ i.e., by the definition of $\{S(t)\}$, $v_2 = S(t)v_1$. Hence, $S(t_2)S(t_1)v_0 = S(t_2 + t_1)v_0$ for all $v_0 \in E$ and the semigroup identity (1.1) is proved.

Considering below particular equations of the form (1.7), we shall only formulate the corresponding theorems on the existence and uniqueness of a solution and specify a space or a set in which the semigroup $\{S(t)\}$ acts. We shall suppose that the operators $S(t)$ are defined by formula (1.9).

1.3 Global attractors for autonomous equations of mathematical physics

1.3.1 2D Navier–Stokes system

The Navier–Stokes system is probably the most popular example of a partial differential equation having a global attractor. A considerable part of the theory of infinite dimensional dynamical systems has been developed from this example.

We consider the autonomous 2D Navier–Stokes system in a bounded domain $\Omega \Subset \mathbb{R}^2$. The system reads

$$\begin{aligned} &\partial_t u + \sum_{i=1}^{2} u^i \partial_{x_i} u = \nu \Delta u - \nabla p + g(x), \\ &(\nabla, u) = 0, \ u|_{\partial\Omega} = 0, \quad (x_1, x_2) \in \Omega, \end{aligned} \tag{1.10}$$

where $u = u(x,t) = (u^1(x,t), u^2(x,t))$ is a velocity vector, $p = p(x,t)$ is a scalar function for the pressure, ν is the kinematic viscosity coefficient, and $g = g(x) = (g^1(x), g^2(x))$ is the forcing term.

By H and $V = H^1$ we denote the closure of the set

$$\mathcal{V} = \left\{ v \mid v \in \left(C_0^\infty(\Omega)\right)^2, \ (\nabla, v) = 0 \right\}$$

in the norms $|\cdot|$ and $\|\cdot\|$ of the spaces $\left(L_2(\Omega)\right)^2$ and $\left(H_0^1(\Omega)\right)^2$, respectively. Recall that

$$\|u\|^2 = |\nabla u|^2 = \sum_{i=1}^{2} \int_\Omega |\nabla u^i(x)|^2 dx.$$

By P we denote the orthogonal projector from $\left(L_2(\Omega)\right)^2$ onto H and its various extensions.

Excluding the pressure, system (1.10) can be written in the form

$$\partial_t u + \nu L u + B(u,u) = g_0(x). \tag{1.11}$$

Here,

$$L = -P\Delta, \ B(u,v) = P \sum_{i=1}^{2} u^i \partial_{x_i} v, \ g_0 = Pg.$$

Let $V' = V^*$ be the dual space for V. The Stokes operator L, considered as an operator on $V \cap \left(H^2(\Omega)\right)^2$, is positive and self-adjoint. Its minimal eigenvalue λ_1 is positive. Suppose that $g(\cdot) \in H$. The initial conditions are posed at $t = 0$:

$$u|_{t=0} = u_0(x), \quad u_0 \in H. \tag{1.12}$$

The operator L is bounded from V into V'.

The form b

$$b(u,v,w) = (B(u,v), w) = \int_\Omega \sum_{i,j=1}^{2} u^i \partial_{x_i} v^j w^j dx$$

is trilinear continuous on V and operator B maps $V \times V$ into V'. The form b satisfies the identities

$$b(u,v,v) = 0, \ b(u,v,w) = -b(u,w,v), \ \forall u,v,w \in V. \tag{1.13}$$

Moreover, the following estimate is valid:

$$|b(u,u,v)| \leq c_0^2 |u| \|u\| \|v\|, \ \forall u, v \in V \tag{1.14}$$

(see [L70, T79]), where the constant c_0 can be taken from the inequality

$$\|f\|_{L_4(\Omega)} \leq c|f|^{1/2} |\nabla f|^{1/2}, \ f \in H_0^1(\Omega), \ c_0 = c. \tag{1.15}$$

The constant c (and c_0) does not depend on Ω. In particular, it follows from (1.14) that

$$|B(u,u)|_{V'} \le c_0^2 \|u\| |u|.$$

Thus, if $u \in L_2(0,T;V) \cap L_\infty(0,T;H)$, then $-\nu Lu - B(u,u) + g(x) \in L_2(0,T;V')$ and equation (1.11) can be considered in the distribution sense of the space $\mathcal{D}'(0,T;V')$ and $\partial_t u \in L_2(0,T;V')$.

Proposition 1.3.1 *Problem (1.11), (1.12) has a unique solution* $u(t) \in C(\mathbb{R}_+; H) \cap L_2^{loc}(\mathbb{R}_+; V)$ *and* $\partial_t u \in L_2^{loc}(\mathbb{R}_+; V')$. *The following estimates hold:*

$$|u(t)|^2 \le |u(0)|^2 e^{-\nu\lambda t} + \nu^{-2}\lambda^{-2}|g|^2. \tag{1.16}$$

$$|u(t)|^2 + \nu \int_0^t \|u(s)\|^2 ds \le |u(0)|^2 + t\nu^{-1}\lambda^{-1}|g|^2, \tag{1.17}$$

$$t\|u(t)\|^2 \le C\left(t, |u(0)|^2\right) \tag{1.18}$$

where $\lambda = \lambda_1$ *is the first eigenvalue of the Stokes operator* L *and* $C(z,R)$ *is a monotone continuous functions of* $z = t$ *and* R.

The existence and uniqueness theorem is a classical result. The detailed proof can be found in [L70, Lio69, T79, BV89, CoF89].

Thus, there exists a semigroup $\{S(t)\}$ acting in $H : S(t) : H \to H$ for $t \ge 0$ that corresponds to problem (1.11), (1.12): $S(t)u_0 = u(t)$, where $u(t)$ is the solution of system (1.11), (1.12).

Proposition 1.3.2 *The semigroup* $\{S(t)\}$ *corresponding to problem (1.11), (1.12) is uniformly* (H,H)*-bounded, compact and* (H,H)*-continuous.*

The detailed proof is given, e.g., in [BV89, T88]. The existence of a bounded absorbing set follows from (1.16). See also Section 2.6.1, where the non-autonomous system is considered. Propositions 1.3.1 and 1.3.2 imply that the semigroup $\{S(t)\}$ satisfies all the conditions of Theorem 1.1.2. The following theorem holds.

Theorem 1.3.1 *The semigroup* $\{S(t)\}$ *corresponding to problem (1.11), (1.12) has the global attractor* $\mathcal{A}$ *that is compact in* H *and coincides with the kernel section:* $\mathcal{A} = \mathcal{K}(0)$.

We consider the following dimensionless number called the (generalized) *Grashof number*

$$G = \frac{|g|}{\nu^2 \lambda_1}.$$

This number plays an important role in the analysis of the structure of the global attractor $\mathcal{A}$. First of all we have the following

Proposition 1.3.3 *Suppose that*

$$G < \frac{1}{c_0^2}, \tag{1.19}$$

where c_0 *is the constant from inequality (1.14). Then equation (1.11) has a unique stationary solution* $z \in V$ *which is globally asymptotically stable, i.e.,*

$$\mathcal{A} = \{z\}.$$

Proof. It is well known that equation (1.11) has a stationary solution z (see, for example [T79]), $\nu Lz + B(z,z) = g$. It follows from (1.17) that

$$\|z\|^2 = |\nabla z|^2 \leq \frac{|g|^2}{\nu^2 \lambda_1}. \tag{1.20}$$

Every solution $u(t)$ of (1.11) can be written as $u(t) = z + v(t)$, where $v(t)$ satisfies the equation

$$\partial_t v + \nu L v + B(v,v) + B(v,z) + B(z,v) = 0.$$

Multiplying by v and using (1.14), (1.13), the inequality $|v| \leq \lambda_1^{-1/2}\|v\|$, and (1.20) we obtain

$$\begin{aligned} \partial_t |v|^2 + 2\nu \|v\|^2 = 2b(v,v,z) &\leq 2c_0^2 |v| \|v\| \|z\| \\ &\leq 2c_0^2 \lambda_1^{-1/2} \|v\|^2 \|z\| \leq 2c_0^2 \lambda_1^{-1} \nu^{-1} |g| \|v\|^2. \end{aligned}$$

Finally,

$$\partial_t |v|^2 + 2\left(\nu - c_0^2 \lambda_1^{-1} \nu^{-1} |g|\right) \|v\|^2 \leq 0$$

and hence

$$\partial_t |v(t)|^2 + \alpha |v(t)|^2 \leq 0,$$

where $\alpha = 2(\nu - c_0^2 \lambda_1^{-1} \nu^{-1} |g|)\lambda_1^{-1} > 0$ since $\frac{|g|}{\nu^2 \lambda_1} = G < c_0^{-2}$. This implies

$$|v(t)|^2 = |u(t) - z|^2 \leq |u(0) - z|^2 e^{-\alpha t}.$$

Consequently, the stationary solution is unique, asymptotically stable, and $\mathcal{A} = \{z\}$. ■

Remark 1.3.1 Inequality (1.15) was originally proved with $c \leq 2^{1/4}$ in [L70]. It is known from [N89] that $c < \left(\frac{16}{27\pi}\right)^{1/4}$. In [CI04] it was proved that the constant c_0^2 in (1.14) can be taken $c_0^2 = \frac{c^2}{\sqrt{2}} = \left(\frac{8}{27\pi}\right)^{1/2}$. Therefore the attractor $\mathcal{A}$ is trivial if $G < 3.2562$.

If the Grashof number $G = \frac{|g|}{\nu^2 \lambda_1}$ is large, then it is very likely from the physical evidence and simulation results that, as $t \to +\infty$, the solutions of the Navier–Stokes system tend to a more complicated attracting set than a stationary solution. Hence, the global attractor $\mathcal{A}$ can have a very complicated structure, possibly, chaotic. See, for example, [FT79, FT82, FT83]. In Section 1.4.2, we shall study upper bounds for the dimension of the global attractors of Navier–Stokes equations which depend of the Grashof numbers. Thus, roughly speaking, the flows can be described by a finite number of parameters, which can be extremely large (but finite) despite the fact that the system itself is an infinite dimensional dynamical system.

1.3.2 Wave equation with dissipation

We consider the following hyperbolic equation with damping (dissipation):

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - f(u) + g(x), \ \ u|_{\partial\Omega} = 0, \ x \in \Omega \Subset \mathbb{R}^n. \tag{1.21}$$

The equation contains a damping term $\gamma\partial_t u$, where $\gamma > 0$. We assume that $g \in L_2(\Omega)$ and the nonlinear function $f(v) \in C^1(\mathbb{R})$ satisfies the conditions

$$F(v) \geq -mv^2 - C_m,\ F(v) = \int_0^v f(w)dw, \tag{1.22}$$

$$f(v)v - \gamma_1 F(v) + mv^2 \geq -C_m,\ \forall\, v \in \mathbb{R}, \tag{1.23}$$

where $m > 0,\ \gamma_1 > 0$, and m is sufficiently small ($m < \lambda_1$, where λ_1 is the first eigenvalue of the operator $-\Delta$ with zero boundary conditions).

Remark 1.3.2 Conditions (1.22) and (1.23) are valid, for example, if

$$\liminf_{|v|\to\infty} \frac{F(v)}{v^2} \geq 0,\ \liminf_{|v|\to\infty} \frac{f(v)v - \gamma_1 F(v)}{v^2} \geq 0. \tag{1.24}$$

Assume that ρ is positive and $\rho < 2/(n-2)$ when $n \geq 3$ and ρ is arbitrary when $n = 1, 2$. We suppose also that

$$|f'(v)| \leq C_0(1 + |u|^\rho). \tag{1.25}$$

The case $\rho < 2/(n-2)$ for equation (1.21) has been studied in [Ha87, GT87] and in other references. The case $\rho = 2/(n-2)$ has been considered in [BV89, L87, ArCaH92] (see also [Fe92, GrP03, PZ06]). We discuss here the case $\rho < 2/(n-2)$.

Remark 1.3.3 Nonlinear hyperbolic equations of the type (1.21) appear in many branches of physics, for example, the dynamics of a Josephson junction driven by a current source is modelled by the sine–Gordon equation of the form (1.21) with

$$f(u) = \beta \sin u.$$

Clearly (1.24) is valid. Another important example is encountered in relativistic quantum mechanics with the nonlinear term

$$f(u) = |u|^\rho u.$$

In this case, evidently, $F(u) = |u|^{\rho+2}/(\rho + 2)$ and inequality (1.24) holds with $\gamma_1 = 1/(\rho + 2)$ (see [T88] and the references therein).

It follows from (1.25) that

$$|f(v)| \leq C_1(1 + |u|^{\rho+1}). \tag{1.26}$$

By the Sobolev embedding theorem,

$$H_0^1(\Omega) \subset L_{2(\rho+1)}(\Omega). \tag{1.27}$$

For $n = 1, 2$ this is valid for any ρ, while for $n \geq 3$, in view the assumptions made, $2(\rho+1) < 2n/(n-2)$, where $2n/(n-2)$ is the critical exponent in the Sobolev theorem.

Now let the function $u \in L_\infty(0, T; H_0^1(\Omega))$ and its derivative $\partial_t u \in L_\infty(0, T; L_2(\Omega))$. Then $\Delta u \in L_\infty(0, T; H^{-1}(\Omega))$ and, due to (1.27), $f(u) \in L_\infty(0, T; L_2(\Omega))$. Therefore, $-\gamma\partial_t u + \Delta u - f(u) + g(x) \in L_\infty(0, T; H^{-1}(\Omega))$ and equation (1.21) can be considered in the space $\mathcal{D}'(0, T; H^{-1}(\Omega))$ in the distribution sense, in particular, $\partial_t^2 u \in L_\infty(0, T; H^{-1}(\Omega))$ (see [Lio69]).

The initial conditions are posed at $t = 0$:

$$u|_{t=0} = u_0(x),\ \partial_t u|_{t=0} = p_0(x). \tag{1.28}$$

Proposition 1.3.4 *If $u_0 \in H_0^1(\Omega)$ and $p_0 \in L_2(\Omega)$, then under the above assumptions problem (1.21), (1.28) has a unique solution $u(t) \in C(\mathbb{R}_+; H_0^1(\Omega))$, $\partial_t u(t) \in C(\mathbb{R}_+; L_2(\Omega))$ and $\partial_t^2 u(t) \in L_\infty(\mathbb{R}_+; H^{-1}(\Omega))$.*

We write $y(t) = (u(t), \partial_t u(t)) = (u(t), p(t))$, $y_0 = (u_0, p_0) = y(0)$ for brevity. We denote by E the space of vector functions $y(x) = (u(x), p(x))$ with finite energy norm $\|y\|_E^2 = |\nabla u|^2 + |p|^2$ in $E = H_0^1(\Omega) \times L_2(\Omega)$. Then $y(t) \in E$ for every $t \geq 0$.

The unique solvability of problem (1.21), (1.28) in the energy space E and properties of its solutions are proved in [Lio69, BV89, T88, H88]. See also [CV02a], where more general cases are studied.

Problem (1.21), (1.28) is equivalent to the following system:

$$\begin{cases} \partial_t u = p \\ \partial_t p = -\gamma p + \Delta u - f(u) + g \end{cases} \quad \begin{cases} u|_{t=0} = u_0 \\ p|_{t=0} = p_0 \end{cases}$$

which can be written in a brief form

$$\partial_t y = A(y), \ y|_{t=0} = y_0. \tag{1.29}$$

Thus, if $y_0 \in E$, then problem (1.21), (1.28) has a unique solution $y(t) \in C_b(\mathbb{R}_+; E)$. This implies that the semigroup $\{S(t)\}$, $S(t)y_0 = y(t)$ is defined in E.

Proposition 1.3.5 *The semigroup $\{S(t)\}$ corresponding to problem (1.21), (1.28) is bounded, asymptotically compact and (E, E)-continuous.*

We will come back to this assertion in Section 2.6.2 studying more general non-autonomous hyperbolic equations.

Finally, we conclude that Theorem 1.1.2 and Proposition 1.3.5 imply

Theorem 1.3.2 *The semigroup $\{S(t)\}$ corresponding to (1.21),(1.28) possesses the global attractor $\mathcal{A}$ that is compact in E and coincides with kernel section: $\mathcal{A} = \mathcal{K}(0)$.*

1.3.3 Ginzburg–Landau equation

This equation serves as a model in many areas of physics and mechanics (see, for example, [KopHo73, KuTs75]). It appears, for example, in the theory of superconductivity. The complex Ginzburg–Landau equation is

$$\partial_t u = (1 + \alpha i)\Delta u + Ru - (1 + i\beta)|u|^2 u, \ x \in \Omega \Subset \mathbb{R}^n. \tag{1.30}$$

We consider the case of periodic boundary conditions in $\Omega =]0, 2\pi[^n$ or zero boundary conditions $u|_{\partial\Omega} = 0$ in an arbitrary domain $\Omega \Subset \mathbb{R}^n$. In equation (1.30) $u = u^1 + iu^2$, $\alpha, \beta \in \mathbb{R}$ are the dispersion parameters, and $R > 0$ is the instability parameter. For $\mathbf{u} = (u^1, u^2)^\top$ we obtain the system

$$\begin{cases} \partial_t u^1 = \Delta u^1 - \alpha \Delta u^2 + Ru^1 - (|u^1|^2 + |u^1|^2)(u^1 - \beta u^2) \\ \partial_t u^2 = \alpha \Delta u^1 + \Delta u^2 + Ru^2 - (|u^1|^2 + |u^1|^2)(\beta u^1 + u^2) \end{cases}. \tag{1.31}$$

or in a more compact form

$$\partial_t \mathbf{u} = a\Delta \mathbf{u} + R\mathbf{u} - \mathbf{f}(\mathbf{u}), \tag{1.32}$$

where the matrix $a = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$ and the function $\mathbf{f}(\mathbf{u}) = |\mathbf{u}|^2 \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix} \mathbf{u}$.

Consider the Jacobi matrix of $\mathbf{f}(\mathbf{u})$

$$\mathbf{f}_{\mathbf{u}}(\mathbf{u}) = \begin{pmatrix} 3(u^1)^2 - 2\beta(u^1)(u^2) + (u^2)^2 & -\beta(u^1)^2 + 2(u^2)(u^1) - 3\beta(u^2)^2 \\ 3\beta(u^1)^2 + 2(u^2)(u^1) + \beta(u^2)^2 & (u^1)^2 + 2\beta(u^1)(u^2) + 3(u^2)^2 \end{pmatrix}. \tag{1.33}$$

We denote by B the matrix of the bilinear form corresponding to the matrix in the right-hand side of (1.33):

$$B = \begin{pmatrix} 3(u^1)^2 - 2\beta(u^1)(u^2) + (u^2)^2 & \beta(u^1)^2 + 2(u^2)(u^1) - \beta(u^2)^2 \\ \beta(u^1)^2 + 2(u^2)(u^1) - \beta(u^2)^2 & (u^1)^2 + 2\beta(u^1)(u^2) + 3(u^2)^2 \end{pmatrix}.$$

The diagonal elements of B are positive if $|\beta| \leq \sqrt{3}$. Moreover,

$$\det B = (3 - \beta^2)\left((u^1)^2 + (u^2)^2\right) = (3 - \beta^2)|\mathbf{u}|^4$$

is also positive. Thus, in this case, the matrix B is positive definite. Therefore

$$\mathbf{f}_{\mathbf{u}}(\mathbf{u})\mathbf{v} \cdot \mathbf{v} \geq 0 \ \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^2, \tag{1.34}$$

if $|\beta| \leq \sqrt{3}$.

We shall use the spaces $\mathbf{H} = L_2(\Omega; \mathbb{C})$, $\mathbf{V} = H_0^1(\Omega; \mathbb{C})$, and $\mathbf{L}_4 = L_4(\Omega; \mathbb{C})$. The Cauchy problem for equation (1.32) with initial data

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(x), \ \mathbf{u}_0(\cdot) \in \mathbf{H}, \tag{1.35}$$

has a unique weak solution $\mathbf{u}(t) := \mathbf{u}(x, t)$ such that

$$\mathbf{u}(\cdot) \in C(\mathbb{R}_+; \mathbf{H}) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{V}) \cap L_4^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{L}_4), \tag{1.36}$$

and the function $\mathbf{u}(t)$ satisfies equation (1.32) in the sense of distributions of the space $\mathcal{D}'(\mathbb{R}_+; \mathbf{H}^{-r})$, where $\mathbf{H}^{-r} = H^{-r}(\Omega; \mathbb{C})$ and $r = \max\{1, n/4\}$ (recall that $n = \dim(\Omega)$). In particular, $\partial_t \mathbf{u}(\cdot) \in L_2(0, M; \mathbf{H}^{-1}) + L_{4/3}(0, M; \mathbf{L}_{4/3})$ for any $M > 0$. The existence of such solution $\mathbf{u}(t)$ is proved, for example, using the Galerkin approximation method (see, e.g. [T88, BV89, CV02a]). The proof of the uniqueness theorem is also standard and relies on inequality (1.34). (We note that, if (1.34) does not hold, the uniqueness theorem for $n \geq 3$ and for arbitrary values of the dispersion parameters α and β is not proved yet, see [Mi02, Mi98, Z00] for important partial uniqueness results).

Any solution $\mathbf{u}(t), t \geq 0$, of system (1.32) satisfies the following differential identity:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}(t)\|^2 + \|\nabla \mathbf{u}(t)\|^2 + \|\mathbf{u}(t)\|_{\mathbf{L}_4}^4 - R\|\mathbf{u}(t)\|^2 = 0, \ \forall t \geq 0, \tag{1.37}$$

where the real function $\|\mathbf{u}(t)\|^2$ is absolutely continuous for $t \geq 0$. Here $\|\cdot\|$ denotes the usual L_2-norm in $\mathbf{H}$.

The proof of (1.37) is analogous to the proof of the corresponding identity for weak solutions of the reaction-diffusion systems considered in [CV96b, CV02a, CV05].

Equation (1.32) generate a semigroup $\{S(t)\}$ in $\mathbf{H}$. This semigroup is $(\mathbf{H}, \mathbf{H})$-continuous and compact (see, e.g., [T88, CV02a]). By Theorem 1.1.1, there exists a global attractor $\mathcal{A}$ of this semigroup. The global attractor describes the long time behaviour of solutions of the Ginzburg–Landau equation. It is known that the dynamics of this system is chaotic for certain values of the parameters, for example, for $\alpha\beta < 0$ (see [Ba–Gis90, D–Ni88]). However, in Section 1.4.2, we show that the dimension of the global attractor of the Ginzburg–Landau equation is finite.

Consider the case $|\beta| > \sqrt{3}$, where condition (1.34) is not longer valid. For low dimensions $n = 1, 2$ it is still possible to construct a semigroup in $\mathbf{H} = (L_2(\Omega))^2$ which has a compact global attractor, see [GHe87, T88]. For $n \geq 3$ one can prove the existence of a global attractor in $\mathbf{L}_p = (L_p(\Omega))^2,\ p > n$, if $(\alpha, \beta) \in \mathcal{P}(n)$, where $\mathcal{P}(n)$ is a subset of $\mathbb{C}$, see [D–Ni88, DGiLe94, Mi97, Mi02] for more details.

Thus we see that without condition (1.34) provided that $|\beta| \leq \sqrt{3}$ it is more difficult to construct a semigroup and to study its global attractor. Fortunately, this obstacle can be eliminated by using another approach that is based on the study of the so-called trajectory attractors (see [CV02a, CV05]). In particular, the method of trajectory attractors works for the Ginzburg–Landau equation with arbitrary n, α, β.

Inhomogeneous Ginzburg–Landau equation

$$\partial_t u = (1 + \alpha i)\Delta u + Ru - (1 + i\beta)|u|^2 u + g(x),\ g \in L_2(\Omega; \mathbb{C}),$$

is also encountered in applications, where, e.g., $g(x) = \delta \exp(ik \cdot x),\ k \in \mathbb{Z}^n,\ \delta > 0$. This equation also generates a semigroup, and Theorem 1.1.1 is applicable.

1.4 Dimension of global attractors

In this section, we present some known results concerning the dimension of global attractors of autonomous evolution equations. These questions have been studied in a number of papers and the corresponding upper and lower dimension estimates have been summend up in [T88] and [BV89] (see also the reviews [Ch93, B03]).

1.4.1 Dimension of invariant sets

We start with definition of the Kolmogorov ε-entropy of a compact set X in a Hilbert (or Banach) space E. We denote by $N_\varepsilon(X, E) = N_\varepsilon(X)$ the minimum number of open balls in E with radius ε which is necessary to cover X :

$$N_\varepsilon(X) := \left\{ \min N \mid X \subset \bigcup_{i=1}^{N} B(x_i, \varepsilon) \right\}.$$

Here $B(x_i, \varepsilon) = \{x \in E \mid \|x - x_i\|_E < \varepsilon\}$ is the ball in E with center x_i and radius ε. Since the set X is compact, we see that $N_\varepsilon(X) < +\infty$ for any $\varepsilon > 0$.

Definition 1.4.1 The *Kolmogorov ε-entropy* of a set X in the space E is the number

$$\mathbf{H}_\varepsilon(X, E) := \mathbf{H}_\varepsilon(X) := \log_2 N_\varepsilon(X). \tag{1.38}$$

For particular sets X, the problem is to study the asymptotic behavior of the quantity $\mathbf{H}_\varepsilon(X)$ as $\varepsilon \to 0+$. This characteristic of compact sets was originally introduced by A.N.Kolmogorov and was studied in the joint work with V.M.Tikhomirov (see [KTi59]). In this paper, the ε-entropy of various classes of functions was investigated. Moreover, an important notion of the *entropy dimension* of a compact set was also defined. This dimension is now often called the *fractal dimension*.

Definition 1.4.2 The (upper) *fractal dimension* of a compact set X in the space E is the number

$$\mathbf{d}_F(X,E) := \mathbf{d}_F(X) := \limsup_{\varepsilon\to 0+} \frac{\mathbf{H}_\varepsilon(X)}{\log_2(1/\varepsilon)}. \tag{1.39}$$

The fractal dimension of a compact set in an infinite dimensional space can be infinite. However, if it is known that $0 < \mathbf{d}_F(X) < +\infty$, then $\mathbf{H}_\varepsilon(X) \approx \mathbf{d}_F(X)\log_2\left(\frac{1}{\varepsilon}\right)$, and therefore, in this case, it is needed $N_\varepsilon(X) \approx \left(\frac{1}{\varepsilon}\right)^{\mathbf{d}_F(X)}$ points to approximate the set X with precision ε.

Another important characteristic of a compact set X is the *Hausdorff dimension*

$$\mathbf{d}_H(X) := \inf\{d \mid \mu(X,d) = 0\},$$

where $\mu(X,d) = \inf\sum r_i^d$, and the infimum is taken over all the coverings of the set X by balls $B(x_i, r_i)$ with radii $r_i \le \varepsilon$ (see [Thi92]). It is apparent that $\mathbf{d}_H(X) \le \mathbf{d}_F(X)$ and there are examples of sets such that $\mathbf{d}_H(X) = 0$ but $\mathbf{d}_F(X) = +\infty$. In the present paper, we shall consider only the fractal dimension of compact sets, because this dimension is closely connected with ε-entropy of these sets.

Remark 1.4.1 The fractal and Hausdorff dimensions are very fruitful in the study of the structure of various "non-smooth" sets, for example, the self-similar sets or the fractals. The simplest example of such a set is the Cantor set K on the segment $[0,1]$, for which $\mathbf{d}_F(K) = \mathbf{d}_H(K) = \log_3 2 < 1$. The fractal (and Hausdorff) dimension of a compact smooth manifold is equal to its usual dimension, i.e., it is integer. However, the example of the Cantor set shows that the dimension can be non-integer.

We now study the ε-entropy and the fractal dimension of strictly invariant sets and global attractors of autonomous evolution equations of the form (1.7). Let the Cauchy problem (1.7), (1.8) generates a semigroup $\{S(t)\}$ acting in a Hilbert space E (see Section 1.1). Consider a compact set X in E, $X \Subset E$. Let the set X be strictly invariant with respect to $\{S(t)\}$, that is, $S(t)X = X$ for all $t \ge 0$. (For example, $X = \mathcal{A}$, where $\mathcal{A}$ is the global attractor of the semigroup.) We assume that the semigroup $\{S(t)\}$ is *uniformly quasidifferentiable on* X in the following sense: for any $t \ge 0$ and for every $u \in X$ there is a linear bounded operator $L(t,u) : E \to E$ (*quasidifferential*) such that

$$\|S(t)v_1 - S(t)v - L(t,u)(v_1 - v)\|_E \le \gamma(\|v_1 - v\|_E, t)\|v_1 - v\|_E \tag{1.40}$$

for all $v, v_1 \in X$ and the function $\gamma = \gamma(\xi, t) \to 0+$ as $\xi \to 0+$ for every fixed $t \ge 0$. We assume that the linear operators $L(t,u)$ are generated by the variational equation for (1.7) that we present in the form

$$\partial_t v = A_u(u(t))v, \quad v|_{t=0} = v_0 \in E, \tag{1.41}$$

where $u(t) = S(t)u_0$, $u_0 \in X$, and $A_u(\cdot)$ is the formal derivative in u of the operator $A(\cdot)$ in (1.7) and the domain E_1 of the operator $A_u(u(t))$ is dense in E. We assume that, for every $u_0 \in X$, the linear problem (1.41) is uniquely solvable for all $v_0 \in E$. By our assumption, the quasidifferentials $L(t,u_0)$ in (1.40) act on a vector v_0 by the role $L(t,u_0)v_0 = v(t)$, where $v(t)$ is the solution of equation (1.41) with initial data v_0.

Let $j \in \mathbb{N}$ and let $L : E_1 \to E$ be a linear (possibly, unbounded) operator. The following number is called the *j-trace* of the operator L:

$$\mathrm{Tr}_j L := \sup_{\{\varphi_i\}_{i=1}^j} \sum_{i=1}^{j} (L\varphi_i, \varphi_i), \tag{1.42}$$

where the infimum is taken over all the orthonormal in E families of vectors $\{\varphi_i\}_{i=1,\dots,j}$ belonging to E_1 and (ψ, φ) denotes the scalar product in E of vectors ψ and φ.

Definition 1.4.3 We set

$$\tilde{q}_j := \limsup_{T\to+\infty} \sup_{u_0\in X} \frac{1}{T} \int_0^T \mathrm{Tr}_j A_u(u(t))dt, \ j = 1, 2, \dots\ , \tag{1.43}$$

where $u(t) = S(t)u_0$.

Theorem 1.4.1 *We assume that the semigroup* $\{S(t)\}$ *acting in* E *has a compact, strictly invariant set* X *and is uniformly quasidifferentiable on* X. *Let the following inequalities hold:*

$$\tilde{q}_j \le q_j, \quad j = 1, 2, 3, \dots,$$

where the numbers $\tilde{q}_j$ *are defined in (1.43). We assume that the function* q_j *is concave in* j *(like* $\cap$*). Let* m *be the smallest integer such that* $q_{m+1} < 0$, *(then, clearly,* $q_m \ge 0$*). We set*

$$d = m + \frac{q_m}{q_m - q_{m+1}}. \tag{1.44}$$

Then, the set X *has the finite fractal dimension and*

$$\mathbf{d}_F(X) \le d. \tag{1.45}$$

Besides, for every $\delta > 0$, *there exist real numbers* $\eta \in (0, 1)$ *and* $\varepsilon_0 > 0$ *such that the following inequality holds for the* ε*-entropy* $\mathbf{H}_\varepsilon(X)$ *of the set* X*:*

$$\mathbf{H}_\varepsilon(X) \le (d + \delta) \log_2 \left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(X), \ \forall\varepsilon < \varepsilon_0. \tag{1.46}$$

The complete proof of this theorem is given in [CV02a]. The proof is based on the study of the volume contraction properties under the action of the quasidifferentials of the semigroup operators. Estimates for the Hausdorff dimension of invariant sets that are similar to (1.45) were originally proved in [DuO80] for a finite dimensional space E and corresponding results were generalized for an infinite dimensional space E in [CoFT85, T88] (see also [Il82, BV83a, BV89]).

We note that estimate (1.46) for the ε-entropy of $\mathcal{A}$ follows from (1.45) and so it may appear that it gives no new information concerning the global attractor. However, studying non-autonomous equations, where global attractors have infinite dimension in a generic case (see Chapter 3), estimates for the ε-entropy of global attractors become more informal and even constitutive. This is why we include estimate (1.46) to this key theorem.

Remark 1.4.2 In applications, the numbers q_j usually have the form $q_j = \varphi(j)$, where $\varphi = \varphi(x), x \geq 0$, is a smooth concave function. Consider its root $d^* : \varphi(d^*) = 0$. Evidently, $d \leq d^*$, since φ is a concave function. When d is large, then the root d^* is very close to d given by the formula (1.44). In some cases, d^* is expressed in a simpler way than d. So, in particular examples we shall use d^* instead of d as the upper bound in (1.45) for the fractal dimension of attractors and, in this case, in (1.46) we can take $\delta = d^* - d$ if this value is positive.

In the recent work [CI04], the estimates (1.46) and (1.45) has been proved for the exact values $q_j = \tilde{q}_j$ without the concavity assumption for the function $\tilde{q}_j$ in j. The number

$$d_L := m + \frac{\tilde{q}_m}{\tilde{q}_m - \tilde{q}_{m+1}}$$

is conventionally called the (global) Lyapunov dimension of the set X (see [KapY79, EdFT91]). In the works [DuO80, CoF85, T88], it was proved that $d_H(X) \leq d_L(X)$. In [CI04] it was shown that $d_F(X) \leq d_L(X)$. The similar result was obtained earlier in [BlIl99], namely, it was proved that if $\tilde{q}_m < 0$ for some $m \in \mathbb{N}$, then $\mathbf{d}_F(X) \leq m$ (see also [Hu96]).

The books [BV89, T88, H88] contains many examples of evolution equations of mathematical physics and mechanics. For all the problems, the global attractors were constructed and upper estimates were proved for the Hausdorff and fractal dimension of these attractors. In the next sections, we present fractal dimensions estimates for global attractors of autonomous equations of mathematical physics considered in Section 1.3.

1.4.2 Dimension estimates for autonomous equations

2D Navier–Stokes system

We consider the 2D Navier–Stokes system

$$\partial_t u = -\nu L u - B(u, u) + g, \ (\nabla, u) = 0, \ u|_{\partial\Omega} = 0, \tag{1.47}$$

$$u|_{t=0} = u_0, \ u_0 \in H, \tag{1.48}$$

where $g \in H$. Problem (1.47), (1.48) defines the semigroup $\{S(t)\}$ acting in H (see Section 1.3.1). By Theorem 1.3.1, this semigroup has the global attractor $\mathcal{A}$ and the set $\mathcal{A}$ is bounded in V and compact in H.

Theorem 1.4.2 *The fractal dimension of the global attractor $\mathcal{A}$ of problem (1.47), (1.48) satisfies the estimate*

$$\mathbf{d}_F \mathcal{A} \leq c \frac{|g||\Omega|}{\nu^2}, \tag{1.49}$$

where c depends on the shape of Ω ($c(\lambda\Omega) = c(\Omega)$ for all $\lambda > 0$).

The Kolmogorov ε-entropy of $\mathcal{A}$ satisfies the inequality

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leq c \frac{|g||\Omega|}{\nu^2} \log_2 \left(\frac{\varepsilon_0}{\eta\varepsilon} \right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}), \ \forall \varepsilon < \varepsilon_0, \tag{1.50}$$

where η and ε_0 are some small positive numbers.

Proof. The semigroup $\{S(t)\}$ is uniformly quasidifferentiable on $\mathcal{A}$ in H and its quasidifferential is the operator $L(t,u_0)v_0 = v(t),\ v_0 \in H$, where $v(t)$ is the solution of the corresponding variation equation

$$\partial_t v = -\nu L - B(u(t),v) - B(v,u(t)) := A_u(u(t))v,\ \ v|_{t=0} = v_0.$$

(See [BV83a, BV89]). We have to estimate the j-trace of $A_u(u(t))$. Note that for all $v \in V$ we have

$$(A_u(u(t))v,v) = \nu\|v\|^2 - (B(v,u(t)),v), \tag{1.51}$$

taking into account that $(B(u,v),v) = 0$ for $u,v \in V$.

Let $\varphi_1,\dots,\varphi_j \in V$ be an arbitrary orthonormal family in H. Using (1.51), we have

$$\begin{aligned}\sum_{i=1}^{j}(A_u(u(t))\varphi_i,\varphi_i) &= -\nu\sum_{i=1}^{j}|\nabla\varphi_i|^2 - \sum_{i=1}^{j}(B(\varphi_i,u(t)),\varphi_i)\\ &= -\nu\sum_{i=1}^{j}|\nabla\varphi_i|^2 - \int_\Omega\sum_{i=1}^{j}\sum_{k,l=1}^{2}\varphi_i^k\partial_{x_k}u^l(t)\varphi_i^l dx\end{aligned}$$

$$\le -\nu\sum_{i=1}^{j}|\nabla\varphi_i|^2 \ + \ \int_\Omega \rho(x)|\nabla u(t)|dx \le -\nu\sum_{i=1}^{j}|\nabla\varphi_i|^2 + |\rho||\nabla u(t)|, \tag{1.52}$$

where $\rho(x) = \sum_{i=1}^{j}|\varphi_i(x)|^2$ (see [CoFT85, T88]).

Since the functions from V vanish on $\partial\Omega$, we extend these functions by zero outside Ω. Then we obtain the functions $\varphi_i(x),\ x \in \mathbb{R}^2$ belonging to $\left(H^1(\mathbb{R}^2)\right)^2$ that are orthonormal in $\left(L_2(\mathbb{R}^2)\right)^2$. The following result from [LibTh76] is extremely essential.

Lemma 1.4.1 ***(Lieb–Thirring inequality)*** *Let $\varphi_1,\dots,\varphi_j \in \left(H^1(\mathbb{R}^n)\right)^m$ be an orthonormal family of vectors in $\left(L_2(\mathbb{R}^n)\right)^m$. Then for $\rho(x) = \sum_{i=1}^{j}|\varphi_i(x)|^2$ the following inequality holds:*

$$\int_{\mathbb{R}^n}(\rho(x))^{1+2/n}\,dx \le C_{m,n}\sum_{i=1}^{j}\int_{\mathbb{R}^n}|\nabla\varphi_i|^2dx, \tag{1.53}$$

where $C_{m,n}$ depends only on m and n.

Remark 1.4.3 It was proved in [I93] that for $m=2,\ n=2$ in the case $\mathrm{div}\varphi_i = 0$ the following inequality holds: $C_{2,2} \le 2$.

By the variational principle

$$\sum_{i=1}^{j}|\nabla\varphi_i|^2 \ge \lambda_1 + \lambda_2 + \ldots + \lambda_j, \tag{1.54}$$

where $\lambda_1,\lambda_2,\ldots$ are the ordered eigenvalues of the operator L. It is known that $\lambda_i \ge C_0|\Omega|^{-1}i$. Therefore we have

$$\lambda_1 + \lambda_2 + \ldots + \lambda_j \ge C_2\frac{j^2}{|\Omega|},\ \lambda_1 \ge \frac{C_1}{|\Omega|}, \tag{1.55}$$

where C_0, C_1, and C_2 are dimensionless constants that depend on the shape of Ω (see, for example, [Me78]). Using (1.53) with $C_{2,2} = 2$, (1.54), and (1.55), we obtain from (1.52)

$$
\begin{aligned}
&-\nu \sum_{i=1}^{j} |\nabla \varphi_i|^2 + \left(2 \sum_{i=1}^{j} |\nabla \varphi_i|^2 \right)^{1/2} |\nabla u(t)| \\
\leq\ & -\frac{\nu}{2} \sum_{i=1}^{j} |\nabla \varphi_i|^2 + \frac{1}{\nu} |\nabla u(t)|^2 \leq -\frac{\nu C_2 j^2}{2|\Omega|} + \frac{1}{\nu} |\nabla u(t)|^2 .
\end{aligned}
$$

Thus,

$$
\operatorname{Tr}_j(A_u(u(t)) \leq -\frac{\nu C_2 j^2}{2|\Omega|} + \frac{1}{\nu} |\nabla u(t)|^2 .
$$

Then using the estimate

$$
\int_0^t \|u(s)\|^2 ds \leq \frac{|u(0)|^2}{\nu} + \frac{|g|^2}{\nu^2 \lambda_1} t
$$

(see (1.17)) we find that

$$
\tilde{q}_j = \limsup_{T\to\infty} \sup_{u_0 \in \mathcal{A}} \frac{1}{T} \int_0^T \operatorname{Tr}_j(A_u(u(t))dt \leq -\frac{\nu C_2 j^2}{2|\Omega|} + \lim_{T\to\infty} \frac{1}{\nu^2 T} \sup_{u_0\in\mathcal{A}} |u_0|^2 + \frac{|g|^2}{\nu^3 \lambda_1} .
$$

Note that $\sup_{u_0\in\mathcal{A}} |u_0|^2 \leq C_3$, therefore

$$
\tilde{q}_j \leq -\frac{\nu C_2 j^2}{2|\Omega|} + \frac{|g|^2}{\nu^3 \lambda_1} \tag{1.56}
$$

and using the second estimate for λ_1 in (1.55), we find that

$$
\tilde{q}_j \leq -\frac{\nu C_2 j^2}{2|\Omega|} + \frac{|g|^2 |\Omega|}{\nu^3 C_1} =: \varphi(j) = q_j .
$$

We note that the function $\varphi(j)$ is concave in j (like $\cap$). Looking for the root d^* of the equation $\varphi(d) = 0$, we find $d^* = \sqrt{\frac{2}{C_1 C_2}} \frac{|g||\Omega|}{\nu^2}$. Therefore (1.50) and (1.49) immediately follow from Theorem 1.4.1 with $c = \sqrt{\frac{2}{C_1 C_2}}$ (see also Remark 1.4.2). ■

Remark 1.4.4 Using (1.56), estimate (1.49) can be written in the form:

$$
\mathbf{d}_F \, \mathcal{A} \leq c' G, \tag{1.57}
$$

where $G = \frac{|g|}{\nu^2 \lambda_1}$ is the Grashof number and $c' = 2\sqrt{|\Omega|\lambda_1 / C_2}$ depends on the shape of Ω. This estimate was proved in [CoF85, CoFT85] (see also [T88]).

Remark 1.4.5 It was proved in [I96a] that

$$
C_1 \geq 2\pi, \ C_2 \geq \pi,
$$

for every domain Ω with finite measure. Therefore, the constant c in (1.49) satisfies $c \leq 1/\pi$ and for c' in (1.57) we have that $c' \leq 2\sqrt{|\Omega|\lambda_1 / \pi}$. These estimates were improved in [CI04]: $c \leq (2\pi^{3/2})^{-1}$ and $c' \leq \sqrt{|\Omega|\lambda_1} / (\sqrt{2}\pi)$.

Corollary 1.4.1 *Let $g \in H$. Then*

$$\mathbf{d}_F\,\mathcal{A} \le \frac{1}{\sqrt{2\pi}}\,(|\Omega|\lambda_1)^{1/2}\,\frac{|g|}{\nu^2\lambda_1} \le \frac{1}{2\pi^{3/2}}\frac{|g||\Omega|}{\nu^2}. \tag{1.58}$$

Observe that the last estimate in (1.58) contains only the explicit physical parameters of system (1.47) and the estimate $c \le (2\pi^{3/2})^{-1}$ seems the best up-to-date.

Remark 1.4.6 It was proved in Proposition 1.3.3 that $\mathcal{A} = \{z\}$ and, thereby, $\mathbf{d}_F\,\mathcal{A} = 0$ if $G = \frac{|g|}{\nu^2\lambda_1} < \frac{1}{c_0^2}$. Since $\lambda_1 \ge \frac{2\pi}{|\Omega|}$ the last inequality holds if $\frac{|g||\Omega|}{\nu^2} < \frac{2\pi}{c_0^2}$. Using the expression for c_0^2 given in Remark 1.3.1: $c_0^2 = \left(\frac{8}{27\pi}\right)^{1/2}$, we conclude that $\mathcal{A} = \{z\}$ and $\mathbf{d}_F\,\mathcal{A} = 0$ provided that $\frac{|g||\Omega|}{\nu^2} < \left(\frac{27\pi^3}{2}\right)^{1/2} \approx 20.46$.

Remark 1.4.7 Estimates (1.58) and (1.49) are valid for the 2D Navier–Stokes systems in unbounded domains with finite measure (see [I96a] for more details).

Remark 1.4.8 For the 2D Navier–Stokes system (1.47) in $\Omega = [0, 2\pi]^2$ with periodic boundary conditions, estimate (1.57) was improved in [CoFT88], see also [T88]. It was shown there that

$$\mathbf{d}_F\,\mathcal{A} \le c'' G^{2/3}(1 + \log G), \tag{1.59}$$

where $G = \frac{|g|}{\lambda_1\nu^2}$ (note that $\lambda_1 = 1$ in this case). Estimate (1.59) is optimal in some sense (see [Liu93, Zi97]).

Dissipative wave equation

We study the equation

$$\partial_t^2 u + \gamma\partial_t u = \Delta u - f(u) + g(x), \quad u|_{\partial\Omega} = 0, \quad x \in \Omega \Subset \mathbb{R}^3, \tag{1.60}$$

where $\gamma > 0$ (see Section 1.3.2). For brevity, we consider the case $n = 3$. We assume that $g(\,\cdot\,) \in L_2(\Omega)$, $f(v) \in C^2(\mathbb{R};\mathbb{R})$, and f satisfies conditions (1.22), (1.23), and (1.25) with $\rho < 2$. Moreover we assume that

$$|f'(v_1) - f'(v_2)| \le C(|v_1|^{2-\delta} + |u_2|^{2-\delta} + 1)|v_1 - v_2|^\delta,\ 0 \le \delta \le 1. \tag{1.61}$$

The Hilbert space $E = H_0^1(\Omega) \times L_2(\Omega)$ is the phase space for this equation. We also denote the space $E_1 = H^2(\Omega) \times H_0^1(\Omega)$ with norm $\|y\|_{E_1} = (\|u\|_2^2 + \|p\|_1^2)^{1/2}$.

We consider the semigroup $\{S(t)\}$ in E generated by equation (1.61). By Theorem 1.3.2, this semigroup has the global attractor $\mathcal{A} \Subset E$. In the works [BV89, T88] it was proved that the set $\mathcal{A}$ is bounded in E_1 :

$$\|w\|_{E_1} \le M, \quad \forall w \in \mathcal{A},$$

where the constant M is independent of w. Then by the Sobolev embedding theorem

$$\|u(\cdot)\|_{C(\overline{\Omega})} \le M_1, \quad \forall w = (u(\cdot), p(\cdot)) = w(\cdot) \in \mathcal{A}. \tag{1.62}$$

We estimate $\mathbf{d}_F\mathcal{A}$ using Theorem 1.4.1 and the technique described in [GT87] (see also [T88, CV02a]).

Theorem 1.4.3 *For the fractal dimension of the global attractor $\mathcal{A}$ of equation (1.60), the following estimate takes place*

$$\mathbf{d}_F\mathcal{A} \le \frac{C}{\alpha^3}, \tag{1.63}$$

where $\alpha = \min\{\gamma/4, \lambda_1/(2\gamma)\}$ and $C = C(M_1)$ (see (1.62)).

For the ε-entropy of $\mathcal{A}$, the following estimate holds:

$$\mathbf{H}_\varepsilon(\mathcal{A}) \le \frac{C(M_1)}{\alpha^3}\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}), \quad \forall\varepsilon < \varepsilon_0, \tag{1.64}$$

where η, ε_0 are some positive numbers.

Proof. Following [GT87, T88], it is convenient to introduce the new variables

$$w = (u, v) = R_\alpha y = (u, u_t + \alpha u),\ u_t = \partial_t u,\ \alpha = \min\{\gamma/4, \lambda_1/(2\gamma)\},$$

where λ_1 is the first eigenvalue of the operator $-\Delta u,\ u|_{\partial\Omega} = 0$. Using these variables, equation (1.60) is equivalent to the following system:

$$\partial_t w = L_\alpha w - G(w) =: A_\alpha w, \quad w|_{t=0} = w_0, \tag{1.65}$$

where $w_0 \in E$,

$$L_\alpha = \begin{pmatrix} -\alpha I & I \\ \Delta + \alpha(\gamma-\alpha) & -(\gamma-\alpha)I \end{pmatrix}, G(w) = (0, f(u) - g(x)). \tag{1.66}$$

Condition (1.61) implies that the operators $\{S(t)\}$ are uniformly quasidifferentiable on $\mathcal{A}$ and the quasidifferentials $L(t, w_0)z_0 = z(t)$ satisfy the variation equation of problem (1.65):

$$\partial_t z = L_\alpha z - G_w(w)z =: A_{\alpha w}(w(t))z, \quad z|_{t=0} = z_0, \tag{1.67}$$

where $z = (r, q)$ and $G_w(w(t))z = (0, f'(u(t))r)$ (see, e.g., [T88]). We have to estimate the following sum:

$$\sum_{i=1}^{j}(A_{\alpha w}(w(t))\zeta_i, \zeta_i)_E. \tag{1.68}$$

Here $\zeta_i = (r_i, q_i)$ is an arbitrary orthonormal family in E. We estimate the right hand side of (1.68):

$$\begin{aligned}(A_{\alpha w}(w(t))\zeta_i, \zeta_i)_E &= (L_\alpha\zeta_i, \zeta_i) - (f'(u)r_i, q_i) \le -(\alpha/2)\|\zeta_i\|_E^2 + \\ C_0(M_1)\|r_i\|_0\|q_i\|_0 &\le -\alpha/4\left(\|r_i\|_1^2 + \|q_i\|_0^2\right) + (C_1(M_1)/\alpha)\|r_i\|_0^2.\end{aligned} \tag{1.69}$$

The parameter α is chosen in such a way that the operator L_α is negative:

$$(L_\alpha\zeta_i, \zeta_i) \le -\alpha/2\|\zeta_i\|_E^2.$$

Observe that it was essential that

$$\sup\{\|f'(u(t))\|_{C_b} \mid (u(\cdot), \partial_t u(\cdot)) = w(\cdot) \in \mathcal{A}\ \ t \in \mathbb{R}\} \le C_0(M_1) \tag{1.70}$$

(see (1.62)). System ζ_i is orthonormal in E, therefore, it follows from (1.69) that

$$\begin{aligned}\sum_{i=1}^{j}(A_{\alpha w}(w(t))\zeta_i,\zeta_i)_E &\le -(\alpha/4)j+(C_0^2(M_1)/\alpha)\sum_{i=1}^{j}\|r_i\|_0^2\\ \le -(\alpha/4)j+(C_0^2(M_1)/\alpha)\sum_{i=1}^{j}\lambda_i^{-1} &\le -(\alpha/4)j+(C_1(M_1)/\alpha)j^{1/3},\end{aligned}\tag{1.71}$$

where $C_1(M_1)=c_1C_0^2(M_1)$ and $\lambda_i,\ i=1,...,j$, are the first j eigenvalues of the operator $-\Delta u,\ u|_{\partial\Omega}=0$, written in non-decreasing order. It is known that $\lambda_i\ge c_0 i^{2/3}$, therefore, $\sum_{i=1}^{j}\lambda_i^{-1}\le c_1 j^{1/3}$. In the second inequality of (1.71), we have used the inequality

$$\sum_{i=1}^{j}\|r_i\|_0^2\le\sum_{i=1}^{j}\lambda_i^{-1}$$

proved in [T88]. Thus,

$$\mathrm{Tr}_j A_{\alpha w}(w(t))\le\varphi(j)=-(\alpha/4)j+(C_1(M_1)/\alpha)j^{1/3},$$

where the function $\varphi(x)$ is concave. The root of φ is

$$d^*=\frac{8C_1(M_1)^{3/2}}{\alpha^3}=\frac{C(M_1)}{\alpha^3},\ \text{where } C(M)=8C_1(M_1)^{3/2}.$$

Finally we infer (1.64) and (1.63) from Theorem 1.4.1 and Remark 1.4.2. ■

Consider the sine-Gordon equation with $f(u)=\beta\sin(u)$. It is clear that the constant $C_0(M_1)=\beta$ in inequality (1.70) and, therefore, $C_1(M_1)=c_1\beta^2$, that is, $C(M_1)=8c_1^{3/2}\beta^3=c\beta^3$. Thus estimates (1.64) and (1.63) for the sine-Gordon equation has the form

$$\begin{aligned}\mathbf{d}_F(\mathcal{A}) &\le c\frac{\beta^3}{\alpha^3},\\ \mathbf{H}_\varepsilon(\mathcal{A}) &\le c\frac{\beta^3}{\alpha^3}\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right)+\mathbf{H}_{\varepsilon_0}(\mathcal{A}),\ \ \forall\varepsilon<\varepsilon_0,\end{aligned}\tag{1.72}$$

where the constant c depends on Ω.

Ginzburg–Landau equation

We consider the inhomogeneous equation similar to (1.30) from Section 1.3.3

$$\partial_t u=\nu(1+\alpha i)\Delta u+Ru-(1+i\beta)|u|^2u+g(x),\ x\in]0,2\pi[^3=:\mathbb{T}^3,\tag{1.73}$$

with periodic boundary conditions in $\mathbb{T}^3$ and with $g(x)=g^1(x)+ig^2(x)\in L_2(\mathbb{T}^3;\mathbb{C})$. Here ν is a positive parameter. For simplicity, we take $n=3$. We assume that

$$|\beta|\le\sqrt{3}.$$

Then equation (1.73) generates a semigroup $\{S(t)\}$ acting in $\mathbf{H}=(L_2(\mathbb{T}^3))^2$ and having the global attractor $\mathcal{A}$ that is compact in $\mathbf{H}$ (see [T88, CV02a]).

We rewrite equation (1.73) in a vector form (1.32)

$$\partial_t \mathbf{u} = \nu a \Delta \mathbf{u} + R\mathbf{v} - \mathbf{f}(\mathbf{u}) + \mathbf{g}(x), \tag{1.74}$$

where $a = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$, $\mathbf{f}(\mathbf{v}) = |\mathbf{v}|^2 \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix} \mathbf{v}$, and $\mathbf{g}(x) = (g^1(x), g^2(x))^\top$. In [BV83a], it is proved that the semigroup $\{S(t)\}$ is uniformly quasidifferentiable on $\mathcal{A}$ and the corresponding variational equation reads

$$\partial_t \mathbf{v} = \nu a \Delta \mathbf{v} + R\mathbf{v} - \mathbf{f}_{\mathbf{u}}(\mathbf{u})\mathbf{v}, \ \mathbf{v}|_{t=0} = \mathbf{v}_0 \in \mathbf{H}, \tag{1.75}$$

where the matrix $\mathbf{f}_{\mathbf{u}}(\mathbf{u})$ is given in (1.33). It follows from (1.34) that

$$\begin{aligned} \langle \nu a \Delta \mathbf{v} + R\mathbf{v} - \mathbf{f}_{\mathbf{u}}(\mathbf{u})\mathbf{v}, \mathbf{v} \rangle &= -\nu \|\nabla \mathbf{v}\|^2 + R\|\mathbf{v}\|^2 - \langle \mathbf{f}_{\mathbf{u}}(\mathbf{u})\mathbf{v}, \mathbf{v} \rangle \\ &\leq -\nu \|\nabla \mathbf{v}\|^2 + R\|\mathbf{v}\|^2, \ \forall \mathbf{v} \in \mathbf{H}^2. \end{aligned} \tag{1.76}$$

To apply Theorem 1.4.1 and to estimate $\mathbf{d}_F(\mathcal{A})$ we have to study the j-trace of the operator in the right-hand side of (1.75). Using (1.76) we have

$$\begin{aligned} \sum_{i=1}^{j} (A_u(u(t))\varphi_i, \varphi_i) &= \sum_{i=1}^{j} -\nu \|\nabla \varphi_i\|^2 + R\|\varphi_i\|^2 - \langle \mathbf{f}_{\mathbf{u}}(\mathbf{u})\varphi_i, \varphi_i \rangle \\ &\leq \sum_{i=1}^{j} -\nu \|\nabla \varphi_i\|^2 + R\|\varphi_i\|^2 = -\nu \sum_{i=1}^{j} \|\nabla \varphi_i\|^2 + Rj, \end{aligned} \tag{1.77}$$

where $\{\varphi_i, i = 1, \ldots, j\}$ is an arbitrary set of functions from $\mathbf{V} = (H^1(\mathbb{T}^3))^2$ that is orthonormal in $\mathbf{H}$.

By the variational principle

$$\sum_{i=1}^{j} |\nabla \varphi_i|^2 \geq \lambda_1 + \lambda_2 + \ldots + \lambda_j, \tag{1.78}$$

where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of the operator $-\Delta$ in $\mathbf{H}$. It is well known that the eigenvalues of this operators have the form $k_1^2 + k_2^2 + k_3^2$, where $(k_1, k_2, k_3) \in (\mathbb{Z}_+)^3$. Therefore, $\lambda_i \geq C_0 i^{2/3}$ and

$$\lambda_1 + \lambda_2 + \ldots + \lambda_j \geq C_1 j^{5/3} \tag{1.79}$$

for some constants C_0 and C_1. Using (1.78) and (1.79) in (1.77), we obtain

$$Tr_j A_u(u(t)) \leq -\nu C_1 j^{5/3} + Rj = \varphi(j), \ \forall j = 1, 2, \ldots \tag{1.80}$$

The function $\varphi(x) = -\nu C_1 x^{5/3} + Rx$ is concave and it root $d^* = \left(\frac{R}{C_1 \nu}\right)^{3/2}$. Thus, we have proved

Theorem 1.4.4 *The fractal dimension of the global attractor $\mathcal{A}$ of equation (1.73) admits the estimate*

$$\mathbf{d}_F(\mathcal{A}) \leq \left(\frac{R}{C_1 \nu}\right)^{3/2}, \tag{1.81}$$

where C_1 is an absolute constant taken from (1.79) (C_1 can be estimated explicitly, see, e.g. [LiYa83, CV02a]).

The ε-entropy of $\mathcal{A}$ satisfies the inequality

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leq \left(\frac{R}{C_1\nu}\right)^{3/2} \log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}),\ \forall \varepsilon < \varepsilon_0, \tag{1.82}$$

where η and ε_0 are some small positive numbers.

Chapter 2

Attractors of non-autonomous equations

In this chapter, we consider general processes and we study their global attractors. The notion of a process is used to describe the behaviour of non-autonomous dynamical systems. A process is a generalization of the notion of a semigroup that play a key role in the study of autonomous dynamical systems. Non-autonomous dynamical systems and their global attractors are discussed in the books A.Haraux [Ha91], V.V.Chepyzhov and M.I.Vishik [CV02a] (see also D.N.Cheban and D.S.Fakeeh [CheFa94]).

In Section 2.1, we study processes $\{U(t,\tau), t \geq \tau\}$ and their uniform global attractors. Recall, that processes are generated by non-autonomous evolution equations of mathematical physics, when, for example, an external force or some other terms of equation depend explicitly on time t. If the Cauchy problem for this equation is well-posed, then the corresponding process $\{U(t,\tau)\}$ maps the value of a solution $u(\tau)$ at time $\tau \in \mathbb{R}$ into the value of this solution $u(t)$ at time $t \geq \tau$. We give the definition of a general process $\{U(t,\tau)\}$ and we define notions of uniformly absorbing and attracting sets of a process. We study the main properties of ω-limit sets for bounded sets. Then we define the uniform global attractor $\mathcal{A}$ of a process $\{U(t,\tau)\}$. We prove the theorem on the existence of a uniform global attractor of a process using the notion of the ω-limit set. We also define the kernel $\mathcal{K}$ of a process and study its properties.

In Section 2.2, we consider uniform and non-uniform global attractors of a process and compare their properties. In particular, we present an example of a non-autonomous equation given by A.Haraux. This example shows that the uniform global attractor can be larger than the non-uniform one. We also study periodic process for which uniform and non-uniform global attractors always coincide.

In Section 2.4, we introduce the notion of a time symbol $\{\sigma(t), t \in \mathbb{R}\}$ of a non-autonomous equation. Roughly speaking, the time symbol is the collection of all terms of the equation that depend on time. We define the hull $\mathcal{H}(\sigma)$ of a symbol σ. We also define the notion of a translation compact function. We mostly study non-autonomous equations having translation compact symbols $\sigma(t)$. We present translation compactness criterions in various topological spaces that are used in the sequel.

In Section 2.5, we formulate the main theorem on the existence and the structure of the uniform global attractor of a process $\{(U_\sigma(t,\tau\}$ of a non-autonomous equation with translation compact symbol $\sigma(t)$.

In Section 2.6.1, we study the uniform global attractor of the non-autonomous 2D

Navier–Stokes system with translation compact external force. A special attention is given to the case, where the system has a unique bounded complete solution that attracts any other solution as $t \to +\infty$ with exponential rate. In Sections 2.6.2 and 2.6.3, we consider analogous problems for the non-autonomous hyperbolic equation with dissipation and for the non-autonomous complex Ginzburg–Landau equation with translation compact terms.

2.1 Processes and their uniform global attractors

Let E be a complete metric space or a Banach space and let a two-parameter family of operators $\{U(t,\tau), \tau \in \mathbb{R}, t \geq \tau\}$, $U(t,\tau) : E \to E$, be given.

Definition 2.1.1 A family of mappings $\{U(t,\tau)\} := \{U(t,\tau), \tau \in \mathbb{R}, t \geq \tau\}$ in E is said to be a *process* if

1. $U(\tau,\tau) = Id$ for all $\tau \in \mathbb{R}$, where Id is the identity operator;

2. $U(t,s) \circ U(s,\tau) = U(t,\tau)$ for all $t \geq s \geq \tau,\ \tau \in \mathbb{R}$.

As in Chapter 1, by $\mathcal{B}(E)$ we denote the family of all bounded (in the norm of E) sets in E. The process $\{U(t,\tau)\}$ is called (E,E)-*bounded*, if $U(t,\tau)B \in \mathcal{B}(E)$ for every $B \in \mathcal{B}(E)$, for all $\tau \in \mathbb{R}$, and for all $t \geq \tau$. The process $\{U(t,\tau)\}$ is called *uniformly* (E,E)-*bounded* if for every $B \in \mathcal{B}(E)$ there exists $B_1 \in \mathcal{B}(E)$ such that $U(t,\tau)B \subset B_1$ for all $\tau \in \mathbb{R}$, $t \geq \tau$.

The following two notions describe dissipativity properties for non-autonomous dynamical systems. A set $B_0 \subset E$ is said to be *uniformly* (*w.r.t.* $\tau \in \mathbb{R}$) *absorbing* for the process $\{U(t,\tau)\}$ if for any set $B \in \mathcal{B}(E)$ there is a number $h = h(B)$ such that

$$U(t,\tau)B \subseteq B_0 \text{ for all } t \text{ and } \tau,\ t - \tau \geq h. \tag{2.1}$$

A set $P \subset E$ is said to be *uniformly* (*w.r.t.* $\tau \in \mathbb{R}$) *attracting* for the process $\{U(t,\tau)\}$ if, for every $\varepsilon > 0$, the set $\mathcal{O}_\varepsilon(P)$ is uniformly absorbing for this process (here and below $\mathcal{O}_\varepsilon(M)$ denotes an ε-neighborhood of a set M in the space E), that is, for every bounded set $B \in \mathcal{B}(E)$, there exists a number $h = h(\varepsilon, P)$ such that

$$U(t,\tau)B \subseteq \mathcal{O}_\varepsilon(P) \text{ for all } t \text{ and } \tau,\ t - \tau \geq h. \tag{2.2}$$

Property (2.2) can also be formulated in the following manner: for every set $B \in \mathcal{B}(E)$

$$\sup_{\tau \in \mathbb{R}} \operatorname{dist}_E \left(U\left(\tau + h, \tau\right) B, P\right) \to 0 \ \ (h \to +\infty)\,. \tag{2.3}$$

Here $\operatorname{dist}_E(X,Y)$ denotes the Hausdorff distance from the set X to the set Y in the space E (see (1.3)).

A process having a compact uniformly absorbing set is called *uniformly compact* and a process having a compact uniformly attracting set is called *uniformly asymptotically compact*.

We now define the notion of the *uniform global attractor* $\mathcal{A}$ of a process $\{U(t,\tau)\}$.

Definition 2.1.2 A set $\mathcal{A} \subset E$ is called the *uniform (w.r.t. $\tau \in \mathbb{R}$) global attractor* of a process $\{U(t,\tau)\}$ if it is closed in E, is uniformly attracting for the process $\{U(t,\tau)\}$ and satisfies the following *property of minimality*: $\mathcal{A}$ belongs to any closed uniformly attracting set of the process.

It is easy to see that any process has at most one uniform global attractor. The notion of a uniform global attractor was introduced in [Ha91] (see also [CV92a, CV93d, CV94a, CV02a]).

For an arbitrary set $B \in \mathcal{B}(E)$, we define the uniform ω-limit set $\omega(B)$:

$$\omega(B) = \bigcap_{h \geq 0} \left[\bigcup_{t-\tau \geq h} U(t,\tau)B \right]_E . \tag{2.4}$$

In (2.4), the square brackets $[\,\cdot\,]_E$ denote the closure in the space E and the union $\bigcup_{t-\tau\geq h}$ is taken for all t,τ such that $\tau \in \mathbb{R}$ and $t \geq \tau + h$ (compare with (1.4)).

Proposition 2.1.1 *If a process $\{U(t,\tau)\}$ in E has a compact uniformly attracting set P, then for any $B \in \mathcal{B}(E)$*

(i) $\omega(B) \neq \emptyset$, *$\omega(B)$ is compact in E, and $\omega(B) \subseteq P$;*

(ii) $\sup_{\tau\in\mathbb{R}} \operatorname{dist}_E(U(h+\tau,\tau)B, \omega(B)) \to 0$ $(h \to +\infty)$;

(iii) *if Y is closed and* $\sup_{\tau\in\mathbb{R}} \operatorname{dist}_E(U(h+\tau,\tau)B, Y)$ $(h \to +\infty)$, *then* $\omega(B) \subseteq Y$.

Proof. From the definition (2.4) of $\omega(B)$, it follows that

$$y \in \omega(B) \Leftrightarrow \left\{ \begin{array}{l} \text{there are sequences } \{x_n\} \subseteq B,\ \{\tau_n\} \subseteq \mathbb{R}, \text{and } \{h_n\} \subset \mathbb{R}_+ \\ \text{such that } h_n \to +\infty \text{ and } U(\tau_n + h_n, \tau_n)x_n \to y \ (n \to \infty). \end{array} \right. \tag{2.5}$$

(i) Let us show that $\omega(B) \neq \emptyset$. For any fixed $\tau \in \mathbb{R}$ and $x \in B$, we consider an arbitrary positive sequence $\{h_n\}$, $h_n \to +\infty$ $(n \to \infty)$. According to the uniformly attracting property (2.3), $\operatorname{dist}_E(U(\tau + h_n, \tau)x, P) \to 0$ $(n \to \infty)$, that is, for some sequence $\{y_n\} \subseteq P$

$$\|U(\tau + h_n, \tau)x - y_n\|_E \to 0 \ (n \to \infty).$$

The set P is compact, therefore we can extract from $\{y_n\}$ a subsequence $\{y_{n'}\}$ converging to a point $y \in P$. Hence $U(\tau + h_{n'}, \tau)x \to y$ $(n' \to \infty)$. Having (2.5), we deduce that the constructed point $y \in \omega(B)$, that is, $\omega(B) \neq \emptyset$. Let us verify that $\omega(B) \subseteq P$. Let $y \in \omega(B)$ and let $\{x_n\} \subseteq B$, $\{\tau_n\} \subseteq \mathbb{R}$, $\{h_n\} \subset \mathbb{R}_+$ be sequences defined in (2.5). Using the uniform attracting property of P (see (2.3)), we have

$$\operatorname{dist}_E(U(\tau_n + h_n, \tau_n)x_n, P) \to 0 \ (n \to \infty).$$

Therefore $\operatorname{dist}_E(y, P) = 0$. The set P is closed, that is, $y \in P$ for all $y \in \omega(B)$ and $\omega(B) \subseteq P$. This implies also that $\omega(B)$ is compact, since $\omega(B)$ is closed by definition (see (2.4)).

(ii) Assume the converse: for some $B \in \mathcal{B}(E)$,

$$\sup_{\tau\in\mathbb{R}} \mathrm{dist}_E(U(\tau+h,\tau)B, \omega(B)) \not\to 0 \ (h \to +\infty).$$

That is, for some sequences $\{x_n\} \subseteq B$, $\{\tau_n\} \subseteq \mathbb{R}$, $\{h_n\} \subset \mathbb{R}_+$ ($h_n \to +\infty$) we have

$$\mathrm{dist}_E(U(\tau_n+h_n,\tau_n)x_n, \omega(B)) \geq \delta > 0 \, \forall n \in \mathbb{N}. \tag{2.6}$$

The uniform attracting property of P implies that

$$\mathrm{dist}_E(U(\tau_n+h_n,\tau_n)x_n, P) \to 0 \ (n \to \infty),$$

So once again, we find a sequence $\{y_n\} \subset P$ such that

$$\|U(\tau_n+h_n,\tau_n)x_n - y_n\|_E \to 0 \ (n \to \infty).$$

The set P is compact and we may assume by refining that $y_n \to y$ $(n \to \infty)$ for some $y \in P$, that is,

$$U(\tau_n+h_n,\tau_n)x_n \to y \ \ (n \to \infty),$$

and it follows from (2.5) that $y \in \omega(B)$. However, (2.6) implies that $\mathrm{dist}_E(y, \omega(B)) \geq \delta > 0$, which is a contradiction.

(iii) Let Y be a closed uniformly attracting set of the process $\{U(t,\tau)\}$. If $y \in \omega(B)$, then, in view of (2.5), for some sequences $\{x_n\} \subseteq B$, $\{\tau_n\} \subseteq \mathbb{R}$, $\{h_n\} \subset \mathbb{R}_+$ we have

$$U(\tau_n+h_n,\tau_n)x_n \to y \ (h_n \to \infty).$$

Since Y is a uniformly attracting set, it follows that

$$\mathrm{dist}_E(U(\tau_n+h_n,\tau_n)x_n, Y) \to 0 \ (n \to \infty)$$

and, consequently, $\mathrm{dist}(y, Y) = 0$, that is, $y \in Y$ for all $y \in \omega(B)$ and, hence, $\omega(B) \subseteq Y$. The proposition is completely proved. ■

Using Proposition 2.1.1, we formulate the following important

Theorem 2.1.1 *If a process $\{U(t,\tau)\}$ is uniformly asymptotically compact, then it has a compact (in E) uniform global attractor $\mathcal{A}$.*

Proof. We claim that the set

$$\mathcal{A} = \left[\bigcup_{n\in\mathbb{N}} \omega(B_n)\right]_E, \tag{2.7}$$

(where $B_n = \{x \in E \mid \|x\|_E \leq n\}$ is the ball in E of radius $n \in \mathbb{N}$) is the required uniform global attractor. Indeed, for the set $\mathcal{A}$ defined in (2.7), we have $\mathcal{A} \subseteq P$ (see Proposition 2.1.1 **(i)**). Moreover, if $B \subseteq \mathcal{B}(E)$, then $B \subseteq B_n$ for some $n \in \mathbb{N}$ and, consequently, $\omega(B) \subseteq \omega(B_n) \subseteq \mathcal{A}$, i.e., $\mathcal{A}$ uniformly attracts $U_\sigma(t,\tau)B$ (Proposition 2.1.1 **(ii)**). At the same time by Proposition 2.1.1 **(iii)**, the set $\omega(B_n)$ belongs to every closed uniformly attracting set. Therefore, the property of minimality is valid for $\mathcal{A}$ defined in (2.7). ■

Remark 2.1.1 To this end, we can not claim that $\mathcal{A} = \omega(P)$, where P is any compact *uniformly attracting* set for $\{U(t,\tau)\}$. We clearly have the inclusion $\omega(P) \subseteq \mathcal{A}$, since $P \subseteq B_N$ for a large N, so $\omega(P) \subseteq \omega(B_N)$ and, therefore, $\omega(P) \subseteq \mathcal{A}$. At the same time in the general case, the inverse inclusion remains unclear because we do not know whether $\omega(B) \subseteq \omega(P)$ for any $B \subseteq \mathcal{B}(E)$. However, if B_0 is a compact *uniformly absorbing* set, then apparently

$$\mathcal{A} = \omega(B_0) = \bigcap_{h \geq 0} \left[\bigcup_{t-\tau \geq h} U(t,\tau) B_0 \right]_E .$$

Having a compact *uniformly attracting* set P, the equality $\mathcal{A} = \omega(P)$ can be also proved under some additional assumptions of continuity for the process $\{U(t,\tau)\}$ (see, for example, Theorem 1.1.1 for an autonomous case and [CV02a] for non-autonomous cases).

Remark 2.1.2 In Theorem 2.1.1, we do not assume that the process $\{U(t,\tau)\}$ is continuous in E. (This assumption was essential in the theorems on the existence of global attractors of semigroups corresponding to autonomous evolution equations.) The reason is that we use only the property of minimality in the definition of a global attractor.

To describe the general structure of the uniform global attractor of a process we need the notion of the *kernel* of the process that generalizes the notion of a kernel of a semigroup.

A function $u(s), s \in \mathbb{R}$, with values in E is said to be a *complete trajectory* of the process $\{U(t,\tau)\}$ if

$$U(t,\tau)u(\tau) = u(t) \text{ for all } t \geq \tau, \tau \in \mathbb{R}. \tag{2.8}$$

A complete trajectory $u(s)$ is called *bounded* if the set $\{u(s),\ s \in \mathbb{R}\}$ is bounded in E.

Definition 2.1.3 The *kernel* $\mathcal{K}$ of the process $\{U(t,\tau)\}$ is the family of all bounded complete trajectories of this process:

$$\mathcal{K} = \{u(\cdot) \mid u \text{ satisfies (2.8) and } \|u(s)\|_E \leq C_u \ \forall s \in \mathbb{R}\} .$$

The set

$$\mathcal{K}(t) = \{u(t) \mid u(\cdot) \in \mathcal{K}\} \subset E,\ t \in \mathbb{R}$$

is called the *kernel section* at time t.

It is not difficult to prove the following

Proposition 2.1.2 *If the process $\{U(t,\tau)\}$ has the global attractor $\mathcal{A}$, then*

$$\bigcup_{t \in \mathbb{R}} \mathcal{K}(t) \subseteq \mathcal{A}. \tag{2.9}$$

Comparing (2.9) with identity (1.6) in autonomous case, we note that, in non-autonomous case, first, $\mathcal{K}(t)$ may depend on time t and, second, the inclusion in (2.9) can be strict, that is, in order to describe the structure of the global attractor $\mathcal{A}$ of a process $\{U(t,\tau)\}$ it is not sufficient to know the structure of the kernel $\mathcal{K}$. The discussion of this problem will be continued in Section 2.5.

2.2 On non-uniform global attractors of processes and Haraux's example

Following A.Haraux [Ha88, Ha91], we define also a (non-uniform) global attractor of a process $\{U(t,\tau)\}$ acting in E. A set P_0 is called a (non-uniform) attracting set of $\{U(t,\tau)\}$ if for any bounded set $B \in \mathcal{B}(E)$ and for any *fixed* $\tau \in \mathbb{R}$

$$\operatorname{dist}_E\left(U\left(t,\tau\right)B, P_0\right) \to 0 \ \left(t \to +\infty\right), \tag{2.10}$$

that is, for any $\varepsilon > 0$, there exists a number $T = T(\tau, B, \varepsilon) \geq \tau$ such that

$$U(t,\tau)B \subseteq \mathcal{O}_\varepsilon(P_0) \text{ for all } t \geq T. \tag{2.11}$$

A process having a compact attracting set is called *asymptotically compact*. Similarly to Definition 2.1.2, we formulate

Definition 2.2.1 A set $\mathcal{A}_0 \subset E$ is called the (non-uniform) *global attractor* of a process $\{U(t,\tau)\}$ if it is closed in E, is attracting for the process $\{U(t,\tau)\}$ and satisfies the *property of minimality*: $\mathcal{A}_0$ belongs to any closed attracting set of the process.

Similarly to Theorem 2.1.1, we prove

Theorem 2.2.1 *If a process $\{U(t,\tau)\}$ is asymptotically compact, then it has a compact (non-uniform) global attractor $\mathcal{A}_0$.*

It is obvious that a uniformly asymptotically compact process $\{U(t,\tau)\}$ is (non-uniformly) asymptotically compact as well and, thereby, $\mathcal{A}_0 \subseteq \mathcal{A}$. However, it was pointed out by A.Haraux that this inclusion can be strict, i.e. the uniform global attractor can be larger than the non-uniform one. We now present the example from [Ha88, Ha91].

We consider the following non-autonomous ordinary differential equation in $\mathbb{R}$:

$$d_t u + a(t)u + u^3 = 0 \ (d_t = d/dt) \tag{2.12}$$

with initial data

$$u|_{t=\tau} = u_\tau, \ u_\tau \in \mathbb{R}, \tag{2.13}$$

where

$$a(t) = \sum_{n=1}^{\infty} n^{-2} \sin(2n^{-4}t). \tag{2.14}$$

The function $a(t)$ is almost periodic (see Example 2.4.1) since it is the uniform limit of almost periodic (and even quasiperiodic) functions. Equation (2.12) generates a process $\{U(t,\tau)\}$ in $\mathbb{R} : U(t,\tau)u_\tau = u(t), \ t \geq \tau, \ \tau \in \mathbb{R}$, where $u(t)$ is a solution of (2.12), (2.13) with initial data u_τ. We set

$$A(t) = \int_0^t a(s)ds = \sum_{n=1}^{\infty} n^2 \sin^2(n^{-4}t), \ t \in \mathbb{R}. \tag{2.15}$$

Let us find a (*non-uniform*) global attractor of the process $\{U(t,\tau)\}$. It follows from (2.12) that

$$d_t u^2 = -2a(t)u^2 - 2u^4 \leq -2a(t)u^2 \tag{2.16}$$

and therefore

$$u^2(t) \leq u^2(\tau)\exp\left(2A(\tau)\right)\exp\left(-2A(t)\right),\ \forall t \geq \tau.$$

Taking $n = [|t|^{1/4}] + 1$ in (2.15) we obtain

$$A(t) \geq c|t|^{1/2}\ \forall t \in \mathbb{R} \tag{2.17}$$

for some $c > 0$. Hence, $u(t) \to 0$ $(t \to +\infty)$ and, moreover, $U(t,\tau)B \to 0$ $(t \to \infty)$ for each fixed $\tau \in \mathbb{R}$ and for any bounded set $B \in \mathcal{B}(\mathbb{R})$. We conclude that the process $\{U(t,\tau)\}$ has a (non-uniform) global attractor $\mathcal{A}_0 = \{0\}$, that is, a single point.

Let us study the *uniform* global attractor of the process $\{U(t,\tau)\}$. First of all, this process is uniformly compact, i.e., it has a compact (bounded in $\mathbb{R}$) uniformly absorbing set. Indeed, the function $a(t)$ is bounded, so,

$$-2a(t)u^2 - 2u^4 \leq 2Ru^2 - 2u^4 \leq -\gamma u^2 + C$$

for appropriate positive $R,\ \gamma$, and C. In view of (2.16), we obtain

$$\begin{aligned} d_t u^2 &\leq -\gamma u^2 + C, \\ u^2(t) &\leq u^2(\tau)\exp\left(-\gamma(t-\tau)\right) + C/\gamma \end{aligned}$$

and hence the set $B_0 = \{|u|^2 \leq 2C/\gamma\}$ is uniformly absorbing for the process $\{U(t,\tau)\}$. The set B_0 is compact and, by Theorem 2.1.1, the uniform global attractor $\mathcal{A}$ exists. Clearly, $\{0\} = \mathcal{A}_0 \subseteq \mathcal{A}$. We claim that $\mathcal{A} \neq \{0\}$.

It is sufficient to prove that there exists a nonzero *bounded* solution $\tilde{u}(t)$ of equation (2.12) defined for all $t \in \mathbb{R}$. Such a solution belongs to the kernel $\mathcal{K}$ of the process $\{U(t,\tau)\}$ and, from (2.9), we have that $\left\{\bigcup_{t\in\mathbb{R}} \tilde{u}(t)\right\} \subseteq \mathcal{A}$, so, $\mathcal{A}$ is large than $\mathcal{A}_0 = \{0\}$.

Integrating (2.16) we obtain

$$\begin{aligned} d_t\left(u^2 e^{2A(t)}\right) + 2u^4 e^{2A(t)} &= 0, \\ d_t\left(v\right) + 2v^2 e^{-2A(t)} &= 0 \end{aligned}$$

where $v(t) = u^2(t)e^{2A(t)}$. Integrating once more, we obtain

$$\frac{1}{v(t)} = \frac{1}{v(0)} + 2\int_0^t e^{-2A(s)}ds.$$

Notice that $e^{-2A(s)} \in L_1(\mathbb{R};\mathbb{R}_+)$ due to (2.17). Finally,

$$\tilde{u}(t) = \pm\left(\frac{e^{-2A(t)}}{\frac{1}{|u_0|^2} + 2\int_0^t e^{-2A(s)}ds}\right)^{1/2},\ t \in \mathbb{R}$$

is the desired solution of (2.12), if $\frac{1}{|u_0|^2} > 2\int_{-\infty}^0 e^{-2A(s)}ds$. The sign of $\tilde{u}$ coincides with the sign of u_0. Indeed, $\tilde{u}$ satisfies equation (2.12) for all $t \in \mathbb{R}$ and is bounded in $\mathbb{R}$.

Notice that in the case of a periodic process, its uniform global attractor coincides with the non-uniform one (see [VC95, CV94b] for more details). For the readers convenience, we now present a simple result concerning the periodic processes.

A process $\{U(t,\tau)\}$ is called *periodic* with period p if

$$U(t+p,\tau+p)=U(t,\tau),\ \forall t\geq\tau,\ \tau\in\mathbb{R}. \tag{2.18}$$

Having a periodic process $\{U(t,\tau)\}$, to prove that a set P is uniformly attracting for $\{U(t,\tau)\}$ it is sufficient to show, instead (2.3), the limit relation

$$\sup_{\tau\in[0,p)}\operatorname{dist}_E\left(U\left(\tau+h,\tau\right)B,P\right)\to 0\ \ (h\to+\infty)\,. \tag{2.19}$$

Indeed, for an arbitrary $\tau\in\mathbb{R}$ we have that $\tau=\tau'+np$, where $\tau'\in[0,p)$ and $n\in\mathbb{Z}$. Therefore, by periodicity,

$$U\left(h+\tau,\tau\right)B=U\left(h+\tau'+np,\tau'+np\right)B=U\left(h+\tau',\tau'\right)B$$

and (2.19) implies (2.3). Using this observation, we have

Theorem 2.2.2 *If a periodic process $\{U(t,\tau)\}$ is uniformly bounded and has a compact (non-uniformly) attracting set, then it is uniformly asymptotically compact. In particular, the process $\{U(t,\tau)\}$ has both uniform and non-uniform global attractors $\mathcal{A}$ and $\mathcal{A}_0$ which coincide, $\mathcal{A}=\mathcal{A}_0$.*

Proof. Let $P_0\Subset E$ be a compact attracting set of the periodic process $\{U(t,\tau)\}$ with period p. It follows from Theorem 2.2.1 that this process has a (non-uniform) global attractor $\mathcal{A}_0$.

Consider an arbitrary bounded set $B\in\mathcal{B}(E)$. Since the process $\{U(t,\tau)\}$ is uniformly bounded the set

$$\tilde{B}=\bigcup_{\tau\in[0,p)}U(p,\tau)B\in\mathcal{B}(E).$$

The set P_0 is (non-uniformly) attracting, therefore, for $\tau=p$,

$$\operatorname{dist}_E\Big(U\left(t,p\right)\tilde{B},P_0\Big)\to 0\ \ (t\to+\infty) \tag{2.20}$$

We now observe that, for all $\tau\in[0,p)$,

$$U(t,\tau)B=U(t,p)U(p,\tau)B\subseteq U(t,p)\tilde{B},\ \forall t\geq p.$$

Then, from (2.20), we conclude that

$$\sup_{\tau\in[0,p)}\operatorname{dist}_E\left(U\left(\tau+h,\tau\right)B,P_0\right)\leq\operatorname{dist}_E\Big(U\left(t,p\right)\tilde{B},P_0\Big)\to 0\ \ (t\to+\infty)$$

and relation (2.19) is proved for the set P_0. Therefore, the process $\{U(t,\tau)\}$ is uniformly asymptotically compact. Repeating the above reasoning for $\mathcal{A}_0$ in place of P_0, we conclude that the set $\mathcal{A}_0$ is uniformly attracting. At the same time, $\mathcal{A}_0$ is the minimal uniformly attracting set since it is minimal (non-uniformly) attracting. Thus, $\mathcal{A}_0=\mathcal{A}$ is the uniform global attractor of the periodic process $\{U(t,\tau)\}$. ■

In our paper, we study mostly uniform global attractors of process corresponding to non-autonomous evolution equations.

2.3 Cauchy problem and the corresponding process

We now explain how to construct a process corresponding to a non-autonomous evolution equation. We consider a non-autonomous evolution equation of the form:

$$\partial_t u = A(u,t),\ t \geq \tau\ (\tau \in \mathbb{R}). \tag{2.21}$$

Here $A(u,t)$ denotes a nonlinear operator $A(\cdot,t) : E_1 \to E_0$ for every $t \in \mathbb{R}$, where E_1 and E_0 are Banach spaces such that $E_1 \subseteq E_0$. We study solutions $u(t)$ of this equation that are defined for all $t \geq \tau$. For $t = \tau$ we consider the initial condition:

$$u(\tau) = u|_{t=\tau} = u_\tau,\ u_\tau \in E, \tag{2.22}$$

where E is a Banach space such that $E_1 \subseteq E \subseteq E_0$. We assume that for every $\tau \in \mathbb{R}$ and for all $u_\tau \in E$, the Cauchy problem (2.21), (2.22) has a unique solution $u(t)$ such that $u(t) \in E$ for all $t \geq \tau$. The meaning of the expression "the function $u(t)$ is a solution of problem (2.21), (2.22)" should be clarified in each particular example. Similarly to a solution of autonomous equation (1.7), solutions $u(t)$, $\tau \leq t \leq T$, of (2.21) are taken from the class $\mathcal{F}_{\tau,T}$ of functions satisfying the conditions $u \in L_\infty(\tau,T;E)$ and $u \in L_p(\tau,T;E_1)$. We also assume that $A(u,t) \in L_q(\tau,T;E_0)$ for some q, $1 < q < \infty$, and $\partial_t u \in L_q(\tau,T;E_0)$. Equality (2.21) holds in the space $L_q(\tau,T;E_0)$. Thus, a function $u(t)$ from $\mathcal{F}_{\tau,T}$ should satisfy (2.21) in the distribution space $\mathcal{D}'(]\tau,T[;E_0)$ (for the details, see [Lio69, BV89, CV02a]). In order to assign a meaning to the initial condition (2.22), various embedding theorems can be used (see, e.g., [LioM68, T79]).

We study the following two-parametric family of operators $\{U(t,\tau)\}$, $t \geq \tau$, $\tau \in \mathbb{R}$, generated by problem (2.21), (2.22) and acting in E by the formula

$$U(t,\tau)u_\tau = u(t),\ t \geq \tau,\ \tau \in \mathbb{R}, \tag{2.23}$$

where $u(t)$ is a solution of (2.21), (2.22) with initial data $u_\tau \in E$. Since the Cauchy problem (2.21), (2.22) is uniquely solvable, the family of operators $\{U(t,\tau)\}$ satisfies the properties from Definition 2.1.1. Thus, the constructed family of operators $\{U(t,\tau)\}$ is called the process corresponding to problem (2.21), (2.22).

In the next sections, we are going to study global attractors of processes corresponding to various non-autonomous dissipative evolution equations arising in mathematical physics.

2.4 Time symbols of non-autonomous equations

General Theorem 2.1.1 is applicable to processes generated by non-autonomous evolution equations. However, this basic theorem adds little to the knowledge of the structure of the uniform global attractor. To say more we have to study some extra properties of processes. In this connection, the notion of a kernel of a process is very useful (see Definition 2.1.3). Recall that the kernel of equation (2.21) is the union of all bounded complete solutions $u(t), t \in \mathbb{R}$, of this equation that are defined on the entire time axis $\{t \in \mathbb{R}\}$.

Having the global attractor $\mathcal{A}$ of non-autonomous equation (2.21), we always have inclusion (2.9). However, in the generic case, this inclusion can be strict, that is, there exist points of the global attractor $\mathcal{A}$ that are not values of bounded complete trajectories of the original equation (2.21) (see Remark 2.6.2). Nevertheless, we shall show that such points lie on the complete trajectories of "contiguous" equations. To describe these "contiguous" equations we introduce the notion of *time symbol* of the equation under the consideration. Speaking informally, the time symbol reflects the dependence on time of the right-hand side of a non-autonomous equation. We assume that all the terms of equation (2.21) that depend explicitly on time t can be written as a function $\sigma(t), t \in \mathbb{R}$, with values in an appropriate Banach space Ψ. We now rewrite equation (2.21) itself in the form:

$$\partial_t u = A_{\sigma(t)}(u),\ t \geq \tau\ (\tau \in \mathbb{R}). \tag{2.24}$$

The function $\sigma(t)$ is said to be the *time symbol* of the equation. In applications, $\sigma(t)$ consists of the coefficients and terms of the equation that depend on time. For example, in the non-autonomous Navier–Stokes system

$$\partial_t u + \nu L u + B(u, u) = g(x, t)$$

with time dependent external force $g(x, t) \in C_{\mathrm{b}}(\mathbb{R}; H)$ the time symbol is $\sigma(t) = g(x, t)$. (This example will be studied in Section 2.6.1 in great details.)

We assume that the symbol $\sigma(t)$, as a function of time t, belongs to an enveloping space

$$\Xi := \{\xi(t), t \in \mathbb{R} \mid \xi(t) \in \Psi \text{ for almost all } t \in \mathbb{R}\},$$

equipped with a Hausdorff topology. In the above example of the 2D Navier–Stokes system, $\Psi = H$ and the space $\Xi = C_{\mathrm{b}}(\mathbb{R}; H)$ can be taken as an enveloped space of this non-autonomous equation. Recall that a function $g(x, t) \in C_{\mathrm{b}}(\mathbb{R}; H)$ if

$$\|g(\cdot, \cdot)\|_{C_{\mathrm{b}}(\mathbb{R};H)} := \sup\{\|g(\cdot, t)\|_H,\ t \in \mathbb{R}\} < +\infty.$$

We assume that the translation group $\{T(h), h \in \mathbb{R}\}$ acting by the formula $T(h)\xi(t) = \xi(h+t)$ is continuous in Ξ. This assumption is clearly holds for the space $\Xi = C_{\mathrm{b}}(\mathbb{R}; H)$.

The symbol of the original equation (2.21) is denoted by $\sigma_0(t)$. Along with this equation having the symbol $\sigma_0(t)$ we consider equations (2.24) with symbols $\sigma_h(t) = \sigma_0(t + h)$ for any $h \in \mathbb{R}$. Moreover, we also consider the equations with symbols $\sigma(t)$ that are limits of the sequences of the form $\sigma_{h_n}(t) = \sigma_0(t + h_n)$ as $n \to \infty$ in the space Ξ. The resulting family of symbols forms the *hull* $\mathcal{H}(\sigma_0)$ of the original symbol $\sigma_0(t)$ in the space Ξ.

Definition 2.4.1 The set

$$\mathcal{H}(\sigma_0) := [\{\sigma(t + h) \mid h \in \mathbb{R}\}]_\Xi \tag{2.25}$$

is called the *hull* $\mathcal{H}(\sigma)$ of the function $\sigma(t)$ in the space Ξ , where $[\cdot]_\Xi$ denotes the closure in the topological space Ξ.

We are going to study equations of the form (2.21) and (2.24), whose symbols $\sigma(t)$ are *translation compact* functions in Ξ (see also [CV95a, CV95b, CV95c, CV02a]).

Definition 2.4.2 A function $\sigma(t) \in \Xi$ is called *translation compact* (*tr.c.*) *in* Ξ, if the hull $H(\sigma)$ is compact in Ξ.

Consider the main examples of translation compact functions that we shall use in this paper.

Example 2.4.1 Let $\Xi = C_b(\mathbb{R};\mathcal{M})$, where $\mathcal{M}$ is a complete metric space. Let $\sigma_0(s)$ be an *almost periodic (a.p.)* function with values in $\mathcal{M}$. It is well-known that, by the Bochner–Amerio criterion, an a.p. function $\sigma_0(s)$ possesses the following characteristic property: the set of all its translations $\{\sigma_0(s+h) = T(h)\sigma_0(s) \mid h \in \mathbb{R}\}$ forms a precompact set in $C_b(\mathbb{R};\mathcal{M})$ (see, e.g., [AP71, LevZh78]). The closure in $C_b(\mathbb{R};\mathcal{M})$ of this set is said to be the hull $\mathcal{H}(\sigma_0)$ of the function $\sigma_0(s)$ (see (2.25)). Thus, by Definition 2.4.2, $\sigma_0(s)$ is a tr.c. function in $C_b(\mathbb{R};\mathcal{M})$. If the function $\sigma_0(s)$ is almost periodic, then any function $\sigma(s) \in \mathcal{H}(\sigma_0)$ is a.p. as well. Evidently, the time translation group $\{T(h) \mid h \in \mathbb{R}\}$ is continuous in $C_b(\mathbb{R};\mathcal{M})$.

Example 2.4.2 Let $\Xi = L_p^{loc}(\mathbb{R};\mathcal{E})$, where $p \geq 1$ and $\mathcal{E}$ is a Banach space. The space $L_p^{loc}(\mathbb{R};\mathcal{E})$ consists of functions $\xi(t), t \in \mathbb{R}$ with values in $\mathcal{E}$ that are p-power locally integrable in the Bochner sense, that is,

$$\int_{t_1}^{t_2} \|\xi(t)\|_{\mathcal{E}}^p dt < +\infty,\ \forall [t_1,t_2] \subset \mathbb{R}.$$

We consider the following convergence topology in the space $L_p^{loc}(\mathbb{R};\mathcal{E})$. By the definition, $\xi_n(t) \to \xi(t)$ $(n \to \infty)$ in $L_p^{loc}(\mathbb{R};\mathcal{E})$ if

$$\int_{t_1}^{t_2} \|\xi_n(t) - \xi(t)\|_{\mathcal{E}}^p dt \to 0 \ (n \to \infty)$$

for every interval $[t_1,t_2] \subset \mathbb{R}$. The space $L_p^{loc}(\mathbb{R};\mathcal{E})$ is countably normable, metrizable, and complete. Consider tr.c. functions in the space $L_p^{loc}(\mathbb{R};\mathcal{E})$. We have the following criterion (see, for example, [CV02a]): a function $\sigma_0(t)$ is tr.c. in $L_p^{loc}(\mathbb{R};\mathcal{E})$ if and only if **(i)** for any $h \geq 0$ the set $\left\{\int_t^{t+h} \sigma_0(s)ds \mid t \in \mathbb{R}\right\}$ is precompact in $\mathcal{E}$ and **(ii)** there exists a positive function $\beta(s) \to 0$ $(s \to 0+)$ such that

$$\int_t^{t+1} \|\sigma_0(s) - \sigma_0(s+l)\|_{\mathcal{E}}^p ds \leq \beta(|l|),\ \forall t \in \mathbb{R}.$$

From this criterion, it follows that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|\sigma_0(s)\|_{\mathcal{E}}^p ds < +\infty,\ \forall t \in \mathbb{R}, \tag{2.26}$$

for any tr.c. function in $L_p^{loc}(\mathbb{R};\mathcal{E})$.

It is obvious that the translation group $\{T(h) \mid h \in \mathbb{R}\}$ is continuous in $L_p^{loc}(\mathbb{R};\mathcal{E})$.

Example 2.4.3 In a similar way, we define translation compact functions in the space $C^{loc}(\mathbb{R};\mathcal{E})$ that consists of continuous functions $\xi(t), t \in \mathbb{R}$ with values in $\mathcal{E}$. The space $C^{loc}(\mathbb{R};\mathcal{E})$ is equipped with the local uniform convergence topology on every interval $[t_1,t_2] \subset \mathbb{R}$ (see [CV02a]). From the Arzelá–Ascoli compactness theorem, we obtain the following criterion: a function $\sigma_0(t)$ is tr.c. in $C^{loc}(\mathbb{R};\mathcal{E})$ if and only if ***(i)*** the set

$\{\sigma_0(h) \mid h \in \mathbb{R}\}$ is precompact in $\mathcal{E}$ and ***(ii)*** $\sigma_0(t)$ is uniformly continuous on $\mathbb{R}$, i.e., there exists a positive function $\alpha(s) \to 0+ \ (s \to 0+)$ such that

$$\|\sigma_0(t_1) - \sigma_0(t_2)\|_{\mathcal{E}} \leq \alpha(|t_1 - t_2|), \ \forall t_1, t_2 \in \mathbb{R}.$$

(See [CV02a] for more details). In particular, any tr.c. function in $C^{loc}(\mathbb{R}; \mathcal{E})$ is bounded in $\mathcal{E}$. The translation group $\{T(h) \mid h \in \mathbb{R}\}$ is clearly continuous in $C^{loc}(\mathbb{R}; \mathcal{E})$.

Example 2.4.4 Almost periodic functions with values in $\mathcal{E}$, that is, tr.c. functions in the space $C_b(\mathbb{R}; \mathcal{E})$, are also tr.c. in $C^{loc}(\mathbb{R}; \mathcal{E})$.

Example 2.4.5 Inside the class of a.p. functions, we consider a subclass of quasiperiodic functions. A function $\sigma_0(t) \in C(R; E)$ is said to be *quasiperiodic (q.p.)* if it has the form:

$$\sigma_0(t) = \phi\left(\alpha_1 t, \alpha_2 t, \ldots, \alpha_k t\right) = \phi\left(\bar{\alpha} t\right), \tag{2.27}$$

where the function $\phi\left(\bar{\omega}\right) = \phi\left(\omega_1, \omega_2, \ldots, \omega_k\right)$ is continuous and 2π-periodic with respect to each argument $\omega_i \in \mathbb{R}$:

$$\phi\left(\omega_1, \ldots, \omega_i + 2\pi, \ldots, \omega_k\right) = \phi\left(\omega_1, \ldots, \omega_i, \ldots, \omega_k\right), \ i = 1, \ldots, k.$$

We denote by $\mathbb{T}^k = [\mathbb{R} \bmod 2\pi]^k$ the k-dimensional torus. Then $\phi \in C(\mathbb{T}^k; \mathcal{E})$. We assume that the real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ in (2.27) are rationally independent (otherwise we can reduce the number of independent arguments ω_i in the representation (2.27)). It follows easily that the hull of the q.p. function $\sigma_0(t)$ in $C(\mathbb{R}; \mathcal{E})$ is the following set:

$$\left\{\phi(\bar{\alpha} t + \bar{\omega}_1) \mid \bar{\omega}_1 \in \mathbb{T}^k\right\} = \mathcal{H}(\sigma_0), \ \bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k). \tag{2.28}$$

Consequently the set $\mathcal{H}(\sigma_0)$ is a continuous image of the k-dimensional torus $\mathbb{T}^k$. For $k = 1$, we obtain a periodic function: $\sigma_0(t + 2\pi) = \sigma_0(t)$.

In [CV02a] other examples of tr.c. functions in $C(\mathbb{R}; \mathcal{E})$ are given which are not a.p. or q.p. functions.

2.5 On the structure of uniform global attractors

We now consider a family of equations (2.24) with symbols $\sigma(t)$ from the hull $\mathcal{H}(\sigma_0)$ of the symbol $\sigma_0(t)$ of the original equation:

$$\partial_t u = A_{\sigma(t)}(u), \ \sigma \in \mathcal{H}(\sigma_0), \tag{2.29}$$

with initial data

$$u|_{t=\tau} = u_\tau. \tag{2.30}$$

We assume that the function $\sigma_0(t)$ is tr.c. in the topological space Ξ. For simplicity, we assume that the set $\mathcal{H}(\sigma_0)$ is a complete metric space. In all examples given above, this assumption holds. We suppose that, for every symbol $\sigma \in \mathcal{H}(\sigma_0)$, the Cauchy problem (2.29), (2.30) has a unique solution for any $\tau \in \mathbb{R}$ and for every initial condition $u_\tau \in E$. Thus, we have the family of processes $\{U_\sigma(t, \tau)\}, \sigma \in \mathcal{H}(\sigma_0)$, acting in the space E.

The family of processes $\{U_\sigma(t, \tau)\}, \sigma \in \mathcal{H}(\sigma_0)$, is called $(E \times \mathcal{H}(\sigma_0), E)$*-continuous* if for any t and $\tau, t \geq \tau$ the mapping $(u, \sigma) \mapsto U_\sigma(t, \tau) u$ is continuous from $E \times \mathcal{H}(\sigma_0)$ into E.

Proposition 2.5.1 *If the process $\{U_{\sigma_0}(t,\tau)\}$ has a compact uniformly attracting set P and the family $\{U_\sigma(t,\tau)\}, \sigma \in \mathcal{H}(\sigma_0)$, corresponding to (2.29) is $(E \times \mathcal{H}(\sigma_0), E)$-continuous, then, for every $\sigma \in \mathcal{H}(\sigma_0)$, the set P is also uniformly attracting for the process $\{U_\sigma(t,\tau)\}$. Moreover, $\mathcal{A}_\sigma \subseteq \mathcal{A} = \mathcal{A}_{\sigma_0}$, where $\mathcal{A}_\sigma$ is the uniform global attractor of the process $\{U_\sigma(t,\tau)\}$ (the inclusion $\mathcal{A}_\sigma \subseteq \mathcal{A}_{\sigma_0}$ can be strict).*

The proof can be found in [CV94a, CV02a].

Remark 2.5.1 A tr.c. function σ_0 in Ξ is called *recurrent* if for every $\sigma \in \mathcal{H}(\sigma_0)$ the hull $\mathcal{H}(\sigma) = \mathcal{H}(\sigma_0)$. For example, any almost periodic function is recurrent. If in Proposition 2.5.1 the tr.c. symbol σ_0 is recurrent (e.g. almost periodic), then clearly $\mathcal{A}_\sigma = \mathcal{A}_{\sigma_0} = \mathcal{A}$ for every $\sigma \in \mathcal{H}(\sigma_0)$. In this case, the uniform global attractor $\mathcal{A}$ describes the limit behaviour of solutions of the entire family of equations (2.29).

We note that the following *translation identity* holds for the family of processes corresponding to (2.29):

$$U_{T(h)\sigma}(t,\tau) = U_\sigma(t+h,\tau+h),\ \forall h \geq 0, t \geq \tau, \tau \in \mathbb{R}. \tag{2.31}$$

Here $T(h)\sigma(t) = \sigma(t+h)$. This translation identity follows directly from the uniqueness of the solution $u(t)$ of problem (2.29), (2.30). To prove (2.31) we replace $\sigma(s)$ in equation (2.29) by $T(h)\sigma(s) = \sigma(s+h)$ and change the variable $t+h = t_1$. Identity (2.31) means that a shift by h of the argument of the symbol $\sigma(s)$ in problem (2.29), (2.30) is equivalent to solving equation (2.29) with symbol $\sigma(s)$ at time $t+h$ with initial data

$$u|_{t=\tau+h} = u_\tau.$$

We now consider a particular case of the symbol $\sigma_0(t)$ of equation (2.29) such that the translation semigroup $\{T(h) \mid h \geq 0\}$ maps it into itself: $T(h)\sigma_0(t) = \sigma_0(t+h) \equiv \sigma_0(t)$ for all $h \geq 0$. In other words, $\sigma_0(t)$ does not depend on t : $\sigma_0(t) = \sigma_0$ for any $s \in \mathbb{R}$, where $\sigma_0 \in \Psi$. In this case by (2.31) the corresponding process $\{U_{\sigma_0}(t,\tau)\}$ satisfies the equality $U_{\sigma_0}(t,\tau) = U_{\sigma_0}(t+h,\tau+h) = U_{\sigma_0}(t-\tau,0)$ for all $h \geq 0,\ t \geq \tau,\ \tau \in \mathbb{R}$. Thus, the process $\{U_{\sigma_0}(t,\tau)\}$ is completely described by the set of one-parameter mappings $S(t) = U_{\sigma_0}(t,0),\ t \geq 0$. Evidently $\{S(t)\}$ forms a semigroup corresponding to the autonomous equation with the constant symbol $\sigma(t) = \sigma_0$. Such equations were treated in Chapter 1. We conclude that semigroups generated by autonomous evolution equations are particular cases of processes generated by non-autonomous equations.

Having the family of non-autonomous equations (2.29), we consider the extended phase space $E \times \mathcal{H}(\sigma_0)$. Using identity (2.31), we construct the semigroup $\{\mathcal{S}(h), h \geq 0\}$ acting in the space $E \times \mathcal{H}(\sigma_0)$ by the formula:

$$\mathcal{S}(h)(u,\sigma) = (U_\sigma(h,0)u, T(h)\sigma)\,, h \geq 0. \tag{2.32}$$

We now prove that the family of mappings $\{\mathcal{S}(h)\}$ form a semigroup in $E \times \mathcal{H}(\sigma_0)$. It is sufficient to verify the semigroup relation:

$$\begin{aligned}
\mathcal{S}(h_1+h_2)(u,\sigma) &= (U_\sigma(h_1+h_2,0)u, T(h_1+h_2)\sigma) \\
&= (U_\sigma(h_1+h_2,h_2)U_\sigma(h_2,0)u, T(h_1)T(h_2)\sigma) \\
&= \big(U_{T(h_2)\sigma}(h_1,0)U_\sigma(h_2,0)u, T(h_1)(T(h_2)\sigma)\big) \\
&= \mathcal{S}(h_1)\left(U_\sigma(h_2,0)u, T(h_2)\sigma\right) = \mathcal{S}(h_1)\mathcal{S}(h_2)(u,\sigma).
\end{aligned}$$

Here we have used the process property **2.** from Definition 2.1.1 and the translation identity (2.31). It is also obvious that $\mathcal{S}(0) = Id$ is the identity mapping.

We denote by Π_1 and Π_2 the projectors acting from $E \times \mathcal{H}(\sigma_0)$ onto E and $\mathcal{H}(\sigma_0)$ by the formulae:

$$\Pi_1(u, \sigma) = u, \ \Pi_2(u, \sigma) = \sigma.$$

We now formulate the main theorem on the structure of the global attractor of equation (2.21) with tr.c. symbol $\sigma_0(t)$. We denote by $\{U_{\sigma_0}(t, \tau)\}$ the corresponding original process with this symbol σ_0.

Theorem 2.5.1 *We assume that the function $\sigma_0(t)$ is translation compact in Ξ. Let the process $\{U_{\sigma_0}(t, \tau)\}$ be asymptotically compact and let the corresponding family of processes $\{U_\sigma(t, \tau)\}, \sigma \in \mathcal{H}(\sigma_0)$ be $(E \times \mathcal{H}(\sigma_0), E)$-continuous. Then the semigroup $\{\mathcal{S}(h)\}$ acting in $E \times \mathcal{H}(\sigma_0)$ by formula (2.32) has the global attractor $\mathfrak{A}$, $\mathcal{S}(h)\mathfrak{A} = \mathfrak{A}$ for all $h \geq 0$. Moreover*

(i) $\Pi_2\mathfrak{A} = \mathcal{H}(\sigma_0)$,

(ii) $\Pi_1\mathfrak{A} = \mathcal{A}$ *is the global attractor of the process* $\{U_{\sigma_0}(t, \tau)\}$,

(iii) *the global attractor $\mathcal{A}$ has the following representation:*

$$\mathcal{A} = \bigcup_{\sigma \in \mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(0) = \bigcup_{\sigma \in \mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(t), \tag{2.33}$$

where $\mathcal{K}_\sigma$ is the kernel of the process $\{U_\sigma(t, \tau)\}$ with symbol $\sigma \in \mathcal{H}(\sigma_0)$. Here t is any fixed number. The kernel $\mathcal{K}_\sigma$ is non-empty for every $\sigma \in \mathcal{H}(\sigma_0)$.

The detailed proof of Theorem 2.5.1 can be found in [CV94a, CV02a]. The existence of the global attractor $\mathfrak{A}$ follows from the general Theorem 1.1.1. To apply this theorem, we have to verify the conditions of asymptotic compactness and continuity of the semigroup $\{\mathcal{S}(h)\}$ acting in $E \times \mathcal{H}(\sigma_0)$ by formula (2.32). Let P be a compact uniformly (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) attracting set for the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$. Obviously, the set $P \times \mathcal{H}(\sigma_0)$ is a compact (in $E \times \Xi$) attracting set for the extended semigroup $\{\mathcal{S}(h), h \geq 0\}$. Clearly, the semigroup $\{\mathcal{S}(h)\}$ is continuous since the family $\{U_\sigma(t, \tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$ is $(E \times \mathcal{H}(\sigma_0), E)$-continuous and the translation semigroup $\{T(h)\}$ is continuous by assumption. Therefore by Theorem 1.1.1, the set

$$\mathfrak{A} = \omega(P \times \mathcal{H}(\sigma_0)) = \bigcap_{h \geq 0} \left[\bigcup_{\eta \geq h} \mathcal{S}(\eta)(P \times \mathcal{H}(\sigma_0)) \right]_{E \times \Xi} \tag{2.34}$$

is the global attractor of the semigroup $\{\mathcal{S}(h)\}$ and the first assertion of Theorem 2.5.1 is proved. We recall that

$$\mathfrak{A} = \{\gamma(0) \ | \ \gamma(\cdot) \text{ is a complete bounded trajectory of } \{\mathcal{S}(h)\}\} \tag{2.35}$$

(see (1.6) from Theorem 1.1.2). Using this representation, we prove the rest assertions of Theorem 2.5.1 (see [CV02a] for the details).

Remark 2.5.2 Using formula (2.33), it is easy to show that $\mathcal{A} = \omega(P)$, where P is an arbitrary compact uniformly attracting set of the process $\{U_{\sigma_0}(t,\tau)\}$ (see Remark 2.1.1).

Remark 2.5.3 If the time symbol $\sigma_0(t)$ is periodic with period p, $\sigma_0(t+p) = \sigma_0(t)$, then the corresponding process $\{U_{\sigma_0}(t,\tau)\}$ is also periodic with period p. In this case, the uniform and non-uniform attractors coincide, $\mathcal{A}_0 = \mathcal{A}$ (see Theorem 2.2.2 and [VC95, CV94b]). Moreover, the hull $\mathcal{H}(\sigma_0) = \{\sigma_0(t+h) \mid h \in [0,p)\}$ and formula (2.33) can be written in a simpler form

$$\mathcal{A} = \bigcup_{h\in[0,p)} \mathcal{K}_{\sigma_0}(h)$$

where $\mathcal{K}_{\sigma_0}$ is the kernel of the original periodic process $\{U_{\sigma_0}(t,\tau)\}$ (compare with (2.9)).

2.6 Uniform global attractors for non-autonomous equations of mathematical physics

In this section, the general theory of uniform global attractors of processes corresponding to abstract non-autonomous equations of the form (2.21) and (2.24) will be applied to some important evolution equation from mathematical physics.

2.6.1 2D Navier–Stokes system with time dependent force

We consider the following non-autonomous 2D Navier–Stokes system with time-dependent external force:

$$\begin{aligned} \partial_t u &= -\nu L u - B(u,u) + g_0(x,t),\ (\nabla, u) = 0, \\ u|_{\partial\Omega} &= 0,\ x = (x_1,x_2) \in \Omega \Subset \mathbb{R}^2. \end{aligned} \tag{2.36}$$

We use the notations from Section 1.3.1, where the autonomous 2D Navier–Stokes system (1.11) is considered with time-independent external force $g_0(x)$.

We assume that the external force $g_0(\cdot,t) \in H$ for almost every $t \in \mathbb{R}$ and g_0 has a finite norm in the space $L_2^b(\mathbb{R};H)$, that is,

$$\|g_0\|^2_{L_2^b(\mathbb{R};H)} = \|g_0\|^2_{L_2^b} := \sup_{t\in\mathbb{R}} \int_t^{t+1} |g_0(\cdot,s)|^2 ds < +\infty. \tag{2.37}$$

Consider the following initial conditions for equations (2.36):

$$u|_{t=\tau} = u_\tau,\ u_\tau \in H\ (\tau \in \mathbb{R}). \tag{2.38}$$

Problem (2.36), (2.38) has a unique solution $u(t) \in C(\mathbb{R}_\tau;H) \cap L_2^b(\mathbb{R}_\tau;V)$ such that $\partial_t u \in L_2^b(\mathbb{R}_\tau;H_{-1})$, $\mathbb{R}_\tau = [\tau,+\infty)$ (see [Lio69, L70, T88, BV89, CV02a]). A solution $u(t)$ from this space satisfies equation (2.36) in the distribution sense of the space

$\mathcal{D}'(\mathbb{R}_\tau; H_{-1})$. Moreover, the following estimates hold:

$$\begin{aligned} |u(t)|^2 &\le |u(\tau)|^2 e^{-\nu\lambda(t-\tau)} + \lambda^{-1}(1+(\nu\lambda)^{-1})\|g_0\|^2_{L_2^b}, \quad (2.39)\\ |u(t)|^2 + \nu\int_\tau^t \|u(s)\|^2 ds &\le |u(\tau)|^2 + (\nu\lambda)^{-1}\int_\tau^t |g_0(s)|^2 ds, \quad (2.40)\\ (t-\tau)\|u(t)\|^2 &\le C\left(t-\tau, |u(\tau)|^2, \int_\tau^t |g_0(s)|^2 ds\right) \quad (2.41) \end{aligned}$$

where $\lambda = \lambda_1$ is the first eigenvalue of the Stokes operator L and $C(z, R, R_1)$ is a monotone continuous functions of $z = t-\tau$, R, and R_1 (see [CV02a]).

Consequently, problem (2.36), (2.38) generate the process $\{U_{g_0}(t,\tau)\}$ acting in H by the formula $U_{g_0}(t,\tau)u_\tau = u(t)$, where $u(t)$ is a solution of (2.36), (2.38).

It follows from (2.39) that the process $\{U_{g_0}(t,\tau)\}$ has the uniformly absorbing set B_0 :

$$B_0 = \{u \in H \mid |u| \le 2R_0\}, \ R_0^2 = (\nu\lambda)^{-1}\left(1+(\nu\lambda)^{-1}\right)\|g_0\|^2_{L_2^b},$$

Besides, due to inequality (2.41), the set

$$B_1 = \bigcup_{\tau\in\mathbb{R}} U_{g_0}(\tau+1,\tau)B_0 \tag{2.42}$$

is also uniformly absorbing. Moreover, B_1 is bounded in $V = H_1$ and, therefore, compact in H (see [Lio69, CV02a]). Thus, the process $\{U_{g_0}(t,\tau)\}$ is uniformly compact in H. Applying Theorem 2.1.1 from Section 2.1, we conclude that the process $\{U_{g_0}(t,\tau)\}$ has the global attractor $\mathcal{A}$ and the set $\mathcal{A}$ is bounded in V. Moreover, using Remark 2.1.1, we observe that the global attractor can be constructed by the formula

$$\mathcal{A} = \omega(B_0) = \bigcap_{h\ge 0}\left[\bigcup_{t-\tau\ge h} U_{g_0}(t,\tau)B_0\right]_H.$$

We now assume that the function $g_0(\cdot,t) =: g_0(t)$ is translation compact in the space $L_2^{loc}(\mathbb{R}; H)$. The corresponding necessary and sufficient conditions are given in Section 2.4. Another sufficient condition is as follows: a function $g_0(t)$ is translation compact in $L_2^{loc}(\mathbb{R}; H)$, if $g_0 \in L_2^b(\mathbb{R}; H_1)$ and $\partial_t g_0 \in L_2^b(\mathbb{R}; H_{-1})$, where $H_{-1} = (H_1)^*$, that is,

$$\begin{aligned} \|g_0\|^2_{L_2^b(\mathbb{R};H_1)} &:= \sup_{t\in\mathbb{R}}\int_t^{t+1} \|g_0(\cdot,s)\|^2 ds \le M_1 < +\infty,\\ \|\partial_t g_0\|^2_{L_2^b(\mathbb{R};H_{-1})} &:= \sup_{t\in\mathbb{R}}\int_t^{t+1} \|\partial_t g_0(\cdot,s)\|^2_{H_{-1}} ds \le M_{-1} < +\infty. \end{aligned}$$

(see [CV02a]). We denote by $\mathcal{H}(g_0)$ the hull of the function g_0 in the space $L_2^{loc}(\mathbb{R}; H)$. It is clear that

$$\|g\|^2_{L_2^b} \le \|g_0\|^2_{L_2^b} \le M \tag{2.43}$$

for every function $g \in \mathcal{H}(g_0)$.

The symbol of equation (2.36) is the function $g_0(t) = \sigma_0(t)$. We note that, for every symbol $g \in \mathcal{H}(g_0)$, the corresponding problems (2.36), (2.38) (with external force g in place of g_0) are uniquely solvable and their solutions $u_g(t)$ satisfy inequalities (2.39) – (2.41) as well. Hence, the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, acting H is defined. In [CV02a], it is proved that this family is $(H \times \mathcal{H}(g_0))$-continuous. Therefore, Theorem 2.5.1 implies that

$$\mathcal{A} = \bigcup_{g \in \mathcal{H}(g_0)} \mathcal{K}_g(0), \tag{2.44}$$

where $\mathcal{K}_g$ is the kernel of the process $\{U_g(t,\tau)\}$, which consists of all the bounded complete solutions $u_g(t), t \in \mathbb{R}$, of the 2D Navier–Stokes system with external force $g(t)$. The kernel $\mathcal{K}_g$ is non-empty for every $g \in \mathcal{H}(g_0)$. Notice that

$$\begin{aligned} \mathcal{A} \;&\subset\; B_0 = B_{R_0}(0),\ R_0^2 = (\nu\lambda)^{-1}\left(1 + (\nu\lambda)^{-1}\right)\|g_0\|^2_{L_2^b}, && (2.45)\\ \mathcal{A} \;&\subset\; B_1,\ B_1 = \{u \in V \mid \|v\| \le R'\}, && (2.46)\end{aligned}$$

where R' depends on ν, λ, and $\|g_0\|^2_{L_2^b}$. In particular we conclude from (2.44) that

$$\|u(t)\| \le R',\ \forall t \in \mathbb{R}, \tag{2.47}$$

for every function $u_g(\cdot) \in \mathcal{K}_g,\ g \in \mathcal{H}(g_0)$.

We now consider the important particular case of system (2.36). Similarly to autonomous case, we define the Grashof number G for the non-autonomous 2D Navier–Stokes system by the formula

$$G := \frac{\|g_0\|_{L_2^b}}{\lambda\nu^2}.$$

Proposition 2.6.1 *We assume that G satisfies the following inequality:*

$$G < \frac{1}{c_0^2}, \tag{2.48}$$

where the constant c_0 is taken from inequality (1.14) (compare with (1.19)). Then, for every $g \in \mathcal{H}(g_0)$, the Navier–Stokes system

$$\partial_t u = -\nu L u - B(u,u) + g(t) \tag{2.49}$$

has the unique solution $z_g(t), t \in \mathbb{R}$, bounded in H, that is, the kernel $\mathcal{K}_g$ consists of the unique trajectory $z_g(t)$. This solution $z_g(t)$ is exponentially stable, i.e., for every solution $u_g(t)$ of equation (2.49) the following inequality holds:

$$|u_g(t) - z_g(t)| \le C_0 |u_\tau - z_g(\tau)| e^{-\beta(t-\tau)},\ \forall t \ge \tau, \tag{2.50}$$

where $u_g(t) = U_g(t,\tau)u_\tau$ (the constants C_0 and β are independent of u_τ and τ).

Proof. By (2.44), at least one bounded solution $z_g(t) := z(t)$ exists. Let $u_g(t) := u(t)$ be an arbitrary solution of (2.49). The function $w(t) = u(t) - z(t)$ satisfies the equation

$$\partial_t w + \nu L w + B(w, w+z) + B(z,w) = 0.$$

Multiplying by w and using the identities $(B(z,w),w) = (B(w,w),w) = 0$ (see (1.13)) and inequality (1.14), we obtain that

$$\begin{aligned} \partial_t |w|^2 + 2\nu\|w\|^2 = 2(B(w,z),w) \;&\le\; 2c_0^2 |w| \|w\| \|z\| \\ &\le\; \nu\|w\|^2 + c_0^4 \nu^{-1} |w|^2 \|z\|^2. \end{aligned}$$

Since $\lambda|w|^2 \le \|w\|^2$, we have

$$\partial_t|w|^2 + \nu\lambda|w|^2 \le \partial_t|w|^2 + \nu\|w\|^2 \le c_0^4\nu^{-1}|w|^2\|z\|^2. \tag{2.51}$$

Consequently,

$$\partial_t|w|^2 + \left(\nu\lambda - c_0^4\nu^{-1}\|z(t)\|^2\right)|w|^2 \le 0. \tag{2.52}$$

Multiplying this inequality by $\exp\left\{\int_\tau^t (\nu\lambda - c_0^4\nu^{-1}\|z(s)\|^2)\, ds\right\}$ and integrating over $[\tau, t]$, we obtain

$$\begin{aligned}
|w(t)|^2 &\le |w(\tau)|^2 \exp\left\{\int_\tau^t \left(-\nu\lambda + c_0^4\nu^{-1}\|z(s)\|^2\right) ds\right\} \\
&= |w(\tau)|^2 \exp\left\{-\nu\lambda(t-\tau) + c_0^4\nu^{-1}\int_\tau^t \|z(s)\|^2 ds\right\}.
\end{aligned} \tag{2.53}$$

By (2.40) we find that

$$\begin{aligned}
\int_\tau^t \|z(s)\|^2 ds &\le \nu^{-1}|z(\tau)|^2 + \left(\nu^2\lambda\right)^{-1}\int_\tau^t |g(s)|^2 ds \\
&\le \nu^{-1}|z(\tau)|^2 + \left(\nu^2\lambda\right)^{-1}(t-\tau+1)\|g\|^2_{L_2^b} \\
&\le \nu^{-1}|z(\tau)|^2 + \left(\nu^2\lambda\right)^{-1}(t-\tau+1)\|g_0\|^2_{L_2^b}.
\end{aligned}$$

Since $z(\tau) \in \mathcal{A}_{\mathcal{H}(g_0)}$, it follows from (2.45) that

$$|z(\tau)|^2 \le (\nu\lambda)^{-1}\left(1 + (\nu\lambda)^{-1}\right)\|g_0\|^2_{L_2^b} = R_0^2.$$

Hence,

$$\begin{aligned}
\int_\tau^t \|z(s)\|^2 ds &\le \left(\nu^{-1}R_0^2 + \left(\nu^2\lambda\right)^{-1}\|g_0\|^2_{L_2^b}\right) + \left(\nu^2\lambda\right)^{-1}(t-\tau)\|g_0\|^2_{L_2^b} \\
&= R_1^2 + \left(\nu^2\lambda\right)^{-1}(t-\tau)\|g_0\|^2_{L_2^b},
\end{aligned}$$

where $R_1^2 = \nu^{-1}R_0^2 + (\nu^2\lambda)^{-1}\|g_0\|^2_{L_2^b}$. Substituting this estimate into (2.53) we obtain the inequality

$$|w(t)|^2 \le |w(\tau)|^2 C_0 \exp\left(-\beta(t-\tau)\right),$$

where

$$\beta = \nu\lambda - c_0^4\left(\nu^3\lambda\right)^{-1}\|g_0\|^2_{L_2^b} \text{ and } C_0 = \exp\left(c_0^4\nu^{-1}R_1^2\right).$$

Notice that

$$\nu^{-4}\lambda^{-2}\|g_0\|^2_{L_2^b} = G^2 < \frac{1}{c_0^4}$$

and therefore $\beta = \nu\lambda c_0^4\left(c_0^{-4} - \nu^{-4}\lambda^{-2}\|g_0\|^2_{L_2^b}\right) > 0$. This implies that

$$|w(t)|^2 = |u(t) - z(t)|^2 \le |u(\tau) - z(\tau)|^2 C_0 e^{-\beta(t-\tau)}.$$

Inequality (2.50) is proved. Let us show that such a function $z(t)$ is unique. If there are two bounded complete solutions $z_1(t)$ and $z_2(t)$, $t \in \mathbb{R}$, then by (2.50)

$$|z_1(t) - z_2(t)|^2 \le |z_1(\tau) - z_2(\tau)|^2 C_0 e^{-\beta(t-\tau)} \le C_1 C_0 e^{-\beta(t-\tau)}.$$

Fixing t and letting $\tau \to -\infty$ we obtain $|z_1(t) - z_2(t)|^2 = 0$ for all $t \in \mathbb{R}$. $\blacksquare$

Properties (2.50) and (2.44) implies that the set

$$\mathcal{A} = [\{z_{g_0}(t) \mid t \in \mathbb{R}\}]_H = \bigcup_{g \in \mathcal{H}(g_0)} \{z_g(0)\} \tag{2.54}$$

is the global attractor of the original equation (2.36) under condition (2.48).

Remark 2.6.1 In the work [CI04] it is shown that $c_0^2 < \left(\frac{8}{27\pi}\right)^{1/2}$ (see also Remark 1.3.1). Therefore, formula (2.54) holds for $G < 3.2562$.

Remark 2.6.2 It is easy to construct examples of functions $g_0(x,t)$ satisfying (2.48) such that the set $\{z_{g_0}(t) \mid t \in \mathbb{R}\}$ is not closed in H. Nevertheless, the set $\mathcal{A}$ is always closed and to describe this set we need to consider all the functions $z_g(t)$ from the kernels of equations with external forces $g \in \mathcal{H}(g_0)$.

Remark 2.6.3 Inequality (2.50) implies that, under condition (2.48), the global attractor $\mathcal{A}$ of system (2.36) is exponential, i.e. it attracts bounded sets of initial data with exponential rate.

We now formulate corollaries for some special cases of functions $g \in \mathcal{H}(g_0)$.

Corollary 2.6.1 *Let the function $g(t)$ in (2.49) be periodic with period p. Then the function $z_g(t)$ has the period p as well.*

Proof. Consider the corresponding bounded complete trajectory $z_g(t)$. Consider the function $z_g(t+p)$ that is, obviously, also a bounded complete trajectory of (2.49) with external force $g(t+p) \equiv g(t)$. Therefore, this function belongs to the kernel $\mathcal{K}_g$, which consists of the unique trajectory $z_g(t)$. Hence, $z_g(t+p) \equiv z_g(t)$. $\blacksquare$

Corollary 2.6.2 *If a function $g(t) \in \mathcal{H}(g_0)$ is almost periodic, then the function $z_g(t)$ is almost periodic as well.*

Proof. Consider the function $w(t) = z(t) - z(t+p)$, where $z(t) := z_g(t)$ and p is an arbitrary fixed number. Similarly to (2.52) we obtain the following inequality:

$$\partial_t |w|^2 + \left(\nu\lambda - c_0^4 \nu^{-1} \|z(t)\|^2\right) |w|^2 \le 2|w| \cdot |g(t) - g(t+p)|,$$

which implies that

$$\partial_t |w|^2 + \left(\nu\lambda - c_0^4 \nu^{-1} \|z(t)\|^2 - \delta\right) |w|^2 \le \delta^{-1} |g(t) - g(t+p)|^2. \tag{2.55}$$

Here δ is a fixed positive number specified below. We also get from inequality (2.40) that

$$\begin{aligned} \nu \int_\tau^t \|z(s)\|^2 ds &\le |z(\tau)|^2 + (\nu\lambda)^{-1} \int_\tau^t |g(s)|^2 ds \le |z(\tau)|^2 + (\nu\lambda)^{-1} (t - \tau + 1) \|g\|_{L_2^b}^2 \\ &\le |z(\tau)|^2 + (\nu\lambda)^{-1} (t - \tau + 1) \|g_0\|_{L_2^b}^2. \end{aligned} \tag{2.56}$$

Since $z(\tau) \in \mathcal{A}$, we have from (2.45)

$$|z(\tau)|^2 \leq (\nu\lambda)^{-1} \left(1 + (\nu\lambda)^{-1}\right) \|g_0\|^2_{L^b_2} = R_0^2.$$

Consequently due to (2.56) we obtain:

$$\begin{aligned}
\int_\tau^t \|z(s)\|^2 ds &\leq \left(\nu^{-1}R_0^2 + (\nu^2\lambda)^{-1}\|g_0\|^2_{L^b_2}\right) + (\nu^2\lambda)^{-1}(t-\tau)\|g_0\|^2_{L^b_2} \\
&= R_1^2 + (\nu^2\lambda)^{-1}(t-\tau)\|g_0\|^2_{L^b_2}, \qquad (2.57)
\end{aligned}$$

where $R_1^2 = \nu^{-1}R_0^2 + (\nu^2\lambda)^{-1}\|g_0\|^2_{L^b_2}$.

We denote $\alpha(t) = \nu\lambda - c_0^4\nu^{-1}\|z(t)\|^2 - \delta$. Multiplying equation (2.55) by $\exp\left\{\int_\tau^t \alpha(s)ds\right\}$ and integrating over $[\tau, t]$ we find that

$$|w(t)|^2 \leq |w(\tau)|^2 e^{-\int_\tau^t \alpha(s)ds} + \frac{1}{\delta}\int_\tau^t |g(\theta) - g(\theta+p)|^2 e^{-\int_\theta^t \alpha(s)ds} d\theta. \qquad (2.58)$$

Using (2.57) we have

$$\begin{aligned}
-\int_\theta^t \alpha(s)ds &\leq c_0^4\nu^{-3}\lambda^{-1}\|g_0\|^2_{L^b_2}(t-\theta) - (\nu\lambda - \delta)(t-\theta) + c_0^4\nu^{-1}R_1^2 \\
&= -\left(\nu\lambda - c_0^4\nu^{-3}\lambda^{-1}\|g_0\|^2_{L^b_2} - \delta\right)(t-\theta) + R_2^2 \\
&= -(\beta - \delta)(t-\theta) + R_2^2, \qquad (2.59)
\end{aligned}$$

where $R_2^2 = c_0^4\nu^{-1}R_1^2$ and $\beta = \nu\lambda - c_0^4\nu^{-3}\lambda^{-1}\|g_0\|^2_{L^b_2}$. We note that $\nu^{-4}\lambda^{-2}\|g_0\|^2_{L^b_2} = G^2 < c_0^{-4}$ (see (2.48)) and therefore $\beta = \nu\lambda - c_0^4\nu^{-3}\lambda^{-1}\|g_0\|^2_{L^b_2} > 0$. We now set $\delta = \beta/2$. Then (2.58) implies that

$$|w(t)|^2 \leq |w(\tau)|^2 e^{R_2^2} e^{-\beta(t-\tau)/2} + \frac{2}{\beta} e^{R_2^2} \int_\tau^t |g(\theta) - g(\theta+p)|^2 e^{-\beta(t-\theta)/2} d\theta. \qquad (2.60)$$

Let the number p be an ε-period of the function g, i.e., $|g(\theta) - g(\theta+p)| \leq \varepsilon$ for all $\theta \in \mathbb{R}$. Then by (2.60) we have

$$\begin{aligned}
|w(t)|^2 &\leq |w(\tau)|^2 C_2 e^{-\beta(t-\tau)/2} + C_2 \frac{2}{\beta}\varepsilon^2 \int_\tau^t e^{-\beta(t-\theta)/2} d\theta \\
&\leq |w(\tau)|^2 C_2 e^{-\beta(t-\tau)/2} + C_2\left(\frac{2\varepsilon}{\beta}\right)^2 \left(1 - e^{-\beta(t-\tau)/2}\right) \\
&\leq |w(\tau)|^2 C_2 e^{-\beta(t-\tau)/2} + C_2\left(\frac{2\varepsilon}{\beta}\right)^2, \text{ where } C_2 = e^{R_2^2}. \qquad (2.61)
\end{aligned}$$

Notice that $|w(\tau)| \leq C'$ for all $\tau \in \mathbb{R}$. Therefore using (2.61) and letting $\tau \to -\infty$ we obtain the inequality

$$|w(t)| = |z(t) - z(t+p)| \leq \varepsilon \frac{2\sqrt{C_2}}{\beta}. \qquad (2.62)$$

Hence, p is also $\varepsilon\frac{2\sqrt{C_2}}{\beta}$-period of the function $z(t)$. Then it is straightforward that the function $z(t)$ is almost periodic. ■

We now study the case, where the function $g_0(t)$ is quasiperiodic, that is,

$$g_0(x,t) = \phi\left(x, \alpha_1 t, \dots, \alpha_k t\right) = \phi\left(x, \bar{\alpha} t\right), \tag{2.63}$$

$\phi\left(\cdot, \bar{\omega}\right) \in C^{Lip}(\mathbb{T}^k; H)$, $\bar{\omega} = (\omega_1, \dots, \omega_k)$, and real numbers $(\alpha_1, \dots, \alpha_k) = \bar{\alpha}$ are rationally independent (see Example 2.4.5).

Proposition 2.6.2 *Let the condition (2.48) hold and let the function $g_0(t)$ be quasiperiodic. Then the corresponding function $z_0(t) = z_{g_0}(t)$ (unique by Theorem 2.6.1) is also quasiperiodic, that is, there exists a function $\Phi\left(x, \bar{\omega}\right) \in C^{Lip}(\mathbb{T}^k; H)$ such that $z_0(x,t) = \Phi\left(x, \alpha_1 t, \dots, \alpha_k t\right)$ and the frequencies $(\alpha_1, \dots, \alpha_k)$ are the same as for the function $g_0(x,t)$.*

Proof. Consider the external force $g_{\bar{\omega}}(x,t) = \phi\left(x, \bar{\alpha} t + \bar{\omega}\right)$, where $\bar{\omega} \in \mathbb{T}^k$. It is obvious that $g_{\bar{\omega}} \in \mathcal{H}(g_0)$ (see (2.28)). By (2.48) to each such external force $g_{\bar{\omega}}$ there corresponds the unique bounded complete trajectory $z_{\bar{\omega}}(x,t)$ of the Navier–Stokes equation with external force $g_{\bar{\omega}}(x,t)$ that satisfies (2.50). We set

$$\Phi\left(x, \bar{\omega}\right) = z_{\bar{\omega}}(x, 0) \tag{2.64}$$

and prove that Φ is the desired function. First of all, we note that

$$z_{\bar{\omega}}(x, t+h) = z_{\bar{\alpha}h+\bar{\omega}}(x,t). \tag{2.65}$$

This follows from the uniqueness of the bounded complete trajectory $z_{\bar{\alpha}h+\bar{\omega}}(x,t)$ corresponding to the function $g_{\bar{\alpha}h+\bar{\omega}}(x,t)$ and it is easy to see that the function $z_{\bar{\omega}}(x,t+h)$ satisfies the Navier–Stokes system with external force $\phi\left(x, \bar{\alpha}(t+h) + \overline{\omega}\right) = g_{\bar{\alpha}h+\bar{\omega}}(x,t)$. By (2.64) we conclude that

$$z_{\bar{\omega}}(x,h) = \Phi(x, \bar{\alpha}h + \bar{\omega}),$$

that is, $z_{\bar{\omega}}(x,t) = \Phi(x, \bar{\alpha}t + \bar{\omega})$ for all $t \in \mathbb{R}$.

We now demonstrate that $\Phi(x, \bar{\omega}) = \Phi(x, \omega_1, \dots, \omega_k)$ has the period 2π with respect to each argument ω_i. This property follows from the uniqueness of bounded complete trajectories because

$$\Phi(x, \bar{\omega} + 2\pi\bar{e}_i) = z_{\bar{\omega}+2\pi\bar{e}_i}(x,0) = z_{\bar{\omega}}(x,0) = \Phi(x,\bar{\omega}).$$

Here $\{\bar{e}_i, i = 1, \dots, k\}$ is the standard basis in $\mathbb{R}^k$. It only remains to verify the Lipschitz condition with respect to $\bar{\omega} \in \mathbb{T}^k$ for the function Φ. We set $w(t) = z_{\bar{\omega}_1}(t) - z_{\bar{\omega}_2}(t)$. Similarly to (2.60) we prove the inequality

$$|w(t)|^2 \le |w(\tau)|^2 C_2 e^{-\beta(t-\tau)/2} + \frac{2}{\beta} C_2 \int_\tau^t |g_{\bar{\omega}_1}(\theta) - g_{\bar{\omega}_2}(\theta)|^2 e^{-\beta(t-\theta)/2} d\theta. \tag{2.66}$$

The function ϕ satisfies the inequality

$$|\phi\left(\bar{\omega}_1\right) - \phi\left(\bar{\omega}_2\right)| \le \varkappa |\bar{\omega}_1 - \bar{\omega}_2| \; \forall \bar{\omega}_1, \bar{\omega}_2 \in \mathbb{T}^k.$$

Therefore

$$|g_{\bar{\omega}_1}(\theta) - g_{\bar{\omega}_2}(\theta)| \le \varkappa|\bar{\omega}_1 - \bar{\omega}_2|. \tag{2.67}$$

Hence, from (2.66) and (2.67) similarly to (2.61) and (2.62) we obtain that

$$|w(t)| = |z_{\bar{\omega}_1}(t) - z_{\bar{\omega}_2}(t)| \le \varkappa\frac{2\sqrt{C_2}}{\beta}|\bar{\omega}_1 - \bar{\omega}_2|,$$

and finally by (2.64)

$$|\Phi(\cdot,\bar{\omega}_1) - \Phi(\cdot,\bar{\omega}_2)| = |z_{\bar{\omega}_1}(0) - z_{\bar{\omega}_2}(0)| \le \varkappa\frac{2\sqrt{C_2}}{\beta}|\bar{\omega}_1 - \bar{\omega}_2|, \tag{2.68}$$

that is, $\Phi(x,\bar{\omega}) \in C^{Lip}(\mathbb{T}^k; H)$. ■

Corollary 2.6.3 *Under the assumptions of Theorem 2.6.2 the global attractor $\mathcal{A}$ of the Navier–Stokes system is a Lipschitz-continuous image of the k-dimensional torus:*

$$\mathcal{A} = \Phi(\mathbb{T}^k) \tag{2.69}$$

and the set $\mathcal{A}$ attracts solutions of the equation with exponential rate (see (2.50)). Recall that $\Phi(\cdot,\bar{\omega}) = \Phi(\cdot,\bar{\alpha}t+\bar{\omega})|_{t=0} = z_{\bar{\omega}}(x,t)|_{t=0},\ \bar{\omega} \in \mathbb{T}^k$.

Remark 2.6.4 If follows from (2.69) that the uniform global attractor $\mathcal{A}$ of the Navier–Stokes system with quasiperiodic external force g_0 satisfying (2.48) and (2.63) is finite dimensional and $\mathbf{d}_F(\mathcal{A}) \le k$,where $d_F(\mathcal{A})$ is the fractal dimension of $\mathcal{A}$ (see Section 1.4.1). It is easy to give examples of external forces satisfying (2.48) and (2.63) such that

$$\mathbf{d}_F(\mathcal{A}) = k$$

(see, e.g. [CV94a]). Thus, the dimension of global attractors $\mathcal{A}$ of non-autonomous Navier–Stokes systems may grow to infinity as $k \to \infty$, while the Grashow numbers (or Reynolds numbers) of these systems remain bounded. Moreover, there are almost periodic external forces such that $\mathbf{d}_F(\mathcal{A}) = \infty$ (see Section 2.7). These phenomenons do not occur in autonomous case, where the dimension of the global attractor is always less than the multiple of the Grashow number (see Theorem 1.4.2 and (1.57)). In Chapter 3, we study the Kolmogorov ε-entropy and the fractal dimension of uniform global attractors of non-autonomous equations in great details.

2.6.2 Non-autonomous damped wave equations

We study the non-autonomous wave equation with damping

$$\partial_t^2 u + \gamma\partial_t u = \Delta u - f_0(u,t) + g_0(x,t),\ u|_{\partial\Omega} = 0,\ x \in \Omega \Subset \mathbb{R}^n, \tag{2.70}$$

where $\gamma\partial_t u$ is a dissipation term ($\gamma > 0$). The autonomous case was considered in Section 1.3.2. We assume that the function $f_0(v,t) \in C^1(\mathbb{R}\times\mathbb{R};\mathbb{R})$ and the following inequalities hold:

$$F_0(v,t) \ge -mv^2 - C_m,\ F_0(v,t) := \int_0^v f_0(w,t)dw, \tag{2.71}$$

$$f_0(v,t)v - \gamma_1 F_0(v,t) + mv^2 \ge -C_m,\ \forall\,(v,t) \in \mathbb{R}\times\mathbb{R}, \tag{2.72}$$

where $m > 0$ and $\gamma_1 > 0$. The number m is sufficiently small.

Let ρ be a positive number such that $\rho < 2/(n-2)$ when $n \geq 3$ and ρ is arbitrary large when $n = 1, 2$. We suppose that

$$\begin{aligned} |\partial_v f_0(v,t)| &\leq C_0(1+|v|^\rho),\ |\partial_t f_0(v,t)| \leq C_0(1+|v|^{\rho+1}), \qquad (2.73)\\ \partial_t F_0(v,t) &\leq \delta^2 F_0(v,t) + C_1,\ \forall\,(v,t) \in \mathbb{R}\times\mathbb{R}, \qquad (2.74)\end{aligned}$$

where δ is sufficiently small.

Remark 2.6.5 Let $f_0(v,t) = f(v)\varphi(t)$, where, for example, $f(v) = |v|^\rho v$ or $f(v) = R + \beta\sin(v)$, $|\beta| < R$), and $\varphi(s)$ is a positive bounded continuous function such that

$$\varphi'(t) \leq \delta^2\varphi(t)\ \forall t \in \mathbb{R},$$

then $f_0(v,s)$ satisfies (2.71) – (2.74).

It follows from (2.73) that

$$|f_0(v,t)| \leq C_0'(1+|v|^{\rho+1}),\ |F_0(v,s)| \leq C_0'(1+|v|^{\rho+2}). \qquad (2.75)$$

Concerning the term $g_0(x,t)$, we assume that $g_0 \in L_2^b(\mathbb{R}; L_2(\Omega))$.

The initial conditions are posed at $t = \tau$:

$$u|_{t=\tau} = u_\tau(x),\ \partial_t u|_{t=\tau} = p_\tau(x),\ \tau \in \mathbb{R}. \qquad (2.76)$$

Proposition 2.6.3 *If $u_\tau \in H_0^1(\Omega)$ and $p_\tau \in L_2(\Omega)$, then problem (2.70), (2.76) has a unique solution $u(t) \in C(\mathbb{R}_\tau; H_0^1(\Omega))$, $\partial_t u(t) \in C(\mathbb{R}_\tau; L_2(\Omega))$, and $\partial_t^2 u(t) \in L_2^{loc}(\mathbb{R}_\tau; H^{-1}(\Omega))$.*

The proof is given, e.g., in [T88, H88, BV89, CV02a].

We set $y(t) = (u(t), \partial_t u(t)) = (u(t), p(t))$ and $y_\tau = (u_\tau, p_\tau) = y(\tau)$ for brevity. We denote by E the space of vector functions $y(x) = (u(x), p(x))$ with finite energy norm

$$\|y\|_E^2 = \|(u,p)\|_E^2 = |\nabla u|^2 + |p|^2$$

in the space $E = H_0^1(\Omega) \times L_2(\Omega)$. Recall that $|\cdot|$ denotes the norm in $L_2(\Omega)$. It follows from Proposition 2.6.3 that $y(t) \in E$ for all $t \geq 0$.

Problem (2.70), (2.76) is equivalent to the system

$$\left\{\begin{array}{l} \partial_t u = p \\ \partial_t u = -\gamma p + \Delta u - f_0(u,t) + g_0(x,t), \end{array}\right. \quad \left\{\begin{array}{l} u|_{t=\tau} = u_\tau \\ p|_{t=\tau} = p_\tau, \end{array}\right.$$

which can be rewritten in the operator form

$$\partial_t y = A_{\sigma_0(t)}(y),\ y|_{t=\tau} = y_\tau, \qquad (2.77)$$

for an appropriate operator $A_{\sigma_0(t)}(\,\cdot\,)$, where $\sigma_0(t) = (f_0(v,t), g_0(x,t))$ is the symbol of equation (2.77) (see Section 2.4). If $y_\tau \in E$ then, by Proposition 2.6.3, problem (2.77) has a unique solution $y(t) \in C_b(\mathbb{R}_\tau; E)$. This implies that the process $\{U_{\sigma_0}(t,\tau)\}$ given by the formula $U_{\sigma_0}(t,\tau)y_\tau = y(t)$ is defined in E.

Proposition 2.6.4 *The processes $\{U_{\sigma_0}(t,\tau)\}$ corresponding to problem (2.77) is uniformly bounded and the estimate*

$$\|y(t)\|_E^2 \le C_1 \|y_\tau\|_E^{\rho+2} \exp(-\beta(t-\tau)) + C_2,\ \beta > 0, \tag{2.78}$$

holds, where $y(t) = U_{\sigma_0}(t,\tau)y_\tau$ and the constants C_1 and C_2 are independent of y_τ.

The proof is given in [CV02a].

It follows from Proposition 2.6.4 that the process $\{U_{\sigma_0}(t,\tau)\}$ has a bounded (in E) uniformly absorbing set B_0,

$$B_0 = \left\{ y = (u,p) \mid \|y\|_E^2 \le 2C_2 \right\},$$

i.e., $U_{\sigma_0}(t,\tau)B \subseteq B_0$ for $t-\tau \ge h(B)$ for every $B \in \mathcal{B}(E)$. The following result is more complicated (for the proof, see [CV02a]).

Proposition 2.6.5 *The process $\{U_{\sigma_0}(t,\tau)\}$ corresponding to problem (2.77) is uniformly asymptotically compact in E.*

It follows from Theorem 2.1.1 and Proposition 2.6.5 that the process $\{U_{\sigma_0}(t,\tau)\}$ has the global attractor $\mathcal{A}$ and the set $\mathcal{A}$ is compact in E.

We now define the enveloped space Ξ for the symbol $\sigma_0(t) = (f_0(v,t), g_0(x,t))$ of equation (2.77). We suppose that $g_0(x,t)$ is a tr.c. function in $L_2^{loc}(\mathbb{R}; L_2(\Omega))$, the function $f_0(v,t)$ satisfies (2.71)-(2.74), and the function $(f_0(v,t), \partial_t f_0(v,t))$ is tr.c. in $C(\mathbb{R};\mathcal{M})$. Here $\mathcal{M}$ is the space of the functions

$$\left\{ (\psi(v), \psi_1(v)),\ v \in \mathbb{R} \mid (\psi,\psi_1) \in C(\mathbb{R};\mathbb{R}^2) \right\}$$

endowed with the following norm:

$$\|(\psi,\psi_1)\|_{\mathcal{M}} = \sup_{v\in\mathbb{R}} \left\{ \frac{|\psi(v)| + |\psi_1(v)|}{|v|^{\rho+1}+1} + \frac{|\psi'(v)|}{|v|^{\rho}+1} \right\}. \tag{2.79}$$

Evidently, $\mathcal{M}$ is a Banach space. The function $\sigma_0(t) = (f_0(v,t), g_0(x,t))$ is clearly tr.c. in $\Xi = C(\mathbb{R};\mathcal{M}) \times L_2^{loc}(\mathbb{R}; H)$.

Consider the hull $\mathcal{H}(\sigma_0)$ of the symbol σ_0 in the space Ξ. It is easy to show that for any $\sigma(t) = (f(v,t), g(x,t)) \in \mathcal{H}(\sigma_0)$, the function $f(v,t)$ satisfies inequalities (2.71) – (2.74) with the same constants as $f_0(v,t)$.

Thus, problem (2.77) is well posed for all $\sigma \in \mathcal{H}(\sigma_0)$ and generates the family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \mathcal{H}(\sigma_0)$, acting in E. The following assertion is proved in [CV02a].

Proposition 2.6.6 *The family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, corresponding to (2.77) is $(E \times \mathcal{H}(\sigma_0), E)$-continuous.*

Applying now Theorem 2.5.1, we have

Theorem 2.6.1 *If the symbol $\sigma_0(t) = (f_0(v,t), g_0(x,t))$ is tr.c. in the space $\Xi = C(\mathbb{R};\mathcal{M}) \times L_2^{loc}(\mathbb{R}; L_2(\Omega))$, then the process $\{U_{\sigma_0}(t,\tau)\}$ corresponding to problem (2.77) has the uniform global attractor*

$$\mathcal{A} = \bigcup_{\sigma \in \mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(0),$$

where $\mathcal{K}_\sigma$ is the kernel of the process $\{U_\sigma(t,\tau)\}$ with symbol $\sigma \in \mathcal{H}(\sigma_0)$. The kernel $\mathcal{K}_\sigma$ is non-empty for all $\sigma \in \mathcal{H}(\sigma_0)$. Besides, we have the following formula

$$\mathcal{A} = \omega(B_0) = \bigcap_{h \geq 0} \left[\bigcup_{t-\tau \geq h} U(t,\tau) B_0 \right]_E .$$

We now consider a particular case of equation (2.70), namely, the following sine-Gordon type equation with dissipation:

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - f(u) + g_0(x,t), \ u|_{\partial\Omega} = 0, \ x \in \Omega. \tag{2.80}$$

Here $\Omega \Subset \mathbb{R}^n$, $\gamma > 0$, $f \in C(\mathbb{R})$, $g_0(\cdot,t) \in L_2^{loc}(\mathbb{R}; L_2(\Omega))$. We suppose for the function $f(u)$ the inequalities

$$|f(v)| \leq C, \ \forall v \in \mathbb{R}, \tag{2.81}$$
$$|f(v_1) - f(v_2)| \leq K|v_1 - v_2|, \ \forall v_1, v_2 \in \mathbb{R}. \tag{2.82}$$

Remark 2.6.6 For $f(u) = K\sin(u)$, equation (2.80) is the sine-Gordon equation with dissipation (see [T88]).

We assume that the external force $g(x,t)$ satisfies the condition

$$\|g_0\|_{L_2^b}^2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|g_0(s)\|_{L_2(\Omega)}^2 ds < +\infty. \tag{2.83}$$

As before for equation (2.80), we consider the Cauchy problem with initial conditions

$$u|_{t=\tau} = u_\tau \in H_0^1(\Omega), \ \partial_t u|_{t=\tau} = p_\tau \in L_2(\Omega). \tag{2.84}$$

Similarly to Proposition 2.6.3, we prove that, for any given $u_\tau(x) \in H_0^1(\Omega)$ and $p_\tau(x) \in L_2(\Omega)$, problem (2.80), (2.84) has a unique solution $u(t) \in C(\mathbb{R}_\tau; H_0^1(\Omega))$, $\partial_t u(t) \in C(\mathbb{R}_\tau; L_2(\Omega))$, and $\partial_t^2 u(t) \in L_2^{loc}(\mathbb{R}_\tau; H^{-1}(\Omega))$. (see, e.g. [T88, H88, BV89, CV02a]). Denoting $y(t) = (u(t), p(t)) = (u(t), \partial_t u(t))$ and $y_\tau = (u_\tau, p_\tau)$, we observe that $y(t) \in C(\mathbb{R}_\tau; E)$, $y(\tau) = y_\tau$. Then, problem (2.80), (2.84) has the form of evolution equation

$$\begin{cases} \partial_t u = p \\ \partial_t p = -\gamma p + \Delta u - f(u) + g_0(x,t), \end{cases}, \begin{cases} u|_{t=\tau} = u_\tau \\ p|_{t=\tau} = p_\tau \end{cases}. \tag{2.85}$$

(see (2.77)). The time symbol of this system is now a one component function $\sigma_0(t) = g_0(\cdot,t)$ with values in $L_2(\Omega)$. Since (2.85) has a unique solution it defines via $y(t) = U_{g_0}(t,\tau)y_\tau$ a process $\{U_{g_0}(t,\tau)\}$ acting in E. We study the uniform global attractor $\mathcal{A}$ of this process. Propositions hold for the process $\{U_{g_0}(t,\tau)\}$ (with $\rho = 0$) and we obtain from Theorem 2.1.1

Proposition 2.6.7 *Under the conditions (2.81), (2.82), and (2.83), problem (2.85) has the global attractor $\mathcal{A}$ and the set $\mathcal{A}$ is compact in E.*

See also [CV02a, CV94a, CVW05]. We note that the process $\{U_{g_0}(t,\tau)\}$ is not uniformly compact but only uniformly *asymptotically* compact since a hyperbolic equation does not smooth its solutions in time.

In order to study the structure of the global attractor $\mathcal{A}$ we assume that the function $g_0(x,t)$ is tr.c in the space $L_2^{loc}(\mathbb{R};L_2(\Omega))$. Consider its hull $\mathcal{H}(g_0)$. For any symbol $g\in\mathcal{H}(g_0)$, problem (2.85) with g instead of g_0 generates the process $\{U_g(t,\tau)\}$ in E. In [CV02a], it is proved that the family of processes $\{U_g(t,\tau)\}, g\in\mathcal{H}(g_0)$, is $(E\times\mathcal{H}(g_0),E)$-continuous. Using Theorem 2.5.1, we obtain the following result.

Proposition 2.6.8 *Let the function $g_0(x,t)$ be tr.c. in $L_2^{loc}(\mathbb{R};L_2(\Omega))$. Then the global attractor $\mathcal{A}$ of the process $\{U_{g_0}(t,\tau)\}$ can be represented by the formula*

$$\mathcal{A}=\bigcup_{g\in\mathcal{H}(g_0)}\mathcal{K}_g(0), \tag{2.86}$$

where $\mathcal{K}_g$ is the kernel of equation (2.85) with symbol $g\in\mathcal{H}(g_0)$. The kernel $\mathcal{K}_g$ is non-empty for every g.

We now specify the case when the global attractor $\mathcal{A}$ has a simple structure and is exponentially attracting. We denote by λ the first eigenvalue of the Laplacian on $H_0^1(\Omega)$. We have the following

Theorem 2.6.2 *Let the Lipschitz constant K in (2.82) satisfy the inequality*

$$K<\lambda; \tag{2.87}$$

and let the dissipation rate γ in (2.80) satisfy

$$\gamma^2>\gamma_0^2:=2\left(\lambda-\sqrt{\lambda^2-K^2}\right). \tag{2.88}$$

Then for every $g\in\mathcal{H}(g_0)$, equation (2.85) with external force g has a unique bounded (in E) solution $z(t)=(w(t),\partial_t w(t))$ for all $t\in\mathbb{R}$. Moreover, for any solution $y(t)=U_g(t,\tau)y_\tau$ of equation (2.85), the following inequality holds:

$$\|y(t)-z(t)\|_E\le C\|y_\tau-z(\tau)\|_E e^{-\beta(t-\tau)}, \tag{2.89}$$

where $C>0$ and $\beta>0$ are independent of y_τ.

Proof. For the readers convenience, we repeat the arguments from [CVW05]. In what follows all the relations can be justified using the Galerkin approximation method (see [Lio69, T88, BV89]). Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of (2.80) having the external force $g\in\mathcal{H}(g_0)$. Then the difference $w(x,t):=u_1(x,t)-u_2(x,t)$ solves

$$\partial_t^2 w+\gamma\partial_t w=\Delta w-(f(u_1)-f(u_2))\ \text{ in } \Omega \text{ and } w|_{\partial\Omega}=0, \tag{2.90}$$

We rewrite this equation in the form

$$\partial_t\left(\partial_t w+\alpha w\right)+(\gamma-\alpha)(\partial_t w+\alpha w)-\Delta w-\alpha(\gamma-\alpha)w=-\left(f(u_1)-f(u_2)\right). \tag{2.91}$$

Here α is a suitable parameter to be chosen later on. Multiplying equation (2.91) by $v = \partial_t w + \alpha w$ and integrating the result over Ω, we obtain after employing integration by parts and using condition (2.82), the inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(|v|^2 + |\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2\right) &+ (\gamma-\alpha)|v|^2 + \\ \alpha\left(|\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2\right) &= -\left(f(u_1) - f(u_2), v\right) \leq K|w||v|. \quad (2.92)\end{aligned}$$

We now choose $\alpha > 0$ such that

$$\alpha(\gamma - \alpha) < \lambda. \tag{2.93}$$

Then, using Poincaré inequality $\lambda|w|^2 \leq |\nabla w|^2$, we obtain

$$\lambda|w|^2 - \alpha(\gamma-\alpha)|w|^2 \leq |\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2,$$

that is,

$$|w|^2 \leq \frac{|\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2}{\lambda - \alpha(\gamma-\alpha)}. \tag{2.94}$$

Using (2.94) and (2.92) we find that

$$\frac{1}{2}\frac{d}{dt}\left(X^2+Y^2\right) + \left\{(\gamma-\alpha)X^2 + \alpha Y^2 - \frac{K}{\sqrt{\lambda - \alpha(\gamma-\alpha)}}XY\right\} < 0, \tag{2.95}$$

where we denote $X^2 = |v|^2 = |\partial_t w + \alpha w|^2$ and $Y^2 = |\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2$.

The quadratic form $\{\ldots\}$ is positive definite provided $\alpha > 0$, $\gamma - \alpha > 0$, and the determinant

$$\alpha(\gamma-\alpha) - \frac{K}{4\left(\lambda - \alpha(\gamma-\alpha)\right)} > 0. \tag{2.96}$$

We set $\varrho = \alpha(\gamma-\alpha)$. Inequality (2.96) is equivalent to

$$\varrho^2 - \lambda\varrho + \frac{K^2}{4} < 0. \tag{2.97}$$

Since we assume that $K < \lambda$ the quadratic inequality (2.97) is satisfied for every ϱ with

$$\frac{\lambda - \sqrt{\lambda^2 - K^2}}{2} < \varrho < \frac{\lambda + \sqrt{\lambda^2 - K^2}}{2}. \tag{2.98}$$

We note that, from (2.98), it follows that $\varrho < \lambda$, i.e., $\alpha(\gamma-\alpha) < \lambda$ and condition (2.93) is satisfied. Thus, we have to produce a number $\alpha > 0$ that satisfies the inequalities

$$\frac{\lambda - \sqrt{\lambda^2 - K^2}}{2} < \alpha(\gamma-\alpha) < \frac{\lambda + \sqrt{\lambda^2 - K^2}}{2}. \tag{2.99}$$

Such an α always exists if the maximum of $\alpha(\gamma-\alpha)$ with respect to α is greater than the left bound in (2.99), i.e., when

$$\frac{\gamma^2}{4} > \frac{\lambda - \sqrt{\lambda^2 - K^2}}{2}. \tag{2.100}$$

This inequality just coincides with assumption (2.88). Consequently, taking an α that satisfies both inequalities in (2.99), we obtain that the quadratic form $\{\ldots\}$ in (2.95) is positive definite and

$$(\gamma-\alpha)X^2+\alpha Y^2-\frac{K}{\sqrt{\lambda-\alpha(\gamma-\alpha)}}XY\geq\beta\left(X^2+Y^2\right),\ \beta>0, \tag{2.101}$$

where β depends explicitly on γ, λ, and K.

Then (2.95) becomes

$$\frac{1}{2}\frac{d}{dt}\left(X^2+Y^2\right)+\beta\left(X^2+Y^2\right)<0$$

and the Gronwall's inequality yields

$$X^2(t)+Y^2(t)\leq\left(X^2(\tau)+Y^2(\tau)\right)e^{-2\beta(t-\tau)}. \tag{2.102}$$

We see that the expression $X^2+Y^2=|\partial_t w+\alpha w|^2+|\nabla w|^2-\alpha(\gamma-\alpha)|w|^2$ is equivalent to the norm $\|y_1-y_2\|_E^2=|\partial_t w|^2+|\nabla w|^2$. Hence, (2.102) implies the inequality

$$\|y_1(t)-y_2(t)\|_E^2\leq C^2\|y_1(\tau)-y_2(\tau)\|_E^2e^{-2\beta(t-\tau)},\ \forall t\geq\tau, \tag{2.103}$$

with some constant $C=C(\gamma,\lambda,\alpha)$.

By Proposition 2.6.8, the kernel $\mathcal{K}_g$ of (2.85) is non-empty, i.e. there is a bounded (in E) solution $z(t)=z_g(t), t\in\mathbb{R}$, of system (2.85).

If we substitute this solution $z(t)$ into (2.103), then we obtain for any other solution $y(t)=U_g(t,\tau)y_\tau$ the estimate

$$\|y(t)-z(t)\|_E\leq C\|y_\tau-z(\tau)\|_Ee^{-\beta(t-\tau)},\ \forall t\geq\tau. \tag{2.104}$$

This inequality also implies that $z(t)$ is the unique bounded complete trajectory of the process $\{U_g(t,\tau)\}$ corresponding to (2.85). ■

In conclusion, we formulate some corollaries from Theorem 2.6.2 that can be proved in analogous manner as for the corresponding propositions for the 2D Navier–Stokes system in Section 2.6.1 (see Corollaries 2.6.1 – 2.6.3 and Proposition 2.6.2).

Corollary 2.6.4 *Under conditions (2.87) and (2.88), the global attractor of equation (2.85) has the forms*

$$\mathcal{A}=[\{z_{g_0}(t)\mid t\in\mathbb{R}\}]_E=\bigcup_{g\in\mathcal{H}(g_0)}\{z_g(0)\}. \tag{2.105}$$

Corollary 2.6.5 *The constructed global attractor $\mathcal{A}$ is exponential, i.e., for every bounded set $B\subset E$*

$$\operatorname{dist}_E\left(U_{g_0}(t,\tau)B,\mathcal{A}\right)\leq C\|B\|_Ee^{-\beta(t-\tau)}\ \forall t\geq\tau, \tag{2.106}$$

where $\|B\|_E=\sup\{\|y\|_E\mid y\in B\}$.

Corollary 2.6.6 *If the function $g(t)$ is periodic with period p, then $z_g(t)$ is also periodic with period p.*

Corollary 2.6.7 *If $g(t)$ is almost periodic, then $z_g(t)$ is almost periodic as well.*

Proof. Similarly to (2.102), we establish that the function $w(t) = z_g(t) - z_g(t+p)$ satisfies the inequality

$$\frac{d}{dt}\left(X^2+Y^2\right)+2\beta\left(X^2+Y^2\right)\le 2|g(t)-g(t+p)||v|, \tag{2.107}$$

where $X^2 = |v(t)|^2 = |\partial_t w(t) + \alpha w(t)|^2$ and $Y^2 = |\nabla w(t)|^2 - \alpha(\gamma-\alpha)|w(t)|^2$. Using the estimate

$$2|g(t)-g(t+p)||v| \le \beta X^2 + \beta^{-1}|g(t)-g(t+p)|^2, \tag{2.108}$$

we find that

$$\frac{d}{dt}\left(X^2+Y^2\right)+\beta\left(X^2+Y^2\right)\le \beta^{-1}|g(t)-g(t+p)|^2. \tag{2.109}$$

If now a number p is an ε-period of the function g, i.e., $|g(t)-g(t+p)| < \varepsilon$ for all $t \in \mathbb{R}$, then from (2.109) we obtain that

$$X^2(t)+Y^2(t) \le \left(X^2(\tau)+Y^2(\tau)\right)e^{-\beta(t-\tau)}+\frac{\varepsilon^2}{\beta^2}.$$

Fixing t and letting $\tau \to -\infty$, we have that

$$\|z_g(t)-z_g(t+p)\|_E^2 \le C\left(X^2(t)+Y^2(t)\right)\le C\frac{\varepsilon^2}{\beta^2},\ \forall t\in\mathbb{R}, \tag{2.110}$$

that is, the number p is an $\varepsilon\sqrt{C}/\beta$-period of the function z_g and, thereby, $z_g(t)$ is almost periodic. ∎

We now assume that the function $g_0(t)$ is quasiperiodic and it has k rationally independent frequencies, i.e.,

$$g_0(t) = \phi(x,\alpha_1 t,\dots,\alpha_k t) = \phi(x,\bar{\alpha}t), \tag{2.111}$$

where $\phi \in C^{lip}(\mathbb{T}^k; L_2(\Omega))$, $\bar{\alpha} = (\alpha_1,\dots,\alpha_k) \in \mathbb{R}$ (see Example 2.4.5).

Proposition 2.6.9 *If $g_0(t)$ is a quasiperiodic function of the form (2.111), then the corresponding function $z_{g_0}(t)$ is also quasiperiodic. Thus, there exists a function $\Phi \in C^{lip}(\mathbb{T}^k; L_2(\Omega))$ such that*

$$z_{g_0}(t) = \Phi(x,\alpha_1 t,\dots,\alpha_k t).$$

The proof is analogous to the proof of Proposition 2.6.2.

Corollary 2.6.8 *If the external force $g_0(t)$ has the form (2.111), then the global attractor $\mathcal{A}$ of equation (2.80) is a Lipschitz continuous image of a k-dimensional torus $\mathbb{T}^k$:*

$$\mathcal{A} = \Phi(\mathbb{T}^k) \text{ and } \mathbf{d}_F(\mathcal{A}) \le k.$$

Remark 2.6.7 It is easy to construct external forces $g_0(t)$ of the form (2.111) such that $\mathbf{d}_F(\mathcal{A}) = k$. Moreover, there exist almost periodic external forces such that $\mathbf{d}_F(\mathcal{A}) = \infty$ (see Section 2.7).

Remark 2.6.8 Changing in (2.80) the time variable $t = t'/\gamma$, we obtain the equation

$$\varepsilon\partial_t^2 u + \partial_t u = \Delta u - f(u) + g_0(x,t),\ u|_{\partial\Omega} = 0,$$

where $\varepsilon = \gamma^{-2}$. The above result are applicable to this equation provided that

$$|f'(u)| < \lambda, \forall u \in \mathbb{R},\ \text{and } 0 < \varepsilon < \varepsilon_0 := 2^{-1}\left(\lambda - \sqrt{\lambda^2-k^2}\right)^{-1}.$$

2.6.3 Non-autonomous Ginzburg–Landau equation

We consider shortly the following non-autonomous generalization of the Ginzburg–Landau equation from Section 1.3.3 with zero boundary conditions (periodic boundary conditions can be treated in a similar way):

$$\partial_t u = (1 + i\alpha_0(t))\Delta u + R_0(t)u - (1 + i\beta_0(t))|u|^2 u + g_0(x,t),\ u|_{\partial\Omega} = 0. \tag{2.112}$$

Here $u = u^1(x,t) + iu^2(x,t)$ is an unknown complex function and $x \in \Omega \Subset \mathbb{R}^n$. The coefficients $\alpha_0(t)$, $\beta_0(t)$, and $R_0(t)$ are given real functions belonging to the space $C_b(\mathbb{R})$. We assume that

$$|\beta_0(t)| \le \sqrt{3},\ \forall t \in \mathbb{R}. \tag{2.113}$$

The phase space for equation (2.112) is $\mathbf{H} = L_2(\Omega;\mathbb{C})$. The norm in $\mathbf{H}$ is denoted by $\|\cdot\|$. We denote also $\mathbf{V} = H_0^1(\Omega;\mathbb{C})$ and $\mathbf{L}_4 = L_4(\Omega;\mathbb{C})$. We assume that the function $g_0(x,t) = g_0^1(x,t) + ig_0^2(x,t)$ belongs to the space $L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})$, that is,

$$\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})} := \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \|g_0(\cdot,s)\|^2 ds. \tag{2.114}$$

Recall that equation (2.112) is equivalent to the following system for the vector-function $\mathbf{u} = (u^1, u^2)^\top$:

$$\partial_t \mathbf{u} = \begin{pmatrix} 1 & -\alpha_0(t) \\ \alpha_0(t) & 1 \end{pmatrix} \Delta\mathbf{u} + R_0(t)\mathbf{u} - \begin{pmatrix} 1 & -\beta_0(t) \\ \beta_0(t) & 1 \end{pmatrix} |\mathbf{u}|^2\mathbf{u} + \mathbf{g}_0(x,t),$$

where $\mathbf{g}_0 = (g_0^1, g_0^2)^\top$.

Under the above assumption, the Cauchy problem for equation (2.112) with initial data

$$u|_{t=\tau} = u_\tau(x),\ u_\tau(\cdot) \in \mathbf{H},\ \tau \in \mathbb{R}, \tag{2.115}$$

has a unique weak solution $u(t) := u(x,t)$ belonging to the space

$$u(\cdot) \in C_{\mathrm{b}}(\mathbb{R}_\tau;\mathbf{H}) \cap L_2^{\mathrm{b}}(\mathbb{R}_\tau;\mathbf{V}) \cap L_4^{\mathrm{b}}(\mathbb{R}_\tau;\mathbf{L}_4),$$

and the function $u(t)$ satisfies (2.112) in a weak distribution sense (see [T88, BV89, CV02a]).

Any solution $u(t), t \ge \tau$, of equation (2.112) satisfies the differential identity

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|^4_{\mathbf{L}_4} - R(t)\|u(t)\|^2 = \langle g_0(t), u(t)\rangle\,,\ \forall t \ge \tau, \tag{2.116}$$

The function $\|u(t)\|^2$ is absolutely continuous for $t \ge \tau$. We note that the parameters $\alpha_0(t)$ and $\beta_0(t)$ are missing in this identity. The proof of (2.116) is analogous to the proof of the corresponding identities for weak solutions of general reaction-diffusion systems considered in [CV02a, CV96b] (see also [CV05]).

Using the standard transformations and the Gronwall lemma, we deduce from (2.116) that any weak solution $u(t)$ of equation (2.112) satisfies the inequality

$$\|u(t)\|^2 \le \|u(\tau)\|^2 e^{-2\lambda(t-\tau)} + C_0^2,\ \forall t \ge \tau. \tag{2.117}$$

where λ is the first eigenvalue of the operator $\{-\Delta u,\ u|_{\partial\Omega} = 0\}$ and the constant C_0 depends on $\|R_0\|_{C_b} = \sup_{t\in\mathbb{R}} |R(t)|$ and $\|g_0\|_{L_2^b(\mathbb{R};\mathbf{H})}$.

Let $\{U(t,\tau)\}$ be the process corresponding to problem (2.112), (2.115) and acting in the space $\mathbf{H}$. Recall that the mappings $U(t,\tau) : \mathbf{H} \to \mathbf{H},\ t \geq \tau, \tau \in \mathbb{R}$, are defined by the formula

$$U(t,\tau)u_\tau = u(t),\ \forall u_\tau \in \mathbf{H}, \tag{2.118}$$

where $u(t), t \geq \tau$, is a solution of equation (2.112) with initial data $u|_{t=\tau} = u_\tau$. It follows from estimates (2.117) that the process $\{U(t,\tau)\}$ has the uniformly absorbing set

$$B_0 = \{v \in \mathbf{H} \mid \|v\| \leq 2C_0\} \tag{2.119}$$

that is bounded in $\mathbf{H}$.

We claim that the process $\{U(t,\tau)\}$ has a *compact* in $\mathbf{H}$ uniformly absorbing set

$$B_1 = \{v \in \mathbf{V} \mid \|v\|_{\mathbf{V}} \leq C_0'\} \tag{2.120}$$

for an appropriate C_0'. For the proof see [CV02a, CV05] and Section 5.1. The set B_1 is bounded in $\mathbf{V}$ and compact in $\mathbf{H}$ since the embedding $\mathbf{V} \Subset \mathbf{H}$ is compact. Thus, the process $\{U(t,\tau)\}$ corresponding to (2.112) is uniformly compact.

Applying Theorem 2.1.1, we conclude that the process $\{U(t,\tau)\}$ has the global attractor $\mathcal{A}$ and the set $\mathcal{A}$ is compact in $\mathbf{H}$, bounded in $\mathbf{V}$, and can be constructed by the formula

$$\mathcal{A} = \omega(B_0) = \bigcap_{h\geq 0}\left[\bigcup_{t-\tau\geq h} U(t,\tau)B_0\right]_{\mathbf{H}}.$$

The time symbol of equation (2.112) is the function

$$\sigma_0(t) = (\alpha_0(t), \beta_0(t), R_0(t), g_0(x,t)), t \in \mathbb{R}.$$

The values of $\sigma_0(t)$ belong to $\Psi = \mathbb{R}^3 \times \mathbf{H}$. We assume that $\beta_0(t)$ satisfies (2.113).

Let the functions $\alpha_0(t)$, $\beta_0(t)$, and $R_0(t)$ be tr.c. in the space $C^{loc}(\mathbb{R})$ and let the function $g_0(x,t)$ be tr.c. in $L_2^{loc}(\mathbb{R};\mathbf{H})$. Then clearly the function $\sigma_0(t)$ is tr.c. in $\Xi = C^{loc}(\mathbb{R};\mathbb{R}^3) \times L_2^{loc}(\mathbb{R};\mathbf{H})$. Consider the hull $\mathcal{H}(\sigma_0)$ of the function $\sigma_0(t)$ in the space $C^{loc}(\mathbb{R};\mathbb{R}^3) \times L_2^{loc}(\mathbb{R};\mathbf{H})$.

Along with equation (2.112), we consider the family of equations

$$\partial_t u = (1 + i\alpha(t))\Delta u + R(t)u - (1 + i\beta(t))|u|^2 u + g(x,t),\ \sigma \in \mathcal{H}(\sigma_0), \tag{2.121}$$

with symbols $\sigma(t) = (\alpha(t), \beta(t), R(t), g(x,t))$, where $\sigma \in \mathcal{H}(\sigma_0)$. We note that for every $\sigma = (\alpha, \beta, R, g) \in \mathcal{H}(\sigma_0)$, the function $\beta(t)$ satisfies inequality (2.113) and $g(x,t)$ satisfies (2.114). Therefore, equations (2.121) generates the family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \mathcal{H}(\sigma_0)$, acting in $\mathbf{H}$ (see [CV02a, CV05]). Recall that $\{U(t,\tau)\} = \{U_{\sigma_0}(t,\tau)\}$ is the process corresponding to the original Ginzburg–Landau equation (2.112). Consider the kernels $\mathcal{K}_\sigma, \sigma \in \mathcal{H}(\sigma_0)$, of equations (2.121). In [CV02a, CV05], it is proved that the family $\{U_\sigma(t,\tau)\}, \sigma \in \mathcal{H}(\sigma_0)$, is $(\mathbf{H}\times\mathcal{H}(\sigma_0);\mathbf{H})$-continuous. Then, by Theorem 2.5.1, we observe that

$$\mathcal{A} = \bigcup_{\sigma\in\mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(0), \tag{2.122}$$

where the kernel $\mathcal{K}_\sigma$ of equation (2.121) is non-empty for all $\sigma \in \mathcal{H}(\sigma_0)$.

In conclusion, we present an example of the Ginzburg–Landau equation having a simple global attractor.

Proposition 2.6.10 *Let the function $\beta_0(t)$ satisfy (2.113) and let the coefficient $R_0(t)$ satisfy the inequality*

$$R_0(t) \leq \lambda - \delta,\ \forall t \in \mathbb{R},\ \ (0 < \delta < \lambda). \tag{2.123}$$

Then, for any $\sigma \in \mathcal{H}(\sigma_0)$, the kernel $\mathcal{K}_\sigma$ of equation (2.121) consists of the unique element $\{z_\sigma(t), t \in \mathbb{R}\}$. Moreover, the function $\{z_\sigma(t), t \in \mathbb{R}\}$ satisfies the following property of exponential attraction of any solutions $\{u_\sigma(t), t \geq \tau\}$ of equation (2.121):

$$\|u_\sigma(t) - z_\sigma(t)\| \leq e^{-\delta(t-\tau)}\|u_\sigma(\tau) - z_\sigma(\tau)\|,\ \forall t \geq \tau. \tag{2.124}$$

Proof. It is established that the kernel $\mathcal{K}_\sigma$ of equation (2.121) is non-empty. So, there exists a bounded complete solution $z_\sigma(t), t \in \mathbb{R}$, of this equation. Consider any other solution $\{u_\sigma(t), t \geq \tau\}$ of equation (2.121). Then the difference $w(t) = u_\sigma(t) - z_\sigma(t)$ satisfies the equation

$$\partial_t w(t) = (1 + i\alpha(t))\Delta w(t) + R(t)w(t) - (1 + i\beta(t))\left(|u(t)|^2 u(t) - |z(t)|^2 z(t)\right). \tag{2.125}$$

We set

$$A(t)v = (1 + i\alpha(t))\Delta v + R(t)v \text{ and } f(t, v) = (1 + i\beta(t))|v|^2 v.$$

Using (2.123), we find that

$$\begin{aligned} \langle A(t)w, w\rangle &= -\langle (1 + i\alpha(t))\nabla w, \nabla w\rangle + \langle R(t)w, w\rangle \\ = -\langle \nabla w, \nabla w\rangle + \langle R(t)w, w\rangle &\leq -\lambda\|w\|^2 + R(t)\|w\|^2 \leq -\delta\|w\|^2. \end{aligned} \tag{2.126}$$

Inequality (2.113) implies that the function $f(t, u)$ is monotone with respect to $u:$

$$\langle f(t, u) - f(t, z), u - z\rangle = \langle f_u'(t, v)(u - z), u - z\rangle = \langle f_u'(t, v)w, w\rangle \geq 0, \tag{2.127}$$

where $v = z + \theta(u - z),\ 0 \leq \theta(x, t) \leq 1$. See (1.34) and [CV02a] for more details.

Multiplying equation (2.125) by w, taking the scalar product in $\mathbf{H}$, and using (2.126) and (2.127), we obtain the inequality

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w(t)\|^2 &= \langle A(t)w, w\rangle - \langle f(t, u) - f(t, z), w\rangle \\ &\leq -\delta\|w\|^2 - \langle f_u'(t, v)w, w\rangle \leq -\delta\|w\|^2. \end{aligned} \tag{2.128}$$

This implies that

$$\|u(t) - z(t)\|^2 = \|w(t)\|^2 \leq e^{-2\delta(t-\tau)}\|w(\tau)\|^2 = e^{-2\delta(t-\tau)}\|u(\tau) - z(\tau)\|^2,\ \forall t \geq \tau,$$

and inequality (2.124) is proved for any function $z_\sigma(t)$ from the kernel $\mathcal{K}_\sigma$ of (2.121).

It follows from inequality (2.124) that $\{z_\sigma(t), t \in \mathbb{R}\}$ is the unique element of the kernel $\mathcal{K}_\sigma$ of equation (2.121). ■

Remark 2.6.9 Property (2.124) of the exponential attraction by the unique trajectory $\{z_\sigma(x,t), t \in \mathbb{R}\}$ of all solutions $\{u_\sigma(x,t), t \geq \tau\}$ of equation (2.121) is the non-autonomous analogue of the exponential stability of the unique stationary point $\{z(x)\}$ of the autonomous equation (2.21) when $R < \lambda$ and $|\beta| \leq \sqrt{3}$.

Finally, we formulate some natural corollaries from Proposition 2.6.10.

Corollary 2.6.9 *Under the assumptions of Proposition 2.6.10, the global attractor* $\mathcal{A}$ *of equation (2.112) has the following forms*

$$\mathcal{A} = \left[\bigcup\nolimits_{t\in\mathbb{R}} \{z_{\sigma_0}(t)\}\right]_{\mathbf{H}} = \bigcup_{\sigma\in\mathcal{H}(\sigma_0)} \{z_\sigma(0)\}\,. \tag{2.129}$$

Moreover, the global attractor $\mathcal{A}$ *is exponential, i.e., for every bounded set* $B \subset \mathbf{H}$

$$\mathrm{dist}_{\mathbf{H}}\left(U_{\sigma_0}(t,\tau)B, \mathcal{A}\right) \leq C\left(\|B\|\right) e^{-\delta(t-\tau)}\ \forall t \geq \tau, \tag{2.130}$$

where $\|B\| = \sup\{\|y\| \mid y \in B\}$.

Corollary 2.6.10 *If the symbol* $\sigma(t)$ *is periodic, then* $z_\sigma(t)$ *is periodic. If* $\sigma(t)$ *is almost periodic, then* $z_\sigma(t)$ *is almost periodic as well. If the initial symbol* $\sigma_0(t)$ *is quasiperiodic of the form*

$$\sigma_0(t) = \phi(\alpha_1 t, \ldots, \alpha_k t) = \phi(\bar{\alpha} t),$$

where $\phi \in C^{lip}(\mathbb{T}^k; \mathbb{R}^3 \times \mathbf{H})$ *and the numbers* $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ *are rationally independent, then the function* $z_{\sigma_0}(t)$ *is also quasiperiodic, i.e., there exists a function* $\Phi \in C^{lip}(\mathbb{T}^k; \mathbf{H})$ *such that*

$$z_{\sigma_0}(t) = \Phi(\alpha_1 t, \ldots, \alpha_k t).$$

Moreover, the global attractor $\mathcal{A}$ *is a Lipschitz continuous image of a* k*-dimensional torus* $\mathbb{T}^k$ *:*

$$\mathcal{A} = \Phi(\mathbb{T}^k) \text{ and } \mathbf{d}_F(\mathcal{A}) \leq k.$$

The proofs are analogous to the proofs of Corollaries 2.6.1 – 2.6.3 and 2.6.4 – 2.6.8.

Remark 2.6.10 There are symbols $\sigma_0(t)$ satisfying (2.113) and (2.123) such that $\mathbf{d}_F(\mathcal{A}) = k$. Moreover, it is easy to construct almost periodic symbols such that $\mathbf{d}_F(\mathcal{A}) = \infty$.

2.7 On the dimension of global attractors of processes

Dealing with non-autonomous evolution equations, we see that the dimension of their uniform global attractors depends on the dimension of the symbol hulls of these equations. E.g., for evolution equations with quasi-periodic time symbols, the fractal dimension of global attractors depends on the number of rationally independent frequencies of the symbols (see Remarks 2.6.4, 2.6.7, and 2.6.10).

In conclusion we demonstrate, that uniform global attractors of processes corresponding to general non-autonomous evolution equation can have infinite fractal dimension.

Consider a process $\{U(t,\tau)\}$ acting in a Hilbert (or Banach) space E. We assume that the process $\{U(t,\tau)\}$ is uniformly asymptotically compact. Then, by Theorem 2.1.1, this process has the global attractor $\mathcal{A}$. Consider the kernel $\mathcal{K}$ of the process $\{U(t,\tau)\}$. It follows from Proposition 2.1.2 that the set

$$\mathbf{K} = \bigcup_{\tau\in\mathbb{R}} \mathcal{K}(\tau)$$

consisting of all values of all complete trajectories $u \in \mathcal{K}$ of the process belongs to $\mathcal{A}$. Moreover, the closure $\overline{\mathbf{K}}$ of this set in E also belongs to $\mathcal{A}$ since the global attractor is a closed set.

We claim that the set $\overline{\mathbf{K}}$ can have infinite dimension

$$\mathbf{d}_F\overline{\mathbf{K}} = +\infty \tag{2.131}$$

for all the problems described in Section 2.6. For example, for the Navier-Stokes system we set

$$u_0(x,t) = \sum_{j=1}^{\infty} a_j(x)\cos(\mu_j t) + b_j(x)\sin(\mu_j t), \tag{2.132}$$

where $a_j(x) = (a_j^1(x), a_j^2(x))$, $b_j(x) = (b_j^1(x), b_j^2(x))$ are smooth linear independent vector functions such that $a_j|_{\partial\Omega} = 0$, $(\nabla, a_j) = 0$, $b_j|_{\partial\Omega} = 0$, $(\nabla, b_j) = 0$. We assume that series (2.132) and its derivatives with respect to x and t converge rapidly. We also assume that the frequencies μ_j $(j = 1, 2, \ldots)$ are rationally independent real numbers. We set

$$g_0(x,t) = \partial_t u_0(x,t) + \nu L u_0(x,t) + B(u_0(x,t), u_0(x,t)), \tag{2.133}$$

and see that $g_0(\cdot) \in C_b(\mathbb{R}; H)$. System (2.36) with such an external force $g_0(x,t)$ generates a process $\{U(t,\tau)\}$ in H having the compact attractor $\mathcal{A}$ (see Section 2.6.1). The process $\{U(t,\tau)\}$ has at least one complete bounded solution, namely $u_0(t)$, so its kernel $\mathcal{K}$ is non-empty and $u_0 \in \mathcal{K}$. It is easy to show that the projection $u_0^N(x,t)$ of the function $u_0(x,t)$ onto $2N$-dimensional space spanned by the vector functions $\{(a_j(x), b_j(x)) \mid j = 1, \ldots, N\}$ provides a dense winding of the N-dimensional torus $\mathbb{T}^N \subset H$. (Here we use the fact that the frequencies $\{\mu_j\}$ are rationally independent). Therefore, the set $\overline{\operatorname{Im} u_0} = \overline{\{u_0(\cdot,t) : t \in \mathbb{R}\}}$ has the fractal dimension greater than N : $\mathbf{d}_F\overline{\operatorname{Im} u_0} \geq N$ for each $N \in \mathbb{N}$, i.e.,

$$\mathbf{d}_F\overline{\mathrm{Im} u_0} = \infty.$$

Evidently, $\overline{\mathrm{Im} u_0} \subseteq \overline{\mathbf{K}}$ and hence, $\mathbf{d}_F\overline{\mathbf{K}} = +\infty$. We recall that $\overline{\mathbf{K}} \subseteq \mathcal{A}$ and thereby

$$\mathbf{d}_F\mathcal{A} = +\infty.$$

In the next chapter, we study the fractal dimension and the Kolmogorov ε-entropy of uniform global attractors of non-autonomous evolution equations.

Chapter 3

Kolmogorov ε-entropy of global attractors

In the end of the previous chapter, it was shown that the fractal dimension of the global attractor $\mathcal{A}$ of a non-autonomous evolution equation can be infinite. At the same time, global attractors are always compact sets in the corresponding phase spaces. Then it is reasonable to study their Kolmogorov ε-entropy since this quantity is finite for every ε. In this chapter, we find upper estimates for the Kolmogorov ε-entropy of global attractors of non-autonomous evolution equations with translation compact symbols. These estimates are optimal in some sense and generalize the estimates for the ε-entropy of finite dimensional global attractors of the corresponding autonomous equations considered in Section 1.4.

In Section 3.1, we present a general upper estimate for the ε-entropy of the uniform global attractor $\mathcal{A}$ of a process $\{U_\sigma(t,\tau)\}$ corresponding to a non-autonomous equation $\partial_t u = A_{\sigma(t)}(u)$ with translation compact symbol $\sigma(t)$.

In Section 3.2, we consider cases, when the fractal dimension $\mathbf{d}_F\mathcal{A}$ of the uniform global attractor $\mathcal{A}$ is finite. This property holds when, for example, the time symbol $\sigma(t)$ is a quasiperiodic function in time t with k rationally independent frequencies. Then we prove that $\mathbf{d}_F\mathcal{A} \le d+k$ for an appropriate value d depending on the problem under the study. This means that the dimension $\mathbf{d}_F\mathcal{A}$ can grow to infinity as $k \to +\infty$.

In Section 3.3, the mentioned above results are applied to the estimates of the ε-entropy and the fractal dimension of the uniform global attractor of some non-autonomous equations of mathematical physics, namely, the 2D Navier–Stokes system with translation compact external force, damped wave equation with translation compact terms, and the non-autonomous complex Ginzburg–Landau equation.

We have to highlight the fundamental role of the paper [KTi59] in the study of the ε-entropy of compact sets in Hilbert or Banach spaces.

3.1 Estimates for ε-entropy

Using the notations from Chapter 2 and Section 2.5, we consider the family of non-autonomous equations

$$\partial_t u = A_{\sigma(t)}(u),\ u|_{t=\tau} = u_\tau,\ u_\tau \in E, \tag{3.1}$$

with symbols $\sigma(t) \in \mathcal{H}(\sigma_0(t))$. Here E is a Hilbert space. We assume that symbol $\sigma_0(t)$ of the original equation (2.21) is a translation compact function in the space Ξ. The topological space Ξ is assumed to be a Hausdorff space. Usually in applications, $\Xi = C(\mathbb{R};\Psi)$ or $\Xi = L_p^{loc}(\mathbb{R};\Psi)$ $(p \geq 1)$, where Ψ is a Banach space. We can also assume that Ξ is a product of a number of such spaces. The space Ξ is endowed with local uniform convergence topology on every bounded segment from $\mathbb{R}$. By definition, a sequence $\{\sigma_n(\cdot)\}$ converges to an element $\sigma(\cdot)$ as $n \to \infty$ in Ξ if

$$\|\Pi_{t_1,t_2}(\sigma_n(\cdot) - \sigma(\cdot))\|_{\Xi_{t_1,t_2}} \to 0 \ (n \to \infty)$$

for every closed interval $[t_1, t_2] \subset \mathbb{R}$. Here Π_{t_1,t_2} denotes the restriction operator onto the interval $[t_1, t_2]$, Ξ_{t_1,t_2} are the family of Banach spaces generating Ξ, and $\|\xi\|_{\Xi_{t_1,t_2}}$ is the norm of ξ in Ξ_{t_1,t_2}. For example, if $\Xi = C(\mathbb{R};\Psi)$, then $\Xi_{t_1,t_2} = C([t_1, t_2];\Psi)$, and $\sigma_n(\cdot) \to \sigma(\cdot)$ $(n \to \infty)$ in $C(\mathbb{R};\Psi)$ if

$$\max_{s\in[t_1,t_2]} \|\sigma_n(s) - \sigma(s)\|_\Psi \to 0 \text{ as } n \to \infty \tag{3.2}$$

for every $[t_1, t_2] \subset \mathbb{R}$. Similarly, $\sigma_n(\cdot) \to \sigma(\cdot)$ $(n \to \infty)$ in $\Xi = L_p^{loc}(\mathbb{R};\Psi)$ if

$$\int_{t_1}^{t_2} \|\sigma_n(s) - \sigma(s)\|_\Psi^p ds \to 0 \text{ as } n \to \infty \tag{3.3}$$

for all $[t_1, t_2] \subset \mathbb{R}$ (see [CV02a] for more details). In addition, we assume that the norms in the spaces Ξ_{t_1,t_2} satisfy the following condition:

$$\|\Pi_{t_1',t_2'}\xi\|_{\Xi_{t_1',t_2'}} \leq \|\Pi_{t_1,t_2}\xi\|_{\Xi_{t_1,t_2}}, \ \forall\, [t_1', t_2'] \subset [t_1, t_2]. \tag{3.4}$$

The spaces $C([t_1, t_2];\Psi)$ and $L_p^{loc}(t_1, t_2;\Psi)$ clearly satisfy (3.4).

We suppose that, for every $\sigma \in \mathcal{H}(\sigma_0)$, the Cauchy problem (3.1) generates a process $\{U_\sigma(t, \tau)\}$ acting in E, by the formula $U_\sigma(t, \tau)u_\tau = u(t)$, $t \geq \tau, \tau \in \mathbb{R}$, where $u(t)$ is a solution of (3.1) with initial data $u_\tau \in E$.

We assume that the conditions of Theorem 2.5.1 take place. Then the process $\{U_{\sigma_0}(t, \tau)\}$ has the global attractor $\mathcal{A}$ that has the form (2.33).

The problem is to study the ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A}) = \mathbf{H}_\varepsilon(\mathcal{A}, E)$ of the global attractor $\mathcal{A}$ in the space E (see Definition 1.4.1). We are going to estimate $\mathbf{H}_\varepsilon(\mathcal{A})$ using the information on the behaviour of the ε-entropy of the sets

$$\Pi_{0,l}\mathcal{H}(\sigma_0)$$

in the space $\Xi_{0,l}$ (where, e.g., $\Xi_{0,l} = C([0, l];\Psi)$ or $\Xi_{0,l} = L_p^{loc}(0, l;\Psi)$). This behaviour is assumed to be known as $l \to +\infty$ and $\varepsilon \to 0+$. Here $\Pi_{0,l}$ denotes the restriction operator on the segment $[0, l]$.

Let us formulate some additional notions and conditions for the process $\{U_{\sigma_0}(t, \tau)\}$ that we need to formulate the main theorem. First of all, we have to generalize for processes the property of quasidifferentiability (1.40) introduced in Section 1.4.1 for semigroups. Let $\{U(t, \tau)\}$ be a process in E. Consider the kernel $\mathcal{K}$ of this process (see Definition 2.1.3). The kernel sections, clearly, satisfy the following invariance property:

$$U(t, \tau)\mathcal{K}(t) = \mathcal{K}(\tau), \ \forall t \geq \tau, \ \ \tau \in \mathbb{R}. \tag{3.5}$$

Definition 3.1.1 A process $\{U(t,\tau)\}$ in E is called *uniformly quasidifferentiable* on $\mathcal{K}$, if there exists a family of linear bounded operators $\{L(t,\tau,u)\}$, where $u \in \mathcal{K}(\tau)$ and $t \geq \tau,\ \tau \in \mathbb{R}$, such that

$$\|U(t,\tau)u_1 - U(t,\tau)u - L(t,\tau,u)(u_1-u)\|_E \leq \gamma(\|u_1-u\|_E, t-\tau)\|u_1-u\|_E \quad (3.6)$$

for all $u, u_1 \in \mathcal{K}(\tau)$, and $\tau \in \mathbb{R}$, where the function $\gamma = \gamma(\xi, s) \to 0+$ as $\xi \to 0+$ for each fixed $s \geq 0$.

We now assume that the process $\{U_{\sigma_0}(t,\tau)\}$ corresponding to (3.1) is uniformly quasidifferentiable on the kernel $\mathcal{K}_{\sigma_0}$ and its quasidifferentials are generated by the variational equation

$$\partial_t v = A_{\sigma_0 u}(u(t))v,\ \ v|_{t=\tau} = v_\tau,\ v_\tau \in E, \quad (3.7)$$

where $u(t) = U_{\sigma_0}(t,\tau)u_\tau,\ u_\tau \in \mathcal{K}_{\sigma_0}(\tau)$, that is, $L(t,\tau,u_\tau)v_\tau = v(t)$, where $v(t)$ is a solution of problem (3.7) with initial data v_τ. It is assumed that this Cauchy problem is uniquely solvable for all $u_\tau \in \mathcal{K}_{\sigma_0}(\tau)$ and for every $v_\tau \in E$.

Similar to (1.43) we introduce the numbers

$$\tilde{q}_j := \limsup_{T\to+\infty} \sup_{\tau\in\mathbb{R}} \sup_{u_\tau\in\mathcal{K}(\tau)} \frac{1}{T}\int_\tau^{\tau+T} \mathrm{Tr}_j A_{\sigma_0 u}(u(t))dt, \quad (3.8)$$

where $u(t) = U_{\sigma_0}(t,\tau)u_\tau$ and the j-trace $\mathrm{Tr}_j(L)$ of a linear operator L in a Hilbert space E is defined in (1.42).

We also assume that the following Lipschitz condition holds for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, corresponding to (3.1):

$$\begin{aligned} \|U_{\sigma_1}(h,0)u_0 - U_{\sigma_2}(h,0)u_0\|_E &\leq C(h)\|\sigma_1-\sigma_2\|_{\Xi_{0,h}}, \\ \forall\sigma_1,\sigma_2 &\in \mathcal{H}(\sigma_0),\ \forall u_0 \in \mathcal{A},\ \forall h \geq 0. \end{aligned} \quad (3.9)$$

It follows from (3.9) that

$$|U_{\sigma_1}(t,\tau)u_\tau - U_{\sigma_2}(t,\tau)u_\tau| \leq C(|t-\tau|)\|\sigma_1-\sigma_2\|_{\Xi_{\tau,t}}\ \forall\sigma_1,\sigma_2 \in \mathcal{H}(\sigma_0),\ \forall u_\tau \in \mathcal{A},$$

for all $t > \tau$, $\tau \in \mathbb{R}$.

We now are ready to formulate the main theorem of this chapter.

Theorem 3.1.1 *Let the assumptions of Theorem 2.5.1 hold. Assume that the original process $\{U_{\sigma_0}(t,\tau)\}$ is uniformly quasidifferentiable on $\mathcal{K}_{\sigma_0}$, its quasidifferentials are generated by the variational equation* (3.7)*, and the numbers $\tilde{q}_j$* (see (3.8)) *satisfies the inequalities*

$$\tilde{q}_j \leq q_j,\ j = 1,2,3,\ldots \quad (3.10)$$

We also assume that the Lipschitz condition (3.9) *holds for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, and the function q_j is concave in j (like $\cap$). Let m be the smallest number such that $q_{m+1} < 0$ (then $q_m \geq 0$). We denote*

$$d = m + q_m/(q_m - q_{m+1}). \quad (3.11)$$

Then for every $\delta > 0$ there exist numbers $\eta \in (0,1),\ \varepsilon_0 > 0,\ h > 0$ such that

$$\mathbf{H}_\varepsilon(\mathcal{A}) \le (d+\delta)\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon\eta}{4C(h)}}\left(\Pi_{0,h\log_{1/\eta}\left(\frac{\varepsilon_0}{\eta\varepsilon}\right)}\mathcal{H}(\sigma_0)\right),\ \ \forall\varepsilon < \varepsilon_0. \quad (3.12)$$

The value $C(h)$ is the Lipschitz constant from condition (3.9).

Recall that in the right-hand side of (3.12) the value$\mathbf{H}_\epsilon\left(\Pi_{0,l}\mathcal{H}(\sigma_0)\right)$ denotes the ϵ-entropy of the set $\mathcal{H}(\sigma_0)$ restricted to the interval $[0,l]$ and this ϵ-entropy is measured in the space $\Xi_{0,l}$ (e.g., in $C([0,l];\Psi)$ or $L_p^{loc}(0,l;\Psi)$). The complete proof of this theorem is given in [CV93e, CV02a].

Remark 3.1.1 Comparing inequality (3.12) with estimate (1.46) in the autonomous case, we observe that the term $(d+\delta)\log_2(\varepsilon_0/\eta\varepsilon)$ corresponds to the upper estimate for the ε-entropy of the kernel sections $\mathcal{K}(\tau)$ and in particular

$$d_F\mathcal{K}(\tau) \le d,\ \forall\tau \in \mathbb{R},\ \text{(see [CV02a])}.$$

Remark 3.1.2 We note that, when δ is small, inequality (3.12) is optimal with respect to the estimate of the ε-entropy of the kernel sections. However, another important parameter, namely h, in (3.12) approaches infinity as $\delta \to 0+$. This parameter controls the denominator in $\epsilon = \frac{\varepsilon\eta}{4C(h)}$, where the function $C(h)$ is the Lipschitz constant in (3.9) which usually grows exponentially when h goes to infinity. So, if the hull $\mathcal{H}(\sigma_0)$ is infinite dimensional, then the ϵ-entropy of $\mathcal{H}(\sigma_0)$ can grow extremely rapidly as $\varepsilon \to 0+$ and faster than the value $D\log\left(\frac{1}{\epsilon}\right)$ for arbitrary D. Thus, it is reasonable to optimize estimate (3.12) with respect to small values of h. The following theorem presents a result in this direction. The proof is given in [CV02a].

Theorem 3.1.2 *Let the assumptions of Theorem 3.1.1 be valid and*

$$\tilde{q}_j \le q_j,\ j = 1, 2, \ldots.$$

Assume that

$$\tfrac{q_j}{j} \to -\infty\ (j \to \infty). \quad (3.13)$$

Then for any $h > 0$ there exist $D > 0$ and $\varepsilon_0 > 0$ such that

$$\mathbf{H}_\varepsilon(\mathcal{A}) \le D\log_2\left(\tfrac{2\varepsilon_0}{\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon}{8C(h)}}\left(\Pi_{0,h\log_2\left(\frac{2\varepsilon_0}{\varepsilon}\right)}\mathcal{H}(\sigma_0)\right) \quad (3.14)$$

for all $\varepsilon \le \varepsilon_0$. (Usually in applications, $C(h)$ approaches 1 as $h \to +0$).

We now consider a particular case, where $\sigma_0(t)$ is an almost periodic function, that is, the hull $\mathcal{H}(\sigma_0)$ is compact in $C_b(\mathbb{R};\Psi)$ with respect to the topology of uniform convergence on $\mathbb{R}$. The norm in $C_b(\mathbb{R};\Psi)$ is given by

$$\|\xi\|_{C_b(\mathbb{R};\Psi)} := \sup_{t\in\mathbb{R}}\|\xi(t)\|_\Psi.$$

Since

$$\|\xi\|_{C([0,l];\Psi)} \le \|\xi\|_{C_b(\mathbb{R};\Psi)},\ \forall l > 0,$$

we clearly have that

$$\mathbf{H}_\epsilon(\Pi_{0,l}\mathcal{H}(\sigma_0); C([0,l];\Psi)) \le \mathbf{H}_\epsilon(\mathcal{H}(\sigma_0); C_b(\mathbb{R};\Psi)) = \mathbf{H}_\epsilon(\mathcal{H}(\sigma_0)),\ \forall l > 0, \quad (3.15)$$

and Theorems 3.1.1 and 3.1.2 imply

Corollary 3.1.1 *We assume that the function $\sigma_0(t)$ is almost periodic. Let the assumptions of Theorem 3.1.1 be valid. Then*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leq (d+\delta)\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon\eta}{4C(h)}}\left(\mathcal{H}(\sigma_0)\right), \ \ \forall\varepsilon < \varepsilon_0, \tag{3.16}$$

where $\mathbf{H}_\epsilon\left(\mathcal{H}(\sigma_0)\right)$ is the ϵ-entropy of the hull $\mathcal{H}(\sigma_0)$ in the space $C_b(\mathbb{R};\Psi)$.

Corollary 3.1.2 *Under the assumptions of Theorem 3.1.2, let $\mathcal{H}(\sigma_0) \Subset C_b(\mathbb{R};\Psi)$. Then*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leq D\log_2\left(\frac{2\varepsilon_0}{\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}\left(\mathcal{A}\right) + \mathbf{H}_{\frac{\varepsilon}{8C(h)}}\left(\mathcal{H}(\sigma_0)\right), \ \forall\varepsilon < \varepsilon_0. \tag{3.17}$$

Remark 3.1.3 If it is known that $\mathcal{H}(\sigma_0) \Subset L_p^b(\mathbb{R};\Psi)$, i.e., $\sigma_0(t)$ is an almost periodic functions in the Stepanov sense, then estimates (3.16) and (3.17) are also valid. In this case, $\mathbf{H}_\epsilon\left(\mathcal{H}(\sigma_0)\right)$ denotes the ϵ-entropy of $\mathcal{H}(\sigma_0)$ in the space $L_p^b(\mathbb{R};\Psi)$ measured in the norm

$$\|f\|_{L_p^b(\mathbb{R};\Psi)} := \left(\sup_{t\in\mathbb{R}}\int_t^{t+1}\|f(s)\|_\Psi^p ds\right)^{1/p}.$$

The estimate (3.16) shows that for a generic almost periodic function $\sigma_0(t)$ having infinite number rationally independent frequencies, the main contribution to the estimate for the ε-entropy of the global attractor $\mathcal{A}$ is made by the ε/L-entropy of the hull $\mathcal{H}(\sigma_0)$, where $L = \frac{4C(h)}{\eta}$. However, if the function $\sigma_0(t)$ has a finite number of frequencies, i.e., it is quasiperiodic, then the contribution of this quantity is comparable with the contribution of the term $d\log_2\left(\frac{\varepsilon_0}{\alpha\eta}\right)$. This leads to the finite dimensionality of the global attractor of the non-autonomous equation under the consideration. We discuss this question in the next section.

In conclusion, we consider two more important characteristics of a compact set X in the space E introduced in [KTi59]. The number

$$\mathbf{df}(X,E) = \mathbf{df}(X) := \limsup_{\epsilon\to 0+}\frac{\log_2\left(\mathbf{H}_\varepsilon(X)\right)}{\log_2\log_2\left(1/\varepsilon\right)} \tag{3.18}$$

is called the *functional dimension* of the set X in E and the number

$$\mathbf{q}(X,E) = \mathbf{q}(X) := \limsup_{\epsilon\to 0+}\frac{\log_2\left(\mathbf{H}_\varepsilon(X)\right)}{\log_2\left(1/\varepsilon\right)} \tag{3.19}$$

is called its *metric order* in E. It is easy to see that $\mathbf{df}(X) = 1$, $\mathbf{q}(X) = 0$, if $\mathbf{d}_F(X) < +\infty$. Thus the values $\mathbf{df}(X)$ and $\mathbf{q}(X)$ characterize infinite dimensional sets. Some examples of calculations of these values are given in [KTi59] (see also [VC98, VC03]).

Using Corollaries 3.1.1 and 3.1.2, we obtain

Corollary 3.1.3 *Let $\sigma_0(t)$ be an almost periodic function, then*

$$\mathbf{df}(\mathcal{A},E) \leq \mathbf{df}\left(\mathcal{H}(\sigma_0), C_b(\mathbb{R};\Psi)\right), \tag{3.20}$$

$$\mathbf{q}(\mathcal{A},E) \leq \mathbf{q}\left(\mathcal{H}(\sigma_0), C_b(\mathbb{R};\Psi)\right). \tag{3.21}$$

3.2 The cases of finite fractal dimension of global attractors

In this section, we study the fractal dimension of the uniform global attractor $\mathcal{A}$ of the process $\{U_{\sigma_0}(t,\tau)\}$ corresponding to (2.21) and its dependence on the fractal dimension of the hull $\mathcal{H}(\sigma_0)$. Recall that the fractal dimension $\mathbf{d}_F(X) = \mathbf{d}_F(X,E)$ of a compact set $X \Subset E$ in a Banach space E is the number

$$\mathbf{d}_F(X) := \limsup_{\varepsilon\to 0+} \frac{\mathbf{H}_\varepsilon(X)}{\log_2(1/\varepsilon)}.$$

We start with a very important example of a quasiperiodic symbol $\sigma_0(t)$ (see Example 2.4.5):

$$\sigma_0(t) = \varphi(\alpha_1 t, \alpha_2 t, ..., \alpha_k t) = \varphi(\bar{\alpha}s),$$

where $\varphi(\bar{\omega}), \bar{\omega} = (\omega_1, ..., \omega_k)$, is a 2π-periodic function in each argument ω_i, $i = 1, ..., k$; $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k), \alpha_i \in \mathbb{R}$, $\{\alpha_i\}$ are rationally independent numbers. We assume that $\varphi(\bar{\omega})$ is a Lipschitz continuous function on the k-dimensional torus $\mathbb{T}^k = [\mathbb{R} \bmod 2\pi]^k$ with values in a Banach space Ψ, $\varphi \in C^{lip}(\mathbb{T}^k;\Psi)$, i.e.,

$$\|\varphi(\bar{\omega}_1) - \varphi(\bar{\omega}_2)\|_\Psi \le L|\bar{\omega}_1 - \bar{\omega}_2|_{\mathbb{T}^k}\ \forall \bar{\omega}_1, \bar{\omega}_2 \in \mathbb{T}^k. \tag{3.22}$$

Here $|\cdot|_{\mathbb{T}^k}$ denotes the usual Euclidean norm in $\mathbb{R}^k$. By (2.28) the hull $\mathcal{H}(\sigma_0)$ of the function $\sigma_0(t)$ in the space $C_b(\mathbb{R};\Psi)$ coincides with

$$\{\varphi(\bar{\alpha}s + \bar{\theta}) \mid \bar{\theta} \in \mathbb{T}^k\} = \mathcal{H}(\sigma_0). \tag{3.23}$$

Proposition 3.2.1 *If $\sigma_0(t)$ is a quasiperiodic function, then*

$$\mathbf{H}_\epsilon(\mathcal{H}(\sigma_0)) := \mathbf{H}_\epsilon(\mathcal{H}(\sigma_0), C_b(\mathbb{R};\Psi)) \le \mathbf{H}_{L\epsilon}(\mathbb{T}^k) \le k\log_2\left(\frac{2}{L\epsilon}\right) \tag{3.24}$$

for all $\epsilon < L^{-1}$ and

$$\mathbf{d}_F(\mathcal{H}(\sigma_0)) = \mathbf{d}_F(\mathcal{H}(\sigma_0), C_b(\mathbb{R};\Psi)) \le k.$$

Proof. If $\sigma_1, \sigma_2 \in \mathcal{H}(\sigma_0)$, then by (3.23) $\sigma_i = \varphi(\bar{\alpha}s + \bar{\theta}_i)$ for some $\bar{\theta}_i \in \mathbb{T}^k,\ i = 1,2$ and by (3.22)

$$\begin{aligned}\|\sigma_1 - \sigma_2\|_{C_b(\mathbb{R};\Psi)} &:= \sup_{t\in\mathbb{R}} \|\sigma_1(t) - \sigma_2(t)\|_\Psi \\ &= \sup_{t\in\mathbb{R}} \|\varphi(\bar{\alpha}t + \bar{\theta}_1) - \varphi(\bar{\alpha}t + \bar{\theta}_2)\|_\Psi \le L|\bar{\theta}_1 - \bar{\theta}_2|_{\mathbb{T}^k}.\end{aligned}$$

Therefore

$$N_\epsilon(\mathcal{H}(\sigma_0)) \le N_{L\epsilon}(\mathbb{T}^k).$$

It is known that the torus $\mathbb{T}^k$ with Euclidean metric can be covered by at most $\left(\frac{2}{\varepsilon}\right)^k$ balls of radius $\varepsilon < 1$ (see, e.g., [ConSl88]). Hence,

$$\begin{aligned} N_\epsilon(\mathcal{H}(\sigma_0)) &\le \left(\frac{2}{L\epsilon}\right)^k, \\ \mathbf{H}_\epsilon(\mathcal{H}(\sigma_0)) &\le k\log_2\left(\frac{2}{L\epsilon}\right),\ \forall \epsilon < L^{-1}, \end{aligned}$$

and consequently

$$\mathbf{d}_F(\mathcal{H}(\sigma_0)) := \limsup_{\epsilon\to 0+} \frac{\mathbf{H}_\epsilon(\mathcal{H}(\sigma_0))}{\log_\epsilon(1/\varepsilon)} \le k,$$

which complete the proof. ■

Remark 3.2.1 In the generic case, $\mathcal{H}(\sigma_0)$ is clearly a Lipschitz continuous manifold in $C_b(\mathbb{R};\Psi)$ isometric to the torus $\mathbb{T}^k$, therefore $\mathbf{d}_F(\mathcal{H}(\sigma_0)) = k$.

Theorem 3.2.1 *Let in the assumptions of Theorem 3.1.1 the function $\sigma_0(t)$ be quasi-periodic of the form $\sigma_0(t) = \phi(\alpha_1 t, \alpha_2 t, \dots, \alpha_k t) = \phi(\bar{\alpha} t)$, where $\phi(\omega_1, \omega_2, \dots, \omega_k) = \phi(\bar{\omega}) \in C^{\mathrm{Lip}}(\mathbb{T}^k;\Psi)$. Then the estimate* (3.16) *becomes*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \le (d+\delta)\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + k\log_2\left(\frac{8C(h)}{L\eta\varepsilon}\right), \quad \forall \varepsilon < \varepsilon_0, \tag{3.25}$$

where L is the Lipschitz constant from inequality (3.22). Moreover,

$$\mathbf{d}_F(\mathcal{A}) \le d + k. \tag{3.26}$$

Proof. Indeed, inequality (3.16) along with (3.24) gives

$$\begin{aligned}
\mathbf{H}_\varepsilon(\mathcal{A}) &\le (d+\delta)\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon\eta}{4C(h)}}(\mathcal{H}(\sigma_0)) \\
&\le (d+\delta)\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + k\log_2\left(\frac{8C(h)}{L\eta\varepsilon}\right).
\end{aligned}$$

Passing to the limit in the ratio $\mathbf{H}_\varepsilon(\mathcal{A})/\log_2(1/\varepsilon)$ as $\varepsilon \to 0+$ we obtain

$$\mathbf{d}_F(\mathcal{A}) \le d + \delta + k.$$

Since δ was arbitrary small, we have (3.26). ■

Recall that, in the autonomous case with $k = 0$, estimate (1.45) is an analog of estimate (3.26), where $X = \mathcal{A}$: $\mathbf{d}_F(\mathcal{A}) \le d$. In the non-autonomous case, when $k \ne 0$, estimate $\mathbf{d}_F(\mathcal{A}) \le d + k$ holds, where the number k of rationally independent frequencies of the function $\sigma_0(t)$ is added to the quantity d.

We now generalize Theorem 3.2.1 for more general symbols $\sigma_0(t)$ that are not almost periodic, however, the dimension of the corresponding global attractors $\mathcal{A}$ is also finite.

Let, as before, $\sigma_0(t)$ be a tr.c. function in Ξ and, thereby, its hull $\mathcal{H}(\sigma_0)$ is compact in Ξ. (For example, the space Ξ is $C(\mathbb{R};\Psi)$ or $\Xi = L_p^{loc}(\mathbb{R};\Psi)$.) In [CV02a], it is proved that the value

$$\limsup_{\epsilon\to 0+} \mathbf{H}_\epsilon\left(\Pi_{0,l\log_2(K/\epsilon)}\Sigma\right)/\log_2(1/\epsilon) \tag{3.27}$$

is independent of a number $K > 0$ for any compact subset $\Sigma \Subset \Xi$. We now define the following number for Σ :

$$\mathbf{d}_F^{loc}(\Sigma, l) := \limsup_{\epsilon\to 0+} \mathbf{H}_\epsilon\left(\Pi_{0,l\log_2(1/\epsilon)}\Sigma\right)/\log_2(1/\epsilon) \tag{3.28}$$

depending on a positive parameter l.

Remark 3.2.2 If $\Sigma = \mathcal{H}(\sigma_0)$, where σ_0 is a smooth q.p. function with k independent frequencies, then $\mathbf{d}_F^{loc}(\Sigma, l) \le k$ for any l because $\mathcal{H}(\sigma_0)$ is a Lipschitz continuous image of a k-dimensional torus $\mathbb{T}^k$ (see Proposition 3.2.1).

If for some l the value $\mathbf{d}_F^{loc}(\Sigma, l) < +\infty$ for a set Σ, then we say, by definition, that Σ has a *local fractal dimension* $\mathbf{d}_F^{loc}(\Sigma, l)$ in the topological space $C(\mathbb{R}; \Psi)$.

Theorem 3.2.2 *Under the conditions of Theorem 3.1.1 assume that*

$$\mathbf{d}_F^{loc}(\mathcal{H}(\sigma_0), h_1) < +\infty,$$

where $h_1 = h(\delta)/\log_2(1/\eta)$. *Then, for any* $\delta > 0$,

$$\mathbf{d}_F(\mathcal{A}) \le d + \delta + \mathbf{d}_F^{loc}(\mathcal{H}(\sigma_0), h_1). \tag{3.29}$$

Moreover, if

$$\mathbf{d}_F^{loc}(\mathcal{H}(\sigma_0), h) \le k, \ \forall h > 0,$$

then

$$\mathbf{d}_F(\mathcal{A}) \le d + k.$$

Indeed, dividing (3.12) by $\log_2(1/\varepsilon)$ and changing the variables $\epsilon = \frac{\eta}{4C(h)}\varepsilon$, we find

$$\begin{aligned}
\mathbf{d}_F(\mathcal{A}) &\le (d+\delta) + \limsup_{\epsilon \to 0+} \mathbf{H}_\epsilon \left(\Pi_{0,h\log_{1/\alpha}\left(\frac{\varepsilon_0}{4C(h)\epsilon}\right)} \mathcal{H}(\sigma_0) \right) \Big/ \left(\log_2(1/\epsilon) + \log_2 \frac{\eta}{4C(h)} \right) \\
&= (d+\delta) + \limsup_{\epsilon \to 0+} \frac{\mathbf{H}_\epsilon \left(\Pi_{0,h_1 \log_2(K/\epsilon)} \mathcal{H}(\sigma_0) \right)}{\log_2(1/\epsilon)} = (d+\delta) + \mathbf{d}_F^{loc}(\mathcal{H}(\sigma_0), h_1),
\end{aligned}$$

where $K = \varepsilon_0/(4C(h))$. Here we used the fact that (3.27) is independent of K.

3.3 Applications to non-autonomous equations of mathematical physics

3.3.1 2D Navier–Stokes system

We consider the family of equations

$$\begin{aligned}
&\partial_t u + \nu L u + B(u,u) = g(x,t), \\
&u|_{t=\tau} = u_\tau, \ u_\tau \in H,
\end{aligned} \tag{3.30}$$

(see Section 2.6.1) with external forces $g \in \mathcal{H}(g_0)$. We assume that the original external force $g_0(x,t)$ is a tr.c. function in $L_2^{loc}(\mathbb{R}; H) =: \Xi$. The space $L_2^{loc}(\mathbb{R}; H)$ is endowed with the strong convergence topology on every $[t_1, t_2] \subset \mathbb{R}$. Then clearly $g_0 \in L_2^b(\mathbb{R}; H)$ and

$$\|g\|_{L_2^b}^2 \le \|g_0\|_{L_2^b}^2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} |g_0(s)|^2 ds < \infty, \tag{3.31}$$

for every function $g \in \mathcal{H}(g_0)$ (see (2.37) and (2.43)).

We consider the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, corresponding to problem (3.30) and acting in H. In Section 2.6.1, it was proved that the process $\{U_{g_0}(t,\tau)\}$ has the uniform global attractor $\mathcal{A} \Subset H$ and the set $\mathcal{A}$ has the structure form

$$\mathcal{A} = \bigcup_{g \in \mathcal{H}(g_0)} \mathcal{K}_g(0), \tag{3.32}$$

where $\mathcal{K}_g$ is the kernel of the process $\{U_g(t,\tau)\}$ with external force $g \in \mathcal{H}(g_0)$.

We now study the Kolmogorov ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A})$ of the set $\mathcal{A}$ in H.

In [CV02a], it is proved that the family $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, satisfies the Lipschitz condition (3.9), namely, we have the inequality

$$|U_{g_1}(h,0)u_0 - U_{g_2}(h,0)u_0| \leq C(h)\|g_1 - g_2\|_{L_2(0,h;H)} \ \forall g_1, g_2 \in \mathcal{H}(g_0),\ u_0 \in \mathcal{A}, \tag{3.33}$$

where the Lipschitz constant $C(h)$ depends also on ν, λ_1, and $\|g_0\|^2_{L_2^b}$ and has an exponential growth in h.

Let us discuss the property of quasidifferentiability. In [CV02a], it is proved that the process $\{U_{g_0}(t,\tau)\}$ is uniformly quasidifferentiable on $\mathcal{K}_{g_0}$ and the corresponding variation equation reads

$$\partial_t v = -\nu L v - B(u(t),v) - B(v,u(t)) =: A_{g_0 u}(u(t),t)v, \quad v|_{t=\tau} = v_\tau, \tag{3.34}$$

where $u(t) = U_{g_0}(t,\tau)u_\tau$, $u_\tau \in \mathcal{K}_{g_0}(\tau)$ (the proof is based on the methods from [BV89] and [T88]). Thus, the quasidifferentials are the maps $L(t,\tau;u_\tau) : H \to H$, $L(t,\tau;u_\tau)v_\tau = v(t)$, where $v(t)$ is a solution of (3.34).

Following the scheme described in Section 3.1, we set

$$\tilde{q}_j := \limsup_{T\to\infty} \sup_{\tau\in\mathbb{R}} \sup_{u_\tau \in \mathcal{K}_{g_0}(\tau)} \left(\frac{1}{T} \int_\tau^{\tau+T} \mathrm{Tr}_j A_{g_0 u}(u(s))ds \right),\ j \in \mathbb{N},$$

where $u(t) = U_{g_0}(t,\tau)u_\tau$, and Tr_j is the j-dimensional trace of an operator. Similar to the autonomous case (see the proof of Theorem 1.4.2 in Section 1.4.2), we obtain the following estimate:

$$\int_\tau^t \mathrm{Tr}_j A_{g_0 u}(u(s))ds \leq -\frac{\nu C_2 j^2}{2|\Omega|}(t-\tau) + \frac{1}{\nu^2}|u_\tau|^2 + \frac{1}{\lambda_1 \nu^3}\int_\tau^t |g_0(s)|^2 ds.$$

Therefore

$$\tilde{q}_j \leq -\frac{\nu C_2 j^2}{2|\Omega|} + \frac{|\Omega|}{C_1 \nu^3} M(|g_0|^2) =: \varphi(j) = q_j,\ j = 1,2,\ldots \tag{3.35}$$

where

$$M(|g_0|^2) := \limsup_{T\to\infty} \sup_{\tau\in\mathbb{R}} \left(\frac{1}{T} \int_\tau^{\tau+T} |g_0(t)|^2 dt \right) \leq \|g_0\|^2_{L_2^b} < \infty.$$

The dimensionless constants C_1 and C_2 were defined in (1.55) (see also Remark 1.4.5). The function $\varphi(j)$ in (3.35) is concave with respect to j.

Let m be the smallest integer such that $q_{m+1} = \varphi(m+1) < 0$ (see Theorem 3.1.1). We set

$$d = m + \frac{q_m}{q_m - q_{m+1}}.$$

Let also d^* be the root of the equation $\varphi(x) = 0$, i.e.,

$$d^* = c\frac{M(|g_0|^2)^{1/2}|\Omega|}{\nu^2}, \ c = \left(\frac{2}{C_1 C_2}\right)^{1/2}. \tag{3.36}$$

Then, clearly,

$$d^* \leq c\frac{\|g_0\|_{L_2^b}|\Omega|}{\nu^2},$$

since $M(|g_0|^2) \leq \|g_0\|_{L_2^b}^2$. It is obvious that

$$d \leq d^* \leq c\frac{\|g_0\|_{L_2^b}|\Omega|}{\nu^2}, \tag{3.37}$$

because the function φ is concave (see Remark 1.4.2).

Now Theorem 3.1.1 is applicable and we have the following result.

Theorem 3.3.1 *For any $\delta > 0$ there exist $h > 0, \varepsilon_0 > 0$, and $\eta < 1$ such that*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leq \left(c\frac{\|g_0\|_{L_2^b}|\Omega|}{\nu^2} + \delta\right)\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon\eta}{4C(h)}}\left(\Pi_{0,h\log_{1/\eta}\left(\frac{\varepsilon_0}{\eta\varepsilon}\right)}\mathcal{H}(g_0)\right) \tag{3.38}$$

for all $\varepsilon \leq \varepsilon_0$ (the constant $C(h)$ is taken from (3.33)). Here $\mathbf{H}_\epsilon(\Pi_{0,l}\mathcal{H}(g_0))$ denotes the ϵ-entropy of the set $\Pi_{0,l}\mathcal{H}(g_0)$ in the space $L_2(0, l; H)$.

Remark 3.3.1 The best up-to-date estimate for the constant c in (3.38) is:

$$c \leq \frac{1}{2\pi^{3/2}}$$

(see Remark 1.4.5 and [CI04]).

We note that $\varphi(j)/j \to -\infty$ $(j \to \infty)$ (see (3.35)). Thus, using Theorem 3.1.2 we obtain the following result.

Theorem 3.3.2 *For any $h > 0$ there exist $D > 0$ and $\varepsilon_0 > 0$ such that*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leq D\log_2\left(\frac{2\varepsilon_0}{\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon}{8C(h)}}\left(\Pi_{0,h\log_2\left(\frac{2\varepsilon_0}{\varepsilon}\right)}\mathcal{H}(g_0)\right) \tag{3.39}$$

for all $\varepsilon \leq \varepsilon_0$.

We now consider a special case, where $g_0(x, t)$ is a quasiperiodic function, i.e.,

$$g_0(x, t) = G(x, \alpha_1 t, \alpha_2 t, ..., \alpha_k t) = G(x, \bar{\alpha}t).$$

Here $G(\cdot) \in C^{lip}(\mathbb{T}^k; H)$ and the numbers $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ are rationally independent (see Section 3.2). Thus, $\mathcal{H}(g_0) = \{G(x, \bar{\alpha}t + \bar{\theta}) \mid \bar{\theta} \in \mathbb{T}^k\}$.

It follows from Kronecker–Weyl's theorem (see, e.g., [KoSiFo80]) that

$$\begin{aligned} M(|g_0|^2) &:= \lim_{T\to\infty}\sup_{\bar{\theta}\in\mathbb{T}^k}\left(\frac{1}{T}\int_0^T |G(\cdot, \bar{\theta} + \bar{\alpha}t)|^2 dt\right) \\ &= \frac{1}{|2\pi|^k}\int\cdots\int_{\mathbb{T}^k} |G(\cdot, \omega_1, \ldots, \omega_k)|^2 d\omega_1 \cdots d\omega_k =: \Gamma^2. \end{aligned}$$

Then from (3.36) we conclude that

$$d \leq d^* = c\frac{M(|\mathcal{H}(g_0)|^2)^{1/2}|\Omega|}{\nu^2} = c\frac{\Gamma|\Omega|}{\nu^2}.$$

Using Theorem 3.2.1 we obtain the

Theorem 3.3.3 *The fractal dimension of the uniform attractor $\mathcal{A}$ of the 2D Navier-Stokes system with quasiperiodic external force $g_0(x,s) = G(x,\bar{\alpha}s)$, $G \in C(\mathbb{T}^k; H)$ satisfies the estimate*

$$\mathbf{d}_F\mathcal{A} \leq c\frac{\Gamma|\Omega|}{\nu^2} + k, \tag{3.40}$$

where the dimensionless constant c depends on the shape of Ω ($c(\Omega) = c(\lambda\Omega)$) and admits the following absolute upper bound: $c < \frac{1}{2\pi^{3/2}}$.

Remark 3.3.2 In the autonomous case ($k = 0$) estimate (3.40) becomes the upper bound (1.49) for the fractal dimension of the attractor of the autonomous Navier-Stokes system (where $\Gamma = |g_0|,\ g_0 = g_0(x)$). In the non-autonomous case, the estimate (3.40) contains also the term $k = \dim \mathbb{T}^k$ that is the dimension of the hull $\mathcal{H}(g_0) = \{G(x, \bar{\alpha}s + \bar{\theta}) \mid \bar{\theta} \in \mathbb{T}^k\}$, where k is the number of rationally independent frequencies of the q.p. external force $g_0(x,t)$.

Remark 3.3.3 It was proved in [CV02a] that

$$\mathbf{d}_F\mathcal{K}_g(t) \leq c\frac{\Gamma|\Omega|}{\nu^2}\ \forall t \in \mathbb{R},$$

and since $\mathbf{d}_F\mathcal{H}(g_0) \leq \dim \mathbb{T}^k = k$ we conclude that estimate (3.40) agrees well with the representation (3.32).

Remark 3.3.4 Suppose that functions $G_k(x, \omega_1, \ldots, \omega_k) = G_k(x, \bar{\omega}^k)$, $\bar{\omega}^k \in \mathbb{T}^k$, $k = 1, 2, \ldots$ are given such that

$$\Gamma_k = \left(\frac{1}{|2\pi|^k}\int_{\mathbb{T}^k} |G_k(\,.\,, \bar{\omega}^k)|^2 d\bar{\omega}^k\right)^{1/2} \leq R\ \ \forall\, k \in \mathbb{N}.$$

Assume also that $1/\nu \leq R_1$. Consider the global attractors $\{\mathcal{A}^k\}$ of the 2D Navier–Stokes systems with external forces $g_{0k}(x,t) = G_k(x, \alpha_1 t, \alpha_2 t, ..., \alpha_k t)$, where the sequence $\{\alpha_i\}$ consists of rationally independent numbers, then it follows from (3.40) that

$$\mathbf{d}_F\mathcal{A}^k \leq k + D,\ \forall\, k \in \mathbb{N}, \tag{3.41}$$

where $D = D(R, R_1)$. Therefore, the right-hand side of (3.41) tends to infinity as $k \to \infty$, while the non-autonomous analogues of the Reynolds number Re and Grashof number Gr depending on R, $1/\nu$, and $|\Omega|$ remain bounded.

Let us present an example of external forces $\{\hat{G}_k(x, \bar{\omega}^k)\}$ satisfying the conditions of Remark 3.3.4 such that

$$\mathbf{d}_F\mathcal{A}_{\mathbb{T}^k} \geq k. \tag{3.42}$$

Consider the following function:

$$\hat{u}(x,t) = \sum_{i=1}^{k} \left(a_i(x)\cos(\alpha_i t) + a_{i+k}(x)\sin(\alpha_i t) \right), \tag{3.43}$$

where $a_i(x)$ $(i = 1, \ldots, 2k, \ldots)$ are linearly independent vector-functions, $a_i(x) = (a_i^1(x), a_i^2(x))$, satisfying the conditions: $a_i(x) \in (C^2(\bar{\Omega}))^2$, $(\nabla, a_i(x)) = 0$, $a_i|_{\partial\Omega} = 0$. We assume that the frequencies $(\alpha_1, \ldots, \alpha_k, \ldots)$ are rationally independent. We set

$$\hat{g}_k(x, \bar{\alpha}t) = \partial_t \hat{u} + \nu L\hat{u} + B(\hat{u}, \hat{u}), \tag{3.44}$$

where $\hat{u}(x,t)$ is defined by formula (3.43). Obviously, $\hat{g}_k(x, \alpha t)$ is a q.p. function. The function $\hat{u}(x,t)$ is a complete bounded trajectory of the Navier-Stokes system with external force $\hat{g}_k$. If the coefficients $a_i(x)$ in (3.43) decay rapidly, then $\Gamma_k \le R$ for all $k \in \mathbb{N}$. We note that, $\hat{u}(\cdot,t) \in \mathcal{A}$ for all $t \in \mathbb{R}$. It is easy to see that the trajectory $\hat{u}(\cdot,t)$ provides an everywhere dense winding of a k-dimensional torus $\tilde{\mathbb{T}}^k \subset H$. Therefore, the closure in H : $\overline{\{\hat{u}(t) \mid t \in \mathbb{R}\}} = \tilde{\mathbb{T}}^k$ belongs to $\mathcal{A}$. Hence,

$$\mathbf{d}_F \tilde{\mathbb{T}}^k = k \le \mathbf{d}_F \mathcal{A}.$$

This example shows that the main term k in estimate (3.41) is precise.

3.3.2 Wave equations with dissipation

We consider the non-autonomous wave equation from Section 2.6.2

$$\begin{aligned} &\partial_t^2 u + \gamma \partial_t u = \Delta u - f_0(u,t) + g_0(x,t),\ u|_{\partial\Omega} = 0, \\ &u|_{t=\tau} = u_\tau,\ \partial_t u|_{t=\tau} = p_\tau,\ u_\tau \in H_0^1(\Omega), p_\tau \in L_2(\Omega), \end{aligned} \tag{3.45}$$

where $x \in \Omega \Subset \mathbb{R}^3$. The function $f_0(v,t) \in C^1(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ satisfies conditions (2.71) – (2.74) from Section 2.6.2 and the following inequality that is analogous to (1.61):

$$|f_v(v_1,t) - f_v(v_2,t)| \le C(|v_1|^{2-\delta} + |u_2|^{2-\delta} + 1)|v_1 - v_2|^\delta, \tag{3.46}$$

for all $v_1, v_2 \in \mathbb{R},\ t \in \mathbb{R}$, where $0 < \delta \le 1$.

Moreover, we assume that the function $(f_0(v,t), f_{0t}(v,t))$ is tr.c. in $C(\mathbb{R}; \mathcal{M}_2)$ and the function $g_0(x,t)$ is tr.c. in $L_2^{loc}(\mathbb{R}; L_2(\Omega))$. The norm in the Banach space $\mathcal{M}_2$ is defined in (2.79). The symbol of problem (3.45) is $\sigma_0(t) = (f_0(v,t), g_0(x,t))$. It is clear that the function $\sigma_0(t)$ is tr.c. in $\Xi = C(\mathbb{R}; \mathcal{M}_2) \times L_2^{loc}(\mathbb{R}; L_2(\Omega))$. As usual, $\mathcal{H}(\sigma_0)$ denotes the hull of $\sigma_0(t)$ in Ξ. Consider the family of problems (3.45) with symbols $\sigma(t) = (f(v,t), g(x,t)) \in \mathcal{H}(\sigma_0)$. By Proposition 2.6.3, problem (3.45) generates a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, $U_\sigma(t,\tau) : E \to E$, acting in the energy space $E = H_0^1(\Omega) \times L_2(\Omega)$.

By Propositions 2.6.5 and 2.6.6, the process $\{U_{\sigma_0}(t,\tau)\}$ is uniformly asymptotically compact and the family $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, is $(E \times \mathcal{H}(\sigma_0), E)$-continuous. Propositions 2.6.1 implies that the process $\{U_{\sigma_0}(t,\tau)\}$ has the uniform global attractor

$$\mathcal{A} = \bigcup_{\sigma \in \mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(0),$$

where $\mathcal{K}_\sigma$ is the kernel of $\{U_\sigma(t,\tau)\}$. The set $\mathcal{A}$ is compact in E.

In [CV02a] it is proved that $\mathcal{A}$ is bounded in $E_1 = H^2(\Omega) \times H_0^1(\Omega)$ (recall that $\Omega \Subset \mathbb{R}^3$),

$$\|y\|_{E_1} \le M, \ \ \forall y \in \mathcal{A},$$

where the constant M is independent of y. Then by the Sobolev embedding theorem

$$\|u(\cdot)\|_{C(\overline{\Omega})} \le M_1, \ \ \forall y = (u(\cdot), p(\cdot)) \in \mathcal{A}. \tag{3.47}$$

We study the ε-entropy of the global attractor $\mathcal{A}$ in E.

In [CV02a], it is proved that the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, corresponding to problem (3.45) satisfies the Lipschitz condition (3.6): for any $h > 0$

$$\begin{aligned} |U_{\sigma_1}(h,0)y - U_{\sigma_2}(h,0)y| &\le C(h)\|\sigma_1 - \sigma_2\|_{\Xi_{0,h}} \\ \forall \sigma_1, \sigma_2 &\in \mathcal{H}(\sigma_0),\ y \in \mathcal{A}. \end{aligned} \tag{3.48}$$

Here $\Xi_{0,h} = C([0,h];\mathcal{M}_2) \times L_2^{loc}(0,h;L_2(\Omega))$. Moreover, in [CV02a], an explicit formula for the Lipschitz constant $C(h)$ is presented.

Similar to autonomous case (see the proof of Theorem 1.4.3), we rewrite the problem (3.45) in the form

$$\partial_t w = A(w) = L_\alpha w - G_{\sigma_0(t)}(w), \ \ w|_{t=\tau} = w_\tau, \tag{3.49}$$

where we use the new variable $w = (u,v) = (u, p + \alpha u)$, the operator L_α is defined in (1.66), and $G_{\sigma_0(t)}(w) = (0, f_0(u,t) - g_0(x,t))$. Here α is a real parameter to be chosen later on.

The variational equation for (3.49) has the form

$$\partial_t z = L_\alpha z - G_{\sigma_0 w}(w(t))z := A_{\sigma_0 w}(w(t))z, \ \ z|_{t=\tau} = z_\tau, \ \ z = (r,q), \tag{3.50}$$

where $G_{\sigma_0 w}(w(t))z = (0, f_u(u(t),t)r)$. Similarly to autonomous case (see [BV89]), we prove that the process $\{U_{\sigma_0}(t,\tau)\}$ of problem (3.49) is uniformly quasidifferentiable on the kernel $\mathcal{K}_{\sigma_0}$ and its quasidifferentials are generated by the system (3.50). We set

$$\tilde{q}_j := \limsup_{T\to\infty} \sup_{\tau\in\mathbb{R}} \sup_{w_\tau \in \mathcal{K}_{\sigma_0}(\tau)} \left(\frac{1}{T} \int_\tau^{\tau+T} \mathrm{Tr}_j A_{\sigma_0 w}(w(t))dt \right), \ j = 1, 2, \ldots,$$

where $w(t) = U_{\sigma_0}(t,\tau)w_\tau$. Using the reasoning from the proof of Theorem 1.4.3, we obtain the following estimate for the numbers $\tilde{q}_j$:

$$\tilde{q}_j \le q_j = -(\alpha/4)j + (C(M_1)/\alpha)j^{1/3} =: \varphi(j), \ \forall j \in \mathbb{N}, \tag{3.51}$$

where M_1 is due to the inequality

$$\sup\left\{\|u(\cdot,t)\|_{C(\bar{\Omega})} \mid t \in \mathbb{R},\ (u(\cdot), \partial_t u(\cdot)) \in \mathcal{K}_{\sigma_0}\right\} \le M_1 \text{ (see (3.47))}.$$

The function $\varphi(x), x \ge 0$, in (3.51) is concave and its root is $d^* = 8C_1(M_1)^{3/2}\alpha^{-3} =: C(M_1)\alpha^{-3}$. All the assumptions of Theorem 3.1.1 are verified so we have

Theorem 3.3.4 *For any $\delta > 0$, there exist $h > 0, \varepsilon_0 > 0$, and $\eta < 1$ such that*

$$\mathbf{H}_\varepsilon(\mathcal{A}_0) \leq \left(\frac{C}{\alpha^3} + \delta\right) \log_2 \left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}_0) + \mathbf{H}_{\frac{\varepsilon\eta}{4C(h)}} \left(\Pi_{0,h\log_{1/\eta}\left(\frac{\varepsilon_0}{\eta\varepsilon}\right)} \mathcal{H}(\sigma_0)\right) \quad (3.52)$$

for all $\varepsilon \leq \varepsilon_0$, where $\alpha = \min\{\gamma/4, \lambda_1/(2\gamma)\}$ and $C = C(M_1)$ (see (3.51)). Here $\mathbf{H}_\epsilon(\Pi_{0,l}\mathcal{H}(\sigma_0))$ denotes the ϵ-entropy of the set $\mathcal{H}(\sigma_0)$ measured in the space $\Xi_{0,l} = C([0,l];\mathcal{M}_2) \times L_2^{loc}(0,l;L_2(\Omega))$.

Remark 3.3.5 We cannot apply Theorem 3.1.2 to the hyperbolic equation (3.45) because the function $\varphi(j)$ in (3.51) does not satisfies (3.13).

In conclusion, we consider a hyperbolic equation with quasiperiodic terms. We suppose that

$$\begin{aligned} f_0(v,t) &= \Phi(v, \alpha_1 t, \alpha_2 t, ..., \alpha_k t) = \Phi(v, \bar{\alpha}t), \\ g_0(x,t) &= G(x, \alpha_1 t, \alpha_2 t, ..., \alpha_k t) = G(x, \bar{\alpha}t), \end{aligned}$$

where $\Phi(v,\bar{\omega}) \in C^{lip}(\mathbb{T}^k;\mathcal{M}_2)$ and $G(x,\bar{\omega}) \in C^{lip}(\mathbb{T}^k;L_2(\Omega))$. To obtain inequality (2.74) we assume that

$$|\bar{\alpha}| \leq \varkappa \ll 1,$$

where $\varkappa = \varkappa(\delta)$. Now, if $\Phi(v,\bar{\omega})$ satisfies the inequality

$$|\Phi_{\bar{\omega}}(v,\bar{\omega})| \leq \delta_1^2 \Phi(v,\bar{\omega}) + C_1, \ \forall\, (v,\bar{\omega}) \in \mathbb{R} \times \mathbb{T}^k,$$

then (2.74) is also valid for a small $\varkappa$. It follows that

$$\mathcal{H}(\sigma_0) = \left\{ \left(\Phi(v, \bar{\alpha}t + \bar{\theta}), G(x, \bar{\alpha}s + \bar{\theta})\right) \mid \bar{\theta} \in \mathbb{T}^k \right\}$$

and hence

$$\mathbf{d}_F\left(\mathcal{H}(\sigma_0), C_b(\mathbb{R};\mathcal{M}_2 \times L_2(\Omega)\right) = \mathbf{d}_F \mathcal{H}(\sigma_0) \leq k.$$

Using Theorem 3.2.1 we obtain the

Theorem 3.3.5 *The fractal dimension of the uniform global attractor $\mathcal{A}$ of the hyperbolic equation (3.45) with quasiperiodic symbol $\sigma_0(t) = (\Phi(v,\bar{\alpha}t), G(x,\bar{\alpha}t))$ satisfies the estimate*

$$\mathbf{d}_F \mathcal{A} \leq \frac{C}{\alpha^3} + k. \quad (3.53)$$

To illustrate Theorem 3.3.5 we consider the dissipative sine-Gordon equation with quasiperiodic forcing term

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - \beta \sin(u) + \psi(\bar{\alpha}t) g(x), \ u|_{\partial\Omega} = 0, \ \Omega \Subset \mathbb{R}^3, \quad (3.54)$$

where $\psi \in C^1(\mathbb{T}^k;\mathbb{R})$ and $g \in L_2(\Omega)$. Observe that the constant C in (3.52) and (3.53) does not exceed $c\beta^3$, where c depends on Ω (see (1.69) and (1.70)). For the global attractor $\mathcal{A}$ of problem (3.54) we have the estimate

$$\mathbf{d}_F \mathcal{A} \leq c\frac{\beta^3}{\alpha^3} + k. \quad (3.55)$$

Remark 3.3.6 In the autonomous case $k = 0$, estimates (3.53) and (3.55) coincided with (1.63) and (1.72).

3.3.3 Ginzburg–Landau equation

Here, we continue to study the non-autonomous Ginzburg–Landau equation (2.112) from Section 2.6.3. We consider the family of problems with periodic boundary conditions:

$$\begin{aligned} &\partial_t u = \nu(1+i\alpha)\Delta u + Ru - (1+i\beta(t))|u|^2 u + g(x,t),\ x \in \mathbb{T}^3, \\ &u|_{t=\tau} = u_\tau(x),\ u_\tau \in \mathbf{H} = L_2(\mathbb{T}^3;\mathbb{C}). \end{aligned} \tag{3.56}$$

For simplicity, we assume that coefficients α and R are independent on time. The symbol $\sigma(t) = (\beta(t), g(x,t))$ of (3.56) belongs to the hull $\mathcal{H}(\sigma_0)$ of the original symbol $\sigma_0(t) = (\beta_0(t), g_0(x,t))$. We assume that $\sigma_0(t)$ is a tr.c. function in $C^{loc}(\mathbb{R}_+;\mathbb{R}) \times L_2^{loc}(\mathbb{R}_+;\mathbf{H}) =: \Xi$ and the parameter $\beta_0(t)$ satisfies inequality (2.113).

Similar to the autonomous case (see Section 1.4.2), we rewrite equation (3.56) in a vector form

$$\partial_t \mathbf{u} = \nu a \Delta \mathbf{u} + R\mathbf{u} - \mathbf{f}(\mathbf{u}, \beta(t)) + \mathbf{g}(x,t),\ \mathbf{u}|_{t=\tau} = \mathbf{u}_\tau,\ \mathbf{u}_\tau \in \mathbf{H}. \tag{3.57}$$

where $a = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$, $\mathbf{f}(\mathbf{v},\beta) = |\mathbf{v}|^2 \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix} \mathbf{v}$, and $\mathbf{g}(x) = (g_1(x), g_2(x))^\top$.

We know from Section 2.6.3 that, for every $\sigma \in \mathcal{H}(\sigma_0)$, problem (3.56) has a unique solution $u \in C(\mathbb{R}_\tau;\mathbf{H}) \cap L_2^{loc}(\mathbb{R}_\tau;\mathbf{V}) \cap L_4^{loc}(\mathbb{R}_\tau;\mathbf{L}_4)$. (see also [BV89, CV94a, CV96a]). Thus for a given initial symbol $\sigma_0(t)$, problem (3.56) generates a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, acting in $\mathbf{H}$. It is proved that the process $\{U_{\sigma_0}(t,\tau)\}$ has the uniform global attractor $\mathcal{A}$ and

$$\mathcal{A} = \bigcup_{\sigma \in \mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(0),$$

where $\mathcal{K}_\sigma$ is the kernel of the process $\{U_\sigma(t,\tau)\}$. The set $\mathcal{A}$ is bounded in $\mathbf{V}$.

In [CV02a], Lipschitz condition is established for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$:

$$\begin{aligned} & \|U_{\sigma_1}(h,0)\mathbf{u}_0 - U_{\sigma_2}(h,0)\mathbf{u}_0\|_{\mathbf{H}} \\ \le\ & C(h)\left(\|\beta_1 - \beta_2\|_{C([0,h])} + \|g_1 - g_2\|_{L_2(0,h;H)}\right), \\ & \forall \sigma_1 = (\beta_1, g_1) \in \mathcal{H}(\sigma_0), \sigma_2 = (\beta_2, g_2) \in \mathcal{H}(\sigma_0),\ \mathbf{u}_0 \in \mathcal{A}. \end{aligned} \tag{3.58}$$

Now, to apply Theorem 3.1.1 we have to check that the process $\{U_{\sigma_0}(t,\tau)\}$ corresponding to problem (3.57) with the original symbol $\sigma_0(t)$ is uniformly quasidifferentiable on the kernel $\mathcal{K}_{\sigma_0}$. This fact is proved in [CV02a]. Recall that the variational equation for (3.57) is

$$\partial_t \mathbf{v} = \nu a \Delta \mathbf{v} + R\mathbf{v} - \mathbf{f}_\mathbf{u}(\mathbf{u}(t), \beta(t))\mathbf{v} =: A_{\sigma_0 \mathbf{u}}(\mathbf{u}(t))\mathbf{v},\ \mathbf{v}|_{t=\tau} = \mathbf{v}_\tau \in \mathbf{H}, \tag{3.59}$$

where the Jacobi matrix $\mathbf{f}_\mathbf{u}(\mathbf{u}, \beta)$ is defined in (1.33). Similar to autonomous case, we prove that

$$\begin{aligned} \tilde{q}_j &= \limsup_{T\to\infty} \sup_{\tau\in\mathbb{R}} \sup_{\mathbf{u}_\tau \in \mathcal{K}_{\sigma_0}(\tau)} \left(\frac{1}{T} \int_\tau^{\tau+T} \mathrm{Tr}_j(A_{\sigma_0 \mathbf{u}}(\mathbf{u}(t)) dt \right) \\ &\le -\nu C_1 j^{5/3} + Rj =: \varphi(j) = q_j,\ j = 1, 2, \ldots, \end{aligned}$$

where $\mathbf{u}(t) = U_{\sigma_0}(t,\tau)\mathbf{u}_\tau$ and finally (see (3.11))

$$d \leq d^* = \left(\frac{R}{C_1\nu}\right)^{3/2},$$

where d^* is the root of the equation $\varphi(x) = 0$ and C_1 was defined in (1.79).We conclude that Theorem 3.1.1 is applicable to the Ginzburg–Landau equation (3.57) and we have the following

Theorem 3.3.6 *For any $\delta > 0$ there exist $h > 0, \varepsilon_0 > 0,$ and $\eta < 1$ such that*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leq \left(\left(\frac{R}{C_1\nu}\right)^{3/2} + \delta\right)\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon\eta}{4C(h)}}\left(\Pi_{0,h\log_{1/\eta}\left(\frac{\varepsilon_0}{\eta\varepsilon}\right)}\mathcal{H}(\sigma_0)\right),$$

for all $\varepsilon \leq \varepsilon_0$ ($C(h)$ is defined in (3.58)). Here $\mathbf{H}_\epsilon(\mathcal{H}(\sigma_0)_{0,l})$ denotes the ϵ-entropy of $\mathcal{H}(\sigma_0)$ in $C([0,l]) \times L_2(0,l;\mathbf{H})$.

Theorem 3.1.2 implies the

Theorem 3.3.7 *For any $h > 0$ there exist $D > 0$ and $\varepsilon_0 > 0$ such that*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leq D\log_2\left(\frac{2\varepsilon_0}{\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon}{8C(h)}}\left(\Pi_{0,h\log_2\left(\frac{2\varepsilon_0}{\varepsilon}\right)}\mathcal{H}(\sigma_0)\right)$$

for all $\varepsilon \leq \varepsilon_0$.

Let us study the Ginzburg–Landau equation with quasiperiodic terms

$$\begin{aligned}\beta_0(t) &= B(\alpha_1 t, \alpha_2 t, ..., \alpha_k t) = B(\bar{\alpha}t),\\ g_0(x,t) &= G(x, \alpha_1 t, \alpha_2 t, ..., \alpha_k t) = G(x, \bar{\alpha}t),\end{aligned}$$

where $B(\bar{\omega}) \in C^{lip}(\mathbb{T}^k;\mathbb{R})$, $|B| \leq \sqrt{3}$, and $G(x,\bar{\omega}) \in C^{lip}(\mathbb{T}^k;\mathbf{H})$. We suppose that the numbers $(\alpha_1, \alpha_2, \dots, \alpha_k) =: \bar{\alpha}$ are rationally independent. As we know,

$$\mathcal{H}(\sigma_0) = \left\{\left(B(\bar{\alpha}t + \bar{\theta}), G(x, \bar{\alpha}t + \bar{\theta})\right) \mid \bar{\theta} \in \mathbb{T}^k\right\}$$

and

$$\mathbf{d}_F(\mathcal{H}(\sigma_0), C_b(\mathbb{R}) \times L_2(\mathbb{R};\mathbf{H})) = \mathbf{d}_F\mathcal{H}(\sigma_0) \leq k$$

(see Section 3.2).

Using Theorem 3.2.1 we obtain the

Theorem 3.3.8 *The fractal dimension of the global attractor $\mathcal{A}$ of the Ginzburg–Landau equation with quasiperiodic symbol $\sigma(s) = (B(\bar{\alpha}t), G(x,\bar{\alpha}t))$ satisfies the estimate*

$$\mathbf{d}_F\mathcal{A} \leq \left(\frac{R}{C_1\nu}\right)^{3/2} + k. \tag{3.60}$$

Now similarly to the Navier-Stokes system we consider the sequence of functions $B_k(\bar{\omega}^k)$ and $G_k(x, \bar{\omega}^k)$ which satisfy the above conditions. We denote by $\mathcal{A}(k)$ the corresponding uniform global attractors. Inequality (3.60) implies that

$$\mathbf{d}_F \mathcal{A}(k) \leq k + D, \tag{3.61}$$

where the constant D does not depend on k.

Similarly to the examples from the end of Section 3.3.1 we can construct examples of Ginzburg–Landau equations with terms $B_k(\bar{\omega}^k)$ and $G_k(x, \bar{\omega}^k)$ and with uniform global attractors $\mathcal{A}(k)$ such that

$$k \leq \mathbf{d}_F \mathcal{A}(k).$$

Therefore the main term k in estimate (3.61) is exact.

Chapter 4

Uniform global attractor of non-autonomous 2D Navier–Stokes system with singularly oscillating external force

In this chapter, we study the global attractor $\mathcal{A}^\varepsilon$ of the non-autonomous 2D Navier–Stokes (N.–S.) system with singularly oscillating external force of the form $g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$, $x \in \Omega \Subset \mathbb{R}^2, t \in \mathbb{R}$, $0 < \rho \le 1$. If the functions $g_0(x,t)$ and $g_1(z,t)$ are translation bounded in the corresponding spaces, then it is known that the global attractor $\mathcal{A}^\varepsilon$ is bounded in the space H (see Section 2.6.1). However, its norm $\|\mathcal{A}^\varepsilon\|_H$, as a function of ε, can be unbounded as $\varepsilon \to 0+$ since the magnitude of the external force is growing.

Assuming that the function $g_1(z,t)$ has a divergence representation of the form $g_1(z,t) = \partial_{z_1} G_1(z,t) + \partial_{z_2} G_2(z,t), z = (z_1, z_2) \in \mathbb{R}^2$, where the functions $G_j(z,t) \in L_2^b(\mathbb{R}; Z)$ (see Section 4.2), we prove that the global attractors $\mathcal{A}^\varepsilon$ of the N.–S. system are uniformly bounded: $\|\mathcal{A}^\varepsilon\|_H \le C$ for all $0 < \varepsilon \le 1$.

We also consider the "limiting" 2D N.–S. system with external force $g_0(x,t)$. We find an explicit estimate for the deviation of a solution $u^\varepsilon(x,t)$ of the original N.–S. system from a solution $u^0(x,t)$ of the "limiting" N.–S. system with the same initial data. If the function $g_1(z,t)$ admits the divergence representation and the functions $g_0(x,t)$ and $g_1(z,t)$ are translation compact in the corresponding spaces, then we prove that the global attractors $\mathcal{A}^\varepsilon$ converge to the global attractor $\mathcal{A}^0$ of the "limiting" system as $\varepsilon \to 0+$ in the norm of H. In the last section, we present an explicit estimate for the Hausdorff deviation of $\mathcal{A}^\varepsilon$ from $\mathcal{A}^0$ of the form: $\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \le C(\rho)\varepsilon^{1-\rho}$ in the case, when the global attractor $\mathcal{A}^0$ is exponential (providing that the Grashof number of the "limiting" 2D N.–S. system is small).

Some problems related to the homogenization and averaging of global attractors for the Navier–Stokes systems and for other evolution equations of mathematical physics with rapidly (non-singularly) oscillating coefficients and terms were studied in [HVe90, I96b, I98, VC01, VFi02, VC03, EfZ02, CVW05, CGoV05].

4.1 2D Navier–Stokes system with singularly oscillating force

We consider the non-autonomous 2D N.–S. system of the form

$$\begin{aligned} \partial_t u + u^1\partial_{x_1}u + u^2\partial_{x_2}u &= \nu\Delta u - \nabla p + g_0(x,t) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right), \\ \partial_{x_1}u_1 + \partial_{x_2}u_2 &= 0,\ u|_{\partial\Omega} = 0,\ x := (x_1,x_2) \in \Omega,\ \Omega \Subset \mathbb{R}^2. \end{aligned} \tag{4.1}$$

Here, $u = u(x,t) = (u^1(x,t), u^2(x,t))$ is the velocity vector field, $p = p(x,t)$ is the pressure and ν is the kinematic viscosity. In equation (4.1), ε is a small parameter, $0 < \varepsilon \le 1$, and ρ is fixed, $0 \le \rho \le 1$. We assume that the origin $0 \in \Omega$.

The vector functions $g_0(x,t) = (g_{01}(x,t), g_{02}(x,t)), x \in \Omega, t \in \mathbb{R}$, and $g_1(z,t) = (g_{11}(z,t), g_{12}(z,t)), z \in \mathbb{R}^2, t \in \mathbb{R}$, are given. The function $g_0(x,t) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right)$ is called the *external force*. We assume that, for every fixed ε, this external force belongs to the space $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ (we shall clarify this assumption later on). Under this condition, the Cauchy problem for equation (4.1) is well-studied (see, [Lio69, L70, T79, CoF89, BV89, CV02a] and Section 2.6.1).

As usual, we denote by H and $V = H^1$ the function spaces that are closures of the set $\mathcal{V}_0 := \left\{v \in (C_0^\infty(\Omega))^2 \mid \partial_{x_1}v_1(x) + \partial_{x_2}v_2(x) = 0,\ \forall x \in \Omega\right\}$ in the norms $|\cdot|$ and $\|\cdot\|$ of the spaces $L_2(\Omega)^2$ and $H_0^1(\Omega)^2$, respectively. We recall that

$$\|v\|^2 = |\nabla v|^2 = \int_\Omega \left(|\partial_{x_1}v^1(x)|^2 + |\partial_{x_2}v^1(x)|^2 + |\partial_{x_1}v^2(x)|^2 + |\partial_{x_2}v^2(x)|^2\right) dx.$$

The space $V' = V^*$ is dual to V. We denote by P the orthogonal projector from $L_2(\Omega)^2$ onto H (see Section 1.3.1). We set

$$g^\varepsilon(x,t) = Pg_0(x,t) + \frac{1}{\varepsilon^\rho} Pg_1\left(\frac{x}{\varepsilon}, t\right).$$

Applying the operator P to both sides of equation (4.1), we exclude the pressure $p(x,t)$ and obtain the following equation for the velocity vector field $u(x,t)$:

$$\partial_t u + \nu L u + B(u,u) = g^\varepsilon(x,t), \tag{4.2}$$

where $L = -P\Delta$ is the Stokes operator, $B(u,v) = P\left[u^1\partial_{x_1}v + u^2\partial_{x_2}v\right]$ and $g^\varepsilon(\cdot,t) \in L_2^{loc}(\mathbb{R}; H)$. The Stokes operator L is self-adjoint and the minimal eigenvalue λ_1 of the operator L is positive.

We assume that the function $g_0(\cdot,t) \in L_2(\Omega)^2$ for almost every $t \in \mathbb{R}$ and has a finite norm in the space $L_2^b(\mathbb{R}; L_2(\Omega)^2)$, that is,

$$\|g_0\|^2_{L_2^b(\mathbb{R};L_2(\Omega)^2)} = \|g_0\|^2_{L_2^b} := \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \left(\|g_0(\cdot,s)\|^2_{L_2(\Omega)^2}\right) ds < +\infty. \tag{4.3}$$

To describe the vector function $g_1(z,t), z = (z_1,z_2) \in \mathbb{R}^2, t \in \mathbb{R}$, we use the space $Z = L_2^b(\mathbb{R}_z^2; \mathbb{R}^2)$. By definition, a vector function $\varphi(z) = (\varphi_1(z_1,z_2), \varphi_2(z_1,z_2)) \in Z$, if

$$\|\varphi(\cdot)\|_Z^2 = \|\varphi(\cdot)\|^2_{L_2^b(\mathbb{R}_z^2;\mathbb{R}^2)} := \sup_{(z_1,z_2)\in\mathbb{R}^2} \int_{z_1}^{z_1+1} \int_{z_2}^{z_2+1} |\varphi(\zeta_1,\zeta_2)|^2 d\zeta_1 d\zeta_2 < +\infty.$$

We now assume that the function $g_1(\cdot,t) \in Z$ for almost every $t \in \mathbb{R}$ and has a finite norm in the space $L_2^b(\mathbb{R};Z)$, that is,

$$\begin{aligned}\|g_1(\cdot)\|^2_{L_2^b(\mathbb{R};Z)} &:= \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \left(\|g_1(\cdot,s)\|_Z^2\right) ds \\ &= \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \left(\sup_{(z_1,z_2)\in\mathbb{R}^2} \int_{z_1}^{z_1+1} \int_{z_2}^{z_2+1} |g_1(\zeta_1,\zeta_2,s)|^2 d\zeta_1 d\zeta_2 \right) ds < +\infty. \quad (4.4)\end{aligned}$$

For equation (4.1), we consider the initial data at an arbitrary time $\tau \in \mathbb{R}:$

$$u|_{t=\tau} = u_\tau,\ u_\tau \in H. \tag{4.5}$$

For a fixed $\varepsilon > 0$, the Cauchy problem (4.1) and (4.5) has a unique solution $u(t) := u(x,t)$ in a weak sense, that is, $u(t) \in C(\mathbb{R}_\tau;H) \cap L_2^{\mathrm{loc}}(\mathbb{R}_\tau;V)$, $\partial_t u \in L_2^{\mathrm{loc}}(\mathbb{R}_\tau;V')$, and $u(t)$ satisfies equation (4.1) in the distribution sense of the space $\mathcal{D}'(\mathbb{R}_\tau;V')$, where $\mathbb{R}_\tau = [\tau,+\infty)$ (see [Lio69, L70, CoF89, BV89, CV02a, T88] and Sections 1.3.1, 2.6.1).

Recall that every weak solution $u(t)$ of equation (4.1) satisfies the following energy equality

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 = \langle u(t), g^\varepsilon(t)\rangle\,,\ \forall t \geq \tau, \tag{4.6}$$

where the function $|u(t)|^2$ is absolutely continuous in t (see Section 1.3.1).

We need the following lemma proved in [CV02a].

Lemma 4.1.1 *Let a real function $y(t), t \geq 0,$ be uniformly continuous and satisfy the inequality*

$$y'(t) + \gamma y(t) \leq f(t), \forall t \geq 0, \tag{4.7}$$

where $\gamma > 0,\ f(t) \geq 0$ for all $t \geq 0,$ and $f \in L_1^{loc}(\mathbb{R}_+)$. Suppose also that

$$\int_t^{t+1} f(s)ds \leq M,\ \forall t \geq 0. \tag{4.8}$$

Then

$$y(t) \leq y(0)e^{-\gamma t} + M(1+\gamma^{-1}),\ \forall t \geq 0. \tag{4.9}$$

Using the standard transformations and the Poincaré inequality, we obtain from (4.6) the following differential inequalities:

$$\begin{aligned}\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 &\leq (\nu\lambda_1)^{-1}|g(t)|^2, \quad (4.10)\\ &\Downarrow \\ \frac{d}{dt}|u(t)|^2 + \nu\lambda_1|u(t)|^2 &\leq (\nu\lambda_1)^{-1}|g(t)|^2. \quad (4.11)\end{aligned}$$

Applying Lemma 4.1.1 to (4.11) with $y(t) = |u(t+\tau)|^2,\ f(t+\tau) = (\nu\lambda_1)^{-1}|g^\varepsilon(t)|^2$, $\gamma = \nu\lambda_1,$ and $M = (\nu\lambda_1)^{-1}\|g^\varepsilon\|^2_{L_2^b(\mathbb{R};H)}$, we obtain the following main apriori estimate for a weak solution $u(t)$ of equation (4.1):

$$|u(t+\tau)|^2 \leq |u(\tau)|^2 e^{-\nu\lambda_1 t} + D\|g^\varepsilon\|^2_{L_2^b(\mathbb{R};H)}, \tag{4.12}$$

where $D = (\nu\lambda_1)^{-1}\left(1+(\nu\lambda_1)^{-1}\right)$. Inequality (4.10) implies that

$$|u(t)|^2 + \nu\int_\tau^t \|u(s)\|^2 ds \leq |u(\tau)|^2 + (\nu\lambda_1)^{-1}\int_\tau^t |g^\varepsilon(s)|^2 ds. \tag{4.13}$$

Lemma 4.1.2 *If the function* $\varphi(z) \in Z = L_2^b(\mathbb{R}_z^2;\mathbb{R}^2)$, *then* $\varphi\left(\frac{x}{\varepsilon}\right) \in L_2(\Omega)^2$ *for all* $\varepsilon > 0$ *and*

$$\left\|\varphi\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L_2(\Omega_x)^2} \le C \left\|\varphi(\cdot)\right\|_{L_2^b(\mathbb{R}_z^2;\mathbb{R}^2)}, \tag{4.14}$$

where the constant C *is independent of* ε *and* φ.

Proof. Indeed, changing the variables $\frac{x}{\varepsilon} = z, dx = \varepsilon^2 dz$, we have

$$\begin{aligned}\left\|\varphi\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L_2(\Omega)^2}^2 &= \int_\Omega \left|\varphi\left(\frac{x}{\varepsilon}\right)\right|^2 dx = \varepsilon^2 \int_{\varepsilon^{-1}\Omega} |\varphi(z)|^2 dz \\ &\le C\varepsilon^{-2} \sup_{(z_1,z_2)\in\mathbb{R}^2} \varepsilon^2 \int_{z_1}^{z_1+1}\int_{z_2}^{z_2+1} |\varphi(\zeta_1,\zeta_2)|^2 d\zeta_1 d\zeta_2 = C^2 \left\|\varphi(\cdot)\right\|_{L_2^b(\mathbb{R}_z^2;\mathbb{R}^2)}^2.\end{aligned}$$

Here, in the last inequality, we have used the fact that the domain $\varepsilon^{-1}\Omega$ can be covered by at most $C\varepsilon^{-2}$ unit squares of the form $[z_1, z_1+1]\times[z_2, z_2+1]$, where C depend on the area of the domain Ω only. ■

Corollary 4.1.1 *If the functions* $g_0(x,t) \in L_2^b(\mathbb{R}; L_2(\Omega)^2)$ *and* $g_1(z,t) \in L_2^b(\mathbb{R}; Z)$, *where* $Z = L_2^b(\mathbb{R}_z^2;\mathbb{R}^2)$, *then the external force* $g^\varepsilon(x,t) = Pg_0(x,t) + \frac{1}{\varepsilon^\rho} Pg_1\left(\frac{x}{\varepsilon}, t\right)$ *belongs to the space* $L_2^b(\mathbb{R}; H)$ *and*

$$\|g^\varepsilon\|_{L_2^b(\mathbb{R};H)} \le \|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)} + \frac{C}{\varepsilon^\rho}\|g_1\|_{L_2^b(\mathbb{R};Z)}, \tag{4.15}$$

where the constant C *is independent of* ε.

Inequality (4.15) follows directly from Lemma 4.1.2 and the formulas for the norm (4.3) and (4.4) in the spaces $L_2^b(\mathbb{R}; L_2(\Omega)^2)$ and $L_2^b(\mathbb{R}; Z)$.

We now apply inequality (4.15) in (4.12) and obtain

$$|u(t+\tau)|^2 \le |u(\tau)|^2 e^{-\nu\lambda_1 t} + C_0^2 + \varepsilon^{-2\rho} C_1^2, \tag{4.16}$$

where the constants C_0 and C_1 depend on ν, λ_1, and the norms $\|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)}$ and $\|g_1\|_{L_2^b(\mathbb{R};Z)}$, respectively.

We now consider the process $\{U_\varepsilon(t,\tau)\} := \{U_\varepsilon(t,\tau), t \ge \tau, \tau \in \mathbb{R}\}$ corresponding to problem (4.2) and (4.5) and acting in the space H (see Section 2.6.1). Recall that the mapping $U_\varepsilon(t,\tau) : H \to H$ is defined by the formula

$$U_\varepsilon(t,\tau)u_\tau = u(t),\ \forall u_\tau \in H,\ t \ge \tau,\ \tau \in \mathbb{R}, \tag{4.17}$$

where $u(t)$ is the solution of (4.2), (4.5).

It follows from estimate (4.16) that for every $\varepsilon, 0 < \varepsilon \le 1$, the process $\{U_\varepsilon(t,\tau)\}$ has the *uniformly* (w.r.t. $\tau \in \mathbb{R}$) *absorbing set*

$$B_{0,\varepsilon} = \left\{v \in H \mid |v| \le 2(C_0 + C_1\varepsilon^{-\rho})\right\} \tag{4.18}$$

and the set $B_{0,\varepsilon}$ is bounded in H for a fixed ε. That is, for any bounded (in H) set B, there exists a time $t' = t'(B)$ such that the set $U(t+\tau,\tau)B \subseteq B_{0,\varepsilon}$ for all $t \ge t(B)$ and $\tau \in \mathbb{R}$.

Using the standard argument, we prove that the process $\{U_\varepsilon(t,\tau)\}$ has a compact in H uniformly absorbing set

$$B_{1,\varepsilon} = \big\{v \in V \mid \|v\| \le C_2(\nu, \lambda_1, C_0 + C_1\varepsilon^{-\rho})\big\} \tag{4.19}$$

where $C_2(y_1, y_2, y_3)$ is a positive increasing function in each $y_j, j = 1, 2, 3$ (see inequality (2.41)). So, the process $\{U_\varepsilon(t,\tau)\}$ corresponding to problem (4.1) and (4.5) is uniformly compact and it has a compact uniformly absorbing set $B_{1,\varepsilon}$ (bounded in V) defined in (4.19). Consequently, the process $\{U_\varepsilon(t,\tau)\}$ has the *uniform global attractor* $\mathcal{A}^\varepsilon$ (see Section 2.6.1) and $\mathcal{A}^\varepsilon \subseteq B_{0,\varepsilon} \cap B_{1,\varepsilon}$.

Since $\mathcal{A}^\varepsilon \subseteq B_{0,\varepsilon}$, we conclude from (4.16) and (4.18) that

$$\|\mathcal{A}^\varepsilon\|_H \le (C_0 + C_1\varepsilon^{-\rho}). \tag{4.20}$$

Remark 4.1.1 For $\rho > 0$, the norm in H of the uniform global attractor A^ε of the 2D N.–S. system (4.1) may grow up as $\varepsilon \to 0+$. In the next sections, we present conditions that provide the uniform boundedness of $\mathcal{A}^\varepsilon$ in H with respect to ε. Moreover, we also study the convergence of $\mathcal{A}^\varepsilon$ as $\varepsilon \to 0+$ to the global attractor $\mathcal{A}^0$ of the corresponding "limiting" equation.

Along with the original N.–S. system (4.1), we consider the following "limiting" system

$$\begin{aligned} \partial_t u + u^1\partial_{x_1}u + u^2\partial_{x_2}u &= \nu\Delta u - \nabla p + g_0(x,t), \\ \partial_{x_1}u_1 + \partial_{x_2}u_2 &= 0,\ u|_{\partial\Omega} = 0, \end{aligned} \tag{4.21}$$

without the term depending on ε. Excluding the pressure, we obtain the equivalent equation

$$\partial_t u + \nu Lu + B(u,u) = Pg_0(x,t), \tag{4.22}$$

where, clearly $Pg_0(x,t) \in L_2^b(\mathbb{R}; H)$. Then the Cauchy problem for equation (4.22) also has a unique solution $u(t) := u(x,t)$ (in a weak distribution sense). Hence, there is a "limiting" process $\{U_0(t,\tau)\}$ acting in $H:\ U_0(t,\tau)u_\tau = u(t), t \ge \tau,\ \tau \in \mathbb{R}$, where $u(t)$ is the solution of problem (4.22), (4.5). Similarly to (4.12) and (4.13), we have the inequalities

$$|u(t+\tau)|^2 \le |u(\tau)|^2 e^{-\nu\lambda_1 t} + D\|Pg_0\|^2_{L_2^b(\mathbb{R};H)}, \tag{4.23}$$

$$|u(t)|^2 + \nu\int_\tau^t \|u(s)\|^2 ds \le |u(\tau)|^2 + (\nu\lambda_1)^{-1}\int_\tau^t |Pg_0(s)|^2 ds. \tag{4.24}$$

It follows from (4.16) that

$$|u(t+\tau)|^2 \le |u(\tau)|^2 e^{-\nu\lambda_1 t} + C_0^2, \tag{4.25}$$

which implies that the set

$$B_{0,0} = \{v \in H \mid |v| \le 2C_0\} \tag{4.26}$$

is uniformly absorbing for the process $\{U_0(t,\tau)\}$. (The constant C_0 is the same as in (4.16)). Moreover, this process has a compact (in H) absorbing set

$$B_{1,0} = \{v \in V \mid \|v\| \le C_2(\nu, \lambda_1, C_0)\}. \tag{4.27}$$

Therefore, the process $\{U_0(t,\tau)\}$ is uniformly compact and has a compact global attractor $\mathcal{A}^0$ such that $\mathcal{A}^0 \subset B_{0,0} \cap B_{1,0}$ and

$$\left\|\mathcal{A}^0\right\|_H \le C_0. \tag{4.28}$$

4.2 Divergence condition and some properties of the global attractors $\mathcal{A}^\varepsilon$

We consider the non-autonomous 2D N.–S. system (4.2) with external force $g^\varepsilon(x,t) = Pg_0(x,t) + \frac{1}{\varepsilon^\rho} Pg_1\left(\frac{x}{\varepsilon}, t\right)$. We assume that the function $g_0(x,t), x \in \Omega, t \in \mathbb{R}$, satisfies (4.3), i.e., $\|g_0(\cdot)\|^2_{L_2^b(\mathbb{R};L_2(\Omega)^2)} < +\infty$ and the function $g_1(z,t)$, $z \in \mathbb{R}^2, t \in \mathbb{R}$, satisfies (4.4), i.e., $\|g_1(\cdot)\|^2_{L_2^b(\mathbb{R};Z)} < +\infty$, where $Z = L_2^b(\mathbb{R}^2_z;\mathbb{R}^2)$. We now formulate

Divergence condition. *There exist vector functions $G_j(z,t) \in L_2^b(\mathbb{R};Z)$, $j = 1,2$, such that $\partial_{z_j} G_j(z,t) \in L_2^b(\mathbb{R};Z)$ and*

$$\partial_{z_1} G_1(z_1,z_2,t) + \partial_{z_2} G_2(z_1,z_2,t) = g_1(z_1,z_2,t), \ \forall (z_1,z_2) \in \mathbb{R}^2, t \in \mathbb{R}. \tag{4.29}$$

Theorem 4.2.1 *If the function $g_1(z,t)$ satisfies the divergence condition (4.29), then, for every ρ, $0 \le \rho \le 1$, the global attractors $\mathcal{A}^\varepsilon$ of the 2D N.–S. system are uniformly (w.r.t. $\varepsilon \in]0,1]$) bounded in H, that is,*

$$\|\mathcal{A}^\varepsilon\|_H \le C_2, \ \forall \varepsilon \in]0,1], \tag{4.30}$$

where C_2 is independent of ε.

Proof. Taking the scalar product in H of equation (4.2) and $u(t)$, we have equality (4.6), i.e.,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 &= \langle u(t), g^\varepsilon(t)\rangle \\
&= (g_0(\cdot,t), u(\cdot,t)) + \varepsilon^{-\rho}\left(g_1\left(\frac{\cdot}{\varepsilon}, t\right), u(\cdot,t)\right).
\end{aligned} \tag{4.31}$$

For the first term in (4.31), we have the inequality

$$(g_0(\cdot,t), u(\cdot,t)) \le \frac{1}{4}\nu\|u(t)\|^2 + \frac{1}{\nu\lambda_1}|g_0(t)|^2. \tag{4.32}$$

For the second term in (4.31) using (4.29), we have

$$\begin{aligned}
&\varepsilon^{-\rho}\left(g_1\left(\frac{\cdot}{\varepsilon}, t\right), u(\cdot,t)\right) = \varepsilon^{-\rho}\sum_{j=1}^2 \int_\Omega \left(\partial_{z_j} G_j\left(\frac{x}{\varepsilon}, t\right), u(x,t)\right) dx \\
= \ & \varepsilon^{1-\rho}\sum_{j=1}^2 \int_\Omega \left(\partial_{x_j} G_j\left(\frac{x}{\varepsilon}, t\right), u(x,t)\right) dx = -\varepsilon^{1-\rho}\sum_{j=1}^2 \int_\Omega \left(G_j\left(\frac{x}{\varepsilon}, t\right), \partial_{x_j} u(x,t)\right) dx \\
\le \ & \varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^2 \int_\Omega \left|G_j\left(\frac{x}{\varepsilon}, t\right)\right|^2 dx + \frac{1}{4}\nu\|u(t)\|^2.
\end{aligned} \tag{4.33}$$

In the third equality above we have integrated by parts in x taking into account the zero boundary condition in (4.1). Using (4.33) and (4.32) in (4.31), we have that

$$\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 \le \frac{2}{\nu\lambda_1}|g_0(t)|^2 + 2\varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega \left|G_j\left(\frac{x}{\varepsilon},t\right)\right|^2 dx,$$

and therefore, due to the Poincaré inequality,

$$\frac{d}{dt}|u(t)|^2 + \nu\lambda_1|u(t)|^2 \le h(t), \tag{4.34}$$

where $h(t) = \frac{2}{\nu\lambda_1}|g_0(t)|^2 + 2\varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega \left|G_j\left(\frac{x}{\varepsilon},t\right)\right|^2 dx$.

By the assumptions,

$$\int_t^{t+1} |g_0(t)|^2 ds \le \|g_0(\cdot)\|^2_{L_2^b(\mathbb{R};L_2(\Omega)^2)} = M_0, \ \forall t \in \mathbb{R}. \tag{4.35}$$

It follows from Lemma 4.1.2 that

$$\int_t^{t+1}\int_\Omega \left|G_j\left(\frac{x}{\varepsilon},t\right)\right|^2 dx \le C\left\|G_j(\cdot)\right\|_{L_2^b(\mathbb{R};Z)} = M_j, \ \forall t\in\mathbb{R}, \ j=1,2, \tag{4.36}$$

where C is independent of ε.

Applying Lemma 4.1.1 with $y(t) = |u(t+\tau)|^2$, $\gamma = \nu\lambda_1$, and $M = 2\left(\nu\lambda_1\right)^{-1}M_0 + 2\varepsilon^{2(1-\rho)}\nu^{-1}\left(M_1+M_2\right)$, we obtain the following main estimate for the function $u(t)$:

$$|u(t+\tau)|^2 \le |u(\tau)|^2 e^{-\nu\lambda_1 t} + \left[2\left(\nu\lambda_1\right)^{-1}M_0 + 2\varepsilon^{2(1-\rho)}\nu^{-1}(M_1+M_2)\right]D_1, \tag{4.37}$$

where $D_1 = \left(1 + (\nu\lambda_1)^{-1}\right)$.

Since $0 \le \rho \le 1$ and $0 < \varepsilon \le 1$, inequality (4.37) implies that the process $\{U_\varepsilon(t,\tau)\}$ corresponding to equation (4.1) has a uniformly absorbing set

$$\tilde{B} = \{v \in H \mid |v| \le C_2\}, \tag{4.38}$$

where $C_2^2 = 2\left[2\left(\nu\lambda_1\right)^{-1}M_0 + 2\nu^{-1}(M_1+M_2)\right]D_1$. It is clear, that the global attractor $\mathcal{A}^\varepsilon$ belongs to any absorbing set, i.e.,

$$\|\mathcal{A}^\varepsilon\|_H \le C_2, \ \forall\varepsilon, 0 < \varepsilon \le 1, \tag{4.39}$$

when the divergence condition (4.29) holds and the theorem is proved. ■

We now estimate the deviation of solutions of the original 2D N.–S. system (4.2) from the corresponding solutions of the "limiting" system (4.22).

We supplement equations (4.2) and (4.22) with the same initial data at $t = \tau$:

$$u|_{t=\tau} = u_\tau, \ u^0|_{t=\tau} = u_\tau, \ u_\tau \in \tilde{B}, \tag{4.40}$$

where the absorbing ball $\tilde{B}$ is defined in (4.38). Recall that the set $\tilde{B}$ is independent of ρ, $0 \le \rho \le 1$ and ε, $0 < \varepsilon \le 1$.

Let $u(x,t)$ and $u^0(x,t)$ be the solutions of equations (4.2) and (4.22), respectively, with the same initial data (4.40) taken from the ball $\tilde{B}$. We are going to estimate the deviation of $u(x,t)$ from $u^0(x,t)$ for $t \geq \tau$. We set $w(x,t) = u(x,t) - u^0(x,t)$. For simplicity, we take $\tau = 0$. The function $w(x,t)$ satisfies the equation

$$\partial_t w + \nu L w + B(u,u) - B(u^0,u^0) = \frac{1}{\varepsilon^\rho} P g_1\left(\frac{x}{\varepsilon}, t\right) \tag{4.41}$$

and zero initial data

$$w|_{t=0} = 0. \tag{4.42}$$

We note that

$$B(u,u) - B(u^0,u^0) = B(w,u^0) + B(u^0,w) + B(w,w).$$

Taking the scalar product in H of equation (4.41) and w, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w(t)|^2 \;&+\; \nu\|w(t)\|^2 + \left\langle B(w,u^0), w\right\rangle \\ &+\; \left\langle B(u^0,w), w\right\rangle + \langle B(w,w), w\rangle = \frac{1}{\varepsilon^\rho}\left\langle g_1\left(\frac{\cdot}{\varepsilon}, t\right), w\right\rangle.\end{aligned} \tag{4.43}$$

It follows from (1.13) that $\langle B(u^0,w), w\rangle = 0$ and $\langle B(w,w), w\rangle = 0$. Therefore,

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w(t)\|^2 + \left\langle B(w,u^0(t)), w\right\rangle = \frac{1}{\varepsilon^\rho}\left\langle g_1\left(\frac{\cdot}{\varepsilon}, t\right), w\right\rangle. \tag{4.44}$$

Using the divergence condition, similarly to (4.33), we observe that

$$\begin{aligned}\varepsilon^{-\rho}\left\langle g_1\left(\frac{\cdot}{\varepsilon}, t\right), w\right\rangle &= -\varepsilon^{1-\rho}\sum_{j=1}^{2}\int_\Omega \left(G_j\left(\frac{x}{\varepsilon}, t\right), \partial_{x_j} u(x,t)\right) dx \\ \leq\; &\frac{1}{2}\varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega \left|G_j\left(\frac{x}{\varepsilon}, t\right)\right|^2 dx + \frac{1}{2}\nu\|u(t)\|^2.\end{aligned} \tag{4.45}$$

It follows from (1.13) and (1.14) that

$$|\left\langle B(w,u^0), w\right\rangle| = |\left\langle B(w,w), u^0\right\rangle| \leq c_0^2|w|\|w\|\|u^0\|. \tag{4.46}$$

Then

$$|\left\langle B(w,u^0), w\right\rangle| \leq c_0^2|w|\|u^0\|\|w\| \leq \frac{1}{2}\nu\|w\|^2 + \frac{1}{2}\frac{c_0^4}{\nu}|w|^2\|u^0\|^2. \tag{4.47}$$

Combining (4.45) and (4.47) in (4.44), we find that

$$\frac{d}{dt}|w(t)|^2 \leq \frac{c_0^4}{\nu}|w(t)|^2\|u^0(t)\|^2 + \varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega \left|G_j\left(\frac{x}{\varepsilon}, t\right)\right|^2 dx.$$

We set

$$z(t) = |w(t)|^2,\ \gamma(t) = c_0^4\nu^{-1}\|u^0(t)\|^2$$

and

$$b(t) = \varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega \left|G_j\left(\frac{x}{\varepsilon}, t\right)\right|^2 dx.$$

Then we have the following differential inequality:

$$z'(t) \le b(t) + \gamma(t)z(t), \; z(0) = 0. \tag{4.48}$$

Applying the Gronwall lemma, we obtain:

$$z(t) \le \int_0^t b(s)\exp\left(\int_s^t \gamma(\theta)d\theta\right)ds \le \left(\int_0^t b(s)ds\right)\exp\left(\int_0^t \gamma(s)ds\right). \tag{4.49}$$

Recall that $u^0(t)$ satisfies (4.24) and $u_0 \in \tilde{B}$, i.e.

$$\begin{aligned}\int_0^t \gamma(s)ds &= c_0^4\nu^{-1}\int_0^t \|u^0(t)\|^2 ds \le c_0^2\nu^{-2}\left(|u_0|^2 + (\nu\lambda_1)^{-1}\int_0^t |g_0(s)|^2 ds\right)\\ &\le c_0^4\nu^{-2}\left(C_2^2 + (\nu\lambda_1)^{-1}(t+1)\|g_0(\cdot)\|^2_{L_2^b(\mathbb{R};L_2(\Omega)^2)}\right) \le C_3(t+1). \end{aligned}\tag{4.50}$$

Using (4.36), we see that

$$\begin{aligned}\int_0^t b(s)ds &= \varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^2\int_0^t\int_\Omega \left|G_j\left(\frac{x}{\varepsilon}, s\right)\right|^2 dxds\\ &\le \varepsilon^{2(1-\rho)}\nu^{-1}C(t+1)\sum_{j=1}^2 \|G_j(\cdot)\|_{L_2^b} \le \varepsilon^{2(1-\rho)}\nu^{-1}(t+1)(M_1' + M_2')\end{aligned} \tag{4.51}$$

Replacing (4.50) and (4.51) to (4.49), we find the following inequality

$$\begin{aligned}|w(t)|^2 &\le \varepsilon^{2(1-\rho)}\nu^{-1}(t+1)(M_1' + M_2')e^{C_3(t+1)}\\ &= \varepsilon^{2(1-\rho)}\nu^{-1}(M_1' + M_2')\varepsilon^t e^{C_3(t+1)} = \varepsilon^{2(1-\rho)}C_4^2 e^{2rt},\end{aligned} \tag{4.52}$$

where $C_4^2 = \nu^{-1}(M_1' + M_2')e^{C_3}$, $2r = C_3 + 1$. The constants C_4 and r are independent of ε. Inequality (4.52) holds for all ρ, $0 \le \rho \le 1$. We have proved the following

Theorem 4.2.2 *Let the function $g_1(z,t)$ satisfy the divergence condition (4.29). Then, for every initial data $u_\tau \in \tilde{B}$ (see (4.38)), the difference $w(x,t) = u(x,t) - u^0(x,t)$ of the solutions of the N.–S. equations (4.2) and (4.22), respectively, with initial data (4.40) taken from the ball $\tilde{B}$, satisfies the following inequality:*

$$|w(t)| = |u(t) - u^0(t)| \le \varepsilon^{(1-\rho)}C_4 e^{r(t-\tau)}, \; \forall\varepsilon, 0 < \varepsilon \le 1, \tag{4.53}$$

where the constant C_4 and r are independent of ε, $u_\tau \in \tilde{B}$, and $0 \le \rho \le 1$.

In Section 4.4 using Theorems 4.2.1 and 4.2.2, we prove that the global attractors $\mathcal{A}^\varepsilon$ converge to $\mathcal{A}^0$ in the strong norm of H as $\varepsilon \to 0+$.

4.3 On the structure of the global attractors $\mathcal{A}^\varepsilon$

We start with consideration translation compact (tr.c.) functions with values in the spaces $L_2(\Omega)^2$ and Z. The definition of a tr.c. function in $\Xi = L_p^{loc}(\mathbb{R}; E)$ with values in a Banach space E is given in Section 2.4 (see Example 2.4.2). Below, we consider tr.c. functions in $\Xi = L_p^{loc}(\mathbb{R}; L_2(\Omega)^2)$ and in $\Xi = L_p^{loc}(\mathbb{R}; Z)$.

Consider the vector functions $g_0(x,t), x \in \Omega, t \in \mathbb{R}$, and $g_1(z,t), z \in \mathbb{R}^2, t \in \mathbb{R}$, that appear on the right-hand side of the 2D N.–S. system. We assume that $g_0(x,t) \in L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $g_1(z,t) \in L_2^{\mathrm{loc}}(\mathbb{R}; Z)$.

Proposition 4.3.1 *If the function $g_1(z,t)$ is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R};Z)$, then, for every fixed ε, $0<\varepsilon\le 1$, the function $g_1(x/\varepsilon,t)$ is tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$, $\Omega\Subset\mathbb{R}^2$.*

Proof. We have to establish that the set of function $\{g_1(x/\varepsilon,t+h)\mid h\in\mathbb{R}\}$ is precompact in $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$. Let $\{h_n,\ n=1,2,\ldots\}$ be an arbitrary sequence of real numbers. Since the function $g_1(z,t)$ is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R};Z)$ there is a subsequence $\{h_{n'}\}\subset\{h_n\}$ such that $g_1(z,t+h_{n'})$ converges to a function $\hat g_1(z,t)$ as $n'\to\infty$ in $L_2^{\mathrm{loc}}(\mathbb{R};Z)$ i.e., for every interval $[t_1,t_2]\subset\mathbb{R}$,

$$\int_{t_1}^{t_2}\|g_1(\cdot,s+h_{n'})-\hat g_1(\cdot,s)\|_Z^2ds\to 0\ (n'\to\infty).$$

Using inequality (4.14), we conclude that

$$\int_{t_1}^{t_2}\|g_1(\cdot/\varepsilon,s+h_{n'})-\hat g_1(\cdot/\varepsilon,s)\|_{L_2(\Omega)^2}^2ds\le C^2\int_{t_1}^{t_2}\|g_1(\cdot,s+h_{n'})-\hat g_1(\cdot,s)\|_Z^2ds,$$

that is, $g_1(x/\varepsilon,t+h_{n'})$ converges to $\hat g_1(x/\varepsilon,t)$ as $n'\to\infty$ in $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$. Thus, the set $\{g_1(x/\varepsilon,t+h)\mid h\in\mathbb{R}\}$ is precompact in $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$. ■

Proposition 4.3.2 *Let $g_0(x,t)$ be tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$ and $g_1(z,t)$ be tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R};Z)$. Consider the function $g^\varepsilon(x,t)=g_0(x,t)+\varepsilon^{-\rho}g_1(x/\varepsilon,t)$ as an element of the space $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$. Then this function is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$ and the hull $\mathcal{H}(g^\varepsilon(x,t))$ (in the space $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$) consists of (tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$) functions $\hat g^\varepsilon(x,t)$ of the form $\hat g^\varepsilon(x,t)=\hat g_0(x,t)+\varepsilon^{-\rho}\hat g_1(x/\varepsilon,t)$ for some $\hat g_0(x,t)\in\mathcal{H}(g_0(x,t))$ and $\hat g_1(z,t)\in\mathcal{H}(g_1(z,t))$, where $\mathcal{H}(g_0(x,t))$ and $\mathcal{H}(g_1(z,t))$ are the hulls of the functions $g_0(x,t)$ and $g_1(z,t)$, respectively.*

Proof. It follows from Proposition 4.3.1 that, for a fixed $\varepsilon\in(0,1]$, the function $g^\varepsilon(x,t)=g_0(x,t)+\varepsilon^{-\rho}g_1(x/\varepsilon,t)$ is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$ (as the sum of two tr.c. functions). Let now $\hat g^\varepsilon(x,t)\in\mathcal{H}(g^\varepsilon(x,t))$, i.e., there is a sequence $\{h_n\}$ such that $g^\varepsilon(x,t+h_n)=g_0(x,t+h_n)+\varepsilon^{-\rho}g_1(x/\varepsilon,t+h_n)\to\hat g^\varepsilon(x,t)$ as $n\to\infty$ in $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$. Since the functions $g_0(x,t)$ and $g_1(z,t)$ are tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$ and $L_2^{\mathrm{loc}}(\mathbb{R};Z)$, respectively, we may assume passing to a subsequence $\{h_{n'}\}\subset\{h_n\}$ that $g_0(x,t+h_{n'})\to\hat g_0(x,t)$ in $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$ and $g_1(z,t+h_{n'})\to\hat g_1(z,t)$ in $L_2^{\mathrm{loc}}(\mathbb{R};Z)$ as $n'\to\infty$. Therefore, $g^\varepsilon(x,t+h_{n'})=g_0(x,t+h_{n'})+\varepsilon^{-\rho}g_1(x/\varepsilon,t+h_{n'})\to\hat g_0(x,t)+\varepsilon^{-\rho}\hat g_1(x/\varepsilon,t)$ as $n\to\infty$ in $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$. So,

$$\begin{aligned}\hat g^\varepsilon(x,t)&=\lim_{n\to\infty}\left[g_0(x,t+h_n)+\varepsilon^{-\rho}g_1(x/\varepsilon,t+h_n)\right]\\&=\lim_{n'\to\infty}g_0(x,t+h_{n'})+\lim_{n'\to\infty}\varepsilon^{-\rho}g_1(x/\varepsilon,t+h_{n'})=\hat g_0(x,t)+\varepsilon^{-\rho}\hat g_1(x/\varepsilon,t).\end{aligned}$$

Thus, every function $\hat g^\varepsilon(x,t)\in\mathcal{H}(g^\varepsilon(x,t))$ has the form $\hat g^\varepsilon(x,t)=\hat g_0(x,t)+\varepsilon^{-\rho}\hat g_1(x/\varepsilon,t)$ for some $\hat g_0(x,t)\in\mathcal{H}(g_0(x,t))$ and $\hat g_1(z,t)\in\mathcal{H}(g_1(z,t))$. ■

We now consider equation (4.2)

$$\partial_t u+\nu Lu+B(u,u)=g^\varepsilon(x,t),\tag{4.54}$$

where $g^\varepsilon(x,t)=Pg_0(x,t)+\varepsilon^{-\rho}Pg_1(x/\varepsilon,t)$ and ε is fixed. We assume that the function $g_0(x,t)$ is tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$ and $g_1(z,t)$ is tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R};Z)$. In particular, $g_0(x,t)\in L_2^b(\mathbb{R};L_2(\Omega)^2)$ and $g_1(z,t)\in L_2^b(\mathbb{R};Z)$.

Let $\mathcal{H}(g^\varepsilon)$ be the hull of the function $g^\varepsilon(x,t)$ in the space $L_2^{\mathrm{loc}}(\mathbb{R};H)$:

$$\mathcal{H}(g^\varepsilon) = [\{g^\varepsilon(\cdot,t+h) \mid h \in \mathbb{R}\}]_{L_2^{\mathrm{loc}}(\mathbb{R};H)} . \tag{4.55}$$

Recall that $\mathcal{H}(g^\varepsilon)$ is compact in $L_2^{\mathrm{loc}}(\mathbb{R};H)$ and, by Proposition 4.3.2, each element $\hat{g}^\varepsilon(x,t) \in \mathcal{H}(g^\varepsilon(x,t))$ can be written in the form

$$\hat{g}^\varepsilon(x,t) = P\hat{g}_0(x,t) + \varepsilon^{-\rho}P\hat{g}_1(x/\varepsilon,t) \tag{4.56}$$

for some functions $\hat{g}_0(x,t) \in \mathcal{H}(g_0(x,t))$ and $\hat{g}_1(z,t) \in \mathcal{H}(g_1(z,t))$, where $\mathcal{H}(g_0(x,t))$ and $\mathcal{H}(g_1(z,t))$ are the hulls of the functions $g_0(x,t)$ and $g_1(z,t)$ in $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$ and $L_2^{\mathrm{loc}}(\mathbb{R};Z)$, respectively.

We note that

$$\begin{aligned} \|\hat{g}_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)} &\leq \|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)}, \ \forall \hat{g}_0 \in \mathcal{H}(g_0); \\ \|\hat{g}_1\|_{L_2^b(\mathbb{R};Z)} &\leq \|g_1\|_{L_2^b(\mathbb{R};Z)}, \ \forall \hat{g}_1 \in \mathcal{H}(g_1). \end{aligned}$$

Then it follows easily from Corollary 4.1.1 that

$$\|\hat{g}^\varepsilon\|_{L_2^b(\mathbb{R};H)} \leq \|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)} + \frac{C}{\varepsilon^\rho}\|g_1\|_{L_2^b(\mathbb{R};Z)}, \ \forall g^\varepsilon \in \mathcal{H}(g^\varepsilon), \tag{4.57}$$

where the constant C is independent of g_0, g_1, ρ, and ε (see (4.14) and (4.15)).

It was shown in Section 4.1 that the process $\{U_\varepsilon(t,\tau)\} := \{U_{g^\varepsilon}(t,\tau)\}$ corresponding to equation (4.54) has the uniform global attractor $\mathcal{A}^\varepsilon \subseteq B_{0,\varepsilon} \cap B_{1,\varepsilon}$, (see (4.18) and (4.19)) and

$$\|\mathcal{A}^\varepsilon\|_H \leq (C_0 + C_1\varepsilon^{-\rho}), \tag{4.58}$$

where the constants C_0 and C_1 depend on $\|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)}$ and $\|g_1\|_{L_2^b(\mathbb{R};Z)}$, respectively. We now describe the structure of the attractor $\mathcal{A}^\varepsilon$.

Along with equation (4.54), we consider the family of equations

$$\partial_t\hat{u} + \nu L\hat{u} + B(\hat{u},\hat{u}) = \hat{g}^\varepsilon(x,t), \tag{4.59}$$

with external forces $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$. It is clear that, for every $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, equation (4.59) generates the process $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$ acting in H. We note that the processes $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$ satisfy the similar properties as the process $\{U_{g^\varepsilon}(t,\tau)\}$ corresponding to the 2D N.–S. system (4.54) with original external force $g^\varepsilon(x,t) = Pg_0(x,t) + \varepsilon^{-\rho}Pg_1(x/\varepsilon,t)$. In particular, the sets $B_{0,\varepsilon}$ and $B_{1,\varepsilon}$ are absorbing for each process $\{U_{\hat{g}^\varepsilon}(t,\tau)\}, \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$ (see (4.57)). Moreover, every process $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$ has a uniform global attractor $\mathcal{A}_{\hat{g}^\varepsilon}$ that belongs to the global attractor $\mathcal{A}^\varepsilon = \mathcal{A}_{g^\varepsilon}$ of the 2D N.–S. system (4.54) with initial external force $g^\varepsilon(x,t)$, $\mathcal{A}_{\hat{g}^\varepsilon} \subseteq \mathcal{A}_{g^\varepsilon}$ (the inclusion can be strict, see Proposition 2.5.1).

Proposition 4.3.3 *Let the function $g_0(x,t)$ be tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$ and let $g_1(z,t)$ be tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R};Z)$. Then for any fixed $\varepsilon, 0 < \varepsilon \leq 1$, the family of processes $\{U_{\hat{g}^\varepsilon}(t,\tau)\}, \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, corresponding to equations (4.59) has an absorbing set $B_{1,\varepsilon}$, which is bounded in H and V and satisfies*

$$\|B_{1,\varepsilon}\|_H \leq (C_0 + C_1\varepsilon^{-\rho}). \tag{4.60}$$

The family $\{U_{\hat{g}^\varepsilon}(t,\tau)\}, \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, is $(H \times \mathcal{H}(g^\varepsilon); H)$-continuous, that is, if

$$\hat{g}_n^\varepsilon \to \hat{g}^\varepsilon \ (n \to \infty) \ \text{in} \ L_2^{\mathrm{loc}}(\mathbb{R};H) \ \text{and} \ u_{\tau n} \to u_\tau \ (n \to \infty) \ \text{in} \ H, \tag{4.61}$$

then

$$U_{\hat{g}_n^\varepsilon}(t,\tau)u_{\tau n} \to U_{\hat{g}^\varepsilon}(t,\tau)u_\tau \ (n \to \infty) \ \text{in} \ H. \tag{4.62}$$

The proof of these properties is analogous to the proof given, e.g., in [CV02a], for the case of a non-oscillating tr.c. external force in $L_2^{\mathrm{loc}}(\mathbb{R};H)$).

We denote by $\mathcal{K}_{\hat{g}^\varepsilon}$ the *kernel* of equation (4.59) (and of the process $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$) with external force $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$. Recall that the kernel $\mathcal{K}_{\hat{g}^\varepsilon}$ is the family of all complete solutions $\hat{u}(t), t \in \mathbb{R}$, of (4.59) which are bounded in the norm of H :

$$|\hat{u}(t)| \le M_{\hat{u}},\ \forall t \in \mathbb{R}. \tag{4.63}$$

The set

$$\mathcal{K}_{\hat{g}^\varepsilon}(s) = \{\hat{u}(s) \mid \hat{u} \in \mathcal{K}_{\hat{g}^\varepsilon}\},\ s \in \mathbb{R},$$

(belonging to H) is called the *kernel section* at time $t = s$.

We recall the theorem on the structure of the uniform global attractor $\mathcal{A}^\varepsilon$ of the 2D Navier-Stokes system (4.54) (see also (2.44)) .

Theorem 4.3.1 *If the function $g^\varepsilon(x,t)$ is tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R};H)$, then the process $\{U_{g^\varepsilon}(t,\tau)\}$ corresponding to equations (4.59) has the uniform global attractor $\mathcal{A}^\varepsilon$ and the following identity holds:*

$$\mathcal{A}^\varepsilon = \bigcup_{\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)} \mathcal{K}_{\hat{g}^\varepsilon}(0). \tag{4.64}$$

Moreover, the kernel $\mathcal{K}_{\hat{g}^\varepsilon}$ is non-empty for all $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$.

The proof of Theorem 4.3.1 is given in [CV02a].

We also note that the attractor $\mathcal{A}^\varepsilon$ is given by the following formula

$$\mathcal{A}^\varepsilon = \omega(B_0) = \bigcap_{h \ge 0} \left[\bigcup_{t-\tau \ge h} U_{g^\varepsilon}(t,\tau)B_0 \right],$$

i.e., to construct the attractor $\mathcal{A}^\varepsilon$ of the entire family of processes $\{U_{\hat{g}^\varepsilon}(t,\tau)\}, \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, one can use only the process $\{U_{g^\varepsilon}(t,\tau)\}$ of original equation (4.54) with external force $g^\varepsilon = Pg_0(x,t) + \varepsilon^{-\rho}Pg_1(x/\varepsilon,t)$.

All the above results are also applicable to the "limiting" 2D N.–S. system (4.22)

$$\partial_t u + \nu L u + B(u,u) = g^0(x,t) \tag{4.65}$$

with tr.c. external force $g^0(t) := Pg_0(\cdot,t) \in L_2^{\mathrm{loc}}(\mathbb{R};H)$. Equation (4.65) generates the "limiting" process $\{U_0(t,\tau)\} = \{U_{g^0}(t,\tau)\}$ which has the uniform global attractor $\mathcal{A}^0$ (see the end of Section 4.1).

Consider the family of equations

$$\partial_t \hat{u} + \nu L \hat{u} + B(\hat{u},\hat{u}) = \hat{g}^0(x,t), \tag{4.66}$$

with external forces $\hat{g}^0 \in \mathcal{H}(g^0)$ (the hull $\mathcal{H}(g^0)$ is taken in the space $L_2^{\mathrm{loc}}(\mathbb{R};H)$) and the corresponding family of processes $\{U_{\hat{g}^0}(t,\tau)\}, \hat{g}^0 \in \mathcal{H}(g^0)$.

We note that we can apply Proposition 4.3.3 and Theorem 4.3.1 directly to the equations (4.65) and (4.66) taking the function $g_1(z,t) \equiv 0$. Therefore, the family of processes $\{U_{\hat{g}^0}(t,\tau)\}, \hat{g}^0 \in \mathcal{H}(g^0)$, has a uniformly absorbing set $B_{1,0}$ (bounded in V),

$$\|B_{1,0}\|_H \le C_0, \tag{4.67}$$

and the family $\{U_{\hat{g}^0}(t,\tau)\}, \hat{g}^0 \in \mathcal{H}(g^0)$, is $(H \times \mathcal{H}(g^0); H)$-continuous. Moreover, the attractor $\mathcal{A}^0$ of the "limiting" equation (4.65) has the form

$$\mathcal{A}^0 = \bigcup_{\hat{g}^0 \in \mathcal{H}(g^0)} \mathcal{K}_{\hat{g}^0}(0), \tag{4.68}$$

where $\mathcal{K}_{\hat{g}^0}$ is the kernel of equation (4.66) with external forces $\hat{g}^0 \in \mathcal{H}(g^0)$.

The formulas (4.64) and (4.68) will be important in the next section, where we study the strong convergence of the attractors $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$ as $\varepsilon \to 0+$.

4.4 Convergence of the global attractors $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$

In this section, we consider equations (4.54) and (4.65), where the functions $g_0(x,t)$ and $g_1(z,t)$ are tr.c. in the spaces $L_2^{\text{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $L_2^{\text{loc}}(\mathbb{R}; Z)$, respectively.

We also assume that the function $g_1(z,t)$ satisfies the divergence condition (4.29). Then due to Theorem 4.2.1 the uniform global attractors $\mathcal{A}^\varepsilon$ of equations (4.54) with external forces $g^\varepsilon(x,t) = Pg_0(x,t) + \varepsilon^{-\rho} Pg_1(x/\varepsilon, t)$ are uniformly bounded in H with respect to ε :

$$\|\mathcal{A}^\varepsilon\|_H \le C_2, \ \forall \varepsilon, \ 0 < \varepsilon \le 1, \tag{4.69}$$

where the constant C_2 is independent of ε. We also consider the global attractor $\mathcal{A}^0$ of the "limiting" equation (4.65) with external force $g^0(t) = Pg_0(\cdot, t)$. Clearly, the set $\mathcal{A}^0$ is also bounded in H (see (4.67)).

We need a generalization of Theorem 4.2.2 that can be applied to the solution of entire families of equations (4.59) and (4.66).

We choose an arbitrary element $u_\tau \in \tilde{B}$. Let $\hat{u}(\cdot,t) = U_{\hat{g}^\varepsilon}(t,\tau)u_\tau, \ t \ge \tau$, be the solution of equation (4.59) with external force $\hat{g}^\varepsilon = P\hat{g}_0 + \varepsilon^{-\rho} P\hat{g}_1 \in \mathcal{H}(g^\varepsilon)$. Let also $\tilde{u}^0(\cdot,t) = U_{\tilde{g}^0}(t,\tau)u_\tau, \ t \ge \tau$, be the solution of (4.66) with external force $\tilde{g}^0 \in \mathcal{H}(g^0)$. We assume that the initial data at $t = \tau$ of these two solutions are the same: $\hat{u}(\cdot,\tau) = \tilde{u}^0(\cdot,\tau) = u_0$, and $u_0 \in \tilde{B}$, where the absorbing ball $\tilde{B}$ is defined in (4.38). (Notice that the function $\tilde{g}^0$ can be different from the function $\hat{g}^0 = P\hat{g}_0$ being the first summand in the representation $\hat{g}^\varepsilon = P\hat{g}_0 + \varepsilon^{-\rho} P\hat{g}_1$.) We now consider the difference

$$\hat{w}(x,t) = \hat{u}(x,t) - \tilde{u}^0(x,t), \ t \ge \tau.$$

Proposition 4.4.1 *Let the original functions $g_0(x,t)$ and $g_1(z,t)$ in (4.1) be tr.c. in $L_2^{\text{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $L_2^{\text{loc}}(\mathbb{R}; Z)$, respectively. Let also the function $g_1(z,t)$ satisfy the divergence condition (4.29). We set $g^\varepsilon(x,t) = Pg_0(x,t) + \varepsilon^{-\rho} Pg_1(x/\varepsilon\ ,t)$ and $g^0(x,t) = Pg_0(x,t)$. Then, for every external force $\hat{g}^\varepsilon = P\hat{g}_0 + \varepsilon^{-\rho} P\hat{g}_1 \in \mathcal{H}(g^\varepsilon)$, there exist an external force $\tilde{g}^0 \in \mathcal{H}(g^0)$ such that, for every initial data $u_\tau \in \tilde{B}$ (see (4.38)), the difference*

$$\hat{w}(t) = \hat{u}(t) - \tilde{u}^0(t) = U_{\hat{g}^\varepsilon}(t,\tau)u_\tau - U_{\tilde{g}^0}(t,\tau)u_\tau$$

of the solutions of the 2D N.–S. systems (4.59) and (4.66) with external forces $\hat{g}^\varepsilon(x,t) = P\hat{g}_0(x,t) + \varepsilon^{-\rho} P\hat{g}_1(x/\varepsilon, t)$ and $\tilde{g}^0(x,t)$, respectively, and with the same initial data u_τ satisfies the following inequality:

$$|\hat{w}(t)| = |\hat{u}(t) - \tilde{u}^0(t)| \le \varepsilon^{(1-\rho)} C_4 e^{r(t-\tau)}, \ \forall \varepsilon, 0 < \varepsilon \le 1, \tag{4.70}$$

where the constant C_4 and r are the same as in Theorem 4.2.2 and they are independent of ε and $0 \le \rho \le 1$.

Proof. Consider the functions

$$u(t) = U_{g^\varepsilon}(t,\tau)u_\tau \text{ and } u^0(t) = U_{g^0}(t,\tau)u_\tau, \forall t \geq \tau, \tag{4.71}$$

where $g^\varepsilon(t) = Pg_0(t) + \varepsilon^{-\rho}Pg_1(t)$ and $g^0(t) = Pg_0(t)$ are the original external forces. Using (4.71), we rewrite inequality (4.53) in the form

$$|U_{g^\varepsilon}(t,\tau)u_\tau - U_{g^0}(t,\tau)u_\tau| \leq \varepsilon^{(1-\rho)}C_4 e^{r(t-\tau)}. \tag{4.72}$$

By Theorem 4.2.2, inequality (4.72) holds for all $u_\tau \in \tilde{B}$. We claim that this inequality also hold for the time shifted external forces

$$\begin{aligned} g_h^\varepsilon(t) &= g^\varepsilon(t+h) = Pg_0(t+h) + \varepsilon^{-\rho}Pg_1(t+h), \\ g_h^0(t) &= g^0(t+h) = Pg_0(t+h), \end{aligned}$$

where $h \in \mathbb{R}$ is arbitrary, that is,

$$|U_{g_h^\varepsilon}(t,\tau)u_\tau - U_{g_h^0}(t,\tau)u_\tau| \leq \varepsilon^{(1-\rho)}C_4 e^{r(t-\tau)}, \tag{4.73}$$

where the constants C_4 and r are independent of h. Indeed, for every $h \in \mathbb{R}$, the time shifted function $g_{1h}(z,t) = g_1(z,t+h)$ apparently satisfies the divergence condition (4.29) for the time shifted functions $G_j^h(z,t) = G_j(z,t+h) \in L_2^b(\mathbb{R};Z),\ j = 1,2$. So, (4.73) follows directly from Theorem 4.2.2.

We recall that the family of processes $\{U_{\hat{g}^\varepsilon}(t,\tau)\}, \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, is $(H \times \mathcal{H}(g^\varepsilon); H)$-continuous. In particular, (see (4.61) and (4.62)) for a fixed $u_\tau \in \tilde{B}$, if

$$\hat{g}_n^\varepsilon \to \hat{g}^\varepsilon \ (n \to \infty) \text{ in } L_2^{\mathrm{loc}}(\mathbb{R};H),$$

then

$$U_{\hat{g}_n^\varepsilon}(t,\tau)u_\tau \to U_{\hat{g}^\varepsilon}(t,\tau)u_\tau \ (n \to \infty) \text{ in } H, \tag{4.74}$$

and similarly

$$U_{\hat{g}_n^0}(t,\tau)u_\tau \to U_{\tilde{g}^0}(t,\tau)u_\tau \ (n \to \infty) \text{ in } H, \tag{4.75}$$

when $\hat{g}_n^0 \to \tilde{g}^0$ $(n \to \infty)$ in $L_2^{\mathrm{loc}}(\mathbb{R};H)$ for some $\tilde{g}^0 \in \mathcal{H}(g^0)$.

We now fix the external forces $\hat{g}^\varepsilon = P\hat{g}_0 + \varepsilon^{-\rho}P\hat{g}_1 \in \mathcal{H}(g^\varepsilon)$. The function $\hat{g}^\varepsilon(t)$ is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R};H)$. Therefore, there exists a sequence $\{h_i\} \subset \mathbb{R}$ such that

$$g_{h_i}^\varepsilon \to \hat{g}^\varepsilon \ (n \to \infty) \text{ in } L_2^{\mathrm{loc}}(\mathbb{R};H), \tag{4.76}$$

where $g_{h_i}^\varepsilon(t) = g^\varepsilon(t+h_i)$. Consider now the sequence of external forces $g_{h_i}^0 = g^0(t+h_i)$. Since the function $g^0(t)$ is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R};H)$, there exists a function $\tilde{g}^0 \in \mathcal{H}(g^0)$ such that

$$g_{h_i}^0 \to \tilde{g}^0 \ (n \to \infty) \text{ in } L_2^{\mathrm{loc}}(\mathbb{R};H). \tag{4.77}$$

(Here we have possibly passed to a subsequence of h_i which we label the same). It follows from (4.73) that

$$|U_{g_{h_i}^\varepsilon}(t,\tau)u_\tau - U_{g_{h_i}^0}(t,\tau)u_\tau| \leq \varepsilon^{(1-\rho)}C_4 e^{r(t-\tau)},\ \forall i \in \mathbb{N}. \tag{4.78}$$

Using (4.76) and (4.77) in (4.74) and (4.75), we pass to the limit in (4.78) as $i \to \infty$ and obtain the required inequality:

$$|U_{\hat{g}^\varepsilon}(t,\tau)u_\tau - U_{\tilde{g}^0}(t,\tau)u_\tau| \leq \varepsilon^{(1-\rho)}C_4 e^{r(t-\tau)}. \tag{4.79}$$

So, inequality (4.70) is proved. ■

We are now ready to formulate the main theorem of this chapter.

Theorem 4.4.1 *Let the functions $g_0(x,t)$ and $g_1(z,t)$ in (4.1) be tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$, respectively. Let also the function $g_1(z,t)$ satisfy the divergence condition (4.29). Then the global attractors $\mathcal{A}^\varepsilon$ of equation (4.54) converges to the global attractor $\mathcal{A}^0$ of the "limiting" equation (4.65) in the strong norm of H as $\varepsilon \to 0+$, that is*

$$\mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \to 0 \ (\varepsilon \to 0+). \tag{4.80}$$

Proof. For a given ε, let u^ε be an arbitrary element of $\mathcal{A}^\varepsilon$. By (4.64), there exists a bounded complete solution $\hat{u}^\varepsilon(t), t \in \mathbb{R}$, of equation (4.59) with some external force $\hat{g}^\varepsilon = P\hat{g}_0 + \varepsilon^{-\rho}P\hat{g}_1 \in \mathcal{H}(g^\varepsilon)$,where $\hat{g}_0 \in \mathcal{H}(g_0)$ and $\hat{g}_1 \in \mathcal{H}(g_1)$, such that

$$u^\varepsilon = \hat{u}^\varepsilon(0). \tag{4.81}$$

We consider the point $\hat{u}^\varepsilon(-R)$ which clearly belongs to $\mathcal{A}^\varepsilon$ and hence

$$\hat{u}^\varepsilon(-R) \in \tilde{B}, \tag{4.82}$$

(see (4.38)). Recall that $\tilde{B}$ is the absorbing set and the global attractor $\mathcal{A}^\varepsilon$ belongs to $\tilde{B}$. The number R will be chosen later on.

For the constructed external force $\hat{g}^\varepsilon$, we apply Proposition 4.4.1: there is a "limiting" external force $\tilde{g}^0 \in \mathcal{H}(g^0)$ such that, for any $\tau \in \mathbb{R}$ and for all $u_\tau \in \tilde{B}$, the following inequality holds:

$$|U_{\hat{g}^\varepsilon}(t,\tau)u_\tau - U_{\tilde{g}^0}(t,\tau)u_\tau| \le \varepsilon^{(1-\rho)}C_4 e^{r(t-\tau)}, \ \forall t \ge \tau. \tag{4.83}$$

Consider the "limiting" equation (4.65) with the chosen "limiting" external force $\tilde{g}^0$. We set $\tau = -R$. Let $\tilde{u}^0(t), t \ge -R$, be the solution of this equation with initial data

$$\tilde{u}^0|_{t=-R} = \hat{u}^\varepsilon(-R). \tag{4.84}$$

Taking $-R$ in place of τ and $-R+t$ in place of t, it follows from (4.83) (see also (4.82)) that

$$|\hat{u}^\varepsilon(-R+t) - \tilde{u}^0(-R+t)| \le \varepsilon^{(1-\rho)}C_4 e^{rt}, \ \forall t \ge 0, \tag{4.85}$$

where $\hat{u}^\varepsilon(-R+t) = U_{\hat{g}^\varepsilon}(-R+t, -R)\hat{u}^\varepsilon(-R)$ and $\tilde{u}^0(-R+t) = U_{\tilde{g}^0}(-R+t, -R)\hat{u}^\varepsilon(-R)$.

The set $\mathcal{A}^0$ attracts $U_{\hat{g}^0}(t+\tau,\tau)\tilde{B}$ in H as $t \to +\infty$ (uniformly with respect to $\tau \in \mathbb{R}$ and $\hat{g}^0 \in \mathcal{H}(g^0)$) (see [CV02a]). Then, for any $\delta > 0$, there exist a number $T = T(\delta)$ such that

$$\mathrm{dist}_H(U_{\hat{g}^0}(t+\tau,\tau)\tilde{B}, \mathcal{A}^0) \le \frac{\delta}{2}, \ \forall \tau \in \mathbb{R}, \ \forall \hat{g}^0 \in \mathcal{H}(g^0), \ \forall t \ge T(\delta).$$

Hence, for $\tau = -R$ and $\hat{u}^\varepsilon(-R) \in \tilde{B}$,

$$\mathrm{dist}_H(U_{\hat{g}^0}(-R+t,-R)\hat{u}^\varepsilon(-R), \mathcal{A}^0) \le \frac{\delta}{2}, \ \forall \hat{g}^0 \in \mathcal{H}(g^0), \ \forall t \ge T(\delta).$$

In particular, for the function $\tilde{g}^0$ specified above

$$\mathrm{dist}_H(\tilde{u}^0(-R+t), \mathcal{A}^0) = \mathrm{dist}_H(U_{\tilde{g}^0}(-R+t,-R)\hat{u}^\varepsilon(-R), \mathcal{A}^0) \le \frac{\delta}{2}, \ \forall t \ge T(\delta). \tag{4.86}$$

Recall that $T(\delta)$ is independent of $u^\varepsilon \in \mathcal{A}^\varepsilon$.

It follows from (4.86) and (4.85) that

$$\begin{aligned} \operatorname{dist}_H(\hat{u}^\varepsilon(-R+t), \mathcal{A}^0) &\leq |\hat{u}^\varepsilon(-R+t) - \tilde{u}^0(-R+t)| + \operatorname{dist}_H(\tilde{u}^0(-R+t), \mathcal{A}^0) \\ &\leq \varepsilon^{(1-\rho)} C_4 e^{rt} + \frac{\delta}{2}, \ \forall t \geq T(\delta). \end{aligned} \tag{4.87}$$

We now set $t = R = T(\delta)$ in (4.87) and since $\hat{u}^\varepsilon(0) = u^\varepsilon$ we obtain that

$$\operatorname{dist}_H(u^\varepsilon, \mathcal{A}^0) = \operatorname{dist}_H(\hat{u}^\varepsilon(0), \mathcal{A}^0) \leq \varepsilon^{(1-\rho)} C_4 e^{rT(\delta)} + \frac{\delta}{2}, \ \forall u^\varepsilon \in \mathcal{A}^\varepsilon.$$

Consequently,

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leq \varepsilon^{(1-\rho)} C_4 e^{rT(\delta)} + \frac{\delta}{2}, \ \forall \delta > 0. \tag{4.88}$$

Finally, for an arbitrary $\delta > 0$, we define $\varepsilon_0 = \varepsilon_0(\delta)$ such that $\varepsilon_0^{(1-\rho)} C_4 e^{rT(\delta)} = \delta/2$. Thus, if

$$\varepsilon \leq \varepsilon_0(\delta) = \left(\frac{\delta}{2C_4 e^{rT(\delta)}} \right)^{\frac{1}{1-\rho}},$$

then

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leq \delta.$$

We conclude that

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \to 0 \ (\varepsilon \to 0+).$$

The theorem is proved. ■

4.5 Estimate for the distance from $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$

In this section, we consider the 2D N.–S. system (4.54) when the Grashof number of the corresponding “limiting” N.–S. system (4.65) is small. In this case, the global attractor $\mathcal{A}^0$ is exponential, i.e., $\mathcal{A}^0$ attracts bounded sets of initial data with exponential rate as time tends to infinity. This property allows to estimate explicitly the distance from the global attractor $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$.

We consider the “limiting” system (4.65) with external force $g^0(t) := Pg_0(\cdot, t) \in L_2^{\mathrm{loc}}(\mathbb{R}; H)$. Let the Grashof number G of this 2D N.–S. system satisfy the following inequality:

$$G := \frac{\|g^0\|_{L_2^b}}{\lambda_1 \nu^2} < \frac{1}{c_0^2}, \tag{4.89}$$

where the constant c_0^2 is taken from the inequality (1.14).

Then, by Proposition 2.6.1, the equation (4.65) has the unique solution $z_{g^0}(t), t \in \mathbb{R}$ bounded in H, that is, the kernel $\mathcal{K}_{g^0}$ consists of the unique trajectory $z_{g^0}(t)$. This solution $z_{g^0}(t)$ is exponentially stable, i.e., for every solution $u_{g^0}(t)$ of equation (4.65) the following inequality holds:

$$|u_{g^0}(t+\tau) - z_{g^0}(t+\tau)| \leq C_0 |u_\tau - z_{g^0}(\tau)| e^{-\beta t} \ \forall t \geq 0, \tag{4.90}$$

where $u_{g^0}(t+\tau) = U_{g^0}(t+\tau, \tau) u_\tau$ (in (4.90), C_0 and β are independent of u_τ and τ).

Property (4.90) implies that the set

$$\mathcal{A}^0 = [\{z_{g^0}(t) \mid t \in \mathbb{R}\}]_H = \bigcup_{g \in \mathcal{H}(g^0)} \{z_g(0)\} \tag{4.91}$$

is the global attractor of the equation (4.65) under condition (4.89) (see (2.54)).

Remark 4.5.1 It is shown in [CI04] that inequality (1.14). holds with $c_1^2 = \left(\frac{8}{27\pi}\right)^{1/2} = 0.3071\ldots$. Using the numerical result from [We83], it was also shown in [CI04] that $c_0^2 = 0.2924\ldots$. This value is possibly the best for inequality (1.14). Hence, (4.90) and (4.91) are valid if $G < 3.42$.

Remark 4.5.2 Inequality (4.90) implies that the global attractor $\mathcal{A}^0$ of system (4.65) is exponential under the condition (4.89), i.e., for any bounded set B in H

$$\sup_{\tau \in \mathbb{R}} \operatorname{dist}_H(U_{g^0}(t+\tau, \tau)B, \mathcal{A}^0) \leq C_1(|B|)e^{-\beta t}, \tag{4.92}$$

where C_1 depends on the norm B in H.

We now formulate the following result concerning the distance from $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$.

Theorem 4.5.1 *Under the assumptions of Theorem 4.4.1, we assume that the Grashof number G of the "limiting" 2D N.–S. system satisfies (4.89). Then the Hausdorff distance (in H) from the global attractor $\mathcal{A}^\varepsilon$ of the original 2D N.–S. system (4.54) to the global attractor $\mathcal{A}^0$ of the corresponding "limiting" system (4.65) satisfies the following inequality:*

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leq C(\rho)\varepsilon^{1-\rho}, \ \forall \varepsilon, \ 0 < \varepsilon \leq 1.$$

Here $0 \leq \rho < 1$ and $C(\rho) > 0$ also depends on ν, $\|g_0\|_{L_2^b}$, and $\|g_1\|_{L_2^b}$.

The proof of Theorem 4.5.1 is analogous to the proof of the similar result concerning the complex Ginzburg–Landau equation with singularly oscillating terms (see Section 5.4).

Remark 4.5.3 In this chapter, we consider the non-autonomous 2D N.–S. systems with singularly oscillating external forces and prove some results concerning the behaviour of their global attractors. We have proved analogous theorems for other non-autonomous evolution equations of mathematical physics with singularly oscillating terms, e.g., for the following damped wave equation

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - f(u) + g_0(x,t) + \varepsilon^{-\rho} g_1(x, t/\varepsilon), \ u|_{\partial\Omega} = 0,$$

where

$$\gamma > 0, \ 0 \leq \rho \leq \rho_0, \ 0 < \varepsilon \leq 1, \ t \in \mathbb{R}, \ x \in \Omega \Subset \mathbb{R}^n$$

and the functions $g_0(x,t)$ and $g_1(x,t)$ are tr.c. in the corresponding space (see [VC06])).

In the next chapter, we study the complex Ginzburg–Landau equation with singularly oscillating terms.

Chapter 5

Uniform global attractor of Ginzburg–Landau equation with singularly oscillating terms

In this chapter, we study the global attractor $\mathcal{A}^\varepsilon$ of the non-autonomous complex Ginzburg–Landau (G.–L.) equation with constant dispersion parameters α, β and with singularly oscillating external force of the form $g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$, $x \in \Omega \Subset \mathbb{R}^n$, $n \geq 3$, $0 < \rho \leq 1$. We assume that $|\beta| \leq \sqrt{3}$. In this case, the Cauchy problem for the G.–L. equation has a unique solution and the corresponding process $\{U_\varepsilon(t,\tau)\}$ acting in the space $\mathbf{H} = L_2(\Omega;\mathbb{C})$ has the global attractor $\mathcal{A}^\varepsilon$ (see Sections 1.3.3 and 2.6.3). Along with this G.–L. equation, we consider its "limiting" equation with external force $g_0(x,t)$. We assume that the function $g_1(z,t)$ has the following divergence presentation: $g_1(z,t) = \Sigma_{i=1}^n \partial_{z_1} G_i(z,t)$ $(z = (z_1, \ldots, z_n) \in \mathbb{R}^n_z)$, where the norms of the functions $G_i(z,t)$ are bounded in the space $L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})$, $\mathbf{Z} = L_2^{\mathrm{b}}(\mathbb{R}^n_z;\mathbb{C})$ (see Section 5.1).

We find the estimate for the deviation (in $\mathbf{H}$) of the solutions of the original G.–L. equation from the solutions of the corresponding "limiting" equation with the same initial data.

If the function $g_1(z,t)$ admits the divergence representation and the functions $g_0(x,t)$ and $g_1(z,t)$ are translation compact in the corresponding spaces, then we prove that the global attractors $\mathcal{A}^\varepsilon$ converges to the global attractor $\mathcal{A}^0$ of the "limiting" system as $\varepsilon \to 0+$ in the strong norm of $\mathbf{H}$.

We also study the case where the global attractor $\mathcal{A}^0$ of the "limiting" G.–L. equation is exponential. In such a situation, we prove the estimate for the deviation of the global attractor $\mathcal{A}^\varepsilon$ from $\mathcal{A}^0$: $\mathrm{dist}_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leq C(\rho)\varepsilon^{1-\rho}$ for all ε, $0 < \varepsilon \leq 1$, where the constant $C(\rho)$ is independent of ε.

5.1 Ginzburg–Landau equation with singularly oscillating external force

We consider the following non-autonomous Ginzburg–Landau (G.–L.) equation:

$$\partial_t u = (1+i\alpha)\Delta u + Ru - (1+i\beta)|u|^2 u + g_0(x,t) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right), \ u|_{\partial\Omega} = 0. \qquad (5.1)$$

Here $u = u_1(x,t) + iu_2(x,t)$ is an unknown complex function depending on $x \in \Omega \Subset \mathbb{R}^n$ and $t \in \mathbb{R}$ (see Sections 1.3.3 and 2.6.3). We assume that $0 \in \Omega$ and

$$|\beta| \leq \sqrt{3}. \tag{5.2}$$

In equation (5.1), $0 \leq \rho \leq 1$ and ε is a small positive parameter. We set $\mathbf{H} = L_2(\Omega;\mathbb{C})$ and $\mathbf{Z} = L_2^{\mathrm{b}}(\mathbb{R}^n;\mathbb{C})$. The norm in $\mathbf{H}$ is denoted by $\|\cdot\|_{\mathbf{H}}$. A function $f(z) \in \mathbf{Z} = L_2^{\mathrm{b}}(\mathbb{R}_z^n;\mathbb{C})$ $(z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n)$ if

$$\|f(\cdot)\|_{\mathbf{Z}}^2 = \|f(\cdot)\|_{L_2^{\mathrm{b}}(\mathbb{R}_z^n;\mathbb{C})}^2 := \sup_{z\in\mathbb{R}^n} \int_{z_1}^{z_1+1} \cdots \int_{z_n}^{z_n+1} |f(\zeta_1,\ldots,\zeta_n)|^2 d\zeta_1 \cdots d\zeta_n < +\infty. \tag{5.3}$$

We assume that the function $g_0(x,t) = g_{01}(x,t) + ig_{02}(x,t),\ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, belongs to the space $L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})$ and the function $g_1(z,t) = g_{11}(z,t) + ig_{12}(z,t), z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$, belongs to $L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})$, i.e., the following norms of these functions are finite:

$$\begin{aligned} \|g_0(\cdot,\cdot)\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})}^2 &:= \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \|g_0(\cdot,s)\|_{\mathbf{H}}^2 ds \\ &= \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \left(\int_\Omega |g_0(x,s)|^2 dx \right) ds < +\infty, \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\|g_1(\cdot,\cdot)\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})}^2 := \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \|g_1(\cdot,s)\|_{\mathbf{Z}}^2 ds \\ &= \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \left(\sup_{z\in\mathbb{R}^n} \int_{z_1}^{z_1+1} \cdots \int_{z_n}^{z_n+1} |g_1(\zeta_1,\ldots,\zeta_n,s)|^2 d\zeta_1 \cdots d\zeta_n \right) ds < +\infty, \end{aligned} \tag{5.5}$$

where $z = (z_1, z_2, \ldots, z_n)$.

Equation (5.1) is equivalent to the following system of two equations for the real vector-function $\mathbf{u} = (u_1, u_2)^\top$:

$$\partial_t \mathbf{u} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \Delta \mathbf{u} + R\mathbf{u} - \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix} |\mathbf{u}|^2 \mathbf{u} + \mathbf{g}_0(x,t) + \frac{1}{\varepsilon^\rho} \mathbf{g}_1\left(\frac{x}{\varepsilon}, t\right), \tag{5.6}$$

where $\mathbf{g}_0 = (g_{01}, g_{02})^\top$ and $\mathbf{g}_1 = (g_{11}, g_{12})^\top$.

Under the above assumption for every fixed $\varepsilon,\ 0 < \varepsilon \leq 1$, the Cauchy problem for equation (5.1) with initial data

$$u|_{t=\tau} = u_\tau(x),\ u_\tau(\cdot) \in \mathbf{H}, \tag{5.7}$$

(here, τ is arbitrary and fixed), has a unique solution $u(t) := u(x,t)$ such that

$$\begin{aligned} u(\cdot) &\in C(\mathbb{R}_\tau;\mathbf{H}) \cap L_2^{\mathrm{loc}}(\mathbb{R}_\tau;\mathbf{V}) \cap L_4^{\mathrm{loc}}(\mathbb{R}_\tau;\mathbf{L}_4), \\ \mathbf{V} &= H_0^1(\Omega;\mathbb{C}),\ \mathbf{L}_4 = L_4(\Omega;\mathbb{C}),\ \mathbb{R}_\tau = [\tau,+\infty). \end{aligned} \tag{5.8}$$

and the function $u(t)$ satisfies equation (5.1) in the sense of distributions of the space $\mathcal{D}'(\mathbb{R}_\tau;\mathbf{H}^{-r})$, where $\mathbf{H}^{-r} = H^{-r}(\Omega;\mathbb{C})$ and $r = \max\{1, n/4\}$ (recall that $n = \dim(\Omega)$). In particular, $\partial_t u(\cdot) \in L_2(\tau,T;\mathbf{H}^{-1}) + L_{4/3}(\tau,T;\mathbf{L}_{4/3})$ for any $T > \tau$. The proof of the existence of such solution $u(t)$ uses the Galerkin approximation method (see, e.g. [T88, BV89, CV02a]). The proof of the uniqueness relies on inequality (5.2) (see, e.g., [CV02a]).

We recall that, if (5.2) does not hold, the uniqueness for $n \geq 3$ and for arbitrary values of the dispersion parameters α and β is not proved yet, see [Mi02, Mi98, Z00] for known uniqueness theorems.

For brevity, we set $\|\cdot\| := \|\cdot\|_{\mathbf{H}}$. Any solution $u(t), t \geq \tau$, of equation (5.1) satisfies the following differential identity:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|_{\mathbf{L}_4}^4 - R\|u(t)\|^2 = \langle g^\varepsilon(t), u(t)\rangle\,,\ \forall t \geq \tau, \tag{5.9}$$

where we denote $g^\varepsilon(t) := g_0(x,t) + \varepsilon^{-\rho} g_1\left(\frac{x}{\varepsilon}, t\right)$. The function $\|u(t)\|^2$ is absolutely continuous for $t \geq \tau$. The proof of (5.9) is analogous to the proof of the corresponding identity for weak solutions of the reaction-diffusion systems considered in [CV02a, CV96b] (see also [CV05]).

Using the standard transformations and the Gronwall lemma, we deduce from (5.9) that any solution $u(t)$ of equation (5.1) satisfies the inequality

$$\|u(t+\tau)\|^2 \leq \|u(\tau)\|^2 e^{-2\lambda_1 t} + C_0^2 + C_1^2\varepsilon^{-2\rho},\ \forall t \geq 0,\ \tau \in \mathbb{R}. \tag{5.10}$$

where λ_1 is the first eigenvalue of the operator $\{-\Delta u,\ u|_{\partial\Omega} = 0\}$, the constant C_0 depends on R and $\|g_0\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})}$ and the constant C_1 depends on $\|g_1\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})}$ (see (5.4) and (5.5)). We also use the following inequality:

$$\int_\tau^t \int_\Omega \left|g_1\left(\frac{x}{\varepsilon}, s\right)\right|^2 e^{-\lambda_1(t-s)} dx ds \leq C\|g_1\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})},\ \forall t \geq \tau, \tau \in \mathbb{R}, \tag{5.11}$$

where C is independent of ε. Indeed,

$$\begin{aligned}
&\int_\tau^t \int_\Omega \left|g_1\left(\frac{x}{\varepsilon}, s\right)\right|^2 e^{-\lambda_1(t-s)} dx ds = \int_\tau^t e^{-\lambda_1(t-s)}\left(\varepsilon^n \int_{\varepsilon^{-1}\Omega} |g_1(z,s)|^2 dz\right) ds \\
\leq\ & C' \int_\tau^t e^{-\lambda_1(t-s)} \left(\sup_{z\in\mathbb{R}^n} \int_{z_1}^{z_1+1} \cdots \int_{z_n}^{z_n+1} |g_1(\zeta_1,\ldots,\zeta_n,s)|^2 d\zeta_1\cdots d\zeta_n\right) ds \\
\leq\ & C''(\lambda_1)\|g_1\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})},
\end{aligned}$$

since we can cover the domain $\varepsilon^{-1}\Omega$ by $C'\varepsilon^{-n}$ unit boxes (see the proof of Lemma 4.1.2) and, therefore, (5.11) is proved.

Integrating (5.9) in time from τ to $\tau + t$ and using (5.10), we obtain that

$$\begin{aligned}
&\frac{1}{2}\|u(\tau+t)\|^2 + \int_\tau^{\tau+t} \left(\|\nabla u(s)\|^2 + \|u(s)\|_{\mathbf{L}_4}^4\right) ds \\
\leq\ & \frac{1}{2}\|u(\tau)\|^2 + R\int_\tau^{\tau+t} \|u(s)\|^2 ds + \int_\tau^{\tau+t} \|g^\varepsilon(s)\|\cdot\|u(s)\| ds, \\
&\int_\tau^t \left(\|\nabla u(s)\|^2 + \|u(s)\|_{\mathbf{L}_4}^4\right) ds \\
\leq\ & \frac{1}{2}\|u(\tau)\|^2 + C_2(t+1) + C_3\left(\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})} + \varepsilon^{-2\rho}\|g_1\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})}\right) t,
\end{aligned} \tag{5.12}$$

(see (5.4) and (5.5)).

We consider the process $\{U_\varepsilon(t,\tau)\} := \{U_\varepsilon(t,\tau) \mid t \ge \tau, \tau \in \mathbb{R}\}$ corresponding to problem (5.1), (5.7) and acting in the space $\mathbf{H}$ (see formula (2.118)). It follows from estimates (5.10) that the process $\{U_\varepsilon(t,\tau)\}$ has the uniformly absorbing set

$$B_{0,\varepsilon} = \{v \in \mathbf{H} \mid \|v\| \le 2\left(C_0 + C_1\varepsilon^{-\rho}\right)\} \tag{5.13}$$

that is bounded in $\mathbf{H}$ for every fixed $\varepsilon > 0$.

We now establish that the process $\{U_\varepsilon(t,\tau)\}$ has a compact in $\mathbf{H}$ uniformly absorbing set

$$B_{1,\varepsilon} = \{v \in \mathbf{V} \mid \|v\|_{\mathbf{V}} \le C_0' + C_1'\varepsilon^{-\rho}\}. \tag{5.14}$$

To prove this fact we take the scalar product in $\mathbf{H}$ of equation (5.1) with the term $-t\Delta u$. After the standard transformations, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(t\|\nabla u\|^2\right) - \frac{1}{2}\|\nabla u\|^2 \quad &+ \quad t\|\Delta u\|^2 - Rt\|\nabla u\|^2 \\ -\left\langle (1+i\beta)|u|^2u, t\Delta u\right\rangle \quad &= \quad -\langle g_0, t\Delta u\rangle - \varepsilon^{-\rho}\langle g_1(x/\varepsilon), t\Delta u\rangle . \end{aligned} \tag{5.15}$$

We denote

$$f(\mathbf{v}) = |\mathbf{v}|^2 \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix} \mathbf{v},\ \mathbf{v} = (v_1, v_2).$$

We notice that since $|\beta| \le \sqrt{3}$ the matrix $f_{\mathbf{v}}'(\mathbf{v})$ is positive definite, that is,

$$f_{\mathbf{v}}'(\mathbf{v})\mathbf{w}\cdot\mathbf{w} \ge 0,\ \forall \mathbf{v} = (v_1, v_2), \mathbf{w} = (w_1, w_2),\ \forall t \ge 0 \tag{5.16}$$

(see (1.34)). Therefore, the term in (5.15) containing β is also positive. Indeed,

$$-\left\langle (1+i\beta)|u|^2u, t\Delta u\right\rangle = -\langle f(\mathbf{u}), t\Delta\mathbf{u}\rangle = t\sum_{i=1}^n \int_\Omega \left(f_{\mathbf{u}}'(\mathbf{u})\partial_{x_i}\mathbf{u}, \partial_{x_i}\mathbf{u}\right)dx \ge 0,\ \forall t \ge 0. \tag{5.17}$$

Integrating both sides of (5.15) in t and taking into account (5.17), we have

$$\begin{aligned} &\frac{1}{2}t\|\nabla u(t)\|^2 - \frac{1}{2}\int_0^t \|\nabla u(s)\|^2 ds + \int_0^t s\|\Delta u(s)\|^2 ds - R\int_0^t s\|\nabla u(s)\|^2 ds \\ &\le -\int_0^t \langle g_0(s), s\Delta u(s)\rangle\, ds - \varepsilon^{-\rho}\int_0^t \langle g_1(x/\varepsilon, s), s\Delta u(s)\rangle\, ds. \end{aligned} \tag{5.18}$$

Using (5.12), we obtain from (5.18) the inequality

$$\begin{aligned} \frac{1}{2}t\|\nabla u(t)\|^2 \quad &+ \quad C_5\int_0^t s\|\Delta u(s)\|^2 ds \le R\int_0^t s\|\nabla u(s)\|^2 ds \\ &+ \quad C_6\left(\int_0^t s\|g_0(s)\|^2 ds + \varepsilon^{-2\rho}\int_0^t s\|g_1(x/\varepsilon, s)\|^2 ds\right). \end{aligned} \tag{5.19}$$

Applying in (5.19) an inequality similarly to (5.11), we find that

$$t\|\nabla u(t)\|^2 \le C_7(t\|u(0)\|^2 + t + 1 + t\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})} + t\varepsilon^{-2\rho}\|g_1\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})}).$$

Assuming that $u(0) \in B_{0,\varepsilon}$ and setting $t = 1$, we obtain

$$\|\nabla u(1)\| \leq C_8(1 + \|g_0\|_{L_2^b(\mathbb{R};\mathbf{H})} + \varepsilon^{-\rho}\|g_1\|_{L_2^b(\mathbb{R};\mathbf{Z})}). \tag{5.20}$$

Clearly, the same inequalities holds if we replace 0 and t with τ and $\tau + t$:

$$t\|\nabla u(\tau + t)\|^2 \leq C_7(t\|u(\tau)\|^2 + t + 1 + t\|g_0\|^2_{L_2^b(\mathbb{R};\mathbf{H})} + t\varepsilon^{-2\rho}\|g_1\|^2_{L_2^b(\mathbb{R};\mathbf{Z})}),$$

So, if $u(\tau) \in B_{0,\varepsilon}$, then

$$\|\nabla u(\tau + 1)\| \leq C_8(1 + \|g_0\|_{L_2^b(\mathbb{R};\mathbf{H})} + \varepsilon^{-\rho}\|g_1\|_{L_2^b(\mathbb{R};\mathbf{Z})}),\ \forall \tau \geq 0. \tag{5.21}$$

It follows from (5.21) that the set

$$B_{1,\varepsilon} = \{v \in \mathbf{V} \mid \|v\|_{\mathbf{V}} \leq C_8(1 + \|g_0\|_{L_2^b} + \varepsilon^{-\rho}\|g_1\|_{L_2^b})\} \tag{5.22}$$

is uniformly absorbing for the process $\{U_\varepsilon(t,\tau)\}$ corresponding to the G.–L. equation (5.1). The set $B_{1,\varepsilon}$ is bounded in $\mathbf{V}$ and compact in $\mathbf{H}$ since the embedding $\mathbf{V} \Subset \mathbf{H}$ is compact. Recall that a process having a compact uniformly absorbing set is called *uniformly compact*. We have proved the following

Proposition 5.1.1 *For any fixed $\varepsilon > 0$, the process $\{U_\varepsilon(t,\tau)\}$ corresponding to equation (5.1) is uniformly compact in the space $\mathbf{H}$. It has the compact uniformly absorbing set $B_{1,\varepsilon}$ defined in (5.22).*

Along with the G.–L. equation (5.1), we consider its "limiting" equation

$$\partial_t u^0 = (1 + i\alpha)\Delta u^0 + Ru^0 - (1 + i\beta)|u^0|^2 u^0 + g_0(x,t),\ u^0|_{\partial\Omega} = 0, \tag{5.23}$$

where the coefficients α, β, R and the external force $g_0(x,t)$ are the same as in (5.1). In particular, conditions (5.2) and (5.4) hold. Therefore, the Cauchy problem for this equation with initial data

$$u^0|_{t=\tau} = u_\tau(x),\ u_\tau(\cdot) \in \mathbf{H}, \tag{5.24}$$

also has a unique solution $u^0(x,t)$ and there is the corresponding process $\{U_0(t,\tau)\}$ in $\mathbf{H}$: $U_0(t,\tau)u_\tau = u^0(t), t \geq \tau \in \mathbb{R}$, where $u^0(t), t \geq \tau$, is a solution of equation (5.23) with initial data $u|_{t=\tau} = u_\tau$. Similar to (5.10), the main a priory estimate for equation (5.23) reads

$$\|u^0(\tau + t)\|^2 \leq \|u^0(\tau)\|^2 e^{-2\lambda_1 t} + C_0^2. \tag{5.25}$$

Following the above reasoning, we prove that the process $\{U_0(t,\tau)\}$ has the uniformly absorbing set

$$B_{0,0} = \{v \in \mathbf{H} \mid \|v\| \leq 2C_0\} \tag{5.26}$$

(Comparing with (5.13), we observe that in (5.26) the parameter ε is missing since the term $\varepsilon^{-\rho} g_1(x/\varepsilon, t)$ is missing in equation (5.23).) Moreover, the process also has the uniformly absorbing set

$$B_{1,0} = \{v \in \mathbf{V} \mid \|\nabla v\|_{\mathbf{V}} \leq C_8(1 + \|g_0\|_{L_2^b(\mathbb{R};\mathbf{H})})\} \tag{5.27}$$

which is bounded in $\mathbf{V}$ and compact in $\mathbf{H}$. Consequently, the process $\{U_0(t,\tau)\}$ corresponding to the "limiting" equation (5.23) is uniformly compact in $\mathbf{H}$ and Proposition 5.1.1 holds for the "limit" case $\varepsilon = 0$ as well.

Using this results, it follows easily that the processes $\{U_\varepsilon(t,\tau)\}, \varepsilon > 0$, and $\{U_0(t,\tau)\}$ have the uniform global attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$, respectively (see [CV02a] and Section 2.6.3), that satisfy the inequalities

$$\begin{aligned} \|\mathcal{A}_\varepsilon\|_{\mathbf{H}} &\leq C_0 + C_1\varepsilon^{-\rho}, \\ \|\mathcal{A}_0\|_{\mathbf{H}} &\leq C_0. \end{aligned}$$

However, the formulated above conditions for the function $g_1(z,t)$ is not sufficient to establish that the global attractors $\mathcal{A}_\varepsilon$ are uniformly (with respect to $\varepsilon > 0$) bounded in $\mathbf{H}$.

We now present the assumption that provide the uniform boundedness of global attractors $\mathcal{A}_\varepsilon$ for $0 < \varepsilon \leq 1$. We assume that the function $g_1(z,t)$ satisfies the following

Divergence condition. *There exist vector functions $G_j(z,t) \in L_2^b(\mathbb{R};\mathbf{Z})$, $j = \overline{1,n}$, such that $\partial_{z_j} G_j(z,t) \in L_2^b(\mathbb{R};\mathbf{Z})$ and*

$$\sum_{j=1}^{n} \partial_{z_j} G_j(z,t) = g_1(z,t),\ \forall z \in \mathbb{R}^n,\ t \in \mathbb{R}. \tag{5.28}$$

Theorem 5.1.1 *If the function $g_1(z,t)$ satisfies the divergence condition (5.28), then, for every ρ, $0 \leq \rho \leq 1$, the global attractors $\mathcal{A}^\varepsilon$ of the G.–L. equations are uniformly (with respect to $\varepsilon \in]0,1]$) bounded in $\mathbf{H}$, that is,*

$$\|\mathcal{A}^\varepsilon\|_{\mathbf{H}} \leq C_2,\ \forall \varepsilon \in]0,1]. \tag{5.29}$$

The proof is analogous to the proof of Theorem 4.2.1.

5.2 Deviation estimate for solutions of the G.–L. equation with oscillating external forces from solutions of the "limiting" equation

We consider equation (5.1)

$$\partial_t u = (1+i\alpha)\Delta u + Ru - (1+i\beta)|u|^2 u + g_0(x,t) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right),\ u|_{\partial\Omega} = 0. \tag{5.30}$$

We assume that the coefficients of this equation satisfy conditions (5.2) – (5.5) and $0 < \rho \leq 1$. The corresponding "limiting" equation is

$$\partial_t u^0 = (1+i\alpha)\Delta u^0 + Ru^0 - (1+i\beta)|u^0|^2 u^0 + g_0(x,t),\ u^0|_{\partial\Omega} = 0. \tag{5.31}$$

For $t = \tau$, we consider the same initial data

$$u|_{t=\tau} = u_\tau(x),\ u^0|_{t=\tau} = u_\tau(x),\ u_\tau(\cdot) \in \mathbf{H}. \tag{5.32}$$

Let $u(x,t), t \geq \tau$, and $u^0(x,t), t \geq \tau$, be solutions of problems (5.30), (5.32) and (5.31), (5.32), respectively. We set $w(x,t) = u(x,t) - u^0(x,t)$. The function $w(t) := w(\cdot,t)$ satisfies the equation

$$\partial_t w = (1+i\alpha)\Delta w + Rw - (1+i\beta)\left(|u|^2 u - |u^0|^2 u^0\right) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right), \ w|_{\partial\Omega} = 0, \ (5.33)$$

and has the initial data $w(\tau) = 0$.

Theorem 5.2.1 *Under the divergence condition (5.28), the difference $w(t) = u(\cdot,t) - u^0(\cdot,t)$ of the solutions $u(x,t)$ and $u^0(x,t)$ of equations (5.30) and (5.31), respectively, with common initial data (5.32) satisfies the following inequality:*

$$\|w(t)\| = \|u(\cdot,t) - u^0(\cdot,t)\| \leq C\varepsilon^{(1-\rho)} e^{r(t-\tau)}, \ \forall t \geq \tau, \tag{5.34}$$

where

$$r = \begin{cases} 0, & \text{for } R < \lambda_1 \\ R - \lambda_1 + \delta, & \text{for } R \geq \lambda_1 \end{cases}, \tag{5.35}$$

$\delta > 0$ is arbitrary small, and $C = C(\delta)$ for $R \geq \lambda_1$.

Proof. We assume for simplicity that $\tau = 0$. Taking the scalar product in $\mathbf{H}$ of equation (5.33) and w, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 - R\|w\|^2 \\ + & \left\langle (1+i\beta)\left(|u|^2 u - |u^0|^2 u^0\right), u - u^0 \right\rangle = \varepsilon^{-\rho}\left\langle g_1\left(\frac{x}{\varepsilon}, t\right), w \right\rangle. \end{aligned} \tag{5.36}$$

Since $|\beta| \leq \sqrt{3}$, it follows from (5.16) that

$$\left\langle (1+i\beta)\left(|u|^2 u - |u^0|^2 u^0\right), u - u^0 \right\rangle \geq 0 \tag{5.37}$$

(see also (1.34) and ([CV02a])). From (5.36) and (5.37), we obtain

$$\frac{d}{dt}\|w\|^2 + 2\|\nabla w\|^2 \leq 2R\|w\|^2 + 2\varepsilon^{-\rho}\left\langle g_1\left(\frac{x}{\varepsilon}, t\right), w \right\rangle, \tag{5.38}$$

Applying (5.28), we find that

$$\begin{aligned} & 2\varepsilon^{-\rho}\left\langle g_1\left(\frac{x}{\varepsilon}, t\right), w \right\rangle ds = 2\varepsilon^{-\rho}\sum_{j=1}^n \left\langle \partial_{z_j} G_j\left(\frac{x}{\varepsilon}, t\right), w \right\rangle \\ = & \ 2\varepsilon^{1-\rho}\sum_{j=1}^n \left\langle \partial_{x_j} G_j\left(\frac{x}{\varepsilon}, t\right), w \right\rangle = -2\varepsilon^{1-\rho}\sum_{j=1}^n \left\langle G_j\left(\frac{x}{\varepsilon}, t\right), \partial_{x_j} w \right\rangle \\ \leq & \ \frac{\lambda_1}{2\delta}\varepsilon^{2(1-\rho)}\sum_{j=1}^n \int_\Omega \left| G_j\left(\frac{x}{\varepsilon}, t\right)\right|^2 dx + \frac{2\delta}{\lambda_1}\int_\Omega |\nabla w(x,t)|^2 dx, \ \delta > 0. \end{aligned} \tag{5.39}$$

We claim that

$$\int_\Omega \left| G_j\left(\frac{x}{\varepsilon}, t\right)\right|^2 dx = \varepsilon^n \int_{\varepsilon^{-1}\Omega} |G_j(z,t)|^2 dx \leq C\|G_j(\cdot,t)\|^2_{\mathbf{Z}}. \tag{5.40}$$

Here, we use the n-dimensional analog of Lemma 4.1.2. Hence,

$$\sum_{j=1}^{n} \int_{\Omega} \left| G_j \left(\frac{x}{\varepsilon}, t \right) \right|^2 dx \leq C \sum_{j=1}^{n} \|G_j(\cdot, t)\|_{\mathbf{Z}}^2, \ \forall t \in \mathbb{R}. \tag{5.41}$$

It follows from (5.39) and (5.41) that

$$2\varepsilon^{-\rho} \left\langle g_1 \left(\frac{x}{\varepsilon}, t \right), w \right\rangle \leq \left(\frac{\lambda_1}{2\delta} \varepsilon^{2(1-\rho)} C \right) h(t) + \frac{2\delta}{\lambda_1} \|\nabla w\|^2, \ \delta > 0,$$

where we set

$$h(t) = \sum_{j=1}^{n} \|G_j(\cdot, t)\|_{\mathbf{Z}}^2 .$$

Consequently from (5.38), we have

$$\frac{d}{dt} \|w\|^2 + (2 - 2\delta\lambda_1^{-1}) \|\nabla w\|^2 \leq 2R\|w\|^2 + \left(\frac{\lambda_1}{2\delta} \varepsilon^{2(1-\rho)} C \right) h(t). \tag{5.42}$$

We assume that $\delta < \lambda_1$. From the Poincaré inequality, we conclude that

$$\frac{d}{dt} \|w\|^2 \leq 2(R - \lambda_1 + \delta) \|w\|^2 + \left(\frac{\lambda_1}{2\delta} \varepsilon^{2(1-\rho)} C \right) h(t). \tag{5.43}$$

If now $R \geq \lambda_1$, then $r = R - \lambda_1 + \delta > 0$ and hence

$$\frac{d}{dt} \|w(t)\|^2 \leq r\|w(t)\|^2 + \left(\frac{\lambda_1}{2\delta} \varepsilon^{2(1-\rho)} C \right) h(t), \ \ \|w(0)\|^2 = 0.$$

Applying the Granwall inequality (see (4.48) and (4.49)), we have

$$\|w(t)\|^2 \leq \left(\frac{\lambda_1}{2\delta} \varepsilon^{2(1-\rho)} C \right) \int_0^t h(s) e^{r(t-s)} ds. \tag{5.44}$$

Recall that $G_j(z, t) \in L_2^b(\mathbb{R}; \mathbf{Z})$ since g_1 satisfies the divergence condition. Therefore,

$$\int_t^{t+1} h(s) ds \leq \sum_{j=1}^{n} \|G_j\|_{L_2^b(\mathbb{R}; \mathbf{Z})}^2 =: M. \tag{5.45}$$

and hence

$$\begin{aligned}
\int_0^t h(s) e^{-rs} ds &= \int_0^1 h(s) e^{-rs} ds + \int_1^2 h(s) e^{-rs} ds + \ldots + \int_{[t]}^t h(s) e^{-rs} ds \\
&\leq \int_0^1 h(s) ds + e^{-r} \int_1^2 h(s) ds + \ldots + e^{-[t]} \int_{[t]}^t h(s) ds \\
&\leq M \left(1 + e^{-r} + \ldots + e^{-[t]} \right) \leq M \left(1 + e^{-r} + \ldots \right) \\
&= \frac{M}{1 - e^{-r}} \leq M(1 + r^{-1}).
\end{aligned}$$

Using this estimate in (5.45), we obtain

$$\|w(t)\|^2 \leq \left(\frac{\lambda_1}{2\delta} \varepsilon^{2(1-\rho)} CM(1+r^{-1}) \right) e^{rt}, \ \forall t \geq 0, \tag{5.46}$$

that is,

$$\|w(t)\| \leq C(\delta) \varepsilon^{(1-\rho)} e^{rt},$$

where $r = R - \lambda_1 + \delta$ and $C(\delta) = \left(\delta^{-1} 2^{-1} \lambda_1 CM(1+r^{-1})\right)^{1/2}$.

If $R < \lambda_1$, then $-r_1 = R - \lambda_1 + \delta < 0$ for a sufficiently small $\delta > 0$. Then have from (5.43) that

$$\frac{d}{dt} \|w\|^2 \leq -r_1 \|w\|^2 + \left(\frac{\lambda_1}{2\delta} \varepsilon^{2(1-\rho)} C \right) h(t). \tag{5.47}$$

Using Lemma 4.1.1 and (5.45), we have

$$\|w(t)\|^2 \leq \|w(0)\|^2 e^{-r_1 t} + 2^{-1} \delta^{-1} \lambda_1 CM(1+r_1^{-1}) \varepsilon^{2(1-\rho)}, \ \forall t \geq 0,$$

and since $w(0) = 0$,

$$\|w(t)\| \leq C(\delta) \varepsilon^{(1-\rho)},$$

where $C(\delta) = \left(2^{-1} \delta^{-1} \lambda_1 CM(1+r_1^{-1})\right)^{1/2}$, $r_1 = \lambda_1 - R - \delta > 0$.

Inequality (5.34) is proved. ■

5.3 On the structure of the attractors $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$

We now consider the G.–L. equation (5.30)

$$\partial_t u = (1+i\alpha)\Delta u + Ru - (1+i\beta)|u|^2 u + g^\varepsilon(x,t), \ u|_{\partial\Omega} = 0, \tag{5.48}$$

where ε is fixed and $g^\varepsilon(x,t) = g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$ is the *time symbol* of the equation (see Section 2.4). We assume that the function $g_0(x,t)$ is tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R}; \mathbf{H})$ and $g_1(z,t)$ is tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R}; \mathbf{Z})$. In particular, $g_0(x,t) \in L_2^b(\mathbb{R}; \mathbf{H})$ and $g_1(z,t) \in L_2^b(\mathbb{R}; \mathbf{Z})$.

Let $\mathcal{H}(g^\varepsilon)$ be the hull of the symbol $g^\varepsilon(x,t)$ in the space $L_2^{\mathrm{loc}}(\mathbb{R}; \mathbf{H})$:

$$\mathcal{H}(g^\varepsilon) = \left[\{g^\varepsilon(\cdot, t+h) \mid h \in \mathbb{R}\}\right]_{L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})}. \tag{5.49}$$

Recall that $\mathcal{H}(g^\varepsilon)$ is compact in $L_2^{\mathrm{loc}}(\mathbb{R}; \mathbf{H})$ and each element $\hat{g}^\varepsilon(x,t) \in \mathcal{H}(g^\varepsilon(x,t))$ can be written in the form

$$\hat{g}^\varepsilon(x,t) = \hat{g}_0(x,t) + \varepsilon^{-\rho} \hat{g}_1(x/\varepsilon, t) \tag{5.50}$$

for some functions $\hat{g}_0 \in \mathcal{H}(g_0)$ and $\hat{g}_1 \in \mathcal{H}(g_1)$, where $\mathcal{H}(g_0)$ and $\mathcal{H}(g_1)$ are the hulls of the functions $g_0(x,t)$ and $g_1(z,t)$ in $L_2^{\mathrm{loc}}(\mathbb{R}; \mathbf{H})$ and $L_2^{\mathrm{loc}}(\mathbb{R}; \mathbf{Z})$, respectively (see Proposition 4.3.2 which is also true for the n-dimensional complex spaces $\mathbf{H}$ and $\mathbf{Z}$).

It was shown in Section 5.1 that the process $\{U_\varepsilon(t,\tau)\} := \{U_{g^\varepsilon}(t,\tau)\}$ corresponding to equation (5.48) has the uniform global attractor $\mathcal{A}^\varepsilon \subseteq B_{0,\varepsilon} \cap B_{1,\varepsilon}$, (see (5.13) and (5.14)) and

$$\|\mathcal{A}^\varepsilon\|_{\mathbf{H}} \leq (C_0 + C_1 \varepsilon^{-\rho}). \tag{5.51}$$

We now describe the structure of the attractor $\mathcal{A}^\varepsilon$.

Along with equation (5.48), we consider the family of equations

$$\partial_t \hat{u}^\varepsilon = (1+i\alpha)\Delta \hat{u}^\varepsilon + R\hat{u}^\varepsilon - (1+i\beta)|\hat{u}^\varepsilon|^2\hat{u}^\varepsilon + \hat{g}^\varepsilon(x,t),\ \hat{u}^\varepsilon|_{\partial\Omega} = 0, \tag{5.52}$$

with symbols $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$. It is clear that, for every $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, equation (5.52) generates the process $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$ acting in $\mathbf{H}$. We note that the processes $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$ satisfy the similar properties as the process $\{U_{g^\varepsilon}(t,\tau)\}$ corresponding to the G.–L. equation (5.48) with original symbol $g^\varepsilon(x,t) = g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$. In particular, the sets $B_{0,\varepsilon}$ and $B_{1,\varepsilon}$ are absorbing for each process of the family $\{U_{\hat{g}^\varepsilon}(t,\tau)\}, \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$.

We denote by $\mathcal{K}_{\hat{g}^\varepsilon}$ the *kernel* of equation (5.52) (and of the process $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$) with symbol $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$. Recall that the kernel $\mathcal{K}_{\hat{g}^\varepsilon}$ is the family of all complete solutions $\hat{u}^\varepsilon(t), t \in \mathbb{R}$, of (5.52) which are bounded in the norm of $\mathbf{H}$:

$$|\hat{u}^\varepsilon(t)| \le M_{\hat{u}^\varepsilon},\ \forall t \in \mathbb{R}. \tag{5.53}$$

As usual,

$$\mathcal{K}_{\hat{g}^\varepsilon}(s) = \{\hat{u}^\varepsilon(s) \mid \hat{u}^\varepsilon \in \mathcal{K}_{\hat{g}^\varepsilon}\},\ s \in \mathbb{R},$$

denotes the *kernel section* at time $t = s$ (a set from $\mathbf{H}$).

We recall the theorem on the structure of the uniform global attractor $\mathcal{A}^\varepsilon$ of the G.–L. equation (5.48) (see Section 2.6.3 and (2.122)).

Theorem 5.3.1 *If the function $g^\varepsilon(x,t)$ is tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$, then the process $\{U_{g^\varepsilon}(t,\tau)\}$ corresponding to equations (5.52) has the uniform global attractor $\mathcal{A}^\varepsilon$ and the following identity holds:*

$$\mathcal{A}^\varepsilon = \bigcup_{\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)} \mathcal{K}_{\hat{g}^\varepsilon}(0). \tag{5.54}$$

Moreover, the kernel $\mathcal{K}_{\hat{g}^\varepsilon}$ is non-empty for every $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$.

All the above results are also applicable to the "limiting" G.–L. equation (5.31)

$$\partial_t u^0 = (1+i\alpha)\Delta u^0 + Ru^0 - (1+i\beta)|u^0|^2 u^0 + g^0(x,t),\ u^0|_{\partial\Omega} = 0, \tag{5.55}$$

with tr.c. symbol $g^0(t) := g_0(\cdot,t) \in L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$. Equation (5.55) generates the "limiting" process $\{U_0(t,\tau)\} := \{U_{g^0}(t,\tau)\}$ which has the uniform global attractor $\mathcal{A}^0$ (see Section 5.2).

Consider the family of equations

$$\partial_t \hat{u}^0 = (1+i\alpha)\Delta \hat{u}^0 + R\hat{u}^0 - (1+i\beta)|\hat{u}^0|^2\hat{u}^0 + \hat{g}^0(x,t), \hat{u}^0|_{\partial\Omega} = 0, \tag{5.56}$$

with symbols $\hat{g}^0 \in \mathcal{H}(g^0)$ and the family of processes $\{U_{\hat{g}^0}(t,\tau)\}, \hat{g}^0 \in \mathcal{H}(g^0)$.

Notice that we can apply Theorem 5.3.1 directly to the equations (5.55) and (5.56) taking the function $g_1(z,t) \equiv 0$. Therefore, the attractor $\mathcal{A}^0$ of the "limiting" equation (5.55) has the form

$$\mathcal{A}^0 = \bigcup_{\hat{g}^0 \in \mathcal{H}(g^0)} \mathcal{K}_{\hat{g}^0}(0), \tag{5.57}$$

where $\mathcal{K}_{\hat{g}^0}$ is the kernel of equation (5.56) with symbol $\hat{g}^0 \in \mathcal{H}(g^0)$.

5.4 Convergence of the global attractors $\mathcal{A}^{\varepsilon}$ to $\mathcal{A}^{0}$ and the estimate for their deviation

All the results of Sections 4.3 and 4.4 can be obtained to the G.–L. equation.

We consider equations (5.48) and (5.55), where the functions $g_0(x,t)$ and $g_1(z,t)$ are tr.c. in the spaces $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$ and $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{Z})$, respectively.

We also assume that the function $g_1(z,t)$ satisfies the divergence condition (5.28). Then due to Theorem 5.1.1 the uniform global attractors $\mathcal{A}^{\varepsilon}$ of equations (5.48) with external forces $g^{\varepsilon}(x,t) = g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$ are uniformly (with respect to ε) bounded in $\mathbf{H}$:

$$\|\mathcal{A}^{\varepsilon}\|_{\mathbf{H}} \le C_2, \ \forall \varepsilon, \ 0 < \varepsilon \le 1. \tag{5.58}$$

We also consider the global attractor $\mathcal{A}^0$ of the "limiting" equation (5.55) with external force $g^0(t) = g_0(\cdot, t)$.

We have to generalize Theorem 5.2.1 in order to apply estimate (5.34) to the families of equations (5.52) and (5.56).

Consider an arbitrary initial data $u_\tau \in \mathbf{H}$. Let $\hat{u}^{\varepsilon}(\cdot,t) = U_{\hat{g}^{\varepsilon}}(t,\tau)u_\tau, \ t \ge \tau$, be the solution of equation (5.52) with symbol $\hat{g}^{\varepsilon} = \hat{g}_0 + \varepsilon^{-\rho}\hat{g}_1 \in \mathcal{H}(g^{\varepsilon})$. Let also $\tilde{u}^0(\cdot,t) = U_{\tilde{g}^0}(t,\tau)u_\tau, \ t \ge \tau$, be the solution of (5.56) with symbol $\tilde{g}^0 \in \mathcal{H}(g^0)$ and with the same initial data. (We note that the symbol $\tilde{g}^0$ can be different from the function $\hat{g}^0 = \hat{g}_0$ in the representation $\hat{g}^{\varepsilon} = \hat{g}_0 + \varepsilon^{-\rho}\hat{g}_1$). We now consider the difference

$$\hat{w}(x,t) = \hat{u}^{\varepsilon}(x,t) - \tilde{u}^0(x,t), \ t \ge \tau.$$

Proposition 5.4.1 *Let the original functions $g_0(x,t)$ and $g_1(z,t)$ in (5.1) be tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$ and $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{Z})$, respectively. Let also the function $g_1(z,t)$ satisfy the divergence condition (5.28). We set $g^{\varepsilon}(x,t) = g_0(x,t) + \varepsilon^{-\rho}g_1(x/\varepsilon,t)$ and $g^0(x,t) = g_0(x,t)$. Then, for every symbol $\hat{g}^{\varepsilon} = \hat{g}_0 + \varepsilon^{-\rho}\hat{g}_1 \in \mathcal{H}(g^{\varepsilon})$, there exist a symbol $\tilde{g}^0 \in \mathcal{H}(g^0)$ such that, for every initial data $u_\tau \in \mathbf{H}$, the difference*

$$\hat{w}(t) = \hat{u}^{\varepsilon}(t) - \tilde{u}^0(t) = U_{\hat{g}^{\varepsilon}}(t,\tau)u_\tau - U_{\tilde{g}^0}(t,\tau)u_\tau$$

of the solutions of the G.–L. equations (5.52) and (5.56) with symbols $\hat{g}^{\varepsilon}(x,t) = \hat{g}_0(x,t) + \varepsilon^{-\rho}\hat{g}_1(x/\varepsilon,t)$ and $\tilde{g}^0(x,t)$, respectively, and with the same initial data u_τ satisfies the following inequality:

$$\|\hat{w}(t)\| = \|\hat{u}^{\varepsilon}(\cdot,t) - \tilde{u}^0(\cdot,t)\| \le C\varepsilon^{(1-\rho)}e^{r(t-\tau)}, \ \forall t \ge \tau, \tag{5.59}$$

where the constant C and r are the same as in Theorem 5.2.1 and they are independent of ε and $0 \le \rho \le 1$.

The proof is similar to the proof of Proposition 4.4.1.

We now formulate the analog of Theorem 4.4.1 on the strong convergence of the global attractors $\mathcal{A}_{\varepsilon}$ of the G.–L. equation (5.30) to the global attractor $\mathcal{A}_0$ of the "limiting" equation (5.31) as $\varepsilon \to 0+$.

Theorem 5.4.1 *Let the functions $g_0(x,t)$ and $g_1(z,t)$ in (5.30) be tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$ and $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{Z})$, respectively. Let also the function $g_1(z,t), z \in \mathbb{R}^n$, satisfy the divergence condition (5.28). Then the global attractors $\mathcal{A}^{\varepsilon}$ of equation (5.30) converges* *to the global attractor $\mathcal{A}^0$ of the "limiting" equation (5.31) in the strong norm of* $\mathbf{H}$ *as* $\varepsilon \to 0+$, *that is*

$$\mathrm{dist}_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \to 0 \ (\varepsilon \to 0+). \tag{5.60}$$

The proof is similar to the proof of Theorem 4.4.1.

We now estimate the value $\mathrm{dist}_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0)$ explicitly under the assumption that the global attractor $\mathcal{A}^0$ is exponential using the results of Proposition 2.6.10.

We assume that

$$R \leq \lambda_1 - \varkappa, \ \forall t \in \mathbb{R}, \tag{5.61}$$

where the number $\varkappa > 0$ and λ_1 is the first eigenvalue of the operator $\{-\Delta, \ u|_{\partial\Omega} = 0\}$. Then the global attractor has a simple structure. We reformulate the corresponding results from Section 2.6.3.

Proposition 5.4.2 *Under the assumptions of Theorem 5.4.1, let R satisfy inequality (5.61). Then*

(i) *for every $\hat{g}^0 \in \mathcal{H}(g^0)$, there exists a unique bounded (in* $\mathbf{H}$*) complete solution $z_{\hat{g}^0}(t), t \in \mathbb{R}$, of equation (5.56) with symbol $\hat{g}^0$, i.e., the kernel $\mathcal{K}_{\hat{g}^0}$ consists of the unique element $z_{\hat{g}^0}$, and, in this case, the formula (5.57) for the global attractor $\mathcal{A}^0$ has the form*

$$\mathcal{A}^0 = \bigcup_{\hat{g}^0 \in \mathcal{H}(g^0)} \{z_{\hat{g}^0}(0)\}; \tag{5.62}$$

(ii) *the complete solution $z_{\hat{g}^0}(t), t \in \mathbb{R}$, attracts any solution $\hat{u}_{\hat{g}^0}(t) = U_{\hat{g}^0}(t,\tau)u_\tau, t \geq \tau$, with exponential rate:*

$$\|\hat{u}_{\hat{g}^0}(t) - z_{\hat{g}^0}(t)\| \leq \|\hat{u}_{\hat{g}^0}(\tau) - z_{\hat{g}^0}(\tau)\| e^{-\varkappa(t-\tau)}, \ \forall t \geq \tau, \ \tau \in R, \tag{5.63}$$

and, therefore, the global attractor $\mathcal{A}^0$ is exponential, i.e.,

$$\sup_{\hat{g}^0 \in \mathcal{H}(g^0)} \mathrm{dist}_{\mathbf{H}}(U_{\hat{g}^0}(t,\tau)B, \mathcal{A}) \leq Ce^{-\varkappa(t-\tau)}, \ C = C\left(\|B\|_{\mathbf{H}}\right), \tag{5.64}$$

where B is a bounded (in $\mathbf{H}$*) set of initial data and $\varkappa$ is taken from (5.61).*

Combining Propositions 5.4.1 and 5.4.2, we obtain

Theorem 5.4.2 *Let $0 < \rho < 1$. Then, under the assumptions of Theorem 5.4.1 and (5.61), the Hausdorff distance (in* $\mathbf{H}$*) from the global attractor $\mathcal{A}^\varepsilon$ to the "limiting" global attractor $\mathcal{A}^0$ satisfies the inequality*

$$\mathrm{dist}_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leq C(\rho)\varepsilon^{1-\rho}, \ \forall \varepsilon, \ 0 < \varepsilon \leq 1. \tag{5.65}$$

Proof. We fix ε. Let u^ε be an arbitrary element of $\mathcal{A}^\varepsilon$. By (5.54), there exists a bounded complete solution $\hat{u}^\varepsilon(t), t \in \mathbb{R}$, of equation (5.48) with some symbol $\hat{g}^\varepsilon = \hat{g}_0(x,t) + \varepsilon^{-\rho}\hat{g}_1(x/\varepsilon, t) \in \mathcal{H}(g^\varepsilon)$, such that

$$\hat{u}^\varepsilon(0) = u^\varepsilon. \tag{5.66}$$

We consider the point $\hat{u}^\varepsilon(-T)$ which clearly belongs to $\mathcal{A}^\varepsilon$. So, it follows from (5.58) that

$$\|\hat{u}^\varepsilon(-T)\| \le C_2, \tag{5.67}$$

where C_2 is independent of ε and T.

For the constructed external force $\hat{g}^\varepsilon$, we apply Proposition 5.4.1: there is a "limiting" external force $\tilde{g}^0 \in \mathcal{H}(g^0)$ such that, for any $\tau \in \mathbb{R}$ and for all $u_\tau \in \mathbf{H}$, the following inequality holds:

$$\|U_{\hat{g}^\varepsilon}(t+\tau,\tau)u_\tau - U_{\tilde{g}^0}(t+\tau,\tau)u_\tau\| \le C\varepsilon^{(1-\rho)},\ \forall t \ge 0, \tag{5.68}$$

where $r = 0$ since $R < \lambda_1$ (see (5.35)). Here C is independent of u_τ.

Consider the "limiting" equation (5.56) with the chosen "limiting" external force $\tilde{g}^0$. We set $\tau = -R$. Let $\tilde{u}^0(t), t \ge -T$, be the solution of this equation with initial data

$$\tilde{u}^0|_{t=-T} = \hat{u}^\varepsilon(-T). \tag{5.69}$$

It follows from Proposition 5.4.2 that there is a unique bounded complete solution $z^0(t), t \in \mathbb{R}$, of equation (5.56) with symbol $\tilde{g}^0$ such that

$$\|\tilde{u}^0(-T+t) - z^0(-T+t)\| \le \|\tilde{u}^0(-T) - z^0(-T)\|e^{-\varkappa t},\ \forall t \ge 0. \tag{5.70}$$

Recall that $z^0(t) \in \mathcal{A}^0$ for all $t \in \mathbb{R}$ and therefore

$$\|z^0(-T)\| \le \|\mathcal{A}^0\| \le C', \tag{5.71}$$

where C' is independent of z^0 and T. Using (5.69) and (5.67), we observe that

$$\|\tilde{u}^0(-T)\| = \|\hat{u}^\varepsilon(-T)\| \le C_2. \tag{5.72}$$

From (5.70), (5.71), and (5.72) we obtain

$$\|\tilde{u}^0(-T+t) - z^0(-T+t)\| \le C''e^{-\varkappa t},\ \forall t \ge 0, \tag{5.73}$$

where $C'' = C' + C_2$.

Setting $\tau = -T$ in (5.68), we have that

$$\begin{aligned} & \|\hat{u}^\varepsilon(-T+t) - \tilde{u}^0(-T+t)\| \\ = \ & \|U_{\hat{g}^\varepsilon}(t+\tau,\tau)u_\tau - U_{\tilde{g}^0}(t+\tau,\tau)u_\tau\| \le C\varepsilon^{(1-\rho)},\ \forall t \ge 0. \end{aligned} \tag{5.74}$$

Using (5.73) and (5.74), we find that

$$\begin{aligned} & \|\hat{u}^\varepsilon(-T+t) - z^0(-T+t)\| \\ \le \ & \|\hat{u}^\varepsilon(-T+t) - \tilde{u}^0(-T+t)\| + \|\tilde{u}^0(-T+t) - z^0(-T+t)\| \\ \le \ & C\varepsilon^{(1-\rho)} + C''e^{-\varkappa t}. \end{aligned} \tag{5.75}$$

We now choose T from the equation

$$\varepsilon^{(1-\rho)} = e^{-\varkappa T},\ \text{that is, } T = \frac{1-\rho}{\varkappa}\log\left(\frac{1}{\varepsilon}\right)$$

and we set $t = T$ in (5.75). Then we obtain

$$\|\hat{u}^\varepsilon(0) - z^0(0)\| \leq (C + C'')\, \varepsilon^{(1-\rho)}$$

and hence

$$\mathrm{dist}_{\mathbf{H}}(u^\varepsilon, \mathcal{A}^0) \leq \|u^\varepsilon - z^0(0)\| = \|\hat{u}^\varepsilon(0) - z^0(0)\| \leq C(\rho)\varepsilon^{(1-\rho)},$$

where $C(\rho) = (C + C'')$. Since u^ε is an arbitrary point of $\mathcal{A}^\varepsilon$ we find that

$$\mathrm{dist}_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leq C(\rho)\varepsilon^{(1-\rho)}.$$

The theorem is proved. ∎

Remark 5.4.1 If $R < \lambda_1$, then Proposition 5.4.2 holds also for equation (5.48) with symbols $g^\varepsilon(x,t) = g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$ and for the family of equation (5.52) with symbols $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$ (see Proposition 2.6.10 and Corollary 2.6.9). In particular, the global attractor $\mathcal{A}^\varepsilon$ of equation (5.48) is exponential as well as the global attractor $\mathcal{A}^0$ and the attraction rate is the same.

Remark 5.4.2 In fact, inequality (5.65) holds (with another constant C) for the symmetric distance $\mathrm{dist}^s_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) = \mathrm{dist}_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) + \mathrm{dist}^s_{\mathbf{H}}(\mathcal{A}^0, \mathcal{A}^\varepsilon)$:

$$\mathrm{dist}^s_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leq C_1(\rho)\varepsilon^{1-\rho}, \ \forall \varepsilon, \ 0 < \varepsilon \leq 1.$$

This result relies on the property of the exponential attraction of solutions to the global attractor $\mathcal{A}^\varepsilon$ mentioned in Remark 5.4.1.

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