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Non-autonomous 2D Navier–Stokes System with Singularly Oscillating External Force and its Global Attractor

V. V. Chepyzhov[1,2] **and M. I. Vishik**[1]

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We study the global attractor $\mathcal{A}^\varepsilon$ of the non-autonomous 2D Navier–Stokes (N.–S.) system with singularly oscillating external force of the form $g_0(x,t) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right)$, $x \in \Omega \Subset \mathbb{R}^2, t \in \mathbb{R}, 0 \leqslant \rho \leqslant 1$. If the functions $g_0(x,t)$ and $g_1(z,t)$ are translation bounded in the corresponding spaces, then it is known that the global attractor $\mathcal{A}^\varepsilon$ is bounded in the space H, however, its norm $\|\mathcal{A}^\varepsilon\|_H$ may be unbounded as $\varepsilon \to 0+$ since the magnitude of the external force is growing. Assuming that the function $g_1(z,t)$ has a divergence representation of the form $g_1(z,t) = \partial_{z_1} G_1(z,t) + \partial_{z_2} G_2(z,t), z = (z_1, z_2) \in \mathbb{R}^2$, where the functions $G_j(z,t) \in L_2^b(\mathbb{R}; Z)$ (see Section 3), we prove that the global attractors $\mathcal{A}^\varepsilon$ of the N.–S. equations are uniformly bounded with respect to ε: $\|\mathcal{A}^\varepsilon\|_H \leqslant C$ for all $0 < \varepsilon \leqslant 1$. We also consider the "limiting" 2D N.–S. system with external force $g_0(x,t)$. We have found an estimate for the deviation of a solution $u^\varepsilon(x,t)$ of the original N.–S. system from a solution $u^0(x,t)$ of the "limiting" N.–S. system with the same initial data. If the function $g_1(z,t)$ admits the divergence representation, the functions $g_0(x,t)$ and $g_1(z,t)$ are translation compact in the corresponding spaces, and $0 \leqslant \rho < 1$, then we prove that the global attractors $\mathcal{A}^\varepsilon$ converges to the global attractor $\mathcal{A}^0$ of the "limiting" system as $\varepsilon \to 0+$ in the norm of H. In the last section, we present an estimate for the Hausdorff deviation of $\mathcal{A}^\varepsilon$ from $\mathcal{A}^0$ of the form:$\mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant C(\rho)\varepsilon^{1-\rho}$ in the case, when the global attractor $\mathcal{A}^0$ is exponential (the Grashof number of the "limiting" 2D N.–S. system is small).

KEY WORDS: Non-autonomous 2D Navier–Stokes system; global attractor; singularly oscillating terms; homogenization; translation compact functions.

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[1] Institute for Information Transmission Problems, Russian Academy of Sciences, Bolshoy Karetniy 19, Moscow GSP-4 127994, Russia. E-mails: chep@iitp.ru,vishik@iitp.ru
[2] To whom correspondence should be addressed.

1. INTRODUCTION

The global attractors of the autonomous and non-autonomous 2D Navier–Stokes (N.–S.) systems were studied in a number of papers and monographs (see, e.g. [2,5,14,17,19] and the references therein). Some problems related to the homogenization and averaging of global attractors of the N.–S. systems and other evolution equations of mathematical physics with rapidly (non-singularly) oscillating coefficients and terms were studied in [3,6,8,10,11,12,20–22] and many other papers.

In the present paper, we study the global attractor $\mathcal{A}^\varepsilon$ of the non-autonomous 2D N.–S. system with singularly oscillating external force of the form $g_0(x,t)+\frac{1}{\varepsilon^\rho}g_1\left(\frac{x}{\varepsilon},t\right)$, $x\in\Omega\Subset\mathbb{R}^2, t\in\mathbb{R}, 0\leqslant\rho\leqslant 1$. The main attention is focused on the behaviour of $\mathcal{A}^\varepsilon$ as $\varepsilon\to 0+$.

Excluding the pressure in a standard manner, we consider the non-autonomous 2D N.–S. system

$$\partial_t u+\nu Lu+B(u,u)=Pg_0(\cdot,t)+\frac{1}{\varepsilon^\rho}Pg_1\left(\frac{\cdot}{\varepsilon},t\right),\quad \operatorname{div}u=0,\quad u|_{\partial\Omega}=0, \tag{1.1}$$

where P is the Leray orthogonal projector from $L_2(\Omega)^2$ onto the space H of divergence free vector field with finite L_2-norm.

To begin with, we assume that the vector functions $g_0(x,t)$, $x\in\Omega$, $t\in\mathbb{R}$, and $g_1(z,t)$, $z\in\mathbb{R}^2$, $t\in\mathbb{R}$, are translation bounded in the spaces $L_2^b(\mathbb{R};L_2(\Omega)^2)$ and $L_2^b(\mathbb{R};Z)$, respectively (see Section 2). Then Eq. (1.1) generates a process $\{U_\varepsilon(t,\tau),\ t\geqslant\tau,\tau\in\mathbb{R}\}$ acting in H by the formula $U_\varepsilon(t,\tau)u_\tau=u(t)$, where, for an arbitrary given $u_\tau(\cdot)\in H$, $u(t)=u(\cdot,t), t\geqslant\tau$, is the solution of Eq. (1.1) with initial data $u(\cdot,\tau)=u_\tau(\cdot)$. In Section 2, we show that the process $\{U_\varepsilon(t,\tau)\}$ has the uniform (with respect to (w.r.t.) $\tau\in\mathbb{R}$) global attractor $\mathcal{A}^\varepsilon$ that is bounded in H for every fixed $\varepsilon>0$. Moreover, we prove that

$$\|\mathcal{A}^\varepsilon\|_H:=\sup\left\{\|u\|_H\,|\,u\in\mathcal{A}^\varepsilon\right\}\leqslant C_0+C_1\varepsilon^{-\rho},\quad \forall\varepsilon>0,\quad \rho\geqslant 0$$

and the constants C_0 and C_1 are independent of ε and ρ. Note that the size of the attractor $\mathcal{A}^\varepsilon$ in the space H may grow up to infinity as $\varepsilon\to 0+$.

In Section 3, we assume that the function $g_1(z,t)$ has a divergence representation

$$g_1(z,t)=\partial_{z_1}G_1(z,t)+\partial_{z_2}G_2(z,t),\quad z=(z_1,z_2)\in\mathbb{R}^2,\quad t\in\mathbb{R}, \tag{1.2}$$

where the functions $G_j(z,t)\in L_2^b(\mathbb{R};Z)$ for $j=1,2$. Then we prove the theorem on the uniform boundedness of global attractors $\mathcal{A}^\varepsilon$ with respect to $\varepsilon\in]0,1]$:

$$\|\mathcal{A}^\varepsilon\|_H\leqslant C_2,\ \forall\varepsilon\in]0,1].$$

Along with the original N.–S. system (1.1), we consider its "limiting" system

$$\partial_t u + \nu L u + B(u,u) = P g_0(\cdot,t), \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \tag{1.3}$$

having the uniform global attractor $\mathcal{A}^0$ bounded in H: $\|\mathcal{A}^\varepsilon\|_H \leqslant C_0$.

In Section 4, we study the deviation $w(x,t) = u^\varepsilon(x,t) - u^0(x,t)$ of a solution $u^\varepsilon(x,t)$ of Eq. (1.1) from a solution $u^0(x,t)$ of Eq. (1.3) with the same initial data $u^\varepsilon(x,\tau) = u^0(x,\tau)$. If the function $g_1(z,t)$ satisfies the above divergence condition, then we prove the following estimate:

$$\|w(t)\|_H \leqslant \varepsilon^{(1-\rho)} C e^{r(t-\tau)}, \quad \forall \varepsilon, 0 < \varepsilon \leqslant 1,$$

where the constants C and r are independent of ε and $0 \leqslant \rho \leqslant 1$.

In Section 5, the translation compact (tr.c.) functions $g_0(x,t)$ and $g_1(z,t)$ in the spaces $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$, respectively, are defined. It turns out that the notion of a tr.c. function is very convenient in problems connected with global attractors of non-autonomous evolution equations (see [5]). We formulate the necessary and sufficient conditions for a function to be tr.c. in the corresponding space. Note that almost periodic functions with values in H or Z are tr.c. as well, but the class of tr.c. functions is much wider. Assuming that the functions $g_0(x,t)$ and $g_1(z,t)$ are tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; (\Omega)^2)$ and $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$, respectively, we prove that the function $g^\varepsilon(x,t) = P g_0(x,t) + \frac{1}{\varepsilon^\rho} + P g_1\left(\frac{x}{\varepsilon}, t\right)$ is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; H)$ for every $\varepsilon > 0$.

To describe the structure of the global attractor $\mathcal{A}^\varepsilon$ of Eq. (1.1) we also consider the family of equations

$$\partial_t \hat{u} + \nu L \hat{u} + B(\hat{u}, \hat{u}) = \hat{g}^\varepsilon(x,t), \quad \operatorname{div} \hat{u} = 0, \quad \hat{u}|_{\partial\Omega} = 0 \tag{1.4}$$

with external forces $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$. Here, $\mathcal{H}(g^\varepsilon) = [\{g^\varepsilon(\cdot, t+h)) \mid h \in \mathbb{R}\}]_{L_2^{\mathrm{loc}}(\mathbb{R}; H)}$ is the hull of the function $g^\varepsilon(x,t) = P g_0(x,t) + \frac{1}{\varepsilon^\rho} + P g_1\left(\frac{x}{\varepsilon}, t\right)$ in the space $L_2^{\mathrm{loc}}(\mathbb{R}; H)$ (here and below $[B]_X$, denotes the closure of the set $B \subseteq X$ in the topological space X). For every $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, Eq. (1.4) generates the process $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$ acting in H. Note that the processes $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$ have properties similar to those of the process $\{U_{g^\varepsilon}(t,\tau)\} = \{U_\varepsilon(t,\tau)\}$ corresponding to the 2D N.–S. system (1.1) with original external force $g^\varepsilon(x,t) = P g_0(x,t) + \varepsilon^{-\rho} P g_1(x/\varepsilon, t)$ (see Section 6).

The kernel $\mathcal{K}_{\hat{g}^\varepsilon}$ of Eq. (1.4) is said to be the family of all complete solutions $\hat{u}(t), t \in \mathbb{R}$, of (1.4) which are bounded in the norm of H: $\|\hat{u}(t)\|_H \leqslant M_{\hat{u}}$, $\forall t \in \mathbb{R}$. The set $\mathcal{K}_{\hat{g}^\varepsilon}(s) = \left\{\hat{u}(s) \mid \hat{u} \in \mathcal{K}_{\hat{g}^\varepsilon}\right\}$, $s \in \mathbb{R}$, (belonging to

H) is called the kernel section at time $t=s$. In Section 6 we present the following fact concerning the structure of the global attractor (see [5]):

$$\mathcal{A}^\varepsilon = \bigcup_{\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)} \mathcal{K}_{\hat{g}^\varepsilon}(0).$$

In Section 7, we prove the main theorem of the paper: if the functions $g_0(x,t)$ and $g_1(z,t)$ are tr.c. in the corresponding spaces, the function $g_1(z,t)$ has a divergence representation (1.2), and $0 \leqslant \rho < 1$, then the global attractors $\mathcal{A}^\varepsilon$ of Eq. (1.1) converges to the global attractor $\mathcal{A}^0$ of the "limiting" Eq. (1.3) in the space H as $\varepsilon \to 0+$, that is,

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \to 0 \text{ as } \varepsilon \to 0+. \tag{1.5}$$

In the concluding section, we study the 2D N.–S. system (1.1) when the Grashof number $G := \lambda_1^{-1}\nu^{-2}\left\|g^0\right\|_{L_2^b(\mathbb{R};H)}$ of the "limiting" N.–S. system (1.3) is small. In this case, the global attractor $\mathcal{A}^0$ is exponential, i.e. it attracts bounded sets of initial data with exponential rate. In this section, we present an estimate for the Hausdorff distance from $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$:

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant C(\rho)\varepsilon^{1-\rho}, \quad \forall \varepsilon \in]0,1], \quad 0 \leqslant \rho < 1. \tag{1.6}$$

It is clear that (1.6) implies (1.5) for this particular case. However, (1.5) holds for an arbitrary value of the Grashof number of the "limiting" N.–S. system (1.3).

2. GLOBAL ATTRACTOR OF THE 2D NAVIER–STOKES SYSTEM WITH SINGULARLY OSCILLATING EXTERNAL FORCE

We consider the non-autonomous 2D N.–S. system of the form

$$\begin{gathered}\partial_t u + u^1\partial_{x_1}u + u^2\partial_{x_2}u = \nu\Delta u - \nabla p + g_0(x,t) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right), \\ \partial_{x_1}u_1 + \partial_{x_2}u_2 = 0,\ u|_{\partial\Omega} = 0,\ x := (x_1, x_2) \in \Omega,\ \Omega \Subset \mathbb{R}^2.\end{gathered} \tag{2.1}$$

Here, $x \in \Omega$, $u = u(x,t) = (u^1(x,t), u^2(x,t))$ is the velocity vector field, $p = p(x,t)$ is the pressure and ν is the kinematic viscosity. We assume that the domain Ω is bounded and the origin $0 \in \Omega$. In Eq. (2.1), ε is a small parameter, $0 < \varepsilon \leqslant 1$, and ρ is fixed, $0 \leqslant \rho \leqslant 1$.

The vector functions $g_0(x,t) = (g_{01}(x,t), g_{02}(x,t)), x \in \Omega, t \in \mathbb{R}$, and $g_1(z,t) = (g_{11}(z,t), g_{12}(z,t)), z \in \mathbb{R}^2, t \in \mathbb{R}$, are given. The function $g_0(x,t) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right)$ is called the *external force*. We assume that, for every fixed ε, this external force belongs to the space $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ (we shall clarify

this assumption later on). Under this condition, the Cauchy problem for Eq. (2.1) is well-studied (see, for example [2,5,9,13,16,18]).

As usual, we denote by H and $V = H^1$ the function spaces that are closures of the set $\mathcal{V}_0 := \left\{ v \in \left(C_0^\infty(\Omega)\right)^2 \mid \partial_{x_1} v_1(x) + \partial_{x_2} v_2(x) = 0,\ \forall x \in \Omega \right\}$ in the norms $|\cdot|$ and $\|\cdot\|$ of the spaces $L_2(\Omega)^2$ and $H_0^1(\Omega)^2$, respectively. We recall that

$$\|v\|^2 = |\nabla v|^2 = \int_\Omega \left(|\partial_{x_1} v^1(x)|^2 + |\partial_{x_2} v^1(x)|^2 + |\partial_{x_1} v^2(x)|^2 + |\partial_{x_2} v^2(x)|^2 \right) dx.$$

The space $V' = V^*$ is dual to the space V. We denote by P the orthogonal projector from $L_2(\Omega)^2$ onto H and its different extensions. We set

$$g^\varepsilon(x,t) = P g_0(x,t) + \frac{1}{\varepsilon^\rho} P g_1\left(\frac{x}{\varepsilon}, t\right).$$

In a standard way, applying the operator P to both sides of Eq. (2.1), we exclude the pressure $p(x,t)$ and obtain the following equation for the velocity vector field $u(x,t)$:

$$\partial_t u + \nu L u + B(u,u) = g^\varepsilon(x,t), \tag{2.2}$$

where $L = -P\Delta$ is the Stokes operator, $B(u,v) = P\left[u^1 \partial_{x_1} v + u^2 \partial_{x_2} v\right]$ and $g^\varepsilon(\cdot,t) \in L_2^{loc}(\mathbb{R}; H)$. The Stokes operator L has the domain $V \cap H^2(\Omega)^2$ and is self-adjoint and positive. The minimal eigenvalue λ_1 of the operator L is positive.

We assume that the function $g_0(\cdot,t) \in L_2(\Omega)^2$ for almost every $t \in \mathbb{R}$ and has a finite norm in the space $L_2^b(\mathbb{R}; L_2(\Omega)^2)$, that is,

$$\|g_0\|^2_{L_2^b(\mathbb{R}; L_2(\Omega)^2)} = \|g_0\|^2_{L_2^b} := \sup_{\tau \in \mathbb{R}} \int_\tau^{\tau+1} \left(\|g_0(\cdot,s)\|^2_{L_2(\Omega)^2} \right) ds < +\infty. \tag{2.3}$$

To describe the vector function $g_1(z,t), z = (z_1, z_2) \in \mathbb{R}^2, t \in \mathbb{R}$, we use the space $Z = L_2^b(\mathbb{R}^2_z; \mathbb{R}^2)$. By definition, a vector function $\varphi(z) = (\varphi_1(z_1,z_2), \varphi_2(z_1,z_2)) \in Z$, if

$$\|\varphi(\cdot)\|^2_Z = \|\varphi(\cdot)\|^2_{L_2^b(\mathbb{R}^2_z; \mathbb{R}^2)} := \sup_{(z_1,z_2) \in \mathbb{R}^2} \int_{z_1}^{z_1+1} \int_{z_2}^{z_2+1} |\varphi(\zeta_1,\zeta_2)|^2 d\zeta_1 d\zeta_2 < +\infty.$$

We now assume that the function $g_1(\cdot, t) \in Z$ for almost every $t \in \mathbb{R}$ and has a finite norm in the space $L_2^b(\mathbb{R}; Z)$, that is,

$$\|g_1(\cdot)\|^2_{L_2^b(\mathbb{R};Z)} := \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \left(\|g_1(\cdot, s)\|_Z^2\right) ds$$
$$= \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \left(\sup_{(z_1, z_2)\in\mathbb{R}^2} \int_{z_1}^{z_1+1} \int_{z_2}^{z_2+1} |g_1(\zeta_1, \zeta_2, s)|^2 d\zeta_1 d\zeta_2 \right) ds < +\infty. \tag{2.4}$$

For Eq. (2.1), we consider the initial data at an arbitrary time $\tau \in \mathbb{R}$:

$$u|_{t=\tau} = u_\tau,\ u_\tau \in H. \tag{2.5}$$

Recall that the trilinear form

$$b(u, v, w) = (B(u, v), w) = \int_\Omega \sum_{i,j=1}^{2} u^i(x)\partial_{x_i} v^j(x) w^j(x) dx$$

is continuous on $V \times V \times V$ and the operator $B(u, v)$ maps $V \times V$ to V'. The form $b(u, v, w)$ satisfies the following identities:

$$b(u, v, v) = 0,\ b(u, v, w) = -b(u, w, v), \quad \forall u, v, w \in V. \tag{2.6}$$

Besides, the following inequality holds:

$$|b(u, v, w)| \leqslant c_0^2 |u|^{1/2} \|u\|^{1/2} |v|^{1/2} \|v\|^{1/2} \|w\|, \quad \forall u, v, w \in V \tag{2.7}$$

(see [9, 13, 18]), where the constant c_0 is taken from the Ladyzhenskaya inequality

$$\|f\|_{L_4(\Omega)} \leqslant c_0 |f|^{1/2} |\nabla f|^{1/2}, \quad \forall f \in H_0^1(\Omega). \tag{2.8}$$

We note that the constant c_0 in (2.8) is independent of Ω. It follows from (2.7) that

$$\|B(u, u)\|_{V'} \leqslant c_0^2 |u| \|u\|, \quad \forall u \in V. \tag{2.9}$$

For a fixed $\varepsilon > 0$, the Cauchy problem (2.1) and (2.5) has a unique solution $u(t) := u(x, t)$ in a weak sense, that is, $u(t) \in C(\mathbb{R}_\tau; H) \cap L_2^{\mathrm{loc}}(\mathbb{R}_\tau; V)$, $\partial_t u \in L_2^{\mathrm{loc}}(\mathbb{R}_\tau; V')$, and $u(t)$ satisfies Eq. (2.1) in the distribution sense of the space $\mathcal{D}'(\mathbb{R}_\tau; V')$, where $\mathbb{R}_\tau = [\tau, +\infty)$ (see [2, 5, 9, 13, 16, 19]).

Recall that every weak solution $u(t)$ of Eq. (2.1) satisfies the following energy equality:

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2+\nu\|u(t)\|^2=\left\langle u(t),g^{\varepsilon}(t)\right\rangle,\quad \forall t\geqslant\tau, \tag{2.10}$$

where the function $|u(t)|^2$ is absolutely continuous in t (see, for example [2,5,9,13,16,19]). In the proof of (2.10), property (2.6) is essential.

We need the following lemma proved in [5].

Lemma 2.1. *Let a real function $y(t), t\geqslant 0$, be uniformly continuous and satisfy the inequality*

$$y'(t)+\gamma y(t)\leqslant f(t),\quad \forall t\geqslant 0, \tag{2.11}$$

where $\gamma>0$, $f(t)\geqslant 0$ for all $t\geqslant 0$, and $f\in L_1^{loc}(\mathbb{R}_+)$. Suppose also that

$$\int_t^{t+1} f(s)ds\leqslant M,\quad \forall t\geqslant 0. \tag{2.12}$$

Then

$$y(t)\leqslant y(0)e^{-\gamma t}+M(1+\gamma^{-1}),\quad \forall t\geqslant 0. \tag{2.13}$$

For the reader's convenience, we sketch the proof of the lemma.

Proof. Multiplying (2.11) by $e^{\gamma t}$ and integrating, we have

$$\begin{aligned}&\frac{d}{dt}\left(y(t)e^{\gamma t}\right)\leqslant f(t)e^{\gamma t},\\ &y(t)\leqslant y(0)e^{-\gamma t}+\int_0^t e^{-\gamma(t-s)}f(s)\,ds.\end{aligned} \tag{2.14}$$

We now estimate the integral on the right-hand side of (2.14) as follows:

$$\begin{aligned}\int_0^t f(s)e^{-\gamma(t-s)}ds&\leqslant\int_{t-1}^t f(s)e^{-\gamma(t-s)}ds+\int_{t-2}^{t-1} f(s)e^{-\gamma(t-s)}ds+\cdots\\ &\leqslant\int_{t-1}^t f(s)ds+e^{-\gamma}\int_{t-2}^{t-1} f(s)ds+e^{-2\gamma}\int_{t-3}^{t-2} f(s)ds+\cdots\\ &\leqslant M\left(1+e^{-\gamma}+e^{-2\gamma}+\cdots\right)=M\left(1-e^{-\gamma}\right)^{-1}<M\left(1+\gamma^{-1}\right).\end{aligned} \tag{2.15}$$

Thus, (2.14) and (2.15) imply (2.13). □

Using the standard transformations, we obtain from (2.10) the following estimate:

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2+\nu\|u(t)\|^2=(g^\varepsilon(t),u(t))\leqslant\|g^\varepsilon(t)\|_{V'}\|u(t)\|$$
$$\leqslant\frac{\nu}{2}\|u(t)\|^2+\frac{1}{2\nu}\|g^\varepsilon(t)\|^2_{V'}\leqslant\frac{\nu}{2}\|u(t)\|^2+\frac{1}{2\nu\lambda_1}|g^\varepsilon(t)|^2.$$

Therefore,

$$\frac{d}{dt}|u(t)|^2+\nu\lambda_1|u(t)|^2\leqslant\frac{d}{dt}|u(t)|^2+\nu\|u(t)\|^2\leqslant(\nu\lambda_1)^{-1}|g^\varepsilon(t)|^2. \quad (2.16)$$

Here, we have used the Poincaré inequalities of the form $\|v\|^2_{V'}\leqslant\lambda_1^{-1}|v|^2$ and $\lambda_1|v|^2\leqslant\|v\|^2$. Thus, we have the differential inequality

$$\frac{d}{dt}|u(t)|^2+\nu\lambda_1|u(t)|^2\leqslant(\nu\lambda_1)^{-1}|g^\varepsilon(t)|^2.$$

Applying Lemma 2.1 with $y(t)=|u(t+\tau)|^2$, $f(t)=(\nu\lambda_1)^{-1}|g^\varepsilon(t+\tau)|^2$, $\gamma=\nu\lambda_1$ and $M=(\nu\lambda_1)^{-1}\|g^\varepsilon\|^2_{L_2^b(\mathbb{R};H)}$, we obtain the following main a priori estimate for a weak solution $u(t)$ of Eq. (2.1):

$$|u(t+\tau)|^2\leqslant|u(\tau)|^2e^{-\nu\lambda_1 t}+D\|g^\varepsilon\|^2_{L_2^b(\mathbb{R};H)}, \quad (2.17)$$

where $D=(\nu\lambda_1)^{-1}\left(1+(\nu\lambda_1)^{-1}\right)$. Inequality (2.16) also implies that

$$|u(t)|^2+\nu\int_\tau^t\|u(s)\|^2ds\leqslant|u(\tau)|^2+(\nu\lambda_1)^{-1}\int_\tau^t|g^\varepsilon(s)|^2ds. \quad (2.18)$$

Lemma 2.2. *If the function $\varphi(z)\in Z=L_2^b(\mathbb{R}^2_z;\mathbb{R}^2)$, then $\varphi\left(\frac{x}{\varepsilon}\right)\in L_2(\Omega)^2$ for all $\varepsilon>0$ and*

$$\left\|\varphi\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L_2(\Omega_x)^2}\leqslant C\,\|\varphi(\cdot)\|_{L_2^b(\mathbb{R}^2_z;\mathbb{R}^2)}, \quad (2.19)$$

where the constant C depends on the area of the domain Ω only.

Proof. Indeed, changing the variables $\frac{x}{\varepsilon}=z$, $dx=\varepsilon^2dz$, we have

$$\left\|\varphi\left(\frac{\cdot}{\varepsilon}\right)\right\|^2_{L_2(\Omega)^2}=\int_\Omega\left|\varphi\left(\frac{x}{\varepsilon}\right)\right|^2dx=\varepsilon^2\int_{\varepsilon^{-1}\Omega}|\varphi(z)|^2dz$$
$$\leqslant C^2\varepsilon^{-2}\sup_{(z_1,z_2)\in\mathbb{R}^2}\varepsilon^2\int_{z_1}^{z_1+1}\int_{z_2}^{z_2+1}|\varphi(\zeta_1,\zeta_2)|^2d\zeta_1d\zeta_2=C^2\|\varphi(\cdot)\|^2_{L_2^b(\mathbb{R}^2_z;\mathbb{R}^2)}.$$

Here, in the last inequality, we have used the fact that the domain $\varepsilon^{-1}\Omega$ can be covered by at most $C^2\varepsilon^{-2}$ unit squares of the form $[z_1, z_1+1]\times[z_2, z_2+1]$, where C depends on the area of the domain Ω only. $\square$

Corollary 2.1. *If the functions* $g_0(x,t)\in L_2^b(\mathbb{R}; L_2(\Omega)^2)$ *and* $g_1(z,t)\in L_2^b(\mathbb{R}; Z)$, *where* $Z=L_2^b(\mathbb{R}_z^2;\mathbb{R}^2)$, *then the external force* $g^\varepsilon(x,t)=Pg_0(x,t)+\frac{1}{\varepsilon^\rho}Pg_1\left(\frac{x}{\varepsilon},t\right)$ *belongs to the space* $L_2^b(\mathbb{R}; H)$ *and*

$$\|g^\varepsilon\|_{L_2^b(\mathbb{R};H)}\leqslant\|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)}+\frac{C}{\varepsilon^\rho}\|g_1\|_{L_2^b(\mathbb{R};Z)}, \tag{2.20}$$

where the constant C *is independent of* ε.

Inequality (2.20) follows directly from Lemma 2.2 and the formulas for the norm (2.3) and (2.4) in the spaces $L_2^b(\mathbb{R}; L_2(\Omega)^2)$ and $L_2^b(\mathbb{R}; Z)$.

We now apply inequality (2.20) in (2.17) and obtain

$$|u(t+\tau)|^2\leqslant|u(\tau)|^2e^{-\nu\lambda_1 t}+C_0^2+\varepsilon^{-2\rho}C_1^2, \tag{2.21}$$

where the constants C_0 and C_1 depend on ν,λ_1, and the norms $\|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)}$ and $\|g_1\|_{L_2^b(\mathbb{R};Z)}$, respectively.

We now consider the process $\{U_\varepsilon(t,\tau)\}:=\{U_\varepsilon(t,\tau), t\geqslant\tau, \tau\in\mathbb{R}\}$ corresponding to problem (2.2) and (2.5) and acting in the space H (see [5]). Recall that the mapping $U_\varepsilon(t,\tau): H\to H$ is defined by the formula

$$U_\varepsilon(t,\tau)u_\tau=u(t),\quad \forall u_\tau\in H,\quad t\geqslant\tau,\quad \tau\in\mathbb{R}, \tag{2.22}$$

where $u(t)$ is the solution of (2.2) and (2.5).

It follows from estimate (2.21) that for every $\varepsilon, 0<\varepsilon\leqslant 1$, the process $\{U_\varepsilon(t,\tau)\}$ has the *uniformly* (w.r.t. $\tau\in\mathbb{R}$) *absorbing set*

$$B_{0,\varepsilon}=\left\{v\in H\mid |v|\leqslant 2(C_0+C_1\varepsilon^{-\rho})\right\} \tag{2.23}$$

and the set $B_{0,\varepsilon}$ is bounded in H for a fixed ε. That is, for any bounded (in H) set B, there exists a time $t'=t'(B)$ such that the set $U(t+\tau,\tau)B\subseteq B_{0,\varepsilon}$ for all $t\geqslant t(B)$ and $\tau\in\mathbb{R}$.

Using the standard argument, we prove that the process $\{U_\varepsilon(t,\tau)\}$ has a compact in H uniformly absorbing set

$$B_{1,\varepsilon}=\left\{v\in V\mid \|v\|\leqslant C_2(\nu,\lambda_1,\|g^\varepsilon\|_{L_2^b(\mathbb{R};H)})\right\}, \tag{2.24}$$

where $C_2(y_1,y_2,y_3)$ is a positive increasing function in each y_j, $j=1,2,3$ (see [5, Chapter 6] for more details). Using (2.20) we obtain that

$$B_{1,\varepsilon}=\left\{v\in V\mid \|v\|\leqslant C_2(\nu,\lambda_1,C_0+C_1\varepsilon^{-\rho})\right\} \tag{2.25}$$

and the absorbing set $B_{1,\varepsilon}$ is bounded in V and, therefore, compact in H. Recall that a process having a compact uniformly absorbing set is called *uniformly compact*. We have established the following result.

Proposition 2.1. *For any fixed $\varepsilon > 0$, the process $\{U_\varepsilon(t,\tau)\}$ corresponding to the problem* (2.1) *and* (2.5) *is uniformly compact in the space H and it has a uniformly absorbing set $B_{1,\varepsilon}$ (bounded in V) defined in* (2.25).

It follows from Proposition 2.1 and from the general theorem proved in [5] that the process $\{U_\varepsilon(t,\tau)\}$ has the *uniform global attractor* $\mathcal{A}^\varepsilon$ and $\mathcal{A}^\varepsilon \subseteq B_{0,\varepsilon} \cap B_{1,\varepsilon}$ (see [5, Chapters 4 and 7]). Recall that the set $\mathcal{A}^\varepsilon$ has the following properties:

(i) for any bounded (in H) set B,

$$\sup_{\tau\in\mathbb{R}} \operatorname{dist}_H(U_\varepsilon(t+\tau,\tau)B, \mathcal{A}^\varepsilon) \to 0\,(t\to+\infty); \tag{2.26}$$

(ii) $\mathcal{A}^\varepsilon$ is the minimal closed set that satisfies (2.26).

In (2.26) $\operatorname{dist}_H(X,Y)$ denotes the Hausdorff semi-distance from a set $X \subset H$ to a set $Y \subset H$:

$$\operatorname{dist}_H(X,Y) := \sup_{x\in X} \operatorname{dist}_H(x,Y) = \sup_{x\in X} \inf_{y\in Y} \|x-y\|_H .$$

Since $\mathcal{A}^\varepsilon \subseteq B_{0,\varepsilon}$, we conclude from (2.21) and (2.23) that

$$\|\mathcal{A}^\varepsilon\|_H \leqslant (C_0 + C_1\varepsilon^{-\rho}). \tag{2.27}$$

Remark 2.1. Generally speaking, for $\rho > 0$, the norm in H of the uniform global attractor $\mathcal{A}^\varepsilon$ of the 2D N.–S. system (2.1) may grow up to infinity as $\varepsilon \to 0+$. In the next sections, we present conditions that provide the uniform boundedness of $\mathcal{A}^\varepsilon$ in H with respect to ε. Moreover, we also study the convergence of $\mathcal{A}^\varepsilon$ as $\varepsilon \to 0+$ to the global attractor $\mathcal{A}^0$ of the corresponding "limiting" equation.

Along with the original N.–S. system (2.1), we consider the following "limiting" system

$$\begin{gathered} \partial_t u + u^1\partial_{x_1}u + u^2\partial_{x_2}u = \nu\Delta u - \nabla p + g_0(x,t), \\ \partial_{x_1}u_1 + \partial_{x_2}u_2 = 0,\ u|_{\partial\Omega} = 0 \end{gathered} \tag{2.28}$$

without the term on the right-hand side depending on ε. Excluding the pressure, we obtain the equivalent equation

$$\partial_t u + \nu L u + B(u,u) = P g_0(x,t), \tag{2.29}$$

where, clearly $Pg_0(x,t) \in L_2^b(\mathbb{R}; H)$. Then the Cauchy problem for Eq. (2.29) also has a unique solution $u(t) := u(x,t)$ (in a weak distribution sense). Hence, there is a "limiting" process $\{U_0(t,\tau)\}$ acting in H: $U_0(t,\tau)u_\tau = u(t), t \geqslant \tau,\ \tau \in \mathbb{R}$, where $u(t)$ is the solution of the problem (2.29) and (2.5). Similarly to (2.17) and (2.18), we have the main a priori estimates

$$|u(t+\tau)|^2 \leqslant |u(\tau)|^2 e^{-\nu\lambda_1 t} + D\|Pg_0\|^2_{L_2^b(\mathbb{R};H)}, \tag{2.30}$$

$$|u(t)|^2 + \nu\int_\tau^t \|u(s)\|^2 ds \leqslant |u(\tau)|^2 + (\nu\lambda_1)^{-1}\int_\tau^t |Pg_0(s)|^2 ds. \tag{2.31}$$

It follows from (2.21) that

$$|u(t+\tau)|^2 \leqslant |u(\tau)|^2 e^{-\nu\lambda_1 t} + C_0^2, \tag{2.32}$$

which implies that the set

$$B_{0,0} = \{v \in H \mid |v| \leqslant 2C_0\} \tag{2.33}$$

is uniformly absorbing for the process $\{U_0(t,\tau)\}$. (the constant C_0 is the same as in (2.21)). Moreover, this process has a compact (in H) absorbing set

$$B_{1,0} = \{v \in V \mid \|v\| \leqslant C_2(\nu, \lambda_1, C_0)\}. \tag{2.34}$$

Therefore, the process $\{U_0(t,\tau)\}$ is uniformly compact and Proposition 2.1 holds for the "limiting" case $\varepsilon = 0$ as well. In particular, the process $\{U_0(t,\tau)\}$ also has a compact global attractor $\mathcal{A}^0$ such that $\mathcal{A}^0 \subset B_{0,0} \cap B_{1,0}$ and

$$\left\|\mathcal{A}^0\right\|_H \leqslant C_0. \tag{2.35}$$

In the next sections, we study the convergence of the global attractors $\mathcal{A}^\varepsilon$ of the 2D N.–S. system to the global attractor $\mathcal{A}^0$ of the "limiting" equation as $\varepsilon \to 0+$.

3. DIVERGENCE CONDITION AND BOUNDEDNESS OF $\mathcal{A}^\varepsilon$

We consider the non-autonomous 2D N.–S. system (2.1) written in the equivalent form (2.2). As in Section 2, we assume that the external force $g^\varepsilon(x,t) = Pg_0(x,t) + \frac{1}{\varepsilon^\rho} Pg_1\left(\frac{x}{\varepsilon}, t\right)$ satisfies the following assumptions: the function $g_0(x,t), x \in \Omega, t \in \mathbb{R}$, satisfies (2.3), i.e. $\|g_0(\cdot)\|^2_{L_2^b(\mathbb{R};L_2(\Omega)^2)} < +\infty$

and the function $g_1(z,t)$, $z \in \mathbb{R}^2, t \in \mathbb{R}$, satisfies (2.4), i.e. $\|g_1(\cdot)\|^2_{L_2^b(\mathbb{R};Z)} < +\infty$, where $Z = L_2^b(\mathbb{R}^2_z; \mathbb{R}^2)$.

We now assume that the function $g_1(z,t)$ satisfies the following additional

Divergence condition. There exist vector functions $G_j(z,t) \in L_2^b(\mathbb{R}; Z)$ such that $\partial_{z_j} G_j(z,t) \in L_2^b(\mathbb{R}; Z)$, $j = 1, 2$, and

$$\partial_{z_1} G_1(z_1, z_2, t) + \partial_{z_2} G_2(z_1, z_2, t) = g_1(z_1, z_2, t), \ \forall (z_1, z_2) \in \mathbb{R}^2, t \in \mathbb{R}. \quad (3.1)$$

Theorem 3.1. *If the function $g_1(z,t)$ satisfies the divergence condition* (3.1)*, then, for every ρ, $0 \leqslant \rho \leqslant 1$, the global attractors $\mathcal{A}^\varepsilon$ of the 2D N.–S. system* (2.1) *are uniformly (w.r.t. $\varepsilon \in]0,1]$) bounded in H, that is,*

$$\|\mathcal{A}^\varepsilon\|_H \leqslant C_2, \ \forall \varepsilon \in]0,1], \quad (3.2)$$

where C_2 is independent of ε.

Proof. Taking the scalar product in H of Eq. (2.2) with $u(t)$, we obtain the following inequality:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 &= \langle u(t), g^\varepsilon(t)\rangle \\ &= (g_0(\cdot,t), u(\cdot,t)) + \varepsilon^{-\rho}\left(g_1\left(\frac{\cdot}{\varepsilon}, t\right), u(\cdot,t)\right). \end{aligned} \quad (3.3)$$

For the first term in (3.3), we use the Cauchy inequality and the Poincaré inequality

$$(g_0(\cdot,t), u(\cdot,t)) \leqslant \frac{1}{4}\nu\|u(t)\|^2 + \frac{1}{\nu\lambda_1}|g_0(t)|^2. \quad (3.4)$$

For the second term in (3.3) using (3.1), we have

$$\begin{aligned} \varepsilon^{-\rho}\left(g_1\left(\frac{\cdot}{\varepsilon}, t\right), u(\cdot,t)\right) &= \varepsilon^{-\rho}\sum_{j=1}^{2}\int_\Omega \left(\partial_{z_j} G_j\left(\frac{x}{\varepsilon}, t\right), u(x,t)\right) dx \\ &= \varepsilon^{1-\rho}\sum_{j=1}^{2}\int_\Omega \left(\partial_{x_j} G_j\left(\frac{x}{\varepsilon}, t\right), u(x,t)\right) dx \\ &= -\varepsilon^{1-\rho}\sum_{j=1}^{2}\int_\Omega \left(G_j\left(\frac{x}{\varepsilon}, t\right), \partial_{x_j} u(x,t)\right) dx \\ &\leqslant \varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega \left|G_j\left(\frac{x}{\varepsilon}, t\right)\right|^2 dx + \frac{1}{4}\nu\|u(t)\|^2. \end{aligned} \quad (3.5)$$

In the third equality, we have integrated by parts in x taking into account the zero boundary condition in (2.1). Using (3.5) and (3.4) in (3.3), we have

$$\frac{d}{dt}|u(t)|^2+\nu\|u(t)\|^2\leqslant\frac{2}{\nu\lambda_1}|g_0(t)|^2+2\varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega\left|G_j\left(\frac{x}{\varepsilon},t\right)\right|^2dx$$

and therefore, due to the Poincaré inequality,

$$\frac{d}{dt}|u(t)|^2+\nu\lambda_1|u(t)|^2\leqslant\frac{2}{\nu\lambda_1}|g_0(t)|^2+2\varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega\left|G_j\left(\frac{x}{\varepsilon},t\right)\right|^2dx. \tag{3.6}$$

By the assumptions,

$$\int_t^{t+1}|g_0(t)|^2ds\leqslant\|g_0(\cdot)\|^2_{L_2^b(\mathbb{R};L_2(\Omega)^2)}=M_0,\ \forall t\in\mathbb{R}. \tag{3.7}$$

It follows from Lemma 2.2 that

$$\int_t^{t+1}\int_\Omega\left|G_j\left(\frac{x}{\varepsilon},\tau\right)\right|^2dxd\tau\leqslant C\left\|G_j(\cdot)\right\|^2_{L_2^b(\mathbb{R};Z)}=M_j,\quad\forall t\in\mathbb{R},\quad j=1,2, \tag{3.8}$$

where C is independent of ε.

Applying Lemma 2.1 with $y(t)=|u(t+\tau)|^2$, $f(t)=2(\nu\lambda_1)^{-1}|g_0(t+\tau)|^2+2\varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^2\int_\Omega\left|G_j\left(\frac{x}{\varepsilon},t+\tau\right)\right|^2dx$, $\gamma=\nu\lambda_1$, and $M=2(\nu\lambda_1)^{-1}\|g_0(\cdot)\|^2_{L_2^b(\mathbb{R};L_2(\Omega)^2)}+2\varepsilon^{2(1-\rho)}\nu^{-1}C\sum_{j=1}^2\left\|G_j(\cdot)\right\|^2_{L_2^b(\mathbb{R};Z)}$, we obtain the following main a priori estimate for the function $u(t)$:

$$|u(t+\tau)|^2\leqslant|u(\tau)|^2e^{-\nu\lambda_1t}+\left[2(\nu\lambda_1)^{-1}M_0+2\varepsilon^{2(1-\rho)}\nu^{-1}C(M_1+M_2)\right]D_1, \tag{3.9}$$

where $D_1=\left(1+(\nu\lambda_1)^{-1}\right)$.

Since $0\leqslant\rho\leqslant1$ and $0<\varepsilon\leqslant1$, inequality (3.9) implies that the process $\{U_\varepsilon(t,\tau)\}$ corresponding to Eq. (2.1) has a uniformly absorbing set

$$\tilde{B}=\{v\in H\mid|v|\leqslant C_2,\}, \tag{3.10}$$

where $C_2^2=2\left[2(\nu\lambda_1)^{-1}M_0+2\nu^{-1}C(M_1+M_2)\right]D_1$. It is clear, that the global attractor $\mathcal{A}$ is contained in the absorbing set $\tilde{B}$, i.e.

$$\left\|\mathcal{A}^\varepsilon\right\|_H\leqslant C_2,\ \forall\varepsilon,0<\varepsilon\leqslant1, \tag{3.11}$$

when the divergence condition (3.1) holds and the theorem is proved. □

4. ESTIMATE FOR THE DEVIATION OF SOLUTIONS OF THE ORIGINAL 2D NAVIER–STOKES SYSTEM FROM SOLUTIONS OF THE "LIMITING" SYSTEM

We consider Eq. (2.2)

$$\partial_t u + \nu L u + B(u,u) = P g_0(x,t) + \frac{1}{\varepsilon^\rho} P g_1\left(\frac{x}{\varepsilon}, t\right). \tag{4.1}$$

We assume that $g_0(x,t) \in L_2^b(\mathbb{R}; L_2(\Omega)^2)$ and $g_1(z,t) \in L_2^b(\mathbb{R}; Z)$. Moreover, we assume that the function $g_1(z,t)$ satisfies the divergence condition (3.1).

Along with Eq. (4.1), we consider the corresponding "limiting" Eq. (2.29)

$$\partial_t u^0 + \nu L u^0 + B(u^0, u^0) = P g_0(x,t). \tag{4.2}$$

We supplement Eqs. (4.1) and (4.2) with the same initial data at $t = \tau$:

$$u|_{t=\tau} = u_\tau, \quad u^0|_{t=\tau} = u_\tau,\ u_\tau \in \tilde{B}, \tag{4.3}$$

where the absorbing ball $\tilde{B}$ is defined in (3.10). Recall that the set $\tilde{B}$ is independent of ρ, $0 \leqslant \rho \leqslant 1$ and ε, $0 < \varepsilon \leqslant 1$.

Let $u(x,t)$ and $u^0(x,t)$ be the solutions of Eqs. (4.1) and (4.2), respectively, with the same initial data (4.3) taken from the ball $\tilde{B}$. We are going to estimate the deviation of $u(x,t)$ from $u^0(x,t)$ for $t \geqslant \tau$. We set $w(x,t) = u(x,t) - u^0(x,t)$. For simplicity, we consider the case $\tau = 0$. The function $w(x,t)$ satisfies the equation

$$\partial_t w + \nu L w + B(u,u) - B(u^0, u^0) = \frac{1}{\varepsilon^\rho} P g_1\left(\frac{x}{\varepsilon}, t\right) \tag{4.4}$$

and the zero initial data

$$w|_{t=0} = 0. \tag{4.5}$$

We note that

$$B(u,u) - B(u^0, u^0) = B(w, u^0) + B(u^0, w) + B(w, w).$$

Taking the scalar product in H of Eq. (4.4) with w, we have

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w(t)\|^2 + \Big\langle B(w, u^0), w\Big\rangle + \Big\langle B(u^0, w), w\Big\rangle + \langle B(w,w), w\rangle = \frac{1}{\varepsilon^\rho}\Big\langle g_1\left(\frac{\cdot}{\varepsilon}, t\right), w\Big\rangle. \tag{4.6}$$

It follows from (2.6) that $\langle B(u^0, w), w\rangle = 0$ and $\langle B(u^0, w), w\rangle = 0$. Therefore,

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w(t)\|^2 + \left\langle B(w, u^0(t)), w\right\rangle = \frac{1}{\varepsilon^\rho}\left\langle g_1\left(\frac{\cdot}{\varepsilon}, t\right), w\right\rangle. \tag{4.7}$$

Using the divergence condition, similarly to (3.5), we observe that

$$\begin{aligned}\varepsilon^{-\rho}\left\langle g_1\left(\frac{\cdot}{\varepsilon}, t\right), w\right\rangle &= -\varepsilon^{1-\rho}\sum_{j=1}^{2}\int_\Omega \left(G_j\left(\frac{x}{\varepsilon}, t\right), \partial_{x_j} w(x,t)\right)dx \\ &\leqslant \frac{1}{2}\varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega \left|G_j\left(\frac{x}{\varepsilon}, t\right)\right|^2 dx + \frac{1}{2}\nu\|w(t)\|^2. \end{aligned}\tag{4.8}$$

It follows from (2.7) and (2.6) that

$$\left|\left\langle B(w, u^0), w\right\rangle\right| = \left|\left\langle B(w, w), u^0\right\rangle\right| \leqslant c_0^2|w|\|w\|\|u^0\|. \tag{4.9}$$

Then

$$\left|\left\langle B(w, u^0), w\right\rangle\right| \leqslant c_0^2|w|\|u^0\|\|w\| \leqslant \frac{1}{2}\nu\|w\|^2 + \frac{1}{2}\frac{c_0^4}{\nu}|w|^2\|u^0\|^2. \tag{4.10}$$

Combining (4.8) and (4.10) in (4.7), we find that

$$\frac{d}{dt}|w(t)|^2 \leqslant \frac{c_0^4}{\nu}|w(t)|^2\|u^0(t)\|^2 + \varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega \left|G_j\left(\frac{x}{\varepsilon}, t\right)\right|^2 dx.$$

We set

$$z(t) = |w(t)|^2,\ \gamma(t) = c_0^4\nu^{-1}\|u^0(t)\|^2$$

and

$$b(t) = \varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega \left|G_j\left(\frac{x}{\varepsilon}, t\right)\right|^2 dx.$$

Then we have

$$z'(t) \leqslant b(t) + \gamma(t)z(t),\ z(0) = 0.$$

Applying Gronwall inequality, we obtain:

$$z(t) \leqslant \int_0^t b(s)\exp\left(\int_s^t \gamma(\theta)d\theta\right)ds \leqslant \left(\int_0^t b(s)ds\right)\exp\left(\int_0^t \gamma(s)ds\right). \tag{4.11}$$

Recall that $u^0(t)$ satisfies (2.31) and $u_0 \in \tilde{B}$, i.e.

$$\int_0^t \gamma(s)ds = c_0^4 \nu^{-1} \int_0^t \|u^0(t)\|^2 ds \leqslant c_0^2 \nu^{-2} \left(|u_0|^2 + (\nu\lambda_1)^{-1} \int_0^t |g_0(s)|^2 ds \right)$$
$$\leqslant c_0^4 \nu^{-2} \left(C_2^2 + (\nu\lambda_1)^{-1} (t+1) \|g_0(\cdot)\|^2_{L_2^b(\mathbb{R}; L_2(\Omega)^2)} \right) \leqslant C_3(t+1). \tag{4.12}$$

Using (3.8), we see that

$$\int_0^t b(s)ds = \varepsilon^{2(1-\rho)} \nu^{-1} \sum_{j=1}^{2} \int_0^t \int_\Omega \left| G_j\left(\frac{x}{\varepsilon}, s\right) \right|^2 dx\, ds$$
$$\leqslant \varepsilon^{2(1-\rho)} \nu^{-1} C(t+1) \sum_{j=1}^{2} \left\| G_j(\cdot) \right\|_{L_2^b(\mathbb{R}; Z)} \leqslant \varepsilon^{2(1-\rho)} \nu^{-1} (t+1)(M_1' + M_2') \tag{4.13}$$

Replacing (4.12) and (4.13) to (4.11), we find the following inequality

$$\begin{aligned} |w(t)|^2 &\leqslant \varepsilon^{2(1-\rho)} \nu^{-1} (t+1)(M_1' + M_2') e^{C_3(t+1)} \\ &= \varepsilon^{2(1-\rho)} \nu^{-1} (M_1' + M_2') \varepsilon^t e^{C_3(t+1)} = \varepsilon^{2(1-\rho)} C_4^2 e^{2rt}, \end{aligned} \tag{4.14}$$

where $C_4^2 = \nu^{-1}(M_1' + M_2')e^{C_3}$, $2r = C_3 + 1$. The constants C_4 and r are independent of ε. Inequality (4.14) holds for all ρ, $0 \leqslant \rho \leqslant 1$. We have proved the following

Theorem 4.1. *Let the function $g_1(z,t)$ satisfy the divergence condition* (3.1). *Then, for every initial data $u_\tau \in \tilde{B}$ (see* (3.10)*), the difference $w(x,t) = u(x,t) - u^0(x,t)$ of the solutions of the N.–S. equations* (4.1) *and* (4.2)*, respectively, with common initial data* (4.3) *taken from the ball $\tilde{B}$, satisfies the following inequality:*

$$|w(t)| = |u(t) - u^0(t)| \leqslant \varepsilon^{(1-\rho)} C_4 e^{r(t-\tau)}, \ \forall \varepsilon, 0 < \varepsilon \leqslant 1, \tag{4.15}$$

where the constant C_4 and r are independent of ε, $u_\tau \in \tilde{B}$, and $0 \leqslant \rho \leqslant 1$.

In Section 6 using Theorems 3.1 and 4.1, we will prove that the global attractors $\mathcal{A}^\varepsilon$ converge to $\mathcal{A}^0$ in the strong norm of H as $\varepsilon \to 0+$ provided that $0 \leqslant \rho < 1$.

5. TRANSLATION COMPACT FUNCTIONS WITH VALUES IN $L_2(\Omega)^2$ AND Z

We briefly recall the definition of a translation compact function in the space $L_2^{\mathrm{loc}}(\mathbb{R}; E)$, where E is a Banach space. The detailed description of tr.c. functions in various topological space can be found in [5, Chapter 5]. The space $L_2^{\mathrm{loc}}(\mathbb{R}; E)$ consists of functions $f(t), t \in \mathbb{R}$, such that $f(t) \in E$ for almost all $t \in \mathbb{R}$ and f is locally square integrable in the Bochner sense. In particular, for every interval $[t_1, t_2] \subset \mathbb{R}$

$$\int_{t_1}^{t_2} \|f(s)\|_E^2 ds < +\infty.$$

The space $L_2^{\mathrm{loc}}(\mathbb{R}; E)$ is equipped with the following local convergence topology. By definition, $f_n(t) \to f(t)\ (n \to \infty)$ in $L_2^{\mathrm{loc}}(\mathbb{R}; E)$ if

$$\int_{t_1}^{t_2} \|f_n(s) - f(s)\|_E^2 ds \to 0\ (n \to \infty)$$

for every interval $[t_1, t_2] \subset \mathbb{R}$. This topology is metrizable and the corresponding metric space is complete. A function $\varphi(\cdot) \in L_2^{\mathrm{loc}}(\mathbb{R}; E)$ is said to be *translation compact* (*tr.c.*) in $L_2^{\mathrm{loc}}(\mathbb{R}; E)$ if the set of its translations $\{\varphi(t+h) \mid h \in \mathbb{R}\}$ is precompact in the above local convergence topology. The set

$$\mathcal{H}(\varphi) = [\{\varphi(t+h) \mid h \in \mathbb{R}\}]_{L_2^{\mathrm{loc}}(\mathbb{R}; E)} \tag{5.1}$$

is called the *hull* of the function φ in the space $L_2^{\mathrm{loc}}(\mathbb{R}; E)$. If the function φ is tr.c. $L_2^{\mathrm{loc}}(\mathbb{R}; E)$, then its hull $\mathcal{H}(\varphi)$ is compact in $L_2^{\mathrm{loc}}(\mathbb{R}; E)$. We have the following criterion (see [5]): a function φ is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; E)$ if and only if **(i)** for all $h \geqslant 0$, the set $\left\{\int_t^{t+h} \varphi(s) ds \mid t \in \mathbb{R}\right\}$ is precompact in E and **(ii)** there is a positive function $\beta(s) \to 0+$ as $s \to 0+$ such that

$$\int_t^{t+1} \|\varphi(s) - \varphi(s+l)\|_E^2 ds \leqslant \beta(|l|),\ \forall l \in \mathbb{R}.$$

Notice that

$$\|\varphi\|^2_{L_2^b(\mathbb{R}; E)} := \sup_{t \in \mathbb{R}} \int_t^{t+1} \|\varphi(s)\|_E^2 ds < +\infty \tag{5.2}$$

for every tr.c. function φ in $L_2^{\mathrm{loc}}(\mathbb{R}; E)$, that is, $\varphi \in L_2^{\mathrm{b}}(\mathbb{R}; E)$. At the same time, the condition (5.2) is not sufficient for a function φ to be tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; E)$.

Almost periodic and quasiperiodic functions with values in E (see [1,15]) are tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; E)$. Other examples of tr.c. functions are given in [5].

We shall use tr.c. functions with values in the spaces $L_2(\Omega)^2$, H, and $Z = L_2^b(\mathbb{R}_z^2; \mathbb{R}^2)$ (see Section 6).

Consider the vector functions $g_0(x,t), x \in \Omega, t \in \mathbb{R}$, and $g_1(z,t), z \in \mathbb{R}^2, t \in \mathbb{R}$, that appear on the right-hand side of the 2D N.–S. system. We assume that $g_0(x,t) \in L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $g_1(z,t) \in L_2^{\mathrm{loc}}(\mathbb{R}; Z)$.

Proposition 5.1. *If the function $g_1(z,t)$ is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$, then, for every fixed ε, $0 < \varepsilon \leqslant 1$, the function $g_1(x/\varepsilon, t)$ is tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$, $\Omega \Subset \mathbb{R}^2$.*

Proof. We have to establish that the set of functions $\{g_1(x/\varepsilon, t + h) \mid h \in \mathbb{R}\}$ is precompact in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$. Let $\{h_n, n = 1, 2, \ldots\}$ be an arbitrary sequence of real numbers. Since the function $g_1(z,t)$ is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$ there is a subsequence $\{h_{n'}\} \subset \{h_n\}$ such that $g_1(z, t + h_{n'})$ converges to a function $\hat{g}_1(z,t)$ as $n' \to \infty$ in $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$, i.e. for every interval $[t_1, t_2] \subset \mathbb{R}$,

$$\int_{t_1}^{t_2} \|g_1(\cdot, s + h_{n'}) - \hat{g}_1(\cdot, s)\|_Z^2 ds \to 0 \, (n' \to \infty).$$

Using inequality (2.19) from Lemma 2.2, we conclude that

$$\int_{t_1}^{t_2} \|g_1(\cdot/\varepsilon, s + h_{n'}) - \hat{g}_1(\cdot/\varepsilon, s)\|_{L_2(\Omega)^2}^2 ds \leqslant C^2 \int_{t_1}^{t_2} \|g_1(\cdot, s + h_{n'}) - \hat{g}_1(\cdot, s)\|_Z^2 ds,$$

that is, $g_1(x/\varepsilon, t + h_{n'})$ converges to $\hat{g}_1(x/\varepsilon, t)$ as $n' \to \infty$ in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$. Thus, the set $\{g_1(x/\varepsilon, t + h) \mid h \in \mathbb{R}\}$ is precompact in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$. $\square$

Proposition 5.2. *Let $g_0(x,t)$ be tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $g_1(z,t)$ be tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$. Consider the function $g^\varepsilon(x,t) = g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$ as an element of the space $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$. Then this function is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and the hull $\mathcal{H}(g^\varepsilon(x,t))$ (in the space $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$) consists of (tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$) functions $\hat{g}^\varepsilon(x,t)$ of the form $\hat{g}^\varepsilon(x,t) = \hat{g}_0(x,t) + \varepsilon^{-\rho} \hat{g}_1(x/\varepsilon, t)$ for some $\hat{g}_0(x,t) \in \mathcal{H}(g_0(x,t))$ and $\hat{g}_1(z,t) \in \mathcal{H}(g_1(z,t))$, where $\mathcal{H}(g_0(x,t))$ and $\mathcal{H}(g_1(z,t))$ are the hulls of the functions $g_0(x,t)$ and $g_1(z,t)$, respectively.*

Proof. It follows from Proposition 5.1 that $g^\varepsilon(x,t) = g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$ is a tr.c. function in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ (as the sum of two tr.c. functions). Let now $\hat{g}^\varepsilon(x,t) \in \mathcal{H}(g^\varepsilon(x,t))$, i.e. there is a sequence $\{h_n\}$

such that $g^\varepsilon(x,t+h_n) = g_0(x,t+h_n) + \varepsilon^{-\rho} g_1(x/\varepsilon, t+h_n) \to \hat{g}^\varepsilon(x,t)$ as $n \to \infty$ in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$. Since the functions $g_0(x,t)$ and $g_1(z,t)$ are tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$, respectively, we may assume passing to a subsequence $\{h_{n'}\} \subset \{h_n\}$ that $g_0(x,t+h_{n'}) \to \hat{g}_0(x,t)$ $(n' \to \infty)$ in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $g_1(z,t+h_{n'}) \to \hat{g}_1(z,t)$ as $n' \to \infty$ in $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$. Therefore, $g^\varepsilon(x,t+h_{n'}) = g_0(x,t+h_{n'}) + \varepsilon^{-\rho} g_1(x/\varepsilon, t+h_{n'}) \to \hat{g}_0(x,t) + \varepsilon^{-\rho}\hat{g}_1(x/\varepsilon,t)$ as $n \to \infty$ in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$. Hence,

$$\begin{aligned}\hat{g}^\varepsilon(x,t) &= \lim_{n\to\infty}\left[g_0(x,t+h_n) + \varepsilon^{-\rho} g_1(x/\varepsilon, t+h_n)\right] \\ &= \lim_{n'\to\infty} g_0(x,t+h_{n'}) + \lim_{n'\to\infty} \varepsilon^{-\rho} g_1(x/\varepsilon, t+h_{n'}) = \hat{g}_0(x,t) \\ &\quad + \varepsilon^{-\rho}\hat{g}_1(x/\varepsilon,t).\end{aligned}$$

Thus, every function $\hat{g}^\varepsilon(x,t) \in \mathcal{H}(g^\varepsilon(x,t))$ has the form $\hat{g}^\varepsilon(x,t) = \hat{g}_0(x,t) + \varepsilon^{-\rho}\hat{g}_1(x/\varepsilon,t)$ for some $\hat{g}_0(x,t) \in \mathcal{H}(g_0(x,t))$ and $\hat{g}_1(z,t) \in \mathcal{H}(g_1(z,t))$. □

6. ON THE STRUCTURE OF THE UNIFORM GLOBAL ATTRACTOR $\mathcal{A}^\varepsilon$ OF THE 2D NAVIER–STOKES SYSTEM

We consider Eq. (2.2)

$$\partial_t u + \nu L u + B(u,u) = g^\varepsilon(x,t), \tag{6.1}$$

where $g^\varepsilon(x,t) = P g_0(x,t) + \varepsilon^{-\rho} P g_1(x/\varepsilon, t)$ and ε is fixed. We assume that the function $g_0(x,t)$ is tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $g_1(z,t)$ is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$. In particular, $g_0(x,t) \in L_2^b(\mathbb{R}; L_2(\Omega)^2)$ and $g_1(z,t) \in L_2^b(\mathbb{R}; Z)$ (see Section 5). So, all the results of Section 2 are applicable to Eq. (6.1).

We now consider the hull $\mathcal{H}(g^\varepsilon)$ of the function $g^\varepsilon(x,t)$ in the space $L_2^{\mathrm{loc}}(\mathbb{R}; H)$:

$$\mathcal{H}(g^\varepsilon) = [\{g^\varepsilon(\cdot, t+h) \mid h \in \mathbb{R}\}]_{L_2^{\mathrm{loc}}(\mathbb{R}; H)}. \tag{6.2}$$

Recall that $\mathcal{H}(g^\varepsilon)$ is compact in $L_2^{\mathrm{loc}}(\mathbb{R}; H)$ and each element $\hat{g}^\varepsilon(x,t) \in \mathcal{H}(g^\varepsilon(x,t))$ (being a tr.c. function in $L_2^{\mathrm{loc}}(\mathbb{R}; H)$) can be written in the form

$$\hat{g}^\varepsilon(x,t) = \hat{g}_0(x,t) + \varepsilon^{-\rho}\hat{g}_1(x/\varepsilon,t) \tag{6.3}$$

for some functions $\hat{g}_0(x,t) \in \mathcal{H}(g_0(x,t))$ and $\hat{g}_1(z,t) \in \mathcal{H}(g_1(z,t))$, where $\mathcal{H}(g_0(x,t))$ and $\mathcal{H}(g_1(z,t))$ are the hulls of the functions $g_0(x,t)$ and $g_1(z,t)$ in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$, respectively.

We note that

$$\|\hat{g}_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)} \leqslant \|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)},\ \forall \hat{g}_0 \in \mathcal{H}(g_0),$$
$$\|\hat{g}_1\|_{L_2^b(\mathbb{R};Z)} \leqslant \|g_1\|_{L_2^b(\mathbb{R};Z)},\ \forall \hat{g}_1 \in \mathcal{H}(g_1).$$

Then it follows easily from Corollary 2.1 that

$$\|\hat{g}^\varepsilon\|_{L_2^b(\mathbb{R};H)} \leqslant \|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)} + \frac{C}{\varepsilon^\rho}\|g_1\|_{L_2^b(\mathbb{R};Z)}, \quad \forall g^\varepsilon \in \mathcal{H}(g^\varepsilon), \tag{6.4}$$

where the constant C is independent of g_0, g_1, ρ and ε (see (2.19) and (2.20)).

It was shown in Section 2 that Eq. (6.1) generates the process $\{U_\varepsilon(t,\tau)\} := \{U_{g^\varepsilon}(t,\tau)\}$ in the space H, where every mapping $U_{g^\varepsilon}(t,\tau): H \to H$ acts by the formula $U_\varepsilon(t,\tau)u_\tau = u(t)$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, where u_τ is arbitrary and $u(t)$ is the solution of Eq. (6.1) with initial data $u|_{t=\tau} = u_\tau$. Moreover, it was proved in Section 2 that the process $\{U_{g^\varepsilon}(t,\tau)\}$ has the uniform global attractor $\mathcal{A}^\varepsilon \subseteq B_{0,\varepsilon} \cap B_{1,\varepsilon}$, (see (2.23) and (2.24)) and

$$\|\mathcal{A}^\varepsilon\|_H \leqslant (C_0 + C_1\varepsilon^{-\rho}), \tag{6.5}$$

where the constants C_0 and C_1 depend on $\|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)}$ and $\|g_1\|_{L_2^b(\mathbb{R};Z)}$, respectively. We now describe the structure of the attractor $\mathcal{A}^\varepsilon$.

Along with Eq. (6.1), we consider the family of equations

$$\partial_t \hat{u} + \nu L\hat{u} + B(\hat{u},\hat{u}) = \hat{g}^\varepsilon(x,t) \tag{6.6}$$

with external forces $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$. It is clear that, for every $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, Eq. (6.6) generates a process $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$ acting in H. We note that the processes $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$ satisfy properties similar to those of the process $\{U_{g^\varepsilon}(t,\tau)\}$ corresponding to the 2D N.–S. system (6.1) with original external force $g^\varepsilon(x,t) = Pg_0(x,t) + \varepsilon^{-\rho}Pg_1(x/\varepsilon,t)$. In particular, the sets $B_{0,\varepsilon}$ and $B_{1,\varepsilon}$ are absorbing for each process $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$, $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$ (see (6.4)). Moreover, every process $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$ has a uniform global attractor $\mathcal{A}_{\hat{g}^\varepsilon}$ that is contained in the global attractor $\mathcal{A}^\varepsilon = \mathcal{A}_{g^\varepsilon}$ of the 2D Navier–Stokes system (6.1) with initial external force $g^\varepsilon(x,t)$, $\mathcal{A}_{\hat{g}^\varepsilon} \subseteq \mathcal{A}_{g^\varepsilon}$ (the inclusion can be strict, see [5]).

Proposition 6.1. *Let the function $g_0(x,t)$ be tr.c. in the space $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$ and let $g_1(z,t)$ be tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R};Z)$. Then for any fixed $\varepsilon, 0 < \varepsilon \leqslant 1$, the family of processes $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$, $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, corresponding to Eq.* (6.6) *has an absorbing set $B_{1,\varepsilon}$, which is bounded in H and V and satisfies*

$$\|B_{1,\varepsilon}\|_H \leqslant (C_0 + C_1\varepsilon^{-\rho}). \tag{6.7}$$

The family $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$, $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, *is* $(H \times \mathcal{H}(g^\varepsilon); H)$*-continuous, that is, if*

$$\hat{g}_n^\varepsilon \to \hat{g}^\varepsilon \ (n \to \infty) \ \textit{in} \ L_2^{\mathrm{loc}}(\mathbb{R}; H) \ \textit{and} \ u_{\tau n} \to u_\tau \ (n \to \infty) \ \textit{in} \ H \tag{6.8}$$

then

$$U_{\hat{g}_n^\varepsilon}(t,\tau)u_{\tau n} \to U_{\hat{g}^\varepsilon}(t,\tau)u_\tau \ (n \to \infty) \ \textit{in} \ H. \tag{6.9}$$

The proof of these properties is analogous to the proof given, e.g. in [5, Chapter 6], for the case of a non-oscillating tr.c. external force in $L_2^{\mathrm{loc}}(\mathbb{R}; H)$).

We denote by $\mathcal{K}_{\hat{g}^\varepsilon}$ the *kernel* of equation (6.6) (and of the process $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$) with external force $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$. Recall that the kernel $\mathcal{K}_{\hat{g}^\varepsilon}$ is the family of all complete solutions $\hat{u}(t), t \in \mathbb{R}$, of (6.6) which are bounded in the norm of H:

$$|\hat{u}(t)| \leqslant M_{\hat{u}}, \ \forall t \in \mathbb{R}. \tag{6.10}$$

The set

$$\mathcal{K}_{\hat{g}^\varepsilon}(s) = \left\{\hat{u}(s) \mid \hat{u} \in \mathcal{K}_{\hat{g}^\varepsilon}\right\}, \quad s \in \mathbb{R}$$

(belonging to H) is called the *kernel section* at time $t = s$.

We have the following theorem on the structure of the uniform global attractor $\mathcal{A}^\varepsilon$ of the 2D N.–S. system (6.1).

Theorem 6.1. *If the function* $g^\varepsilon(x,t)$ *is tr.c. in* $L_2^{\mathrm{loc}}(\mathbb{R}; H)$, *then the process* $\{U_{g^\varepsilon}(t,\tau)\}$ *corresponding to Eq.* (6.1) *has the uniform global attractor* $\mathcal{A}^\varepsilon$ *and the following identity holds:*

$$\mathcal{A}^\varepsilon = \bigcup_{\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)} \mathcal{K}_{\hat{g}^\varepsilon}(0). \tag{6.11}$$

Moreover, the kernel $\mathcal{K}_{\hat{g}^\varepsilon}$ *is non-empty for all* $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$.

The proof of Theorem 6.1 is given in [5].

We also note that the attractor $\mathcal{A}^\varepsilon$ is given by the following formula

$$\mathcal{A}^\varepsilon = \bigcap_{t \geqslant 0} \left[\bigcup_{h \geqslant t} \bigcup_{\tau \in \mathbb{R}} U_{g^\varepsilon}(\tau + h; \tau) B_{1,\varepsilon} \right]_H,$$

i.e. to construct the attractor $\mathcal{A}^\varepsilon$ of the entire family of processes $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$, $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, one can use only the process $\{U_{g^\varepsilon}(t,\tau)\}$ of original Eq. (6.1) with external force $g^\varepsilon = Pg_0(x,t) + \varepsilon^{-\rho} Pg_1(x/\varepsilon, t)$.

All the above results are also applicable to the "limiting" 2D N.–S. system (2.29)

$$\partial_t u + \nu L u + B(u, u) = g^0(x, t) \tag{6.12}$$

with tr.c. external force $g^0(t) := Pg_0(\cdot, t) \in L_2^{\mathrm{loc}}(\mathbb{R}; H)$. The corresponding "limiting" process $\{U_0(t, \tau)\} = \{U_{g^0}(t, \tau)\}$ is defined by the formula $U_{g^0}(t, \tau)u_\tau = u(t)$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, where $u(t)$ is a solution of Eq. (6.12) with initial data $u|_{\tau=\tau} = u_\tau$. Since $g^0(t) \in L_2^{\mathrm{b}}(\mathbb{R}; H)$ the process $\{U_{g^0}(t, \tau)\}$ has the uniform global attractor $\mathcal{A}^0$ (see the end of Section 2).

Consider the family of equations

$$\partial_t \hat{u} + \nu L \hat{u} + B(\hat{u}, \hat{u}) = \hat{g}^0(x, t) \tag{6.13}$$

with external forces $\hat{g}^0 \in \mathcal{H}(g^0)$ (the hull $\mathcal{H}(g^0)$ is taken in the space $L_2^{\mathrm{loc}}(\mathbb{R}; H)$) and the corresponding family of processes $\{U_{\hat{g}^0}(t, \tau)\}$, $\hat{g}^0 \in \mathcal{H}(g^0)$.

We note that we can apply Proposition 6.1 and Theorem 6.1 directly to the Eqs. (6.12) and (6.13) taking the function $g_1(z, t) \equiv 0$. Therefore, the family of processes $\{U_{\hat{g}^0}(t, \tau)\}$, $\hat{g}^0 \in \mathcal{H}(g^0)$, has an absorbing set $B_{1,0}$ (bounded in V),

$$\|B_{1,0}\|_H \leqslant C_0, \tag{6.14}$$

and the family $\{U_{\hat{g}^0}(t, \tau)\}$, $\hat{g}^0 \in \mathcal{H}(g^0)$, is $(H \times \mathcal{H}(g^0); H)$-continuous. Moreover, the attractor $\mathcal{A}^0$ of the "limiting" Eq. (6.12) has the form

$$\mathcal{A}^0 = \bigcup_{\hat{g}^0 \in \mathcal{H}(g^0)} \mathcal{K}_{\hat{g}^0}(0), \tag{6.15}$$

where $\mathcal{K}_{\hat{g}^0}$ is the kernel of Eq. (6.13) with external forces $\hat{g}^0 \in \mathcal{H}(g^0)$.

The formulas (6.11) and (6.15) will be very important in the next section, where we study the convergence of the attractors $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$ as $\varepsilon \to 0+$.

7. DIVERGENCE CONDITION AND CONVERGENCE OF GLOBAL ATTRACTORS $\mathcal{A}^\varepsilon$

In this section, we consider Eqs. (6.1) and (6.12) assuming that the functions $g_0(x, t)$ and $g_1(z, t)$ are tr.c. in the spaces $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$, respectively.

We also assume that the function $g_1(z,t)$ satisfies the divergence condition (3.1), that is, there exist vector functions $G_j(z,t) \in L_2^b(\mathbb{R}; Z)$ such that $\partial_{z_j} G_j(z,t) \in L_2^b(\mathbb{R}; Z)$ for $j = 1, 2$, and

$$\partial_{z_1} G_1(z_1, z_2, t) + \partial_{z_2} G_2(z_1, z_2, t) = g_1(z_1, z_2, t), \quad \forall (z_1, z_2) \in \mathbb{R}^2, \quad t \in \mathbb{R}. \tag{7.1}$$

Then due to Theorem 3.1 the uniform global attractors $\mathcal{A}^\varepsilon$ of Eq. (6.1) with external forces $g^\varepsilon(x,t) = Pg_0(x,t) + \varepsilon^{-\rho} Pg_1(x/\varepsilon, t)$ are uniformly bounded in H with respect to ε:

$$\left\| \mathcal{A}^\varepsilon \right\|_H \leqslant C_2, \ \forall \varepsilon, \ 0 < \varepsilon \leqslant 1, \tag{7.2}$$

where the number C_2 is independent of ε. We also consider the global attractor $\mathcal{A}^0$ of the "limiting" Eq. (6.12) with external force $g^0(t) = Pg_0(\cdot, t)$. Clearly , the set $\mathcal{A}^0$ is also bounded in H (see (6.14)).

We need a generalization of Theorem 4.1 that can be applied to the solution of entire families of Eqs. (6.6) and (6.13).

We choose an arbitrary function $u_\tau \in \tilde{B}$, where the absorbing ball $\tilde{B}$ is defined in (3.10). Let $\hat{u}(\cdot, t) = U_{\hat{g}^\varepsilon}(t, \tau) u_\tau$, $t \geqslant \tau$, be the solution of Eq. (6.6) with external force $\hat{g}^\varepsilon = P\hat{g}_0 + \varepsilon^{-\rho} P\hat{g}_1 \in \mathcal{H}(g^\varepsilon)$. Let also $\tilde{u}^0(\cdot, t) = U_{\tilde{g}^0}(t, \tau) u_\tau$, $t \geqslant \tau$, be the solution of (6.13) with external force $\tilde{g}^0 \in \mathcal{H}(g^0)$. We assume that the initial data at $t = \tau$ of these two solutions are identical: $\hat{u}(\cdot, \tau) = \tilde{u}^0(\cdot, \tau) = u_\tau$. (Notice that the function $\tilde{g}^0$ can be different from the function $\hat{g}^0 = P\hat{g}_0$ being the first summand in the representation $\hat{g}^\varepsilon = P\hat{g}_0 + \varepsilon^{-\rho} P\hat{g}_1$). We now consider the difference

$$\hat{w}(x,t) = \hat{u}(x,t) - \tilde{u}^0(x,t), \ t \geqslant \tau.$$

Proposition 7.1. *Let the original functions $g_0(x,t)$ and $g_1(z,t)$ in* (2.1) *be tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$. Let also the function $g_1(z,t)$ satisfy the divergence condition* (7.1)*. We set $g^\varepsilon(x,t) = Pg_0(x,t) + \varepsilon^{-\rho} Pg_1(x/\varepsilon\ , t)$ and $g^0(x,t) = Pg_0(x,t)$. Then, for every external force $\hat{g}^\varepsilon = P\hat{g}_0 + \varepsilon^{-\rho} P\hat{g}_1 \in \mathcal{H}(g^\varepsilon)$, there exist an external force $\tilde{g}^0 \in \mathcal{H}(g^0)$ such that, for every initial data $u_\tau \in \tilde{B}$ (see* (3.10))*, the difference*

$$\hat{w}(t) = \hat{u}(t) - \tilde{u}^0(t) = U_{\hat{g}^\varepsilon}(t, \tau) u_\tau - U_{\tilde{g}^0}(t, \tau) u_\tau$$

of the solutions of the 2D N.–S. systems (6.6) *and* (6.13) *with external forces $\hat{g}^\varepsilon(x,t) = P\hat{g}_0(x,t) + \varepsilon^{-\rho} P\hat{g}_1(x/\varepsilon, t)$ and $\tilde{g}^0(x,t)$, respectively, and with the same initial data $u_\tau \in \tilde{B}$ satisfies the following inequality:*

$$|\hat{w}(t)| = |\hat{u}(t) - \tilde{u}^0(t)| \leqslant \varepsilon^{(1-\rho)} C_4 e^{r(t-\tau)}, \ \forall \varepsilon, 0 < \varepsilon \leqslant 1, \tag{7.3}$$

where the constant C_4 and r are the same as in Theorem 4.1 and they are independent of ε and $0 \leqslant \rho \leqslant 1$.

Proof. Consider the functions

$$u(t)=U_{g^\varepsilon}(t,\tau)u_\tau \text{ and } u^0(t)=U_{g^0}(t,\tau)u_\tau, \quad \forall t\geqslant\tau, \tag{7.4}$$

where $g^\varepsilon(t)=Pg_0(t)+\varepsilon^{-\rho}Pg_1(t)$ and $g^0(t)=Pg_0(t)$ are the original external forces. Using (7.4), we rewrite inequality (4.15) in the form

$$|U_{g^\varepsilon}(t,\tau)u_\tau-U_{g^0}(t,\tau)u_\tau|\leqslant\varepsilon^{(1-\rho)}C_4e^{r(t-\tau)}. \tag{7.5}$$

By Theorem 4.1, inequality (7.5) holds for all $u_\tau\in\tilde{B}$. We claim that this inequality also holds for the time shifted external forces

$$\begin{aligned} g_h^\varepsilon(t) &= g^\varepsilon(t+h)=Pg_0(t+h)+\varepsilon^{-\rho}Pg_1(t+h),\\ g_h^0(t) &= g^0(t+h)=Pg_0(t+h), \end{aligned}$$

where $h\in\mathbb{R}$ is arbitrary, that is,

$$|U_{g_h^\varepsilon}(t,\tau)u_\tau-U_{g_h^0}(t,\tau)u_\tau|\leqslant\varepsilon^{(1-\rho)}C_4e^{r(t-\tau)}. \tag{7.6}$$

Indeed, for every $h\in\mathbb{R}$, the time shifted function $g_{1h}(z,t)=g_1(z,t+h)$ obviously satisfies the divergence condition (7.1) for the time shifted functions $G_{jh}(z,t)=G_j(z,t+h)\in L_2^b(\mathbb{R};Z)$, $j=1,2$. So (7.6) follows directly from Theorem 4.1.

We recall that the family of processes $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$, $\hat{g}^\varepsilon\in\mathcal{H}(g^\varepsilon)$, is $(H\times\mathcal{H}(g^\varepsilon);H)$-continuous. In particular, (see (6.8) and (6.9)) for a fixed $u_\tau\in\tilde{B}$, if

$$\hat{g}_n^\varepsilon\to\hat{g}^\varepsilon\ (n\to\infty) \text{ in } L_2^{\mathrm{loc}}(\mathbb{R};H)$$

then

$$U_{\hat{g}_n^\varepsilon}(t,\tau)u_\tau\to U_{\hat{g}^\varepsilon}(t,\tau)u_\tau\ (n\to\infty) \text{ in } H \tag{7.7}$$

and similarly

$$U_{\hat{g}_n^0}(t,\tau)u_\tau\to U_{\tilde{g}^0}(t,\tau)u_\tau\ (n\to\infty) \text{ in } H. \tag{7.8}$$

when $\hat{g}_n^0\to\tilde{g}^0\ (n\to\infty)$ in $L_2^{\mathrm{loc}}(\mathbb{R};H)$ for some $\tilde{g}^0\in\mathcal{H}(g^0)$.

We now fix the external force $\hat{g}^\varepsilon=P\hat{g}_0+\varepsilon^{-\rho}P\hat{g}_1\in\mathcal{H}(g^\varepsilon)$. The function $\hat{g}^\varepsilon(t)$ is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R};H)$. Therefore, there exists a sequence $\{h_i\}\subset\mathbb{R}$ such that

$$g_{h_i}^\varepsilon\to\hat{g}^\varepsilon\ (i\to\infty) \text{ in } L_2^{\mathrm{loc}}(\mathbb{R};H), \tag{7.9}$$

where $g^{\varepsilon}_{h_i}(t) = g^{\varepsilon}(t + h_i)$. Consider now the sequence of external forces $g^0_{h_i} = g^0(t + h_i)$. Since the function $g^0(t)$ is tr.c. in $L_2^{\mathrm{loc}}(\mathbb{R}; H)$, there exists a function $\tilde{g}^0 \in \mathcal{H}(g^0)$ such that

$$g^0_{h_i} \to \tilde{g}^0 \ (i \to \infty) \text{ in } L_2^{\mathrm{loc}}(\mathbb{R}; H) \tag{7.10}$$

(here we have possibly passed to a subsequence of h_i which we label the same). It follows from (7.6) that

$$|U_{g^{\varepsilon}_{h_i}}(t, \tau)u_\tau - U_{g^0_{h_i}}(t, \tau)u_\tau| \leqslant \varepsilon^{(1-\rho)} C_4 e^{r(t-\tau)}, \quad \forall i \in \mathbb{N}. \tag{7.11}$$

Using (7.9) and (7.10) in (7.7) and (7.8), we pass to the limit in (7.11) as $i \to \infty$ and obtain the required inequality:

$$|U_{\hat{g}^{\varepsilon}}(t, \tau)u_\tau - U_{\tilde{g}^0}(t, \tau)u_\tau| \leqslant \varepsilon^{(1-\rho)} C_4 e^{r(t-\tau)}. \tag{7.12}$$

So, inequality (7.3) is proved. □

We are now ready to formulate the main theorem of the paper.

Theorem 7.1. *Let* $0 \leqslant \rho < 1$ *and the functions* $g_0(x, t), g_1(z, t)$ *in* (2.1) *be tr.c. in* $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$, $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$, *respectively. Let also the function* $g_1(z, t)$ *satisfy the divergence condition* (7.1). *Then the global attractors* $\mathcal{A}^{\varepsilon}$ *of Eq.* (6.1) *converge to the global attractor* $\mathcal{A}^0$ *of the "limiting" Eq.* (6.12) *in the strong norm of* H *as* $\varepsilon \to 0+$, *that is*

$$\mathrm{dist}_H(\mathcal{A}^{\varepsilon}, \mathcal{A}^0) \to 0 \ (\varepsilon \to 0+). \tag{7.13}$$

Proof. For a given ε, let u^{ε} be an arbitrary element of $\mathcal{A}^{\varepsilon}$. By (6.11), there exists a bounded complete solution $\hat{u}^{\varepsilon}(t), t \in \mathbb{R}$, of Eq. (7.1) with some external force $\hat{g}^{\varepsilon} = P\hat{g}_0 + \varepsilon^{-\rho} P\hat{g}_1 \in \mathcal{H}(g^{\varepsilon})$, where $\hat{g}_0 \in \mathcal{H}(g_0)$ and $\hat{g}_1 \in \mathcal{H}(g_1)$, such that

$$u^{\varepsilon} = \hat{u}^{\varepsilon}(0). \tag{7.14}$$

We consider the point $\hat{u}^{\varepsilon}(-R)$ which clearly belongs to $\mathcal{A}^{\varepsilon}$ and hence

$$\hat{u}^{\varepsilon}(-R) \in \tilde{B} \tag{7.15}$$

(see (3.10)). Recall that $\tilde{B}$ is the absorbing set and the global attractor $\mathcal{A}^{\varepsilon}$ belongs to $\tilde{B}$. The number R will be chosen later on.

For the constructed external force $\hat{g}^{\varepsilon}$, we apply Proposition 7.1: there is a "limiting" external force $\tilde{g}^0 \in \mathcal{H}(g^0)$ such that, for any $\tau \in \mathbb{R}$ and for all $u_\tau \in \tilde{B}$, the following inequality holds:

$$|U_{\hat{g}^{\varepsilon}}(t, \tau)u_\tau - U_{\tilde{g}^0}(t, \tau)u_\tau| \leqslant \varepsilon^{(1-\rho)} C_4 e^{r(t-\tau)}, \quad \forall t \geqslant \tau. \tag{7.16}$$

Consider the "limiting" Eq. (6.12) with the chosen "limiting" external force $\tilde{g}^0$. We set $\tau = -R$. Let $\tilde{u}^0(t), t \geqslant -R$, be a solution of this equation with initial data

$$\tilde{u}^0|_{t=-R} = \hat{u}^\varepsilon(-R). \tag{7.17}$$

Taking $-R$ in place of τ and $-R+t$ in place of t, it follows from (7.16) (see also (7.15)) that

$$|\hat{u}^\varepsilon(-R+t) - \tilde{u}^0(-R+t)| \leqslant \varepsilon^{(1-\rho)} C_4 e^{rt}, \quad \forall t \geqslant 0, \tag{7.18}$$

where $\hat{u}^\varepsilon(-R+t) = U_{\hat{g}^\varepsilon}(-R+t, -R)\hat{u}^\varepsilon(-R)$ and $\tilde{u}^0(-R+t) = U_{\tilde{g}^0}(-R+t, -R)\hat{u}^\varepsilon(-R)$.

The set $\mathcal{A}^0$ attracts $U_{\hat{g}^0}(t+\tau, \tau)\tilde{B}$ in H as $t \to +\infty$ (uniformly with respect to $\tau \in \mathbb{R}$ and $\hat{g}^0 \in \mathcal{H}(g^0)$) (see [5]). Then, for any $\delta > 0$, there exist a number $T = T(\delta)$ such that

$$\operatorname{dist}_H(U_{\hat{g}^0}(t+\tau, \tau)\tilde{B}, \mathcal{A}^0) \leqslant \frac{\delta}{2}, \ \forall \tau \in \mathbb{R}, \ \forall \hat{g}^0 \in \mathcal{H}(g^0), \quad \forall t \geqslant T(\delta).$$

Hence, for $\tau = -R$ and $\hat{u}^\varepsilon(-R) \in \tilde{B}$,

$$\operatorname{dist}_H(U_{\hat{g}^0}(-R+t, -R)\hat{u}^\varepsilon(-R), \mathcal{A}^0) \leqslant \frac{\delta}{2}, \ \forall \hat{g}^0 \in \mathcal{H}(g^0), \quad \forall t \geqslant T(\delta).$$

In particular, for the function $\tilde{g}^0$ specified above

$$\operatorname{dist}_H(\tilde{u}^0(-R+t), \mathcal{A}^0) = \operatorname{dist}_H(U_{\tilde{g}^0}(-R+t, -R)\hat{u}^\varepsilon(-R), \mathcal{A}^0) \leqslant \frac{\delta}{2}, \quad \forall t \geqslant T(\delta). \tag{7.19}$$

Recall that $T(\delta)$ is independent of $u^\varepsilon \in \mathcal{A}^\varepsilon$.

It follows from (7.19) and (7.18) that

$$\begin{aligned} \operatorname{dist}_H(\hat{u}^\varepsilon(-R+t), \mathcal{A}^0) &\leqslant |\hat{u}^\varepsilon(-R+t) - \tilde{u}^0(-R+t)| + \operatorname{dist}_H(\tilde{u}^0(-R+t), \mathcal{A}^0) \\ &\leqslant \varepsilon^{(1-\rho)} C_4 e^{rt} + \frac{\delta}{2}, \quad \forall t \geqslant T(\delta). \end{aligned} \tag{7.20}$$

We now set $t = R = T(\delta)$ in (7.20) and since $\hat{u}^\varepsilon(0) = u^\varepsilon$ we obtain that

$$\operatorname{dist}_H(u^\varepsilon, \mathcal{A}^0) = \operatorname{dist}_H(\hat{u}^\varepsilon(0), \mathcal{A}^0) \leqslant \varepsilon^{(1-\rho)} C_4 e^{rT(\delta)} + \frac{\delta}{2}, \quad \forall u^\varepsilon \in \mathcal{A}^\varepsilon.$$

Consequently,

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant \varepsilon^{(1-\rho)} C_4 e^{rT(\delta)} + \frac{\delta}{2}, \quad \forall \delta > 0. \tag{7.21}$$

Finally, for an arbitrary $\delta > 0$, we define $\varepsilon_0 = \varepsilon_0(\delta)$ such that $\varepsilon_0^{(1-\rho)} C_4 e^{rT(\delta)} = \delta/2$. Thus, if

$$\varepsilon \leqslant \varepsilon_0(\delta) = \left(\frac{\delta}{2C_4 e^{rT(\delta)}} \right)^{1/1-\rho}$$

then

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant \delta.$$

We conclude that

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \to 0 \, (\varepsilon \to 0+).$$

The theorem is proved. □

8. ESTIMATE FOR THE DISTANCE BETWEEN ATTRACTORS $\mathcal{A}^\varepsilon$ AND $\mathcal{A}^0$, WHEN THE ATTRACTOR $\mathcal{A}^0$ IS EXPONENTIAL

In this section, we briefly study the important case of the 2D N.–S. system (6.1) when the Grashof number of the corresponding "limiting" N.–S. system (6.12) is small. In this case, the global attractor $\mathcal{A}^0$ is exponential, i.e. it attracts bounded sets of initial data with exponential rate as time tends to infinity. This property allows to estimate explicitly the distance between the global attractors $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$.

We consider the "limiting" system (6.12) with external force $g^0(t) := Pg_0(\cdot, t) \in L_2^{\mathrm{loc}}(\mathbb{R}; H)$. Let the Grashof number G of this 2D N.–S. system satisfy the following inequality:

$$G := \frac{\left\| g^0 \right\|_{L_2^b}}{\lambda_1 \nu^2} < \frac{1}{c_1^2}, \tag{8.1}$$

where the constant c_1^2 is taken from the inequality

$$|(B(v, w), v)| = |(B(v, v), w)| \leqslant c_1^2 |v| \|v\| \|w\|, \tag{8.2}$$

which holds for all $v, w \in V$ (see, (2.6) and (2.7), where we can set $c_1 = c_0$).

Then the Eq. (6.12) has the unique solution $z_{g^0}(t), t \in \mathbb{R}$ bounded in H, that is, the kernel $\mathcal{K}_{g^0}$ consists of the unique trajectory $z_{g^0}(t)$. This solution $z_{g^0}(t)$ is exponentially stable, i.e. for every solution $u_{g^0}(t)$ of Eq. (6.12) the following inequality holds:

$$|u_{g^0}(t+\tau) - z_{g^0}(t+\tau)| \leqslant C_0 |u_\tau - z_{g^0}(\tau)| e^{-\beta t} \, \forall t \geqslant 0, \tag{8.3}$$

where $u_{g^0}(t+\tau)=U_{g^0}(t+\tau,\tau)u_\tau$ (the constants C_0 and β are independent of u_τ and τ). This statement is proved in [6] (see also [5]).

Property (8.3) implies that the set

$$\mathcal{A}^0=\left[\left\{z_{g^0}(t)\,|\,t\in\mathbb{R}\right\}\right]_H=\bigcup_{g\in\mathcal{H}(g^0)}\{z_g(0)\} \tag{8.4}$$

is the global attractor of the Eq. (6.12) under condition (8.1).

Remark 8.1. It is shown in [4] that inequality (8.2) holds with $c_1^2=\left(\frac{8}{27\pi}\right)^{1/2}=0.3071\ldots$. Using the numerical result of Weinstein [23], it was also shown in [4] that $c_1^2=0.2924\ldots$. This value is possibly the best for inequality (8.2). Hence, (8.3) and (8.4) are valid if $G<3.42$.

Remark 8.2. It is easy to construct examples of functions $g^0(x,t)$ satisfying (8.1) such that the set $\left\{z_{g^0}(t)\,|\,t\in\mathbb{R}\right\}$ is not closed in H. However, the set $\mathcal{A}^0$ is always closed and to describe this set we need to consider all the functions $z_{\hat{g}^0}(t)$ from the kernels of equations with external forces $\hat{g}^0\in\mathcal{H}(g^0)$ (see formula (6.15)). Notice that due to (8.1) the function $z_{\hat{g}^0}(t)$ is unique for any $\hat{g}^0\in\mathcal{H}(g^0)$ and it is exponentially stable (see (8.3)).

Remark 8.3. Inequality (8.3) implies that the global attractor $\mathcal{A}^0$ of system (6.12) is exponential under the condition (8.1), i.e. for any bounded set B in H

$$\sup_{\tau\in\mathbb{R}}\operatorname{dist}_H(U_{g^0}(t+\tau,\tau)B,\mathcal{A}^0)\leqslant C_1(|B|)e^{-\beta t}, \tag{8.5}$$

where C_1 depends on the norm B in H.

We now formulate the following result concerning the distance between $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$.

Theorem 8.1. *Under the assumptions of Theorem 7.1, we assume that the Grashof number G of the "limiting" 2D N.–S. system satisfies* (8.1). *Then the Hausdorff distance (in H) from the global attractor $\mathcal{A}^\varepsilon$ of the original 2D N.–S. system* (6.1) *to the global attractor $\mathcal{A}^0$ of the corresponding "limiting" system* (6.12) *satisfies the following inequality*

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon,\mathcal{A}^0)\leqslant C(\rho)\varepsilon^{1-\rho},\quad \forall\varepsilon,\ 0<\varepsilon\leqslant 1.$$

Here $0\leqslant\rho<1$ and $C(\rho)>0$ also depends on ν, $\|g_0\|_{L_2^b}$, and $\|g_1\|_{L_2^b}$.

The proof of Theorem 8.1 will be given in the forthcoming paper. It is analogous to the proof of the similar result concerning the Ginzburg–Landau equation with singularly oscillating terms considered in [7].

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