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ON THE CONVERGENCE OF SOLUTIONS OF THE LERAY-α MODEL TO THE TRAJECTORY ATTRACTOR OF THE 3D NAVIER–STOKES SYSTEM

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Abstract. We study the relations between the global dynamics of the 3D Leray-α model and the 3D Navier–Stokes system. We prove that time shifts of bounded sets of solutions of the Leray-α model converges to the trajectory attractor of the 3D Navier–Stokes system as time tends to infinity and α approaches zero. In particular, we show that the trajectory attractor of the Leray-α model converges to the trajectory attractor of the 3D Navier–Stokes system when $\alpha \to 0+$.

1. **Introduction.** The 3D Navier–Stokes (N.–S.) system for viscous incompressible fluids has the form

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = g(x), \\ \nabla\cdot u = 0,\ x = (x_1, x_2, x_3) \in \mathbb{R}^3, \end{cases} \tag{1}$$

where $u = \big(u^1(x,t), u^2(x,t), u^3(x,t)\big)$ is the unknown velocity field of a fluid pattern at point x and at time t, $p = p(x,t)$ is the unknown pressure, and $g(x) = \big(g^1(x), g^2(x), g^3(x)\big)$ is a given external force. The positive parameter ν is the kinematic viscosity of the fluid.

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In [23], Leray considered the following regularization of system (1) in order to prove the existence of a solution to the Navier–Stokes system in $\mathbb{R}^3$:

$$\begin{cases} \partial_t v_\alpha - \nu\Delta v_\alpha + (u_\alpha \cdot \nabla)v_\alpha + \nabla p_\alpha = g(x), \\ \nabla \cdot v_\alpha = 0,\ x \in \mathbb{R}^3, \end{cases} \tag{2}$$

where $u_\alpha = \Phi_\alpha * v_\alpha$ and Φ_α is a smoothing kernel such that the function u_α approaches v_α, in some sense, as $\alpha \to 0+$. Therefore in some sense, the system (2) converges to the 3D N.–S. system (1).

In this paper, the systems (1) and (2) are supplemented with periodic boundary conditions, i.e., it is assumed that $x = (x_1, x_2, x_3) \in \mathbb{T}^3 := [\mathbb{R} \bmod 2\pi L]^3$ and all the functions in (1) and (2) are periodic in each $x_i,\ i = 1, 2, 3$, with period $2\pi L$.

In [14], a special smoothing kernel was considered, namely, the Green function associated with the Helmholtz operator $I - \alpha^2\Delta$, that is,

$$\begin{aligned} &u_\alpha(x) - \alpha^2\Delta u_\alpha(x) = v_\alpha(x),\ x \in \mathbb{T}^3, \\ &u_\alpha = \Phi_\alpha * v_\alpha = (I - \alpha^2\Delta)^{-1}v_\alpha. \end{aligned} \tag{3}$$

This kernel works as a kind of spatial filter with width α. The parameter α also reflects a sub-grid length scale in the model.

The system (2) and (3) was considered in [14] (see also [18, 19]) as a large eddy simulation sub-grid scale model of 3D turbulence and was called the *Leray-α model*.

This model was inspired by the Navier–Stokes-α model (also known as the viscous Camassa–Holm system or Lagrangian averaged Navier–Stokes-α equations) of turbulence (see [3, 4, 5, 16, 17]). It has been demonstrated analytically and computationally in many works that the Navier–Stokes-α model is a powerful tool in the study of turbulence (see, e.g., [3, 4, 5, 6, 16, 17]). In particular, it was found that the explicit steady analytical solutions of the Navier–Stokes-α equations compare successfully with empirical and numerical data for a wide range of Reynolds numbers in turbulent channel and pipe flows (see [3, 4, 5]). At the same time, the use of the Leray-α system as a closure model for the Reynolds averaged equations in channels and pipes leads to exactly the same reduced system of equations as the Navier–Stokes-α model under the corresponding symmetries (see, e.g., [3, 4, 5]). This comparison means that Leray-α model and the Navier–Stokes-α equations are equally useful as closure models for the mean effects of sub-grid excitations. Along the same lines it is worth mentioning that other α-models such as the Clark-α model [2] and the modified-Leray-α model [21] yield the same reduced system of equations in turbulent channels and pipes and therefore they enjoy the same success stories as the Navier–Stokes-α and Leray-α systems as sub-grid scale models of turbulence.

In [14], the Cauchy problem for the Leray-α model was studied, the global attractor for this system was constructed, an upper bound for the dimension of this global attractor was established in terms of the relevant physical parameters, and some other turbulence related features and characteristics (such as energy spectra and boundary layers) were discussed. We stress that the proved upper estimates for the dimension of the global attractor (the number of the degree of freedom) of the Leray-α model demonstrate the great potential of this model to become a good sub-grid scale large eddy simulation model of the turbulence.

The theory of trajectory attractors for evolution equations of mathematical physics was developed in [8, 11, 13] with an emphasis on equations for which the uniqueness of a solution of the corresponding Cauchy problem is not known, e.g. for the 3D Navier–Stokes system (see also [27, 28]). For such equations, the traditional theory of global attractors is not directly applicable. The trajectory attractors were constructed for a number of important equations and systems of mathematical physics, e.g. for the 3D N.–S. system, for the complex Ginzburg–Landau equation, various reaction-diffusion systems, the dissipative hyperbolic equation with arbitrary polynomial growth of the nonlinear term, and for other equations (see, e.g., [7, 9, 10, 12, 13, 31, 32]).

In the present paper, we study the connection between the long-time dynamics of the Leray-α model and the 3D Navier–Stokes equations as $\alpha \to 0+$. Our purpose is to prove the following main result: bounded (in the energy norm) families B_α of solutions $\{v_\alpha(x,t),\, t \geq 0\},\ 0 < \alpha \leq 1$, of the Leray-$\alpha$ model (2) and (3) converge (in the sense specified below) as $\alpha \to 0+$ to the trajectory attractor $\mathfrak{A}$ of the 3D N.–S. system with periodic boundary conditions. In particular, we show that the trajectory attractor of the Leray-α model converges to the trajectory attractor of the 3D Navier–Stokes system when $\alpha \to 0+$.

In Section 1, we give the definition and the main properties of the trajectory attractor of the 3D N.–S. system. We also define the kernel $\mathcal{K}$ of this system: the set $\mathcal{K}$ is the family of all bounded (in the energy norm) complete trajectories $\{u(t), t \in \mathbb{R}\}$ of system (1) that satisfy the energy inequality. We show that the restriction Π_+ of the kernel $\mathcal{K}$ onto the semi-axis $\mathbb{R}_+$ coincides with trajectory attractor of the 3D N.–S. system: $\mathfrak{A} = \Pi_+\mathcal{K}$.

In Section 2, we present the functional setting of the Leray-α model. The Cauchy problem for (2) and (3) has a unique weak solution and moreover this solution is regular. This is a classical result. Following [14], we establish the existence of the global attractor for the Leray-α model.

In Sections 3 and 4, we prove the main theorem on the convergence of translations $T(h)B_\alpha$ (by definition, $T(h)w(t) = w(t+h)$) of bounded (in the energy norm) families of solutions $\{v_\alpha(x,t),\, t \geq 0\}$ of the Leray-α model to the trajectory attractor $\mathfrak{A}$ of the 3D N.–S. system (1) as $h \to +\infty$ and $\alpha \to 0+$. We deduce from this assertion that the trajectory attractors $\mathfrak{A}_\alpha$ of the Leray-α model converge to the trajectory attractor $\mathfrak{A}$ of the 3D N.–S. system as $\alpha \to 0+$. The analogous statement holds for the kernels $\mathcal{K}_\alpha$ and $\mathcal{K}$ of the corresponding equations. Part of the results presented here has been announced in [33].

In a forthcoming paper, we shall prove similar and new results concerning the relation between the non-autonomous NS-α model (also known as viscous Camassa–Holm equations or Lagrangian averaged Navier–Stokes-α equations, see, e.g., [3, 17] and references therein) and the corresponding 3D Navier–Stokes system.

2. **Trajectory attractor and kernel of the 3D N.–S. system.** We consider the following autonomous 3D N.–S. system with periodic boundary conditions:

$$\partial_t u - \nu\Delta u + \sum_{i=1}^{3} u^i \partial_i u + \nabla p = g_0(x),\ t \geq 0, \tag{4}$$

$$\sum_{i=1}^{3} \partial_i u^i = 0,\ x \in \mathbb{T}^3 := [\mathbb{R} \bmod 2\pi L]^3\,. \tag{5}$$

Here $u = u(x,t) = \big(u^1(x,t), u^2(x,t), u^3(x,t)\big)$ is the unknown vector field in $\mathbb{T}^3$ describing the motion of the fluid, the scalar function $p(x,t)$ is the unknown pressure, and $g_0(x) = \big(g_0^1(x), g_0^2(x), g_0^3(x)\big)$ is a given field of external forces with zero spatial mean, i.e., $\int_{\mathbb{T}^3} g_0(x)dx = 0$. We assume that the function $u(x,t)$ is periodic with respect to $x = (x_1, x_2, x_3) \in \mathbb{T}^3$, and has zero spatial mean, i.e., $\int_{\mathbb{T}^3} u(x,t)dx = 0$.

We denote by H and V the closure of the space $\mathcal{V} = \{v(x) = \left(v^1(x), v^2(x), v^3(x)\right),$ $x \in \mathbb{T}^3 \mid v(x)$ *is a trigonometrical vector polynomial with period* $2\pi L$ *in each* $x_i, i = 1, 2, 3,$ *such that* $\nabla \cdot v = 0$ *and* $\int_{\mathbb{T}^3} v(x)dx = 0\}$ in the norms $|\cdot|$ and $\|\cdot\|$ of the spaces $L_2(\mathbb{T}^3)^3$ and $H^1(\mathbb{T}^3)^3$, respectively. Recall that the orthogonal complement $H^\perp$ of the space H in $L_2(\mathbb{T}^3)^3$ is $\{\nabla p(x) \mid p(\cdot) \in H^1(\mathbb{T}^3)\}$ (see, e.g., [15, 29]). Let $P : L_2(\mathbb{T}^3)^3 \to H$ be the Helmholtz–Leray orthogonal projector onto H and let $A = -P\Delta$ be the Stokes operator with domain $\mathcal{D}(A) = H^2(\mathbb{T}^3)^3 \cap V$. We observe that, in the periodic case, $A = -\Delta$. The operator A is self-adjoint, positive, and has a compact resolvent. Therefore, the space H has an orthonormal basis $\{w_j\}_{j=1}^\infty$ of eigenfunctions of A, that is, $Aw_j = \lambda_j w_j$, where

$$0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_j \to +\infty.$$

We denote by

$$((u, v)) := (A^{1/2}u, A^{1/2}v) = (\nabla u, \nabla v),\ \|u\| := |A^{1/2}u|,\ u, v \in V,$$

the scalar product and the norm in V, respectively. The Poincaré inequality reads

$$|u|^2 \leq \lambda_1^{-1}\|u\|^2,\ \forall u \in V. \tag{6}$$

Let V' be the dual space of V. For any $v \in V'$, we denote by $\langle v, u\rangle$ the value of the functional v from V' on a vector $u \in V$. The operator A is a isomorphism from V to V'. In particular, $((w, u)) = \langle Aw, u\rangle$ for all $w, u \in V$.

We apply the operator P to both sides of equation (4) and obtain an equivalent system:

$$\partial_t u + \nu Au + B(u, u) = g(x),\ t \geq 0. \tag{7}$$

Here, we denote $B(u, v) = P[(u \cdot \nabla)v] = P\sum_{i=1}^3 u^i\partial_i v$ and $g = Pg_0$ (see, e.g., [15, 22, 24, 29]). The operator $B(u, v)$ maps $V \times V$ to V' and satisfies the following inequality:

$$|\langle B(u, v), w\rangle| \leq c|u|^{1/4}\|u\|^{3/4}|v|^{1/4}\|v\|^{3/4}\|w\|,\ \forall u, v, w \in V, \tag{8}$$

where c is an absolute constant (see, e.g., [15, 22, 29]). In particular,

$$\|B(u, u)\|_{V'} \leq c|u|^{1/2}\|u\|^{3/2},\ \forall u \in V. \tag{9}$$

From the standard formula

$$B(u, v) = P\sum_{i=1}^3 \partial_i\left(u^i v\right), \tag{10}$$

(see [24, 29, 15]) it follows that

$$\langle B(u, v), w\rangle = -\langle B(u, w), v\rangle,\ \forall u, v, w \in V, \tag{11}$$

$$\langle B(u, v), v\rangle = 0,\ \forall u, v \in V. \tag{12}$$

Let a function $u(\cdot) \in L_2(0, M; V) \cap L_\infty(0, M; H)$ be given. Therefore, $\nu Au \in L_2(0, M; V')$ and due to (9) we have

$$B(u(\cdot), u(\cdot)) \in L_{4/3}(0, M; V'). \tag{13}$$

Consider the space of distributions $\mathcal{D}'(0, M; V')$. The function $u(\cdot)$ has the time derivative $\partial_t u(\cdot)$ in the space $\mathcal{D}'(0, M; V')$ (see [24]). A function $u(\cdot) \in L_2(0, M; V) \cap L_\infty(0, M; H)$ is called a *weak solution* of equation (7) if it satisfies this equation in the space $\mathcal{D}'(0, M; V')$. Then, clearly, from (13), we have $\partial_t u(\cdot) \in L_{4/3}(0, M; V')$ for any weak solution $u(\cdot)$ of (7) and hence $u(\cdot) \in C([0, M]; V')$. Recall that $u(\cdot) \in$

$L_\infty(0,M;H)$. Then, by the classical lemma from [25] (see also [29]), the function $u(\cdot) \in C_w([0,M];H)$ and the initial data

$$u|_{t=0} = u_0(x) \in H \tag{14}$$

is meaningful for equation (7) in the class of solutions belonging to $L_2(0,M;V) \cap L_\infty(0,M;H)$.

We now formulate the classical theorem on the existence of a weak solution of the Cauchy problem for the 3D N.–S. system in the form we need in the sequel (see [15, 22, 24, 29]).

Theorem 1. *Let $g \in V'$ and $u_0 \in H$. Then for every $M > 0$, there exists a weak solution $u(t)$ of equation (7) from the space $L_2(0,M;V) \cap L_\infty(0,M;H)$ such that $u(0) = u_0$ and $u(t)$ satisfies the energy inequality*

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 \le \langle g, u(t)\rangle\,,\ t \in [0,M]. \tag{15}$$

Inequality (15) means the following: for any function $\psi(\cdot) \in C_0^\infty(]0,M[),\ \psi(t) \ge 0$,

$$-\frac{1}{2}\int_0^M |u(t)|^2\psi'(t)dt + \nu\int_0^M \|u(t)\|^2\psi(t)dt \le \int_0^M \langle g, u(t)\rangle\,\psi(t)dt. \tag{16}$$

The proof of Theorem 1 uses the Galerkin approximation method. For every $m \in \mathbb{N}$, we construct the Galerkin approximation $u_m(x,t) \in C^1([0,M];H^2 \cap V)$ of order m, that is a solution of the corresponding system of ordinary differential equations, and prove the existence of a subsequence $\{m_j\} \subset \{m\}$ such that $u_{m_j}(x,t)$ converges in a weak sense to a weak solution $u(x,t)$ of problem (7) and (14). The Galerkin approximation $u_m(x,t)$ satisfies the energy equality

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu\|u_m(t)\|^2 = \langle g, u_m(t)\rangle\,,\ t \in [0,M]. \tag{17}$$

Passing to a limit in (17) in a weak sense as $m_j \to \infty$, we obtain (15) in the form (16) (see [13, 15, 24]).

Remark 1. For the 3D Navier–Stokes system the question of the uniqueness of a weak solution of problem (7) and (14) remains open. It is also unknown, whether every weak solution satisfies the energy inequality (15) (and what is more, the energy equality). Nevertheless, it is known that every weak solution resulting from the Galerkin approximation method satisfies the energy inequality (15).

Remark 2. Inequality (16) is equivalent to the following assertion: there exists a set $Q \subset [0,M]$ having zero Lebesgue measure such that

$$\frac{1}{2}\left\{|u(t)|^2 - |u(\tau)|^2\right\} + \nu\int_\tau^t \|u(t)\|^2dt \le \int_\tau^t \langle g, u(t)\rangle\,dt \tag{18}$$

for all $\tau \in [0,M]\backslash Q$ and for all $t \ge \tau$ (see, e.g., [13, 15]).

We now construct the trajectory attractor for the N.–S. equation (7). The detailed theory of trajectory attractors can be found in [11, 13]. Here, we only give the key elements of the trajectory attractor construction.

At first, we define the trajectory space $\mathcal{K}^+$ of equation (7). We consider weak solutions $u(t), t \ge 0$, of this equation in the space $L_2^{\mathrm{loc}}(\mathbb{R}_+;V) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H)$, i.e., functions $u(t), t \in [0,M]$, belong to $L_2(0,M;V) \cap L_\infty(0,M;H)$ and satisfy (7) in the space $\mathcal{D}'(0,M;V')$ for any $M > 0$.

Definition 1. The trajectory space $\mathcal{K}^+$ is the set of all weak solutions $u(\cdot)$ of equation (7) in the space $L_2^{\mathrm{loc}}(\mathbb{R}_+;V)\cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H)$ that satisfy the energy inequality (15) for all $t\geq 0$, that is,

$$-\frac{1}{2}\int_0^\infty |u(t)|^2\psi'(t)dt+\nu\int_0^\infty \|u(t)\|^2\psi(t)dt\leq\int_0^\infty \langle g,u(t)\rangle\,\psi(t)dt \tag{19}$$

for all $\psi\in C_0^\infty(\mathbb{R}_+),\psi\geq 0$.

It follows from Theorem 1 that, for any $u_0\in H$, there is a trajectory $u\in\mathcal{K}^+$ such that $u(0)=u_0$. Hence, the trajectory space $\mathcal{K}^+$ is non-empty and sufficiently large.

We need the Banach space

$$\mathcal{F}_+^{\mathrm{b}}=\{v(\cdot)\in L_2^{\mathrm{b}}(\mathbb{R}_+;V)\cap L_\infty(\mathbb{R}_+;H),\ \partial_t v(\cdot)\in L_{4/3}^{\mathrm{b}}(\mathbb{R}_+;V')\}$$

with norm

$$\|v\|_{\mathcal{F}_+^{\mathrm{b}}}=\|v\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;V)}+\|v\|_{L_\infty(\mathbb{R}_+;H)}+\|\partial_t v\|_{L_{4/3}^{\mathrm{b}}(\mathbb{R}_+;V')}, \tag{20}$$

where $\|v\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;V)}=\sup_{t\geq 0}\int_t^{t+1}\|v(s)\|^2ds,\ \|v\|_{L_\infty(\mathbb{R}_+;H)}=\operatorname{ess\,sup}_{t\geq 0}|v(t)|$, and $\|\partial_t v\|^{4/3}_{L_{4/3}^{\mathrm{b}}(\mathbb{R}_+;V')}=\sup_{t\geq 0}\int_t^{t+1}\|v(s)\|^{4/3}_{V'}ds$.

We denote by $\{T(h)\}:=\{T(h),h\geq 0\}$ the translation semigroup acting on a function $\{v(t),t\geq 0\}$ by the formula

$$T(h)v(t)=v(t+h),\ t\geq 0.$$

For example, the semigroup $\{T(h)\}$ acts on $\mathcal{F}_+^{\mathrm{b}}$. We consider $\{T(h)\}$ on the trajectory space $\mathcal{K}^+$ of equation (7). Clearly, if $u(\cdot)\in\mathcal{K}^+$, then $u_h(\cdot)=T(h)u(\cdot)=u(\cdot+h)\in\mathcal{K}^+$ for all $h\geq 0$. Therefore,

$$T(h)\mathcal{K}^+\subseteq\mathcal{K}^+,\ \forall h\geq 0. \tag{21}$$

Our aim is to construct the global attractor of the translation semigroup $\{T(h)\}$ in $\mathcal{K}^+$. We call this attractor the *trajectory attractor* because the semigroup $\{T(h)\}$ acts in the trajectory space $\mathcal{K}^+$. In [13], the following proposition is proved.

Proposition 1. *Let $g\in V'$. Then*

1. *The trajectory space $\mathcal{K}^+\subseteq\mathcal{F}_+^{\mathrm{b}}$;*
2. *for any function $u(\cdot)\in\mathcal{K}^+$,*

$$\|T(h)u(\cdot)\|_{\mathcal{F}_+^{\mathrm{b}}}\leq C_0\|u(\cdot)\|^2_{C_\infty(0,1;H)}e^{-\beta h}+R_0^2,\ \forall h\geq 1, \tag{22}$$

where $\beta=\nu\lambda_1$, the constant C_0 depends on ν, λ_1 and R_0 depends on ν, λ_1, $\|g\|_{V'}$.

We introduce a topology in the space $\mathcal{K}^+$. Consider the space

$$\mathcal{F}_+^{\mathrm{loc}}=\{v(\cdot)\in L_2^{\mathrm{loc}}(\mathbb{R}_+;V)\cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H),\ \partial_t v(\cdot)\in L_{4/3}^{\mathrm{loc}}(\mathbb{R}_+;V')\}.$$

We define on $\mathcal{F}_+^{\mathrm{loc}}$ the following sequential topology which we denote Θ_+^{loc}. By definition, a sequence of functions $\{v_n\}\subseteq\mathcal{F}_+^{\mathrm{loc}}$ converges to a function $v\in\mathcal{F}_+^{\mathrm{loc}}$ in the topology Θ_+^{loc} as $n\to+\infty$ if, for any $M>0$,

$$\begin{aligned} v_n(\cdot)\ &\rightharpoonup\ v(\cdot)\ (n\to\infty)\text{ weakly in }L_2(0,M;V),\\ v_n(\cdot)\ &\rightharpoonup\ v(\cdot)\ (n\to\infty)\text{ weakly-}*\text{ in }L_\infty(0,M;H),\end{aligned}$$

and

$$\partial_t v_n(\cdot)\rightharpoonup\partial_t v(\cdot)\ (n\to\infty)\text{ weakly in }L_{4/3}(0,M;V').$$

It is easy to describe the topology $\Theta_+^{\rm loc}$ in terms of open neighbourhoods and to prove that $\Theta_+^{\rm loc}$ is a Hausdorff topology with a countable base of its topology (however, $\Theta_+^{\rm loc}$ is not metrizable) (see [11, 13]). We note that $\mathcal{F}_+^{\rm b} \subseteq \Theta_+^{\rm loc}$. Besides, any ball $B_R = \{v \in \mathcal{F}_+^{\rm b} \mid \|v\|_{\mathcal{F}_+^{\rm b}} \le R\}$ is compact in $\Theta_+^{\rm loc}$. Hence, the set B_R with topology induced by $\Theta_+^{\rm loc}$ is metrizable and the corresponding metric space is complete. This property makes it possible to construct the trajectory attractor (in the topology $\Theta_+^{\rm loc}$) of the semigroup $\{T(h)\}$ acting on $\mathcal{K}^+$. It follows easily from the definition of the topology $\Theta_+^{\rm loc}$ that the translation semigroup $\{T(h)\}$ is continuous in $\Theta_+^{\rm loc}$. The following assertion is important for us (see the proof in [13]) .

Proposition 2. *The trajectory space $\mathcal{K}^+$ is closed in $\Theta_+^{\rm loc}$.*

In a standard way, we define an attracting set for $\mathcal{K}^+$ in the topology $\Theta_+^{\rm loc}$.

Definition 2. A set $P \subseteq \mathcal{F}_+^{\rm b}$ is called attracting for the space $\mathcal{K}^+$ in the topology $\Theta_+^{\rm loc}$ if, for any bounded (in the norm of $\mathcal{F}_+^{\rm b}$) set $B \subset \mathcal{K}^+$, the set P attracts $T(h)B$ in the topology $\Theta_+^{\rm loc}$ as $h \to +\infty$, that is, for any neighbourhood $\mathcal{O}(P)$ (in $\Theta_+^{\rm loc}$) of the set P, there is a number $h_1 = h_1(B, \mathcal{O})$ such that $T(h)B \subseteq \mathcal{O}(P)$ for all $h \ge h_1$.

We now define the trajectory attractor.

Definition 3. A set $\mathfrak{A} \subset \mathcal{K}^+$ is called the trajectory attractor of the semigroup $\{T(h)\}$ in the topology $\Theta_+^{\rm loc}$ if

1. $\mathfrak{A}$ is bounded in $\mathcal{F}_+^{\rm b}$ and compact in $\Theta_+^{\rm loc}$;
2. $\mathfrak{A}$ is strictly invariant with respect to $\{T(h)\} : T(h)\mathfrak{A} = \mathfrak{A}$, $\forall h \ge 0$;
3. $\mathfrak{A}$ is an attracting set in the topology $\Theta_+^{\rm loc}$ for $\{T(h)\}$ on $\mathcal{K}^+$.

The set $\mathfrak{A}$ is also called the $(\mathcal{F}_+^{\rm b}, \Theta_+^{\rm loc})$-*attractor* of the semigroup $\{T(h)\}|_{\mathcal{K}^+}$ (see also [1]).

It follows from the main inequality (22) that the ball B_{2R_0} in $\mathcal{F}_+^{\rm b}$ is an attracting (and even absorbing) set of the semigroup $\{T(h)\}$ on $\mathcal{K}^+$. The ball B_{2R_0} is clearly compact in $\Theta_+^{\rm loc}$. Therefore, the continuous semigroup $\{T(h)\}$ has a compact attracting set. Consequently, the translation semigroup $\{T(h)\}$ has the trajectory attractor $\mathfrak{A} \subset \mathcal{K}^+ \cap B_{2R_0}$ and moreover

$$\mathfrak{A} = \bigcap_{s>0} \left[\bigcup_{h \ge s} T(h)(\mathcal{K}^+ \cap B_{2R_0}) \right]_{\Theta_+^{\rm loc}},$$

where $[\cdot]_{\Theta_+^{\rm loc}}$ denotes the closure in $\Theta_+^{\rm loc}$. The detailed consideration of global attractors for semigroups in topological spaces can be found in [13] (see also [26]).

To describe the general structure of the trajectory attractor $\mathfrak{A}$ we need the notion of the kernel of equation (7). The *kernel* $\mathcal{K}$ is the set of all weak solutions $u(t), t \in \mathbb{R}$, bounded in the space

$$\mathcal{F}^{\rm b} = \{v(\cdot) \in L_2^{\rm b}(\mathbb{R}; V) \cap L_\infty(\mathbb{R}; H),\ \partial_t v(\cdot) \in L_{4/3}^{\rm b}(\mathbb{R}; V')\}$$

that satisfies an inequality in a similar way to (19) for all $\psi \in C_0^\infty(\mathbb{R}), \psi \ge 0$, and the integrals are taken over the entire time axis $\mathbb{R}$:

$$-\frac{1}{2}\int_{-\infty}^{\infty} |u(t)|^2 \psi'(t)dt + \nu \int_{-\infty}^{\infty} \|u(t)\|^2 \psi(t)dt \le \int_{-\infty}^{\infty} \langle g, u(t)\rangle\, \psi(t)dt. \qquad (23)$$

(The norm in $\mathcal{F}^{\rm b}$ is defined in a similar way to the norm in $\mathcal{F}_+^{\rm b}$ (see (20)) replacing $\mathbb{R}_+$ by $\mathbb{R}$).

We denote by Π_+ the restriction operator onto $\mathbb{R}_+$. It is proved in [13] that the trajectory attractor $\mathfrak{A}$ of the 3D Navier–Stokes system coincides with the restriction of the kernel $\mathcal{K}$ of equation (7) onto $\mathbb{R}_+$:

$$\mathfrak{A} = \Pi_+\mathcal{K}. \tag{24}$$

It is clear that the set $\mathcal{K}$ is bounded in $\mathcal{F}^{\mathrm{b}}$ and compact in Θ^{loc}. The topology Θ^{loc} is defined similar to Θ^{loc}_+ where the intervals $(0, M)$ are replaced by $(-M, M)$.

Notice that the following embeddings are continuous:

$$\Theta^{\mathrm{loc}}_+ \subset L^{\mathrm{loc}}_2(\mathbb{R}_+; H^{1-\delta}), \tag{25}$$

$$\Theta^{\mathrm{loc}}_+ \subset C^{\mathrm{loc}}(\mathbb{R}_+; H^{-\delta}), \text{ for } 0 < \delta \le 1, \tag{26}$$

(see [24, 13]). Hence, the trajectory attractor $\mathfrak{A} = \Pi_+\mathcal{K}$ satisfies the following properties: for any bounded (in $\mathcal{F}^{\mathrm{b}}_+$) set $B \subset \mathcal{K}^+$,

$$\begin{aligned} \mathrm{dist}_{L_2(0,M;H^{1-\delta})}(T(h)B, \mathfrak{A}) &\to 0 \ (h \to +\infty), \\ \mathrm{dist}_{C([0,M];H^{-\delta})}(T(h)B, \mathfrak{A}) &\to 0 \ (h \to +\infty), \end{aligned}$$

where M is an arbitrary positive number.

3. **Leray-α model of viscous incompressible fluid and its global attractor.** We consider the following system subject to periodic boundary conditions:

$$\partial_t v - \nu\Delta v + (u \cdot \nabla)v + \nabla p = g_0(x), \ \nabla \cdot v = 0, \tag{27}$$

$$v = u - \alpha^2\Delta u, \ \ x = (x_1, x_2, x_3) \in \mathbb{T}^3. \tag{28}$$

This system is an approximation of the 3D Navier–Stokes equations considered in the previous section. In system (27) and (28), the unknown functions are the vector fields $v = v(x, t) = (v^1, v^2, v^3)$ or $u = u(x, t) = (u^1, u^2, u^3)$, and the scalar function $p = p(x, t)$. We assume that the functions v, u, p, and g_0 are all periodic with respect to each variable $x_i, i = 1, 2, 3$, with period $2\pi L$ and they have zero spatial mean. In equation (28), α is a fixed positive parameter which is called the sub-grid (filter) length scale of the model (see [14]). For $\alpha = 0$, the function $v = u$ and we obtain exactly the 3D Navier–Stokes system (4) and (5). System (27) and (28) is called the Leray-α model.

We now rewrite system (27) and (28) in an equivalent form using the standard projector P in H and excluding the pressure as in the previous section, where all the necessary notations were defined. We obtain the system

$$\partial_t v + \nu Av + B(u, v) = g(x), \tag{29}$$

$$v = u + \alpha^2 Au. \tag{30}$$

Here as in equation (7), $A = -P\Delta = -\Delta$ denotes the Stokes operator and the bilinear operator $B(u, v) = P(u \cdot \nabla)v$ satisfies properties (8) – (12).

It is obvious that, for every $v \in H$, equation (30) has a unique solution $u \in H^2 \cap V$ such that

$$\|u\|_{H^2} := |Au| \le \frac{1}{\alpha^2}|v|, \ \forall v \in H. \tag{31}$$

Here, we denote $H^2 = H^2(\mathbb{T}^3)^3$. It follows from the embedding theorem in $\mathbb{R}^3$ that $H^2(\mathbb{T}^3) \subset L_\infty(\mathbb{T}^3)$. In particular, we have the energy inequality

$$\|u\|_{L_\infty(\mathbb{T}^3)^3} \le C(\alpha)|u + \alpha^2 Au| = C(\alpha)|v|, \ \forall u \in H^2 \cap V, \tag{32}$$

where $v = u + \alpha^2 Au$. In [14], it was proved that

$$C(\alpha) \leq \frac{5}{4} + \frac{8}{5}\left(\frac{L}{c_1 \alpha}\right)^{3/2}, \tag{33}$$

where c_1 is an absolute positive constant. Notice that $C(\alpha) \to +\infty$ as $\alpha \to 0+$. We obtain from inequality (32) that

$$|B(u,v)| \leq |(u \cdot \nabla)v| \leq c\|u\|_{L_\infty(\mathbb{T}^3)^3}\|v\| \leq C_1(\alpha)|v|\|v\|, \ \forall v \in V, \tag{34}$$

where $v = u + \alpha^2 Au$.

Consider an arbitrary function $v(\cdot) \in L_2(0,M;V) \cap L_\infty(0,M;H)$ and the corresponding function $u(\cdot) \in L_\infty(0,M;H^2)$. Then from (34), we conclude

$$B(u(\cdot), v(\cdot)) \in L_2(0,M;H). \tag{35}$$

(Compare with (13) for the 3D Navier–Stokes system, where $\alpha = 0$).

We study weak solutions $v(x,t)$ of system (29) and (30) belonging to the space $L_2(0,M;V) \cap L_\infty(0,M;H)$, $M > 0$. Then $Av(\cdot) \in L_2(0,M;V')$ and, by (35),

$$\partial_t v(\cdot) \in L_2(0,M;V'). \tag{36}$$

We supplement system (29) and (30) with initial data

$$v|_{t=0} = v_0(x) \in H. \tag{37}$$

We now formulate the theorem on the existence and uniqueness of a weak solution of the Cauchy problem (29), (30), and (37) for the Leray-α model ($\alpha > 0$).

Theorem 2. *Let $\alpha > 0$, $g \in V'$, and $v_0 \in H$. Then problem (29), (30), and (37) has a unique solution $v(\cdot) \in L_2(0,M;V) \cap L_\infty(0,M;H)$ such that $\partial_t v(\cdot) \in L_2(0,M;V')$ and v satisfies the energy equality*

$$\frac{1}{2}\frac{d}{dt}|v(t)|^2 + \nu\|v(t)\|^2 = (g, v(t)), \tag{38}$$

where the function $|v(t)|^2, t \in [0,M]$, is absolutely continuous, its time derivative has the usual sense, and (38) holds for almost every $t \in [0,M]$. Moreover, $v \in C([0,M];H)$.

Proof. The standard Galerkin approximation method is used (see [14]). All the reasonings are quite analogous to the proof of the existence and uniqueness theorem for the 2D Navier–Stokes system (see [1, 15, 29]). Energy equality (38) follows from (36). We take the scalar product of equation (29) with $v(\cdot) \in L_2(0,M;V)$ and use Lemma 1.2 from [29, Ch.3] which implies that

$$2\langle \partial_t v, v\rangle = \frac{d}{dt}|v(t)|^2 \text{ for a.e. } t \in [0,M].$$

Here, we have used also the identity $\langle B(u,v), v\rangle = 0$ (see (11)). Hence, the function $|v(t)|^2$ is absolutely continuous. Recall that the function $v(t), t \in [0,M]$, is weakly continuous in H. Therefore it is strongly continuous in H as well. □

The energy equality implies the main *a priori* estimates of problem (29), (30), and (37).

Proposition 3. *Any solution $v(t)$ of problem (29), (30), and (37) satisfies the following inequalities:*

$$|u(t)|^2 \quad \leq \quad |v(t)|^2 \leq |v(0)|^2 e^{-\nu\lambda_1 t} + \frac{\|g\|_{V'}^2}{\nu^2\lambda_1}, \tag{39}$$

$$\nu\int_t^{t+1} \|u(s)\|^2 ds \quad \leq \quad \nu\int_t^{t+1} \|v(s)\|^2 ds \leq |v(0)|^2 e^{-\nu\lambda_1 t} + \frac{\|g\|_{V'}^2}{\nu^2\lambda_1} + \frac{\|g\|_{V'}^2}{\nu}, \tag{40}$$

where $u(t)$ is the solution of equation (30).

Proof. It follows from (30) that

$$|u(t)|^2 \leq |v(t)|^2, \ \|u(t)\|^2 \leq \|v(t)\|^2, \ \forall t \geq 0. \tag{41}$$

From (38), we have

$$\frac{1}{2}\frac{d}{dt}|v(t)|^2 + \nu\|v(t)\|^2 = (g, v(t)) \leq \frac{\nu}{2}\|v(t)\|^2 + \frac{\|g\|_{V'}^2}{2\nu}.$$

Using the Poincaré inequality (29), we obtain

$$\frac{d}{dt}|v(t)|^2 + \nu\lambda_1|v(t)|^2 \leq \frac{d}{dt}|v(t)|^2 + \nu\|v(t)\|^2 \leq \frac{\|g\|_{V'}^2}{\nu} \tag{42}$$

and, therefore,

$$|v(t)|^2 \leq |v(0)|^2 e^{-\nu\lambda_1 t} + \frac{\|g\|_{V'}^2}{\nu^2\lambda_1}. \tag{43}$$

We have proved (39). We now integrate inequality (42) over $[t, t+1]$ and reach

$$|v(t+1)|^2 + \nu\int_t^{t+1} \|v(s)\|^2 ds \leq |v(t)|^2 + \frac{\|g\|_{V'}^2}{\nu} \leq |v(0)|^2 e^{-\nu\lambda_1 t} + \frac{\|g\|_{V'}^2}{\nu^2\lambda_1} + \frac{\|g\|_{V'}^2}{\nu}.$$

Hence, (40) is also proved. □

We note that estimates (39) and (40) are independent of α.

Let us establish some smoothness properties of solutions of (29) and (30) for $\alpha > 0$.

Proposition 4. *If $g \in H$, then any solution $v(t)$ of problem (29), (30), and (37) satisfies the inequality*

$$t\|v(t)\|^2 + \nu\int_0^t s|Au(s)|^2 ds \leq C_2(\alpha, t, |v(0)|, |g|), \ \forall t \geq 0, \tag{44}$$

where $C_2(\alpha, z, R_1, R_2)$ is a positive monotone function with respect to z, R_1, and R_2 for every fixed $\alpha > 0$.

Proof. At first, we prove inequality (44) for the Galerkin approximation $v_m(t)$ of the exact solution $v(t)$. After that, we obtain (44) for $v(t)$ passing to the limit in the inequality for the Galerkin approximation as $m \to \infty$. For brevity, we sketch the reasoning for the Galerkin system omitting the index m. Multiplying equation (29) by $tAv(t)$, we have

$$\frac{1}{2}\frac{d}{dt}\left\{t\|v(t)\|^2\right\} - \frac{1}{2}\|v(t)\|^2 + \nu t|Av(t)|^2 + t(B(u,v), Av) = t(g, Av(t)). \tag{45}$$

Recall that

$$|(g, Av)| \leq \frac{\nu}{4}|Av(t)|^2 + \frac{1}{\nu}|g|^2. \tag{46}$$

Besides from (34), we conclude that

$$|(B(u,v),Av)| \le C_1(\alpha)|v|\cdot\|v\|\cdot|Av| \le \frac{\nu}{4}|Av(t)|^2 + \frac{C_1^2(\alpha)}{\nu}|v|^2\|v\|^2. \tag{47}$$

Replacing (46) and (47) to (45), we have

$$\frac{d}{dt}\left\{t\|v(t)\|^2\right\} + \nu t|Av(t)|^2 \le \|v(t)\|^2 + t\frac{2C_1^2(\alpha)}{\nu}|v(t)|^2\|v(t)\|^2 + t\frac{2}{\nu}|g|^2. \tag{48}$$

Integrating inequality (48) over $[0,t]$ we reach

$$t\|v(t)\|^2 + \nu\int_0^t s|Au(s)|^2 ds \le \int_0^t \|v(s)\|^2 ds + t\int_0^t \frac{2C_1^2(\alpha)}{\nu}|v(s)|^2\|v(s)\|^2 ds + \frac{t^2}{\nu}|g|^2. \tag{49}$$

It follows from inequality (39) that

$$|v(s)|^2 \le |v(0)|^2 + \frac{\|g\|_{V'}^2}{\nu^2\lambda_1}. \tag{50}$$

Integrating (42) over $[0,t]$, we obtain

$$\nu\int_0^t \|v(s)\|^2 ds \le |v(0)|^2 + t\frac{\|g\|_{V'}^2}{\nu}. \tag{51}$$

Consequently from (49), we conclude that

$$\begin{aligned} t\|v(t)\|^2 + \nu\int_0^t s|Au(s)|^2 ds &\le \frac{|v(0)|^2}{\nu} + t\frac{\|g\|_{V'}^2}{\nu^2} \\ + t\frac{2C_1^2(\alpha)}{\nu}\left(|v(0)|^2 + \frac{\|g\|_{V'}^2}{\nu^2\lambda_1}\right)&\left(\frac{|v(0)|^2}{\nu} + t\frac{\|g\|_{V'}^2}{\nu^2}\right) + \frac{t^2}{\nu}|g|^2 \\ &\le C_2(\alpha, t, |v(0)|, |g|), \end{aligned}$$

where $C_2(\alpha, z, R_1, R_2) = \frac{R_1^2}{\nu} + z\frac{R_2^2}{\nu^2\lambda_1} + z\frac{2C_1^2(\alpha)}{\nu}\left(R_1 + \frac{R_2^2}{\nu^2\lambda_1^2}\right) \times \left(\frac{{R_1}^2}{\nu} + z\frac{R_2^2}{\nu^2\lambda_1}\right) + \frac{z^2}{\nu}R_2^2$. □

Consider the semigroup $\{S_\alpha(t)\} = \{S(t)\}$ acting in H by the formula $S(t)v_0 = v(t)$, where $v_0 \in H$ and $v(t)$ is a solution of problem (29), (30), and (37). It follows from inequality (39) that the semigroup $\{S(t)\}$ has a bounded (in H) absorbing set $P_0 = \{v \mid |v| \le R_0\}$, where $R_0^2 = \frac{2\|g\|_{V'}^2}{\nu^2\lambda_1}$. The set $P_1 = S(1)P_0$ is also absorbing and, due to Proposition 4, P_1 is bounded in V and therefore compact in H. It can be proved that the semigroup $\{S(t)\}$ is continuous in H. All these facts imply the existence of the global attractor $\mathcal{A}_\alpha$, for $\alpha > 0$, of the Leray-α model, that is $\mathcal{A}_\alpha$ is compact in H, strictly invariant with respect to H, and

$$\operatorname{dist}_H(S(t)B, \mathcal{A}_\alpha) \to 0 \text{ as } t \to +\infty$$

for any bounded set $B \subset H$ of initial data (see [1, 15, 20, 28, 30]). Moreover, the set $\mathcal{A}_\alpha$ is bounded in V.

In the next section, we study the behavior of solutions of the Leray-α model as $\alpha \to 0+$. We establish the relation between these solutions and the trajectory attractor of the 3D Navier–Stokes system constructed in Section 1.

4. **The 3D Leray-α model and its relation to the 3D Navier–Stokes system.** We have proved estimates (39) and (40) for the solution $v(t)$ of system (29), (30), and (36). The same estimates hold for the function $u(t)$, where $v(t) = u(t) + \alpha^2 Au(t)$, and the estimates do not depend on α. We need one more estimate for the time derivative $\partial_t v(t)$.

Proposition 5. *Let $g \in V'$. Then the solution $v(t)$ of (29), (30), and (36) satisfies the inequality*

$$\left(\int_t^{t+1} \|\partial_t v(s)\|_{V'}^{4/3} ds\right)^{3/4} \leq C_4 |v(0)|^2 e^{-\beta t} + R_1^2, \tag{52}$$

where $\beta = \lambda_1 \nu$, C_4 depends on λ_1 and ν, and R_1 depends on λ_1, ν, and $\|g\|_{V'}$. The numbers C_4 and R_1 are independent of α.

Proof. Consider the operator $B(u(t), v(t))$, where $v = u + \alpha^2 Au$. We note that

$$|u| \leq |v| \text{ and } \|u\| \leq \|v\|. \tag{53}$$

From inequalities (8) and (53), we conclude that

$$\|B(u,v)\|_{V'} \leq c|u|^{1/4}\|u\|^{3/4}|v|^{1/4}\|v\|^{3/4} \leq c|v|^{1/2}\|v\|^{3/2}. \tag{54}$$

Consequently,

$$\begin{aligned}
\left(\int_t^{t+1} \|B(u(s), v(s))\|_{V'}^{4/3} ds\right)^{3/4} &\leq c\left(\int_t^{t+1} |v(s)|^{2/3}\|v(s)\|^2 ds\right)^{3/4} \\
&\leq c \cdot \operatorname*{ess\,sup}_{s\in[t,t+1]} |v(s)|^{1/2} \left(\int_t^{t+1} \|v(s)\|^2 ds\right)^{3/4} \\
\leq c\left(|v(0)|^2 e^{-\nu\lambda_1 t} + \frac{\|g\|_{V'}^2}{\nu^2\lambda_1}\right)^{1/4} &\left(\frac{1}{\nu}|v(0)|^2 e^{-\nu\lambda_1 t} + \frac{\|g\|_{V'}^2}{\nu^3\lambda_1} + \frac{\|g\|_{V'}^2}{\nu^2}\right)^{3/4} \\
&\leq C_4' |v(0)|^2 e^{-\nu\lambda_1 t} + (R_1')^2 = C_4'|v(0)|^2 e^{-\beta t} + (R_1')^2.
\end{aligned} \tag{55}$$

Using the triangle inequality, it follows from equation (29) that

$$\begin{aligned}
&\left(\int_t^{t+1} \|\partial_t v(s)\|_{V'}^{4/3} ds\right)^{3/4} \\
&\leq \nu\left(\int_t^{t+1} \|Av(s)\|_{V'}^{4/3} ds\right)^{3/4} + \left(\int_t^{t+1} \|B(u(s), v(s))\|_{V'}^{4/3} ds\right)^{3/4} + \|g\|_{V'} \\
&\leq \nu\left(\int_t^{t+1} \|v(s)\|^2 ds\right)^{1/2} + \left(\int_t^{t+1} \|B(u(s), v(s))\|_{V'}^{4/3} ds\right)^{3/4} + \|g\|_{V'} \\
&\leq \nu\left(\frac{1}{\nu}|v(0)|^2 e^{-\nu\lambda_1 t} + \frac{\|g\|_{V'}^2}{\nu^3\lambda_1} + \frac{\|g\|_{V'}^2}{\nu^2}\right)^{1/2} + C_4'|v(0)|^2 e^{-\nu\lambda_1 t} + (R_1')^2 + \|g\|_{V'} \\
&\leq C_4|v(0)|^2 e^{-\beta t} + R_1^2.
\end{aligned}$$

Here, we have used the equality $\|Av\|_{V'} = \|v\|$, the inequality

$$\left(\int_t^{t+1} \|v(s)\|^{4/3} ds\right)^{3/4} \leq \left(\int_t^{t+1} \|v(s)\|^2 ds\right)^{1/2},$$

and estimate (55). □

We now consider the Banach space $\mathcal{F}_+^{\mathrm{b}}$ defined in Section 1. Recall that

$$\mathcal{F}_+^{\mathrm{b}} = \{w(t), t \in \mathbb{R}_+ |\ w(\cdot) \in L_2^{\mathrm{b}}(\mathbb{R}_+; V) \cap L_\infty^{\mathrm{b}}(\mathbb{R}_+; H),\ \partial_t w(\cdot) \in L_{4/3}^{\mathrm{b}}(\mathbb{R}_+; V')\}.$$

From inequalities (39), (40), and (52), we obtain

Corollary 1. *Let $g \in V'$. Then, for any solution $v(t)$ of problem (29), (30), and (36), we have $v(\cdot) \in \mathcal{F}_+^{\mathrm{b}}$ and*

$$\|T(h)v(\cdot)\|_{\mathcal{F}_+^{\mathrm{b}}} \leq C_5 |v(0)|^2 e^{-\beta h} + R_4^2,\ \forall h \geq 0, \tag{56}$$

where C_5 and R_4 are independent of α; C_5 depends on λ_1, ν, and R_4 depends on λ_1, ν, $\|g\|_{V'}$.

Let $v = u + \alpha^2 Au$. Then, together with (53), we have

$$\|\partial_t u\|_{V'} \leq \|\partial_t v\|_{V'}, \tag{57}$$

which implies

Corollary 2. *The function $u(t)$ corresponding to the solution $v(t)$ of problem (29), (30), and (36) satisfies the following inequality:*

$$\|T(h)u(\cdot)\|_{\mathcal{F}_+^{\mathrm{b}}} \leq C_5 |v(0)|^2 e^{-\beta h} + R_4^2,\ \forall h \geq 0, \tag{58}$$

Similar to the trajectory space $\mathcal{K}^+$ of the N.–S. system introduced in Section 1, we define the trajectory space $\mathcal{K}_\alpha^+$ of the Leray-α model (29) and (30). By definition, the space $\mathcal{K}_\alpha^+$ is the union of all weak solutions $v(t) = v(x,t), t \geq 0$, of equations (29) and (30) with initial data $v(0) \in H$. It follows from Corollary 1 that $\mathcal{K}_\alpha^+ \subset \mathcal{F}_+^{\mathrm{b}}$ for all $\alpha > 0$.

We need the energy equality (38) that we rewrite in the equivalent form similar to energy inequality (19) for the 3D N.–S. system.

Proposition 6. *For any $v \in \mathcal{K}_\alpha^+$*

$$-\frac{1}{2}\int_0^\infty |v(t)|^2 \psi'(t)dt + \nu \int_0^\infty \|v(t)\|^2 \psi(t)dt = \int_0^\infty \langle g, v(t)\rangle\, \psi(t)dt \tag{59}$$

for all $\psi \in C_0^\infty(\mathbb{R}_+)$.

To prove (59), we multiply (38) by $\psi(t)$ and integrate by parts.

In the space $\mathcal{K}_\alpha^+$, we consider the topology Θ_+^{loc} defined in Section 1. Recall that $\mathcal{F}_+^{\mathrm{b}} \subset \Theta_+^{\mathrm{loc}}$.

Proposition 7. *Let a sequence of functions $v_{\alpha_n}(t) \in \mathcal{K}_{\alpha_n}^+, n \in \mathbb{N}$, satisfy the following properties:*

1. *$\{v_{\alpha_n}(\cdot)\}$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$;*
2. *$\alpha_n \to 0+$ as $n \to \infty$;*
3. *$v_{\alpha_n}(\cdot) \to v(\cdot)$ in the topology Θ_+^{loc} as $n \to \infty$.*

Then $v(\cdot)$ is a weak solution of the 3D N.–S. system and, moreover, $v \in \mathcal{K}^+$, that is $v(t)$ satisfies the energy inequality (19) for $t \geq 0$.

Proof. By assumption,

$$\|v_{\alpha_n}(\cdot)\|_{\mathcal{F}_+^{\mathrm{b}}} \leq C,\ \forall n \in \mathbb{N}. \tag{60}$$

Hence, due to the convergence $v_{\alpha_n}(\cdot) \to v(\cdot)$ as $n \to \infty$ in Θ_+^{loc},

$$\|v(\cdot)\|_{\mathcal{F}_+^{\mathrm{b}}} \leq C \tag{61}$$

as well.

We denote by $u_{\alpha_n}(t)$ the solution of equation (30) corresponding to $v_{\alpha_n}(t)$. It follows from (61) that

$$\|u_{\alpha_n}(\cdot)\|_{\mathcal{F}^{\mathrm{b}}_+} \le C, \ \forall n \in \mathbb{N}. \tag{62}$$

We prove that $v(\cdot)$ is a weak solution of the 3D N.–S. system on every interval $(0, M)$. The function $v_{\alpha_n}(\cdot)$ satisfies the equation

$$\partial_t v_{\alpha_n} + \nu A v_{\alpha_n} + B(u_{\alpha_n}, v_{\alpha_n}) = g \tag{63}$$

in the space $\mathcal{D}'(0, M; V')$. By assumptions,

$$v_{\alpha_n}(\cdot) \rightharpoonup v(\cdot) \text{ as } n \to \infty \tag{64}$$

weakly in $L_2(0, M; V)$, $*$-weakly in $L_\infty(0, M; H)$, and, in addition,

$$\partial_t v_{\alpha_n}(\cdot) \rightharpoonup \partial_t v(\cdot) \text{ as } n \to \infty \tag{65}$$

weakly in $L_{4/3}(0, M; V')$. Then the convergences in (64) and (65) hold in a weaker topology of the space $\mathcal{D}'(0, M; V')$. From (64) we obtain

$$A v_{\alpha_n}(\cdot) \rightharpoonup A v(\cdot) \text{ as } n \to \infty \tag{66}$$

in $\mathcal{D}'(0, M; V')$.

In order to establish the equality

$$\partial_t v + \nu A v + B(v, v) = g, \tag{67}$$

it is sufficient to prove that the sequence $B(u_{\alpha_n}(\cdot), v_{\alpha_n}(\cdot))$ converges to $B(v(\cdot), v(\cdot))$ in $\mathcal{D}'(0, M; V')$ as $n \to \infty$.

Notice that

$$u_{\alpha_n}(\cdot) \rightharpoonup v(\cdot) \text{ as } n \to \infty \tag{68}$$

weakly in $L_2(0, M; V)$. Indeed, the functions $u_{\alpha_n}(\cdot)$ satisfies the equation

$$u_{\alpha_n} + \alpha_n^2 A u_{\alpha_n} = v_{\alpha_n}. \tag{69}$$

Recall that $\{u_{\alpha_n}(\cdot)\}$ is bounded in $L_2(0, M; V)$. Then, passing to a subsequence, we may assume that $\{u_{\alpha_n}(\cdot)\}$ converges to a function $w(\cdot)$ weakly in $L_2(0, M; V)$:

$$u_{\alpha_n}(\cdot) \rightharpoonup w(\cdot) \text{ as } n \to \infty. \tag{70}$$

Then the sequence $A u_{\alpha_n}(\cdot) \rightharpoonup A w(\cdot)$ weakly in $L_2(0, M; V')$ as $n \to \infty$ and, therefore, $\alpha_n A u_{\alpha_n}(\cdot) \rightharpoonup 0$ weakly in $L_2(0, M; V')$ as $n \to \infty$. Hence in equality (69), we may pass to the limit in the space $L_2(0, M; V')$ and obtain that

$$w = \text{w-}\lim_{n\to\infty} u_{\alpha_n} = \text{w-}\lim_{n\to\infty} v_{\alpha_n} = v. \tag{71}$$

Consequently, (70) and (71) imply (68).

The sequences $\{\partial_t v_{\alpha_n}(\cdot)\}$ and $\{\partial_t u_{\alpha_n}(\cdot)\}$ are bounded in $L_{4/3}(0, M; V')$. Then the Aubin compactness theorem (see [15, 24, 29]) implies that, passing to a subsequence, we may assume that $\{v_{\alpha_n}(\cdot)\}$ and $\{u_{\alpha_n}(\cdot)\}$ both converge to $v(\cdot)$ strongly in $L_2(0, M; H)$. Recall that $L_2(0, M; H) \subseteq L_2(\mathbb{T}^3 \times]0, M[)^3$ and, therefore, we may assume that

$$v_{\alpha_n}(x, t) \to v(x, t), \ u_{\alpha_n}(x, t) \to v(x, t) \text{ for a.e. } (x, t) \in \mathbb{T}^3 \times]0, M[. \tag{72}$$

Identity (10) implies

$$B(u_{\alpha_n}, v_{\alpha_n}) = P \sum_{i=1}^{3} \partial_i \left(u^i_{\alpha_n} \cdot v_{\alpha_n} \right). \tag{73}$$

From (72), we conclude that

$$u^i_{\alpha_n}(x,t)\cdot v_{\alpha_n}(x,t)\to v^i(x,t)\cdot v(x,t)\ (n\to\infty)\text{ for a.e. }(x,t)\in\mathbb{T}^3\times]0,M[.$$

Furthermore, due to (54), $u^i_{\alpha_n}(\cdot)v_{\alpha_n}(\cdot)$ is bounded in the space $L_{4/3}(0,M;H)$ and, moreover, in $L_{4/3}(\mathbb{T}^3\times]0,M[)^3$. Applying Lemma 1.3 from [24, Ch.1, Sec.1] on the weak convergence in L_q-spaces, we conclude that

$$u^i_{\alpha_n}(\cdot)\cdot v_{\alpha_n}(\cdot)\rightharpoonup v^i(\cdot)\cdot v(\cdot)\text{ as }n\to\infty$$

weakly in $L_{4/3}(\mathbb{T}^3\times]0,M[)^3$ and weakly in $L_{4/3}(0,M;H)$. Then, finally,

$$B(u_{\alpha_n}(\cdot),v_{\alpha_n}(\cdot))\rightharpoonup B(v(\cdot),v(\cdot))\text{ as }n\to\infty \tag{74}$$

weakly in $L_{4/3}(0,M;V')$ and, therefore, in $\mathcal{D}'(0,M;V')$. We have proved that $v(\cdot)$ is a weak solution of the 3D N.–S. system.

We have to prove now that the function $v(\cdot)$ satisfies the energy inequality (16) on every interval $(0,M)$. Indeed, the functions $v_{\alpha_n}(\cdot)$ satisfy the energy equality (59), that is

$$-\frac{1}{2}\int_0^M |v_{\alpha_n}(t)|^2\psi'(t)dt+\nu\int_0^M \|v_{\alpha_n}(t)\|^2\psi(t)dt=\int_0^M \langle g,v_{\alpha_n}(t)\rangle\,\psi(t)dt \tag{75}$$

for any function $\psi\in C_0^\infty(]0,M[)$. Let $\psi(t)\geq 0$ for $t\in]0,M[$. We have already proved that $v_{\alpha_n}(\cdot)\to v(\cdot)$ strongly in $L_2(0,M;H)$ as $n\to\infty$ (see the paragraph after (71)). Then, clearly, the real functions $|v_{\alpha_n}(\cdot)|\to|v(\cdot)|$ strongly in $L_2(0,M)$ as $n\to\infty$. In particular, passing to a subsequence, we may assume that

$$|v_{\alpha_n}(t)|^2\to|v(t)|^2\text{ for a.e. }t\in[0,M].$$

Consider the functions $|v_{\alpha_n}(t)|^2\psi'(t),\ t\in[0,M]$. It follows from estimate (39) that these functions have a integrable majorant on $[0,M]$. The Lebesgue dominant convergence theorem implies that

$$\int_0^M |v_{\alpha_n}(t)|^2\psi'(t)dt\to\int_0^M |v(t)|^2\psi'(t)dt\text{ as }n\to\infty. \tag{76}$$

We note that $v_{\alpha_n}(\cdot)\sqrt{\psi(\cdot)}\rightharpoonup v(\cdot)\sqrt{\psi(\cdot)}$ weakly in $L_2(0,M;V)$. Therefore,

$$\int_0^M \|v(t)\|^2\psi(t)dt\leq\liminf_{n\to\infty}\int_0^M \|v_{\alpha_n}(t)\|^2\psi(t)dt. \tag{77}$$

We also have

$$\int_0^M \langle g,v_{\alpha_n}(t)\rangle\,\psi(t)dt\to\int_0^M \langle g,v(t)\rangle\,\psi(t)dt\text{ as }n\to\infty. \tag{78}$$

Using (76)–(78), we pass to the limit in (75) and obtain the inequality

$$-\frac{1}{2}\int_0^M |v(t)|^2\psi'(t)dt+\nu\int_0^M \|v(t)\|^2\psi(t)dt\leq\int_0^M \langle g,v(t)\rangle\,\psi(t)dt \tag{79}$$

for any $\psi\in C_0^\infty(]0,M[)$, $\psi(\cdot)\geq 0$.

We have proved that $v(\cdot)$ is a weak solution of the 3D Navier–Stokes system which satisfies the energy inequality. Thus, $v\in\mathcal{K}^+$. □

Remark 3. The proof that we have produced above is along the lines of the Leray program in which he proved in $\mathbb{R}^3$ that the weak solutions of the 3D N.–S. equations are in some sense the limit of his mollified system, which, in our care, we take it to be the Leray-α. In our study, it is very important that every weak solution of the 3D N.–S. system producing from the Leray-α system by passing to the limit as $\alpha \to 0+$ satisfies the energy inequality (19). We will use this fact in the next sections.

In the next section, we apply Proposition 7 to prove the convergence of solutions of the Leray-α model to the trajectory attractor of the 3D Navier–Stokes system as $\alpha \to 0+$.

5. **Convergence of bounded sets of trajectories of the Leray-α model to the trajectory attractor of the N.–S. system.** We denote by $\mathfrak{A}_0$ the trajectory attractor of the 3D N.–S. system

$$\partial_t u + \nu A u + B(u,u) = g(x) \tag{80}$$

that was constructed in Section 1. Recall that $\mathfrak{A}_0 \subset \mathcal{K}^+$ and the set $\mathfrak{A}_0$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$ and compact in Θ_+^{loc}.

Let $B_\alpha = \{v_\alpha(x,t),\ t \geq 0\}, 0 < \alpha \leq 1$, be a family of solutions (trajectories) of the Leray-α model that are uniformly (with respect to $\alpha \in (0,1]$) bounded in the norm of $\mathcal{F}_+^{\mathrm{b}}$:

$$\partial_t v_\alpha + \nu A v_\alpha + B(u_\alpha, v_\alpha) = g(x),\ v_\alpha = u_\alpha + \alpha^2 A u_\alpha, \tag{81}$$

that is $B_\alpha \subset \mathcal{K}_\alpha^+$ (see Section 2) for $\alpha \in (0,1]$ and, in addition,

$$\|v_\alpha\|_{\mathcal{F}_+^{\mathrm{b}}} = \|v_\alpha\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;V)} + \|v_\alpha\|_{L_\infty^{\mathrm{b}}(\mathbb{R}_+;H)} + \|\partial_t v_\alpha\|_{L_{4/3}^{\mathrm{b}}(\mathbb{R}_+;V')} \leq R,\ \forall v_\alpha \in B_\alpha, \tag{82}$$

where R is an arbitrary fixed positive number. We also denote by

$$\tilde{B}_\alpha = \left\{u_\alpha \in \mathcal{F}_+^{\mathrm{b}} \mid u_\alpha + \alpha^2 A u_\alpha = v_\alpha \in B_\alpha\right\} = (1+\alpha^2 A)^{-1} B_\alpha.$$

Recall that $\|u_\alpha\|_{\mathcal{F}_+^{\mathrm{b}}} \leq \|v_\alpha\|_{\mathcal{F}_+^{\mathrm{b}}}$ (see (41) and (57)). Therefore,

$$\|u_\alpha\|_{\mathcal{F}_+^{\mathrm{b}}} \leq R,\ \forall u_\alpha \in \tilde{B}_\alpha,\ 0 < \alpha \leq 1. \tag{83}$$

We denote by $\mathcal{K}_0$ the kernel of equation (80). Recall that $\mathcal{K}_0$ is the union of all bounded (in the nom $\mathcal{F}^{\mathrm{b}}$) complete weak solutions $\{u(t),\ t \in \mathbb{R}\}$ of equation (80) that satisfy the energy inequality (19). In Section 1, it was shown that $\mathfrak{A}_0 = \Pi_+ \mathcal{K}_0$.

Consider the topology Θ_+^{loc} in the space $\mathcal{F}_+^{\mathrm{loc}}$, which is a weak convergence topology in the corresponding spaces on any bounded interval $[0, M]$.

We now formulate the main theorem of the paper.

Theorem 3. *Let $B_\alpha = \{v_\alpha(x,t),\ t \geq 0\}, 0 < \alpha \leq 1$, be bounded sets of solutions of the Leray-α model (81) that satisfy the inequality*

$$\|B_\alpha\|_{\mathcal{F}_+^{\mathrm{b}}} \leq R,\ 0 < \alpha \leq 1.$$

Then the shifted sets of solutions $\{T(h)B_\alpha\}$ $(T(h)w(t) = w(t+h))$ converge to the trajectory attractor $\mathfrak{A}_0 = \Pi_+ \mathcal{K}_0$ of the 3D N.–S. system (80) in the topology Θ_+^{loc} as $h \to +\infty$ and $\alpha \to 0+$:

$$T(h)B_\alpha \to \mathfrak{A}_0 \textit{ in } \Theta_+^{\mathrm{loc}} \textit{ as } h \to +\infty \textit{ and } \alpha \to 0+\,. \tag{84}$$

The same convergence holds for the corresponding sets $\tilde{B}_\alpha = (1+\alpha^2 A)^{-1} B_\alpha$:

$$T(h)\tilde{B}_\alpha \to \mathfrak{A}_0 \textit{ in } \Theta_+^{\mathrm{loc}} \textit{ as } h \to +\infty \textit{ and } \alpha \to 0+\,. \tag{85}$$

Proof. Clearly, it is sufficient to prove (84). Assume the converse: there exists a neighbourhood (in $\Theta_+^{\rm loc}$) $\mathcal{O}(\mathfrak{A}_0)$ of the set $\mathfrak{A}_0$ and sequences $\alpha_n \to 0+, h_n \to +\infty$ as $n \to \infty$ such that

$$T(h_n)B_{\alpha_n} \not\subset \mathcal{O}(\mathfrak{A}_0). \tag{86}$$

Then, for some solutions $w_{\alpha_n}(\cdot) \in B_{\alpha_n}$, the functions $v_n(t) = T(h_n)w_{\alpha_n}(t) = w_{\alpha_n}(h_n + t)$ do not belong to $\mathcal{O}(\mathfrak{A}_0)$:

$$v_n(\cdot) \notin \mathcal{O}(\mathfrak{A}_0). \tag{87}$$

Notice that the function $v_n(t)$ is a solution of the Leray-α_n system (81) with $\alpha = \alpha_n$ on the interval $[-h_n, +\infty)$, since the equation is autonomous and $v_n(t)$ is a time shift of the solution $w_{\alpha_n}(\cdot)$. Moreover, it follows from (82) that

$$\sup_{t\ge -h_n} |v_n(t)| + \left(\sup_{t\ge -h_n}\int_t^{t+1} \|v_n(s)\|^2 ds\right)^{1/2} + \sup_{t\ge -h_n}\left(\int_t^{t+1} \|v_n(s)\|^{4/3} ds\right)^{3/4} \le R. \tag{88}$$

This inequality implies that the sequence $\{v_n(\cdot)\}$ is weakly compact in $\Theta_{-M,M} = L_2(-M,M;V) \cap L_\infty(-M,M;H) \cap \{\partial_t v \in L_{4/3}(-M,M;V')\}$ for every M, if we consider elements of $\{v_n(\cdot)\}$ with indices n such that $h_n \ge M$. Therefore, for every fixed $M > 0$, we can choose a subsequence $\{n_l\} \subset \{n\}$ such that $\{v_{n_l}(\cdot)\}$ converges weakly in $\Theta_{-M,M}$. Then, using the standard Cantor diagonal procedure, we can construct a function $u(t), t \in \mathbb{R}$, and a subsequence $\{n_l'\}$ such that

$$v_{n_l'}(\cdot) \to u(\cdot) \text{ weakly in } \Theta_{-M,M} \text{ as } n_l' \to \infty \text{ for any } M > 0.$$

From (88), we obtain

$$\sup_{t\in\mathbb{R}} |u(t)| + \left(\sup_{t\in\mathbb{R}}\int_t^{t+1} \|u(s)\|^2 ds\right)^{1/2} + \sup_{t\in\mathbb{R}}\left(\int_t^{t+1} \|u(s)\|^{4/3} ds\right)^{3/4} \le R,$$

that is $u \in \mathcal{F}^{\rm b} = L_2^{\rm b}(\mathbb{R};V) \cap L_\infty(\mathbb{R};H) \cap \{\partial_t u \in L_{4/3}^{\rm b}(\mathbb{R};V')\}$.

We now apply Proposition 7, where we can clearly assume that functions $v_{\alpha_n}(t)$ are defined on the semiaxis $[-M, +\infty)$ instead of $[0, +\infty)$ since the equations are autonomous. We obtain from this proposition that $u(x,t)$ is a weak solution of the 3D N.–S. system for all $t \in \mathbb{R}$ and satisfies the energy inequality, that is $u \in \mathcal{K}_0$, where $\mathcal{K}_0$ is the kernel of equation (80). Then we conclude that $\Pi_+ u \in \Pi_+\mathcal{K}_0 = \mathfrak{A}_0$ and $\Pi_+ v_{n_l'}(\cdot) \to \Pi_+ u(\cdot)$ in $\Theta_+^{\rm loc}$ as $n \to \infty$. In particular for a large n_l', we have

$$\Pi_+ v_{n_l'} \in \mathcal{O}(\Pi_+ u) \subseteq \mathcal{O}(\mathfrak{A}_0). \tag{89}$$

This contradicts (87) and completes the proof. □

In fact, we have proved a slightly stronger assertion.

Corollary 3. *Under the assumptions of Theorem 3, the set $T(h)B_\alpha$ converges towards the kernel $\mathcal{K}_0$ of equation (80) in the weak topology of the space $\Theta_{-M,M}$ as $h \to +\infty$ and $\alpha \to 0+$ for every $M > 0$. The same results holds for the set $T(h)\tilde{B}_\alpha$, where $\tilde{B}_\alpha = (1 + \alpha^2 A)^{-1} B_\alpha$.*

We now reformulate this result in terms of Hausdorff (non-symmetric) semi-distance from a set X to a set Y in a Banach space E

$$\operatorname{dist}_E(X,Y) := \sup_{x\in X} \operatorname{dist}_E(x,Y) = \sup_{x\in X}\inf_{y\in Y} \|x - y\|_E.$$

Recall that the following embeddings are continuous

$$\begin{aligned}\Theta_{-M,M} &\subseteq L_2(-M,M;H^{1-\delta}),\\ \Theta_{-M,M} &\subseteq C([-M,M];H^{-\delta}),\ \forall 0<\delta\le 1.\end{aligned}$$

Hence, we obtain

Corollary 4. *For any fixed $M>0$ and for any sets $B_\alpha\subset\mathcal{K}_\alpha^+$ uniformly bounded in $\mathcal{F}_+^{\mathrm{b}}$, the following limit relations hold:*

$$\begin{aligned}\mathrm{dist}_{L_2(-M,M;H^{1-\delta})}\left(T(h)B_\alpha,\mathcal{K}_0\right) &\to 0+,\\ \mathrm{dist}_{C([-M,M];H^{-\delta})}\left(T(h)B_\alpha,\mathcal{K}_0\right) &\to 0+\ \textit{as } h\to+\infty \textit{ and } \alpha\to 0+\,.\end{aligned}$$

In conclusion, we consider the behaviour of trajectory attractors of the Leray-α model as $\alpha\to 0+$.

We consider the trajectory space $\mathcal{K}_\alpha^+,\alpha>0$, of system (81) that was constructed in Section 2. The translation semigroup $\{T(h)\}$ acts on $\mathcal{K}_\alpha^+$ (recall that $T(h)v_\alpha(t)=v_\alpha(t+h),\ h\ge 0$, where $v_\alpha\in\mathcal{K}_\alpha^+$). It is easy prove that the space $\mathcal{K}_\alpha^+$ is closed in Θ_+^{loc}. Proposition 3 implies that $\mathcal{K}_\alpha^+\subset\mathcal{F}_+^{\mathrm{b}}$ and there exists an absorbing set of the semigroup $\{T(h)\}$ in $\mathcal{K}_\alpha^+$, bounded in $\mathcal{F}_+^{\mathrm{b}}$ and compact in Θ_+^{loc}. Then, similar to Section 1, we prove the existence of the trajectory attractor $\mathfrak{A}_\alpha$ of the Leray-α model for $\alpha>0$, that is, $\mathfrak{A}_\alpha\subset\mathcal{K}_\alpha^+$, $\mathfrak{A}_\alpha$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$ and compact in Θ_+^{loc}; $T(h)\mathfrak{A}_\alpha=\mathfrak{A}_\alpha$ for all $h\ge 0$; and $T(h)B_\alpha\to\mathfrak{A}_\alpha$ in Θ_+^{loc} as $h\to+\infty$ for any bounded set $B_\alpha\subset\mathcal{K}_\alpha^+$. In addition, $\mathfrak{A}_\alpha=\Pi_+\mathcal{K}_\alpha$, where $\mathcal{K}_\alpha$ is the kernel of equation (81). Moreover, it follows from Proposition 3 that the trajectory attractors $\mathfrak{A}_\alpha$ are uniformly (with respect to $\alpha\in(0,1]$) bounded in $\mathcal{F}_+^{\mathrm{b}}$.

Theorem 3 implies

Corollary 5. *The following limit relations hold:*

$$\begin{aligned}\mathfrak{A}_\alpha &\to \mathfrak{A}_0 \textit{ in } \Theta_+^{\mathrm{loc}} \textit{ as } \alpha\to 0+,\\ \mathcal{K}_\alpha &\to \mathcal{K}_0 \textit{ in } \Theta^{\mathrm{loc}} \textit{ as } \alpha\to 0+\,.\end{aligned}$$

To prove Corollary 5, we recall that the family $\{\,\mathfrak{A}_\alpha,0<\alpha\le 1\}$ is uniformly bounded in $\mathcal{F}_+^{\mathrm{b}}$. Thus, we apply (84) for $B_\alpha=\mathfrak{A}_\alpha$ and obtain

$$T(h)\mathfrak{A}_\alpha\to\mathfrak{A}_0 \text{ in } \Theta_+^{\mathrm{loc}} \text{ as } h\to+\infty \text{ and } \alpha\to 0+\,.$$

However, $T(h)\mathfrak{A}_\alpha=\mathfrak{A}_\alpha$ for all $h\ge 0$. Therefore,

$$\mathfrak{A}_\alpha\to\mathfrak{A}_0 \text{ in } \Theta_+^{\mathrm{loc}} \text{ as } \alpha\to 0+\,.$$

Similarly, we prove that

$$\mathcal{K}_\alpha\to\mathcal{K}_0 \text{ in } \Theta^{\mathrm{loc}} \text{ as } \alpha\to 0+\,.$$

Using Corollary 81 we then have

Corollary 6. *For any $M>0$*

$$\begin{aligned}\mathrm{dist}_{L_2(-M,M;H^{1-\delta})}\left(\mathcal{K}_\alpha,\mathcal{K}_0\right) &\to 0+,\\ \mathrm{dist}_{C([-M,M];H^{-\delta})}\left(\mathcal{K}_\alpha,\mathcal{K}_0\right) &\to 0+\ \textit{as } \alpha\to 0+\,.\end{aligned}$$

Finally, we establish the relation between the trajectory attractor $\mathfrak{A}_\alpha$ and the global attractor $\mathcal{A}_\alpha$ of the 3D Leray-α model for a fixed $\alpha>0$. The global attractor $\mathcal{A}_\alpha$ was constructed in [14].

Proposition 8. *The trajectory attractor of the Leray-α model*

$$\mathfrak{A}_\alpha = \{v(t) = S(t)v_0, t \geq 0 \mid v_0 \in \mathcal{A}_\alpha\}.$$

It follows from Proposition 4 that the trajectory attractor $\mathfrak{A}_\alpha$ is bounded in the space $\mathcal{F}_+^{\mathrm{b,s}} = L_2^{\mathrm{b}}(\mathbb{R}_+; H^2) \cap L_\infty(\mathbb{R}_+; V) \cap \{\partial_t w(\cdot) \in L_2^{\mathrm{b}}(\mathbb{R}_+; H)\}$ and $\mathfrak{A}_\alpha$ attracts bounded set of trajectories from $\mathcal{K}_\alpha^+$ in the strong topology of the space $\Theta_+^{\mathrm{loc,s}} = L_2^{\mathrm{loc}}(\mathbb{R}_+; H^2) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; V) \cap \{\partial_t w(\cdot) \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H)\}$. This statements is proved in a similar way to the analogous results for 2D N.–S. system (see [10, 13]). Of course, these properties do not persist when we pass to the limit as $\alpha \to 0+$.

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REFERENCES

[1] A.V.Babin and M.I.Vishik, “Attractors of Evolution Equations,” Nauka, Moscow, 1989. English transl.: North Holland, Amsterdam, 1992.

[2] C.Cao, D.Holm, and E.S.Titi, *On the Clark-α model of turbulence: global regularity and long-time dynamics*, Journal of Turbulence, **6** (2005), 1–11.

[3] S.Chen, C.Foias, D.D.Holm, E.Olson, E.S.Titi, and S.Wynne, *The Camassa–Holm equations as a closure model for turbulent channel and pipe flow*, Phys. Rev. Lett. **81** (1998), 5338–5341.

[4] S.Chen, C.Foias, D.D.Holm, E.Olson, E.S.Titi, and S.Wynne, *A connection between the Camassa–Holm equations and turbulent flows in pipes and channels*, Phys. Fluids, **11** (1999), 2343–2353.

[5] S.Chen, C.Foias, D.D.Holm, E.Olson, E.S.Titi, and S.Wynne, *The Camassa–Holm equations and turbulence*, Physica D, **133** (1999), 49–65.

[6] S.Chen, D.D.Holm, L.G.Margolin, and R.Zhang, *Direct numerical simulations of the Navier–Stokes-alpha model*, Physica D, **133** (1999), 66–83.

[7] V.V.Chepyzhov and A.Miranville, *Trajectory and global attractors of dissipative hyperbolic equations with memory*, Commun. Pure Appl. Anal., **4** (2005), 115–142.

[8] V.V.Chepyzhov and M.I.Vishik, *Trajectory attractors for evolution equations*, C.R.Acad.Sci. Paris, Series I, **321** (1995), 1309-1314.

[9] V.V.Chepyzhov and M.I.Vishik, *Trajectory attractors for reaction-diffusion systems*, Topological Methods in Nonlinear Analysis, Journal of the Juliusz Schauder Center, **7** (1996), 49–76.

[10] V.V.Chepyzhov and M.I.Vishik, *Trajectory attractors for 2D Navier–Stokes systems and some generalizations*, Topological Methods in Nonlinear Analysis, Journal of Juliusz Schauder Center, **8** (1996), 217–243.

[11] V.V.Chepyzhov. and M.I.Vishik, *Evolution equations and their trajectory attractors*, J.Math. Pures Appl., **76** (1997), 913–964.

[12] V.V.Chepyzhov and M.I.Vishik, *Perturbation of trajectory attractors for dissipative hyperbolic equations*, Operator Theory: Adv. Appl., **110** (1999), 33–54.

[13] V.V.Chepyzhov and M.I.Vishik, “Attractors for Equations of Mathematical Physics,” AMS Colloquium Publications. V. 49. Providence: AMS, 2002.

[14] A.Cheskidov, D.D.Holm, E.Olson, and E.S.Titi, *On Leray-α model of turbulence*, Royal Society London, Proc., Series A, Mathematical, Physical & Engineering Sciences, **461** (2005), 629–649.

[15] P.Constantin and C.Foias, “Navier–Stokes Equations,” The University of Chicago Press, Chicago and London, 1989.

[16] C.Foias, D.D.Holm, and E.S.Titi, *The Navier–Stokes-alpha model of fluid turbulence*, Physica D, **152/153** (2001), 505–519.

[17] C.Foias, D.D.Holm, and E.S.Titi, *The three dimensional viscous Camassa–Holm equations, and their relation to the Navier–Stokes equations and turbulence theory*, Journal of Dynamics and Differential Equations, **14** (2002), 1–35.

[18] B.J.Geurts and D.D.Holm, *Fluctuation effects on 3D Lagrangian mean and Eulerian mean fluid motion*, Physica D, **133** (1999), 215–269.

[19] B.J.Geurts and D.D.Holm, *Regularization modeling for large-eddy simulation*, Phys. Fluids, **15** (2003), L13–L16.

[20] J.K.Hale, "Asymptotic Behaviour of Dissipative Systems," Math. Surveys and Mon., 25, Amer. Math. Soc., Providence, RI, 1988.

[21] A.A.Ilyin, E.M.Lunasin, and E.S.Titi, *A modified-Leray-α subgrid scale model of turbulence*, Nonlinearity, **19** (2006), 879–897.

[22] O.A.Ladyzhenskaya, "The Mathematical Theory of Viscous Incompressible Flow," Moscow, Nauka, 1970. English transl.: Gordon and Breach, New York, 1969.

[23] J.Leray, *Essai sur le mouvemenet d'un fluide visqueux emplissant l'espace*, Acta Math., **63** (1934), 193–248.

[24] J.-L.Lions, "Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires," Dunod Gauthier-Villars, Paris, 1969.

[25] J.-L.Lions and E.Magenes, "Problèmes aux Limites Non Homogènes et Applications," volume 1, Paris, Dunod, 1968.

[26] A.Marzocchi and S. Zandonella Necca, *Attractors for dynamical systems in topological spaces*, Discrete Contin. Dyn. Syst. **8** (2002), 585–597.

[27] G.Sell, *Global attractors for the three-dimensional Navier–Stokes equations*, J.Dyn.Diff.Eq., **8** (1996), 1–33.

[28] G.Sell and Y.You, "Dynamics of Evolutionary Equations," Springer-Verlag, New York, 2002.

[29] R.Temam, Navier–Stokes Equations, "Theory and Numerical Analysis," 3rd. rev. ed., North-Holland, Amsterdam, 1984.

[30] R.Temam, "Infinite-dimensional Dynamical Systems in Mechanics and Physics," Applied Mathematics Series, **68**, New York, Springer-Verlag, 1988, 2nd ed. 1997.

[31] M.I.Vishik and V.V.Chepyzhov, *Trajectory and global attractors of three-dimensional Navier–Stokes systems*, Math. Notes, **71** (2002), 177–193.

[32] M.I.Vishik and V.V.Chepyzhov, *Non-autonomous Ginzburg–Landau equation and its attractors*, Sbornik: Mathematics, **196** (2005), 17–42.

[33] M.I.Vishik, E.S.Titi, and V.V.Chepyzhov, *Trajectory attractor approximation of the 3D Navier–Stokes system by a Leray-α model*, Doklady Mathematics, **71** (2005), 92–95.

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