Some remarks on stability of semigroups arising from linear viscoelasticity

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Abstract. An abstract integrodifferential equation arising from linear viscoelasticity is considered. The stability properties of the related $C_0$-semigroup are discussed, in dependence on the form of the convolution (memory) kernel.

Keywords: linear viscoelasticity, memory kernels, $C_0$-semigroups, stability, exponential stability

1. Introduction

Let $(H_0, \langle \cdot , \cdot \rangle, \| \cdot \|)$ be an infinite-dimensional separable Hilbert space, and let $A$ be a strictly positive self-adjoint operator with compact inverse, defined on a domain $\mathcal{D}(A) \subset H_0$. We consider the problem

\[
\begin{aligned}
\partial_{tt} u(t) + Au(t) + \int_0^\infty \mu(s) [Au(t) - Au(t - s)] \, ds &= 0, \quad t > 0, \\
u(t) &= u_0(t), \quad t \leq 0, \\
\partial_t u(0) &= v_0. 
\end{aligned}
\]

Here, $\mu$ is a summable decreasing positive kernel defined on $\mathbb{R}^+ = (0, \infty)$, whose properties will be specified later. When $H_0 = L^2(\Omega)$, with $\Omega \subset \mathbb{R}^n$, and $A = -\Delta$, with $\mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega)$, the above equation describes the dynamics of linear viscoelastic solids.

Introducing the \textit{past history} variable [2]

\[
\eta^t(s) = u(t) - u(t - s), \quad s \geq 0,
\]
problem (1.1) can be rewritten as

\[
\begin{aligned}
\partial_{tt}u(t) + Au(t) + \int_0^\infty \mu(s) A\eta^t(s) \, ds &= 0, \quad t > 0, \\
\partial_t \eta^t(s) + \partial_s \eta^t(s) &= \partial_t u(t), \quad t, s > 0, \\
\eta^t(0) &= 0, \quad t \geq 0, \\
u(0) &= u_0, \quad \partial_t u(0) = v_0, \quad \eta^0(s) = \eta_0(s),
\end{aligned}
\]

where \(u_0 = u_0(0)\) and \(\eta_0(s) = u_0 - u_0(-s)\). Once an appropriated functional setting is established, (1.1) and (1.2) are equivalent. In fact, the latter is more general, in the sense that it provides solutions also for less regular initial past histories (see [13]).

As we will show shortly, (1.2) generates a linear contraction semigroup \(S(t)\) on a suitable Hilbert space \(\mathcal{H}\), the so-called extended phase space, accounting for the past history of the variable \(u\), which, in concrete situations, represents the displacement from equilibrium. It is then of some interest to study in detail the stability properties of \(S(t)\). This was indeed the main focus of several works appeared in the last years (see, e.g., [2–5,9,14,15]). In particular, for sufficiently regular kernels satisfying suitable assumptions (see (3.1) below), exponential stability has been proved, by means of Laplace transform methods [4], semigroup techniques [14,15], and direct energy estimates [9].

The aim of the present work is to establish stability results for a quite general class of kernels, and discuss some necessary conditions for exponential stability in order to hold. We will also carry out a detailed analysis of the asymptotic behavior of \(S(t)\) when particular kernels are considered, precisely, decreasing step functions. In this case, if the steps are proportional to a given length \(\ell\), stability may fail.

**Notation.** For \(\sigma \in \mathbb{R}\), we consider the Hilbert spaces \(H_\sigma = D(A^{\sigma/2})\) and \(L^2\)-weighted spaces \(M_\sigma = L^2_\mu(\mathbb{R}^+; H_\sigma)\), endowed with the inner products

\[
\langle u_1, u_2 \rangle_{H_\sigma} = \langle A^{\sigma/2}u_1, A^{\sigma/2}u_2 \rangle, \quad \langle \eta_1, \eta_2 \rangle_{M_\sigma} = \int_0^\infty \mu(s) \langle A^{\sigma/2}\eta_1(s), A^{\sigma/2}\eta_2(s) \rangle \, ds.
\]

Finally, we define the extended phase space

\[
\mathcal{H} = H_1 \times H_0 \times M_1.
\]

2. **The contraction semigroup**

**Assumptions on \(\mu\).** We assume \(\mu\) to be a monotone (possibly not strictly) decreasing summable function which is piece-wise absolutely continuous. In particular, this implies that \(\mu\) is nonnegative. Precisely, setting \(s_0 = 0\), we suppose that there exists a strictly increasing sequence \(\{s_n\}\) (possibly finite, or even reduced to the sole \(s_0\)) such that, for all \(n \geq 1\), \(\mu\) has jumps at \(s = s_n\), and it is absolutely continuous on each interval \(I_n = (s_{n-1}, s_n)\) and on the interval \(I_\infty = (s_\infty, \infty)\), where \(s_\infty = \sup_n s_n\), in the case when \(\{s_n\}\) is an infinite sequence and \(s_\infty < \infty\). In order to ensure dissipativity, we also require that

\[
\kappa = \int_0^\infty \mu(s) \, ds > 0.
\]
Under these conditions, $\mu'$ exists nonpositive almost everywhere. Finally, for $n \geq 1$, we denote

$$\mu_n = \mu(s_n^-) - \mu(s_n^+) \quad \text{and} \quad \mu_\infty = \mu(s_\infty^-) - \mu(s_\infty^+),$$

being $\mu_\infty$ defined only if $s_\infty < \infty$.

**Remark 2.1.** In fact, we could even consider more general kernels. Namely, we could allow many limit points of jumps, and even infinitely many. What we actually need is that the set of limit points have null measure.

We define the linear operator $T$ on $M_1$ with domain

$$D(T) = \{ \eta \in M_1 \mid \frac{\partial}{\partial s} \eta \in M_1, \eta(0) = 0 \},$$

acting as

$$T\eta = -\frac{\partial}{\partial s} \eta, \quad \eta \in D(T),$$

where $\frac{\partial}{\partial s}$ is the distributional derivative with respect to $s$. The operator $T$ is the infinitesimal generator of the right-translation $C_0$-semigroup $\Sigma(t)$ on $M_1$ (see [13]). Precisely,

$$\Sigma(t)\eta = \begin{cases} 0, & 0 < s \leq t, \\ \eta(s - t), & s > t. \end{cases}$$

Introducing (here and in the sequel) the 3-component vectors

$$z(t) = [u(t), v(t), \eta(t)]^\top \quad \text{and} \quad z_0 = [u_0, v_0, \eta_0]^\top \in \mathcal{H},$$

problem (1.2) translates into the linear evolution equation in $\mathcal{H}$

$$\begin{cases} \frac{d}{dt} z = Lz, \\ z(0) = z_0, \end{cases} \tag{2.1}$$

where the linear operator $L$ is defined as

$$L \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} -A(u + \int_0^\infty \mu(s)\eta(s) \, ds) \\ v \\ T\eta + v \end{pmatrix},$$

with domain

$$D(L) = \left\{ z \in \mathcal{H} \mid \begin{array}{l} v \in H_2, \\ u + \int_0^\infty \mu(s) A\eta(s) \, ds \in H_0, \\ \eta \in D(T) \end{array} \right\}.$$
Theorem 2.2. Problem (2.1) generates a $C_0$-semigroup $S(t) = e^{t L}$ of bounded linear operators on $\mathcal{H}$. Moreover, $S(t)$ is a contraction, that is,

$$\|S(t)\|_{L(\mathcal{H})} \leq 1, \quad \forall t \geq 0,$$

where $L(\mathcal{H})$ is the Banach space of bounded linear operator on $\mathcal{H}$.

The proof of the result follows from the Lumer–Phillips Theorem [19, Theorem 4.3]. Indeed (cf. [13]),

$$\langle Lz, z \rangle_{\mathcal{H}} = \langle T\eta_0, \eta_0 \rangle_{\mathcal{M}_1} = \frac{1}{2} \int_0^\infty \mu'(s)\|A^{1/2}\eta_0(s)\|^2 \, ds - \frac{1}{2} \sum_n \mu_n \|A^{1/2}\eta_0(s_n)\|^2 \leq 0, \quad (2.2)$$

for every $z \in D(L)$, where it is understood that the sum includes the value $n = \infty$ if $s_\infty < \infty$. Also, it is standard matter to show that $\mathbb{I} - L$ maps $D(L)$ onto $\mathcal{H}$ (see [3,11], where a similar case is treated). In particular, (2.2) yields

$$\frac{d}{dt} \|S(t)z\|_{\mathcal{H}} = \int_0^\infty \mu'(s)\|A^{1/2}\eta^t(s)\|^2 \, ds - \sum_n \mu_n \|A^{1/2}\eta^t(s_n)\|^2 \leq 0, \quad (2.3)$$

for every $z \in D(L)$. Concerning the third component of the solution $S(t)z$, we have the explicit representation formula (see [18])

$$\eta^t(s) = \begin{cases} u(t) - u(t-s), & 0 < s \leq t, \\ \eta_0(s-t) + u(t) - u_0, & s > t, \end{cases} \quad (2.4)$$

that holds for every $z \in \mathcal{H}$.

3. Exponential stability

The exponential stability of $S(t)$, or of semigroups related to similar models with memory, has been investigated by many authors (see, e.g., [3–5,9,14,15]). This amounts to proving the existence of two constants of $M \geq 1$ and $\varepsilon > 0$ such that

$$\|S(t)\|_{L(\mathcal{H})} \leq M e^{-\varepsilon t}, \quad \forall t \geq 0,$$

or, equivalently (cf. [19]), the existence of $t_0 > 0$ such that

$$\|S(t_0)\|_{L(\mathcal{H})} < 1.$$

We also mention that some authors studied the exponential decay of $u(t)$ and $\partial_t u(t)$ of the original system (1.1), taking $u(t) = 0$ for $t < 0$ (see, e.g., [6,16]).
**Remark 3.1.** Lack of exponential stability implies that there is no decay pattern valid for all initial data. Indeed, assume that there exists a positive function \( \Psi(t) \) decreasing to zero such that \( \| S(t) z \|_H \leq K z \Psi(t) \), for some \( K > 0 \) (depending on \( z \)). Then, the linear operators

\[ A_t = \frac{1}{\Psi(t)} S(t), \quad t \geq 0, \]

fulfill the estimate

\[ \sup_{t \geq 0} \| A_t z \|_H \leq K z, \quad \forall z \in H, \]

and by the Uniform Boundedness Theorem we conclude that \( \| S(t) \|_{L(H)} \leq K \Psi(t) \) for some \( K > 0 \).

In the above cited works, exponential stability is obtained assuming that \( \mu' \) is continuous and the condition

\[ \mu'(s) + \delta \mu(s) \leq 0, \quad \forall s > 0, \quad (3.1) \]

holds for some \( \delta > 0 \), which is the same to require that

\[ \mu(t + s) \leq e^{-\delta t} \mu(s), \quad \forall t \geq 0, \forall s > 0. \quad (3.2) \]

Indeed, (3.2) follows from (3.1) by applying the Gronwall lemma. Conversely, by (3.2) there holds

\[ \mu'(s) = \lim_{t \to 0} \frac{\mu(t + s) - \mu(s)}{t} \leq -\delta \mu(s). \]

Notice that (3.2) implies the exponential decay of the kernel \( \mu \) at infinity. The energy equality (2.2) together with condition (3.1) yield

\[ \langle T \eta_0, \eta_0 \rangle_{\mathcal{M}_1} \leq -\frac{\delta}{2} \| \eta \|_{\mathcal{M}_1}^2, \quad \forall \eta_0 \in \mathcal{D}(T), \]

or, equivalently,

\[ \| \Sigma(t) \|_{L(\mathcal{M}_1)} \leq e^{-\frac{\delta t}{2}}. \]

In this respect, with our more general assumptions for \( \mu \), nothing changes. In fact, we could consider (3.1) almost everywhere, or even replace \( \mu' \) with the distributional derivative \( D \mu \) of \( \mu \) (to take into account the jumps), and require the nonpositivity of the distribution \( D \mu + \delta \mu \). In any case, it appears of some interest to understand if (3.1) can be weakened. Indeed, it would seem quite natural to expect exponential stability assuming only the exponential decay of the kernel. Let us then begin our analysis studying the relationship between the decay properties of \( S(t) \) and \( \Sigma(t) \).

**Theorem 3.2.** If \( S(t) \) is exponentially stable, then so is \( \Sigma(t) \), with at least the same convergence rate.
Proof. Choose $z = [0, 0, \eta_0]^\top \in \mathcal{H}$. By assumption, 
\[
\max \{\|A^{1/2}u(t)\|, \|\eta^f\|_{\mathcal{M}_1}\} \leq \|S(t)z\|_{\mathcal{H}} \leq M e^{-\varepsilon t}\|\eta_0\|_{\mathcal{M}_1}.
\]
On the other hand, exploiting (2.4)
\[
\|\eta^f\|^2_{\mathcal{M}_1} \geq \int_t^\infty \|A^{1/2}\eta_0(s-t) + A^{1/2}u(t)\|^2 \, ds \\
\geq \frac{1}{2} \int_t^\infty \mu(s)\|A^{1/2}\eta_0(s-t)\|^2 \, ds - \|A^{1/2}u(t)\|^2 \int_t^\infty \mu(s) \, ds \\
\geq \frac{1}{2} \int_t^\infty \mu(s)\|A^{1/2}\eta_0(s-t)\|^2 \, ds - M^2 \kappa e^{-2\varepsilon t}\|\eta_0\|^2_{\mathcal{M}_1} \\
= \frac{1}{2} \|\Sigma(t)\eta_0\|^2_{\mathcal{M}_1} - M^2 \kappa e^{-2\varepsilon t}\|\eta_0\|^2_{\mathcal{M}_1}.
\]
Hence,
\[
\|\Sigma(t)\eta_0\|_{\mathcal{M}_1} \leq M \sqrt{2(1 + \kappa)} e^{-\varepsilon t}\|\eta_0\|_{\mathcal{M}_1},
\]
as claimed. \(\square\)

In light of the above result, in order to prove lack of exponential stability of $S(t)$ we can concentrate our attention on $\Sigma(t)$.

**Theorem 3.3.** The semigroup $\Sigma(t)$ is exponentially stable if and only if there exist $C \geq 1$ and $\delta > 0$ such that
\[
\mu(t+s) \leq C e^{-\delta t}\mu(s),
\]
for every $t \geq 0$ and almost every $s > 0$.

**Proof.** If (3.3) holds, we get at once that, for every $\eta_0 \in \mathcal{M}_1$,
\[
\|\Sigma(t)\eta\|^2_{\mathcal{M}_1} = \int_t^\infty \mu(s)\|A^{1/2}\eta_0(s-t)\|^2 \, ds = \int_0^\infty \mu(t+s)\|A^{1/2}\eta_0(s)\|^2 \, ds \\
\leq C e^{-\delta t} \int_0^\infty \mu(s)\|A^{1/2}\eta_0(s)\|^2 \, ds = C e^{-\delta t}\|\eta_0\|^2_{\mathcal{M}_1}.
\]
Conversely, assume that
\[
\|\Sigma(t)\|^2_{L(\mathcal{M}_1)} \leq C e^{-\delta t}.
\]
Then, for every $\eta_0 \in \mathcal{M}_1$ there holds
\[
\|\Sigma(t)\eta_0\|^2_{\mathcal{M}_1} - C e^{-\delta t}\|\eta_0\|^2_{\mathcal{M}_1} = \int_0^\infty [\mu(t+s) - C e^{-\delta t}\mu(s)]\|A^{1/2}\eta_0(s)\|^2 \, ds \leq 0.
\]
For any fixed \( t \), let
\[
\mathcal{A}_t = \{ s \in \mathbb{R}^+ \mid \mu(t + s) - C e^{-\delta t} \mu(s) > 0 \}.
\]
Set then \( \eta_0(s) = \chi_{\mathcal{A}_t}(s) u_\ast \), with \( u_\ast \in H^1 \) such that \( \| A^{1/2} u_\ast \| = 1 \). We conclude that
\[
\int_{\mathcal{A}_t} [\mu(t + s) - C e^{-\delta t} \mu(s)] \, ds = 0,
\]
which yields the desired conclusion. \( \square \)

The next example goes along the line of Remark 3.1.

**Example 3.4.** Consider
\[
\mu(s) = \frac{1}{1 + s^2},
\]
which does not satisfy (3.3). For \( \nu \in (0, 1) \), take \( \eta_0(s) = s^{(1-\nu)/2} u_\ast \), with \( u_\ast \in H^1 \) such that \( \| A^{1/2} u_\ast \| = 1 \). Then, for \( t \geq 1 \),
\[
\| \Sigma(t) \eta_0 \|_{M_1}^2 = \int_0^\infty \frac{s^{1-\nu}}{1 + (t + s)^2} \, ds = \frac{1}{t^\nu} \int_0^\infty \frac{s^{1-\nu}}{\nu + (1 + s)^2} \, ds \geq \frac{1}{3t^\nu} \| \eta_0 \|_{M_1}^2.
\]
Thus, the convergence rate to zero of \( \| \Sigma(t) \eta_0 \|_{M_1}^2 \) can be slower of \( t^{-\nu} \), for every \( \nu \in (0, 1) \). Refining the example, it is possible to find, for instance, logarithmic rates.

Combining Theorem 3.2 and Theorem 3.3 we have, in particular,

**Corollary 3.5.** If \( S(t) \) is exponentially stable, then \( \mu \) has an exponential decay at infinity.

**Remark 3.6.** It should be noted that the above results depend only on the representation formula for \( \eta \). Hence, Corollary 3.5 still holds if we consider, for instance, the equation
\[
\partial_{tt} u(t) + \partial_t u(t) + A u(t) + \int_0^\infty \mu(s) [Au(t - s) - Au(t)] \, ds = 0,
\]
which, when \( \mu \equiv 0 \), is exponentially stable. A similar issue was noted in [6], for a Volterra-type equation.

Corollary 3.5 establishes a necessary condition. But, as a matter of fact, relation (3.3) is stronger. Indeed, it is not difficult to construct an exponentially decaying kernel \( \mu \) that does not satisfy (3.3).

**Example 3.7.** Let \( \mu \) be a smooth kernel satisfying our general assumptions and with the following properties:
- \( 0 < \mu(s) \leq e^{-s} \);
- \( \mu \) is constant on any interval \([n^2, n^2 + n + 1]\), \( n \in \mathbb{N} \).
Then, if we assume (3.3) to hold, for almost every \( s \in [n^2, n^2 + 1] \) we get
\[
\mu(s) = \mu(s + n^2 + n + 1 - s) \leq C e^{-\delta(n^2+n+1-s)} \mu(s) \leq C e^{-\delta n} \mu(s),
\]
so obtaining the inequality
\[
1 \leq C e^{-\delta n},
\]
which is clearly false for \( n \) large enough.

We conclude that the exponential decay of the kernel is not enough in order to have exponential stability of \( S(t) \). Of course, it remains for the moment open whether or not the (necessary) stronger condition (3.3) is sufficient as well.

4. Stability

We now want to examine the stability of \( S(t) \), namely, to prove under which conditions we can ensure that \( \|S(t)z\|_{\mathcal{H}} \to 0 \) as \( t \to \infty \) for every initial datum \( z \in \mathcal{H} \). Observe that \( \Sigma(t) \) is always stable. Indeed, for any \( \eta_0 \in \mathcal{M}_1 \),
\[
\|\Sigma(t)\eta_0\|_{\mathcal{M}_1}^2 = \int_0^\infty \mu(t+s)\|A^{1/2}\eta_0(s)\|^2 \, ds,
\]
and the claim is a straightforward consequence of the Dominated Convergence Theorem.

Assume next that we are given a reflexive Banach space \( \mathcal{V} \subset \mathcal{H} \), with continuous and dense embedding (but not necessarily compact). Then we have

**Lemma 4.1.** Suppose that for every \( z \in \mathcal{V} \) the following hold.

(i) \( \|S(t)z\| = \|z\| \) for all \( t > 0 \) implies that \( z = 0 \).

(ii) The set \( \bigcup_{t \geq t_0} S(t)z \) is relatively compact in \( \mathcal{H} \) and bounded in \( \mathcal{V} \), for some \( t_0 \geq 0 \).

Then \( S(t) \) is stable.

**Proof.** Let \( z \in \mathcal{V} \). From (ii), the \( \omega \)-limit set of \( z \), defined as
\[
\omega(z) = \cap_{t \geq 0} \overline{\bigcup_{\tau \geq t} S(\tau)z_0},
\]
is nonempty. Choosing then an arbitrary \( \zeta \in \omega(z) \), there exists \( t_n \to \infty \) such that \( S(t_n)z \to \zeta \) in \( \mathcal{H} \). We now claim that \( \zeta \in \mathcal{V} \). Indeed, since \( S(t_n)z \) is bounded in \( \mathcal{V} \) (at least for \( n \) large enough), there is a subsequence \( t_{n_j} \) such that \( S(t_{n_j})z \to \xi \in \mathcal{V} \) weakly in \( \mathcal{V} \). But this implies that \( S(t_{n_j})z \to \xi \) weakly in \( \mathcal{H} \), so that \( \xi = \zeta \). Since \( S(t) \) is a contraction semigroup, \( \|S(t)z\|_{\mathcal{H}} \) is a decreasing function of \( t \). In particular,
\[
\|\zeta\|_{\mathcal{H}} = \lim_{n \to \infty} \|S(t_n)z\|_{\mathcal{H}} = \lim_{n \to \infty} \|S(t + t_n)z\|_{\mathcal{H}} = \lim_{n \to \infty} \|S(t)S(t_n)z\|_{\mathcal{H}} = \|S(t)\zeta\|_{\mathcal{H}},
\]
for every $t \geq 0$. By (i), we conclude that $\zeta = 0$, and, consequently, $\omega(z) = \{0\}$. This forces the convergence $S(t)z \to 0$ as $t \to \infty$.

Let now $z \in \mathcal{H}$. For any $\varepsilon > 0$, there exists $z_* \in \mathcal{V}$ such that $\|z - z_*\|_{\mathcal{H}} \leq \varepsilon/2$. Besides, there is a $t_* = t_*(z_*)$ such that $\|S(t)z_*\|_{\mathcal{H}} \leq \varepsilon/2$, for all $t \geq t_*$. Then, since $S(t)$ is a contraction,

$$\|S(t)z\|_{\mathcal{H}} \leq \|S(t)(z - z_*)\|_{\mathcal{H}} + \|S(t)z_*\|_{\mathcal{H}} \leq \|z - z_*\|_{\mathcal{H}} + \frac{\varepsilon}{2} \leq \varepsilon, \quad \forall t \geq t_*,$$

i.e., $S(t)z \to 0$ as $t \to \infty$, for all $z \in \mathcal{H}$.

In order to apply the above lemma, we introduce the subspace of $\mathcal{H}$

$$\mathcal{V} = H_2 \times H_1 \times [\mathcal{M}_2 \cap \mathcal{D}(T)] \subset \mathcal{D}(\mathbb{L}),$$

which is a reflexive Banach space when endowed with the norm

$$\|(u, v, \eta)^\top\|_{\mathcal{V}}^2 = \|Au\|^2 + \|A^{1/2}v\|^2 + \|\eta\|_{\mathcal{M}_2}^2 + \|T\eta\|_{\mathcal{M}_1}^2.$$

The embedding $\mathcal{V} \subset \mathcal{H}$ is clearly continuous and dense, but not compact, due to the third component (see [18] for a counterexample to compactness). We now show that condition (ii) of Lemma 4.1 holds for every $z \in \mathcal{V}$ (for any kernel $\mu$ in the considered class). Let then $z = [u_0, v_0, \eta_0]^\top \in \mathcal{V}$ be fixed. Approximate $z$ in $\mathcal{V}$ with a sequence

$$z_n = [u_{0n}, v_{0n}, \eta_{0n}]^\top \in H_3 \times H_2 \times [\mathcal{M}_3 \cap \mathcal{D}(T \circ A^{1/2})].$$

Setting $\tilde{z}_n = [A^{1/2}u_{0n}, A^{1/2}v_{0n}, A^{1/2}\eta_{0n}]^\top$, by (2.3) we get

$$\frac{d}{dt} \|S(t)z_n\|^2_{H_2 \times H_1 \times \mathcal{M}_2} = \frac{d}{dt} \|S(t)\tilde{z}_n\|^2_{\mathcal{H}} \leq 0,$$

so that

$$\|S(t)z_n\|_{H_2 \times H_1 \times \mathcal{M}_2} \leq \|z_n\|_{H_2 \times H_1 \times \mathcal{M}_2}.$$

Hence, letting $n \to \infty$, we conclude that

$$\|S(t)z\|_{H_2 \times H_1 \times \mathcal{M}_2} \leq \|z\|_{H_2 \times H_1 \times \mathcal{M}_2}, \quad \forall z \in \mathcal{V}.$$ 

By (2.4), we learn that

$$T\eta^t(s) = \begin{cases} -\partial_t u(t - s), & 0 < s \leq t, \\ T\eta_0(s - t), & s > t, \end{cases}$$

which entails at once the estimate

$$\sup_{t \geq 0} \|T\eta^t\|_{\mathcal{M}_1} < \infty.$$
This means that $\bigcup_{t \geq 0} S(t)z$ is bounded in $V$. In order to get the required compactness in $H$, we need a uniform control of the “tails” of $\eta^t$. Arguing exactly as in [10], this is obtained noting that

$$\lim_{y \to \infty} \left[ \sup_{s \geq 1} \int_{(0, 1/y) \cup (y, \infty)} \mu(s) \left\| A^{1/2} \eta^t(s) \right\|^2 ds \right] = 0,$$

and taking advantage of the following slight generalization of [18, Lemma 5.5]:

**Lemma 4.2.** Let $K \subset D(T)$ be such that

$$\sup_{\eta \in K} \left[ \| \eta \|_{M_2} + \| T\eta \|_{M_1} \right] < \infty, \quad \lim_{y \to \infty} \left[ \sup_{\eta \in K} \int_{(0, 1/y) \cup (y, \infty)} \mu(s) \left\| A^{1/2} \eta^t(s) \right\|^2 ds \right] = 0.$$

Then $K$ is relatively compact in $M_1$.

Therefore, the remaining part of condition (ii) follows for $t_0 = 1$.

**Remark 4.3.** Since $V \subset D(L)$, whenever $z \in V$, on account of (2.3), the condition $\| S(t)z \|_H = \| z \|_H$ for all $t > 0$ reads

$$\int_0^\infty \mu'(s) \left\| A^{1/2} \eta^t(s) \right\|^2 ds - \sum_n \mu_n \left\| A^{1/2} \eta^t(s_n) \right\|^2 = 0, \quad \forall t \geq 0.$$

**Remark 4.4.** Notice that, if $\| S(t)z \|_H = \| z \|_H$ for all $t > 0$, it is enough to show that $u(t)$ is constant to reach the conclusion that $z = 0$. Indeed, in that case, we have $u(t) = u_0$, $\partial_t u(t) = v_0 = 0$ and, from the representation formula (2.4),

$$\eta^t(s) = \begin{cases} 0, & 0 < s \leq t, \\ \eta_0(s - t), & s > t. \end{cases}$$

The energy equality then yields

$$\| S(t)z \|^2_H = \| u_0 \|^2 + \| \Sigma(t)\eta_0 \|^2_{M_1} = \| u_0 \|^2 + \| \eta_0 \|^2_{M_1}.$$

Since we know that $\| \Sigma(t)\eta_0 \|_{M_1} \to 0$, we conclude that $\eta_0 = 0$. Hence, the first equation of (1.2) simply reads $Au_0 = 0$, which yields $u_0 = 0$.

In view of the above remarks and Lemma 4.1, we can state a more convenient sufficient condition for stability.

**Lemma 4.5.** Assume that, for every $z \in V$, the equality

$$\int_0^\infty \mu'(s) \left\| A^{1/2} \eta^t(s) \right\|^2 ds - \sum_n \mu_n \left\| A^{1/2} \eta^t(s_n) \right\|^2 = 0, \quad \forall t \geq 0,$$  \hspace{1cm} (4.1)

implies that $u(t) = u_0$ for all $t \geq 0$. Then $S(t)$ is stable.
Remark 4.6. The solution \( u(t) \) corresponding to a datum \( z \in \mathcal{V} \) is \( \tau \)-periodic, for every \( \tau \) belonging to the set
\[
\mathcal{P} = \{ s \in \mathbb{R}^+ \mid \eta_t^\tau(s) = 0, \forall t \geq 0 \}.
\]
Indeed, in view of (2.4), if \( \tau \in \mathcal{P} \), then
\[
u(t) = u(t - \tau), \quad \forall t \geq \tau.
\]
In particular, if (4.1) holds for some \( z \in \mathcal{V} \), then \( s_n \in \mathcal{P} \) for every \( n \geq 1 \), so that \( u(t) \) has period \( s_n \).

In order to state the main result on the stability of \( S(t) \), we need first a definition.

**Definition 4.7.** Let \( \ell \in \mathbb{R}^+ \). We say that a kernel \( \mu \) is \( \ell \)-paced if there exist a strictly increasing sequence \( \{k_n\} \subset \mathbb{N} \) and a positive sequence \( \{\gamma_n\} \), strictly decreasing until, possibly, \( \gamma_n = 0 \), such that
\[
\mu(s) = \sum_{n=1}^{\infty} \gamma_n \chi_{[\ell k_n, \ell k_n - \ell)}(s),
\]
having set \( k_0 = 0 \) (clearly, \( \mu_n = \gamma_n - \gamma_{n+1} \)).

**Remark 4.8.** The “pace” \( \ell \) of the kernel \( \mu \) is not uniquely determined. It is however clear there exists the largest possible one, call it \( \ell^* \), which is the greatest common divisor of \( \{\ell k_n\} \). In that case, provided that (4.1) holds, \( \ell^* \) is a period of \( u(t) \).

The above definition includes the particular case when \( \mu \) has only a finite number \( N \) of steps. There, \( \gamma_n = 0 \) for \( n \geq N \), and the choice of the related \( k_n \) is arbitrary.

In view of our general assumptions, there holds
\[
\sum_{n=1}^{\infty} \ell \gamma_n(k_n - k_{n-1}) = \kappa.
\]

Then we have

**Theorem 4.9.** If the kernel \( \mu \) is not \( \ell \)-paced, then \( S(t) \) is stable.

**Proof.** Let \( z \in \mathcal{V} \) be such that (4.1) holds. From Lemma 4.5, stability follows once we show that \( u(t) \) is constant. Due to Remark 4.6, \( u(t) \) is \( \tau \)-periodic, for every \( \tau \in \mathcal{P} \). Hence, if \( \mathcal{P} \) contains two rationally independent numbers, then \( u(t) \) must be constant. We shall distinguish two situations.

\( \diamond \) Assume first that \( \mu \) is not a step function. Then, since \( \mu \) is absolutely continuous on every open interval \( (s_{n-1}, s_n) \), from (4.1) the set \( \mathcal{P} \) has positive Lebesgue measure, and it certainly contains two rationally independent numbers.

\( \diamond \) Conversely, let \( \mu \) be a step function (either with a finite or an infinite number of steps). Then \( u(t) \) is periodic of period \( s_n \) and so it is constant, unless the \( s_n \) are of the form
\[
s_n = \begin{cases} 
\beta, & n = 1, \\
\beta(p_n + r_n), & n > 1,
\end{cases}
\]
for some \( \beta \in \mathbb{R}^+ \), \( p_n \in \mathbb{N} \) and \( r_n \in [0, 1) \cap \mathbb{Q} \). Thus, \( u(t) \) is periodic of period \( \beta r_n \) as well. If there are infinitely many different \( r_n \), then there is a Cauchy subsequence \( \{ r_{n_k} \} \). But since \( \beta |r_{n_k+1} - r_{n_k}| \) is again a period of \( u(t) \), we learn that \( u(t) \) has an arbitrarily small period and, consequently, is constant.

In summary, the only case where we cannot prove stability is when the sequence \( \{ r_n \} \) belongs to a finite set \( \{ q_1, \ldots, q_N \} \subset [0, 1) \cap \mathbb{Q} \). Setting \( q_k = a_k/b_k \), with \( a_k \in \mathbb{N} \cup \{0\} \) and \( b_k \in \mathbb{N} \), and denoting

\[
d = \frac{1}{b_1 \ldots b_N},
\]

we have that

\[
p_n + r_n = k_n d,
\]

for some \( k_n \in \mathbb{N} \). Finally, defining \( k_1 = 1/d \) and \( \ell = \beta d \), we conclude that \( s_n = \ell k_n \), that is, \( \mu \) is an \( \ell \)-paced kernel. \( \Box \)

5. The \( \ell \)-paced kernel

In this section, we focus on the \( \ell \)-paced kernel (4.2), which is not covered by Theorem 4.9. A particular case is the one-step kernel, namely,

\[
\mu(s) = \gamma_1 \chi_{[0,\ell]}(s).
\]

This, in principle, seems to be a promising candidate kernel in order to have exponential stability of \( S(t) \). Indeed, it satisfies condition (3.3), for the norm of the related semigroup \( \Sigma(t) \) is zero for \( t \geq \ell \). Instead, as we will see, somehow unexpected situations may occur, depending on the relations among \( \ell, \kappa \) and the eigenvalues of \( A \).

Let then \( \{ \alpha_m \} \) be the sequence of the eigenvalues of \( A \), with relative eigenvectors \( \{ e_m \} \). Under our general assumptions, \( \{ \alpha_m \} \) is a strictly positive sequence, that can be ordered in such a way to tend (increasingly) to infinity.

Denoting

\[
\omega_m = \omega_m(\ell, \kappa, \alpha_m) = \frac{\ell}{\pi} \sqrt{\alpha_m(1 + \kappa)},
\]

we have the following results, valid for all \( \ell \)-paced kernels.

**Proposition 5.1.** If \( \omega_m \in \mathbb{Q} \) for some \( m \in \mathbb{N} \), then \( S(t) \) is not stable, and admits periodic orbits.

**Proof.** Let \( m \in \mathbb{N} \) be such that

\[
\frac{\ell}{2\pi} \sqrt{\alpha_m(1 + \kappa)} = \frac{p}{q},
\]

for some \( p, q \in \mathbb{N} \). Then \( \alpha_m(1 + \kappa) \) is a rational multiple of \( \pi^2 \), which implies that \( \omega_m \) is a rational multiple of \( \ell \). Hence, \( \omega_m \in \mathbb{Q} \) for some \( m \in \mathbb{N} \), and the proposition follows. \( \Box \)
for some \( p, q \in \mathbb{N} \). Up to replacing \( \ell \) with \( \ell/q \), which is still a pace of \( \mu \) provided that we change the sequence \( k_n \) appearing in (4.2) into \( qk_n \), the above equality becomes

\[
\frac{\ell}{2\pi} \sqrt{\alpha_m(1 + \kappa)} = p.
\]

Define

\[
\beta = \sqrt{\alpha_m(1 + \kappa)} = \frac{2p\pi}{\ell}.
\]

Then,

\[
S(t)[0, \beta e_m, e_m \sin \beta s]^\top = [u(t), \partial_t u(t), \eta^t(s)]^\top,
\]

where

\[
u(t) = e_m \sin \beta t, \quad \partial_t u(t) = \beta e_m \cos \beta t, \quad \eta^t(s) = u(t) - u(t - s).
\]

Indeed, it is easily verified that

\[
\partial_t \eta^t(s) = -\partial_s \eta^t(s) + \partial_t u(s).
\]

Moreover

\[
\partial_{tt} u(t) + A u(t) + \int_0^\infty \mu(s) A u(t - s) \, ds = -\beta^2 u(t) + \alpha_m(1 + \kappa) u(t) = 0,
\]

and

\[
- \int_0^\infty \mu(s) A u(t - s) \, ds = -\sum_{n=1}^\infty \gamma_n \int_{\ell k_{n-1}}^{\ell k_n} A u(t - s) \, ds
\]

\[
= -\alpha_m e_m \sum_{n=1}^\infty \gamma_n \int_{\ell k_{n-1}}^{\ell k_n} \sin \beta(t - s) \, ds
\]

\[
= \frac{\alpha_m}{\beta} e_m \sum_{n=1}^\infty \gamma_n [\cos(\beta \ell k_{n-1}) - \cos(\beta t - \beta \ell k_n)] = 0,
\]

since \( \beta \ell \) is a multiple of \( 2\pi \). \( \square \)

Therefore, in this situation, the system exhibits solutions which have a purely elastic behavior. A similar feature was noted in [7,8] (see also [5]), for some regular nonmonotone kernels.
**Proposition 5.2.** If $\omega_m \not\in \mathbb{Q}$ for every $m \in \mathbb{N}$, then $S(t)$ is stable.

**Proof.** Let $z \in V$ be such that (4.1) holds. It is convenient (though not essential) to consider the representation (4.2) of $\mu$ for which $\ell$ is the largest possible pace, so that, from Remark 4.6, $u(t)$ has period $\ell$. Using (2.4) and (4.1), we get that

$$\eta_0(s) = u_0 - u(\ell k_n - s), \quad \forall s \in [0, \ell k_n]. \tag{5.1}$$

Introduce the function

$$\phi^t(s) = \eta^t(s) - u(t) = \begin{cases} -u(t - s), & 0 < s \leq t, \\ \eta_0(s - t) - u_0, & s > t. \end{cases}$$

Our first task is to show that the integral

$$\int_0^\infty \mu(s)A\phi^t(s)\,ds$$

is in fact independent of $t$. Let then $t > 0$ be fixed. Choosing $N$ large enough such that $t < \ell k_N$, there holds

$$\int_0^{\ell k_N} \mu(s)A\phi^t(s)\,ds = -\int_0^t \mu(s)Au(t - s)\,ds + \int_t^{\ell k_N} \mu(s)[\eta_0(s - t) - u_0]\,ds.$$ 

But $u(t - s) = u(\ell k_N + t - s)$. Besides, (5.1) entails that

$$\eta_0(s - t) - u_0 = -u(\ell k_N + t - s), \quad \forall s \in [t, \ell k_N].$$

Therefore

$$\int_0^{\ell k_N} \mu(s)A\phi^t(s)\,ds = -\sum_{n=1}^N \gamma_n \int_{\ell k_{n-1}}^{\ell k_n} Au(\ell k_N + t - s)\,ds = -\sum_{n=1}^N \gamma_n \int_{\ell k_{n-1}}^{\ell k_n} Au(\ell k_N + t - s)\,ds.$$ 

Appealing again to the periodicity of $u(t)$,

$$\psi_n = \gamma_n \int_{\ell k_{n-1}}^{\ell k_n} Au(\ell k_N + t - s)\,ds$$

is an element of $H_{-1}$ independent of $t$. In addition, using the contracting property of the semigroup $S(t)$, we obtain

$$\| A^{-1/2} \psi_n \| \leq \ell \gamma_n (k_n - k_{n-1}) \| z \|_H.$$ 

Hence, setting

$$\psi = -\sum_{n=1}^\infty \psi_n,$$
it follows that $\psi \in H_{-1}$ and
\[ \|\psi\|_{H_{-1}} \leq \kappa \|\varphi\|_H. \]

So we end up with
\[ \int_0^\infty \mu(s)A\psi^t(s)\,ds = \lim_{N \to \infty} \int_0^{\ell k N} \mu(s)A\psi^t(s)\,ds = \psi, \]
that is, the integral in the left-hand side of the above equality is independent of time. Consequently, the first equation of (1.2) reads
\[ \partial_{tt} u(t) + (1 + \kappa)Au(t) + \int_0^\infty \mu(s)\phi^t(s)\,ds = \partial_{tt} u(t) + (1 + \kappa)Au(t) + \psi = 0. \]

Naming
\[ \zeta = \frac{1}{1 + \kappa} A^{-1} \psi \quad \text{and} \quad \xi = u + \zeta, \]
we rewrite the equation as
\[ \partial_{tt} \xi(t) + (1 + \kappa)A\xi(t) = 0. \]

We now consider the scalar function $\beta_m(t) = \langle \xi(t), e_m \rangle$, which clearly satisfies the ordinary differential equation
\[ \beta''_m(t) + \alpha_m(1 + \kappa)\beta_m(t) = 0. \]

Solving this equation, we find that
\[ \beta_m(t) = A_m \sin \sqrt{\alpha_m(1 + \kappa)} t + B_m \cos \sqrt{\alpha_m(1 + \kappa)} t, \]
for some $A_m, B_m \in \mathbb{R}$. In particular, $\beta_m(t)$ is periodic of period $2\pi / \sqrt{\alpha_m(1 + \kappa)}$. On the other hand, $\beta_m(t)$ is also periodic of period $\ell$, for
\[ \beta_m(t) = \langle u(t), e_m \rangle + \langle \zeta, e_m \rangle. \]

Since $2\pi / \sqrt{\alpha_m(1 + \kappa)}$ and $\ell$ are rationally independent, we conclude that $\beta_m(t)$ is constant. But this is true for every $m \in \mathbb{N}$; hence $\xi(t)$ is constant, and, in turn, so is $u(t)$. The claim then follows from Lemma 4.5. □
6. The step-kernel: lack of exponential stability

We conclude the paper analyzing the exponential stability of $S(t)$ when $\mu$ is a step-function, that is, when it has the form

$$\mu(s) = \sum_{n=1}^{\infty} \gamma_n \chi_{[s_{n-1},s_n]}(s), \quad (6.1)$$

were $s_0 = 0$ and $\{s_n\}$ is the strictly increasing sequence of jumps, while $\{\gamma_n\}$ is a positive sequence, strictly decreasing until, possibly, $\gamma_n = 0$. To keep uniformity of notation, as before, if the number of steps is finite and equal to $N$, then $\gamma_n = 0$ for $n > N$, and the relative $s_n$ is chosen arbitrarily. We denote

$$\gamma_\infty = \inf_{n \in \mathbb{N}} \gamma_n.$$ 

The value $\gamma_\infty$ represents the height of the “last jump” when $\mu$ has infinitely many jumps, hence it is zero unless the $s_n$ accumulate to $s_\infty < \infty$. In that case, $\gamma_\infty$ can be strictly positive.

In the preceding section, we showed that for this kind of kernel the semigroup $S(t)$ is stable, unless $\mu$ is $\ell$-paced and $\omega_m(\ell, \kappa, \alpha_m) \in \mathbb{Q}$ for some $m \in \mathbb{N}$, in which case there exist periodic trajectories. It then interesting to see if it is exponentially stable as well.

We shall consider one-dimensional viscoelasticity, although the following results hold for more general settings (see the end of the section). Thus, fix $H_0 = L^2(0, \pi)$, and let

$$A = -\frac{d^2}{dx^2}, \quad \text{with } D(A) = H^2(0, \pi) \cap H^1_0(0, \pi).$$

In this framework, $\alpha_m = m^2$, $e_m = e_m(x) = \sin mx$ and $\omega_m = \ell m \sqrt{(1 + \kappa)}/\pi$. It is convenient to introduce

$$\omega = \frac{\omega_m}{m} = \frac{\ell}{\pi} \sqrt{(1 + \kappa)}.$$ 

In view of Proposition 5.1 and Proposition 5.2, the semigroup $S(t)$ is stable if and only if $\omega \not\in \mathbb{Q}$.

As we will show, when $\omega \not\in \mathbb{Q}$, exponential stability never occurs, even if the necessary condition (3.3) is satisfied (for instance, when the number of steps is finite). To this end, we need some preliminaries. The first is a classical result due to Dirichlet on simultaneous Diophantine approximation. For the reader’s convenience, we include a short proof.

**Lemma 6.1.** Let $a_1, \ldots, a_N$ be real numbers. Then, for every $\varepsilon > 0$, there exist $m \in \mathbb{N}$ and $p_n \in \mathbb{Z}$, with $m \geq 1/\varepsilon$, such that

$$|a_n m - p_n| \leq \varepsilon,$$

for every $n = 1, \ldots, N$. 
Proof. Choose $M \in \mathbb{N}$ such that $M \geq 1/\varepsilon$. Split the unit cube $[0, 1]^N \subset \mathbb{R}^N$ into $M^N$ cubes of edges $1/M$. Denoting $\bar{a} = (a_1, \ldots, a_N)$, consider $M^N + 1$ “vector-pigeons”

$$\frac{j\bar{a}}{M} = (\frac{ja_1}{M}, \ldots, \frac{ja_N}{M}), \quad j = M, 2M, \ldots, (M^N + 1)M,$$

where $\frac{x}{M} = x - [x]$. Two of them, say, $\frac{j_1\bar{a}}{M}$ and $\frac{j_2\bar{a}}{M}$, with $j_1 > j_2$, occupy the same pigeonhole. Setting $m = j_1 - j_2$ and $p_n = [j_1a_n] - [j_2a_n]$, we obtain the desired conclusion. \boxed{

In the sequel, we shall assume that $\mu$ is a step-function of the form (6.1), but not $\ell$-paced with $\omega \in \mathbb{Q}$. The half-Fourier transform of $\mu$ is defined as

$$\hat{\mu}(\lambda) = \int_0^\infty e^{-i\lambda s} \mu(s) \, ds.$$

We have

**Lemma 6.2.** For every $m \in \mathbb{N}$, denote

$$c_m = m\hat{\mu}(m\sqrt{1 + \kappa}).$$

Then, $c_m \neq 0$ for every $m$, and there is a sequence $\{m_j\} \subset \mathbb{N}$ such that

$$\lim_{j \to \infty} c_{m_j} = 0.$$

**Proof.** By direct a computation,

$$c_m = \frac{1}{i\sqrt{1 + \kappa}} \sum_{n=1}^\infty \gamma_n \left( e^{-is_n - im\sqrt{1 + \kappa}} - e^{-is_n - im\sqrt{1 + \kappa}} \right).$$

Exploiting the convergence $\gamma_n \downarrow \gamma_\infty$, there holds

$$c_m = \frac{1}{i\sqrt{1 + \kappa}} \left( \gamma_1 - \sum_{n=1}^\infty (\gamma_n - \gamma_{n+1}) e^{-is_n - im\sqrt{1 + \kappa}} - \gamma_\infty e^{-is_\infty - im\sqrt{1 + \kappa}} \right),$$

where it is understood that the last term in the right-hand side disappears when $\gamma_\infty = 0$.

Assume now that $c_m = 0$ for some $m \in \mathbb{N}$. Then,

$$\gamma_1 = \sum_{n=1}^\infty (\gamma_n - \gamma_{n+1}) e^{-is_n - im\sqrt{1 + \kappa}} + \gamma_\infty e^{-is_\infty - im\sqrt{1 + \kappa}}.$$

On the other hand, we also have that

$$\gamma_1 = \sum_{n=1}^\infty (\gamma_n - \gamma_{n+1}) + \gamma_\infty.$$
Since $\gamma_n - \gamma_{n+1} \geq 0$ and $\gamma_\infty \geq 0$, equality of the above two expressions occurs only if the numbers $s_n m \sqrt{1 + \kappa}$ and $s_\infty m \sqrt{1 + \kappa}$ are all multiples of $2\pi$, against the assumption that $\mu$ is not $\ell$-paced with $\omega \in \mathbb{Q}$.

In light of the above representation of $\gamma_1$, we rewrite $c_m$ more conveniently as

$$c_m = \frac{1}{i\sqrt{1 + \kappa}} \left( \sum_{n=1}^{\infty} (\gamma_n - \gamma_{n+1})(1 - e^{-is_n m \sqrt{1 + \kappa}}) + \gamma_\infty(1 - e^{-is_\infty m \sqrt{1 + \kappa}}) \right).$$

Fix $\varepsilon > 0$ small, and choose $N = N(\varepsilon) \in \mathbb{N}$ large enough such that

$$\gamma_N - \gamma_\infty = \sum_{n=N}^{\infty} (\gamma_n - \gamma_{n+1}) \leq \frac{\varepsilon}{4}.$$

Then,

$$|c_m| \leq \gamma_\infty |1 - e^{-is_\infty m \sqrt{1 + \kappa}}| + \sum_{n=1}^{N-1} (\gamma_n - \gamma_{n+1}) |e^{-is_n m \sqrt{1 + \kappa}} - 1| + \frac{\varepsilon}{2}.$$

Besides, on account of Lemma 6.1, we can find $m = m(\varepsilon) \in \mathbb{N}$, $p_n = p_n(\varepsilon) \in \mathbb{Z}$ and $p_\infty = p_\infty(\varepsilon) \in \mathbb{Z}$ such that

$$|s_n m \sqrt{1 + \kappa} - 2p_n \pi| \leq \frac{\varepsilon}{4\gamma_1},$$

for all $n = \infty, 1, \ldots, N - 1$. For this particular $m$,

$$|e^{-is_n m \sqrt{1 + \kappa}} - 1| = |e^{-i(s_n m \sqrt{1 + \kappa} - 2p_n \pi)} - 1| \leq |s_n m \sqrt{1 + \kappa} - 2p_n \pi| \leq \frac{\varepsilon}{4\gamma_1},$$

where we used the elementary inequality $|e^{i\vartheta} - 1| \leq |\vartheta|$, for $\vartheta \in \mathbb{R}$. Therefore,

$$|c_m| \leq \frac{\varepsilon \gamma_\infty}{4\gamma_1} + \frac{\varepsilon}{4\gamma_1} \sum_{n=1}^{N} (\gamma_n - \gamma_{n+1}) + \frac{\varepsilon}{2} \leq \varepsilon,$$

as desired. \hfill \Box

We can now state and prove the main result of this section.

**Theorem 6.3.** If $\mu$ is a step-function of the form (6.1), then the semigroup $S(t)$ is not exponentially stable.

**Proof.** We assume that $\mu$ is not $\ell$-paced with $\omega \in \mathbb{Q}$, otherwise $S(t)$ is not even stable. Using a procedure devised in [17], and further developed in [12], we will show that the necessary condition in order to have exponential stability (see [20])

$$\inf_{\lambda \in \mathbb{R}} \| (i\lambda I - \mathbb{L}) z \|_{\mathcal{H}} \geq \varepsilon \| z \|_{\mathcal{H}}, \quad \forall z \in \mathcal{D}(\mathbb{L}), \ varepsilon > 0,$$

(6.2)
fails to hold. For \( \lambda \in \mathbb{R} \) and \( z_m = [0, 0, m^{-1} e_m]\top \in \mathcal{H} \), we consider the (complex) equation for the unknown variable \( z = [u, v, \eta]\top \)

\[
(i\lambda I - L)z = z_m.
\]

Note that \( \|z_m\|_\mathcal{H} = \sqrt{\kappa} \). If we look for a solution of the form

\[
u = \frac{\rho}{i\lambda} e_m, \quad v = \rho e_m, \quad \eta(s) = \varphi(s)e_m,
\]

with \( \rho \in \mathbb{C} \) and \( \varphi \in L^2_\mu(\mathbb{R}^+, \mathbb{C}) \) such that \( \varphi(0) = 0 \), the above system turns into

\[
\begin{cases}
\lambda^2 \rho - m^2 \rho - i\lambda m^2 \int_0^\infty \mu(s)\varphi(s) \, ds = 0, \\
i\lambda \varphi(s) + \partial_s \varphi(s) - \rho = \frac{1}{m}.
\end{cases}
\]

An integration of the second equation gives

\[
\varphi(s) = \frac{1}{i\lambda} \left( \frac{1}{m} + \rho \right) \left( 1 - e^{-i\lambda s} \right).
\]

Substituting \( \varphi \) into the first equation, we obtain

\[
\rho(\lambda^2 - m^2 - m^2 \kappa) - m(\kappa - \hat{\mu}(\lambda) - m\rho\hat{\mu}(\lambda)) = 0.
\]

We now apply Lemma 6.2: choosing \( \lambda = m_j \sqrt{1 + \kappa} \), we get that

\[
|\rho| = \frac{|m_j \kappa - c_{m_j}|}{m_j |c_{m_j}|} \sim \frac{\kappa}{|c_{m_j}|} \to \infty, \quad \text{as} \ j \to \infty,
\]

and the corresponding solution \( z \) satisfies

\[
\|z\|_\mathcal{H} \geq \|v\| = |\rho| \to \infty, \quad \text{as} \ j \to \infty,
\]

so that (6.2) is violated. \( \square \)

We now come back to an abstract operator \( A \), and see under which conditions we can establish lack of exponential stability when \( \mu \) is a step function of the form (6.1). There holds

**Theorem 6.4.** Assume that there exists a sequence \( \{\beta_j\} \) formed by eigenvalues of \( A \) for which the following holds: for every \( N \in \mathbb{N} \) there are sequences of integers \( \{p_{n,j}\} \), for \( n = 1, \ldots, N - 1 \), and (only if \( \gamma_\infty > 0 \), which implies \( s_\infty < \infty \)) \( \{p_{n,\infty}\} \) such that

\[
\lim_{j \to \infty} \left| \frac{s_n \sqrt{\beta_j(1 + \kappa)}}{2\pi} - p_{n,j} \right| = 0, \quad \forall n = \infty, 1, \ldots, N - 1.
\]  

Then the relative semigroup \( S(t) \) is not exponentially stable.
Proof. We proceed exactly as in the proof of Theorem 6.3, replacing $m_j$ with $\sqrt{\beta_j}$, choosing $\lambda = \sqrt{\beta_j(1 + \kappa)}$, and redefining $c_{m_j}$ as

$$\tilde{c}_j = \sqrt{\beta_j} \hat{\mu}(\sqrt{\beta_j(1 + \kappa)}).$$

Condition (6.3) clearly permits to recast the proof of Lemma 6.2, which yields the needed convergence $\tilde{c}_j \to 0$. Note that the case where $\tilde{c}_j = 0$ for some $j$ corresponds to an $\ell$-paced kernel for which there exist periodic orbits. □

Thus, given $A$, one has to look if (6.3) is satisfied. This really amounts to knowing the asymptotic expansion of (some of) the eigenvalues of $A$, and having at disposal suitable simultaneous Diophantine approximations. For instance, if the spectrum of $A$ includes

$$\beta_m = \gamma^2 m^2 + o(m),$$

for some $\gamma > 0$ and every $m$ large enough, then (6.3) holds true for some subsequence $m_j$. Indeed,

$$\frac{s_n \sqrt{\beta_m(1 + \kappa)}}{2\pi} = \frac{s_n \gamma \sqrt{1 + \kappa}}{2\pi} m + o(1),$$

and the claim follows from Lemma 6.1.

Example 6.5. Let $A$ be the negative Laplacian with Dirichlet boundary conditions on a domain $\Omega = \prod_{i=1}^{N} [0, L_i] \subset \mathbb{R}^N$ (for $N = 1$, this is one-dimensional viscoelasticity). For every $m \in \mathbb{N}$, the numbers $\pi^2 m^2 / L_i^2$ are eigenvalues of $A$; thus, from the above discussion, we are in the hypotheses of Theorem 6.4. The same is true when $\Omega$ is a ball of $\mathbb{R}^3$ of radius $R > 0$. Indeed, among the eigenvalues of $A$, the ones corresponding to the radial eigenfunctions are given by $\pi^2 m^2 / R^2$, for $m \in \mathbb{N}$.

To treat the case of the disk, a more involved argument is needed. The eigenvalues corresponding to the radial eigenfunctions are the solutions to the equation

$$J_0(R \sqrt{\lambda}) = 0,$$

where $J_0$ is the Bessel function of the first kind of order 0. The zeros of the above equations are given by (see, e.g., [21])

$$\beta_m = \frac{\pi^2}{R^2} \left( m^2 - \frac{m}{2} \right) + O(1), \quad m \in \mathbb{N}. $$

Thus,

$$\frac{s_n \sqrt{\beta_m(1 + \kappa)}}{2\pi} = \frac{s_n}{2R} \left( m - \frac{1}{4} \right) \sqrt{1 + \kappa} + o(1).$$
Let us denote by $\mathcal{L}_R$ the set of all finite linear combinations over the rational field of the numbers $s_n\sqrt{1+\kappa}/R$ (including $s_\infty\sqrt{1+\kappa}/R$ if $s_\infty < \infty$). If

$$\mathcal{L}_R \cap (\mathbb{Q} \setminus \{0\}) = \emptyset,$$

then we can find a subsequence $m_j$ for which (6.3) holds. This is a consequence of the following result on simultaneous inhomogeneous Diophantine approximation.

**Lemma 6.6.** Let $a_1, \ldots, a_N$ be real numbers whose linear combinations over the rational field are not rational numbers except, possibly, 0. Then, for every $\varepsilon > 0$, there exist $m \in \mathbb{N}$ and $p_n \in \mathbb{Z}$, with $m \geq 1/\varepsilon$, such that

$$\left| a_n \left( m - \frac{1}{4} \right) - p_n \right| \leq \varepsilon,$$

for every $n = 1, \ldots, N$.

**Proof.** If 0 is missed in the rational linear combinations, then 1, $a_1, \ldots, a_N$ are rationally independent, and the lemma is just a particular instance of a general result due to Kronecker (see [1, Chapter 3]), which says that the sequence $\frac{j}{\bar{a}}$ is dense in the unit cube $[0, 1]^N$, where $\bar{a} = (a_1, \ldots, a_N)$ (in fact, it is even true that $\frac{j}{\bar{a}}$ is uniformly distributed). If 1, $a_1, \ldots, a_N$ are not rationally independent, then there are 1, $a_1, \ldots, a_K$ rationally independent, for some $K < N$ (possibly, $K = 1$), and

$$a_n = \sum_{i=1}^{K} q_{n,i} a_i \quad (n > K),$$

with $q_{n,i} \in \mathbb{Q}$. Calling $d$ the greatest common divisor of the denominators of $q_{n,i}$, and defining $\tilde{a}_n = a_n/d$, we have

$$a_n = \begin{cases} 
\tilde{a}_n d, & n \leq K, \\
\sum_{i=1}^{K} k_{n,i} \tilde{a}_i, & n > K,
\end{cases}$$

where $k_{n,i} = q_{n,i}d$ are now integers. At this point, we apply the result to $\tilde{a}_1, \ldots, \tilde{a}_K$, and this clearly yields the claim. \(\square\)

Observe that, given $\mu$, condition (6.4) is satisfied for all radii $R > 0$, except countably many. In particular, if $\mu$ is $\ell$-paced, then (6.4) reads

$$\frac{\ell \sqrt{1+\kappa}}{R} \notin \mathbb{Q}.$$

In conclusion, we have proved
Proposition 6.7. Let $A$ be the negative Laplacian with Dirichlet boundary conditions on the planar disk of radius $R > 0$. Then the related semigroup $S(t)$ is not exponentially stable for all the values of $R$ except countably many.

Obviously, the very same argument applies if we replace the operator $A$ appearing in (1.1) with $\gamma^2 \hat{A}$, for some $\gamma > 0$, where $\hat{A}$ is an abstract strictly positive self-adjoint operator with compact inverse whose spectrum includes the numbers

$$m^2 + \sigma m + O(1),$$

for some $\sigma \in \mathbb{R}$ and every $m$ large enough. In that case, the sufficient condition in order to find a subsequence $m_j$ satisfying (6.3) is exactly (6.4) with $L_{\pi/\gamma}$ in place of $L_R$.

In concrete PDE examples, $\gamma$ is related to the size of the domain; thus, we expect to have a situation similar to the one of the disk.

Acknowledgements

We thank Filippo Gazzola for fruitful discussions.

This work has been partially supported by the Russian Foundation of Basic Researches project no. 05-01-00390, the Civilian Research & Development Foundation (CRDF) Grant no. RUM1-2654-MO-05, and the Russian Science Support Foundation.

References


