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Trajectory and global attractors for evolution equations with memory

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Abstract

Our aim in this note is to analyze the relation between two notions of attractors for the study of the long time behavior of equations with memory, namely, the global attractor in the so-called past history approach, and the more recently proposed notion of trajectory attractor.

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1. Introduction

The long time behavior of equations with memory has been much studied in recent years. The main difficulty, in treating such a problem, is that, due to the presence of the memory term (in general the time convolution of a linear operator applied to the unknown function with a suitable memory kernel), the system is nonlocal; furthermore, the values of the unknown function are known for all negative times.

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It is worth noting that similar features are present in retarded evolution equations (see, for instance, [1–3, 8,9,18]).

A first method allowing one to overcome these difficulties is based on an idea of Dafermos [12] (see also [13]). This idea consists in adding a new variable, called the past history, accounting for all the past values of the unknown function (in the sense that its values for negative times are incorporated in the initial datum of the past history). One then obtains an autonomous system in an extended phase space, for which the existence of the global attractor can be studied. We refer the reader to [10,11,14–17] and the references therein for the application of this framework to many situations.

Recently, a second approach, based on the notion of a trajectory attractor [6] (see also [7]), has been proposed in [4,5]. The idea, roughly speaking, consists in working on the phase space suggested by the initial data, namely, a space of semi-trajectories. One then considers the translation semigroup acting on this phase space and calls the trajectory attractor for the problem with memory the global attractor associated with the translation semigroup.

Now, although this second approach may work for more general forms of equations with memory, it is natural, when both approaches can be applied to a given problem, to study their mutual relationships. More precisely, our aim in this note is to examine in detail the connection between the two kinds of attractors obtained by both approaches.

Notation. We set $\mathbb{R}^- = (-\infty, 0]$ and $\mathbb{R}^+ = [0, \infty)$. If $\mathcal{S}$ is a space of functions on $\mathbb{R}$, we denote by $\mathcal{S}^-$ and $\mathcal{S}^+$ the restrictions of $\mathcal{S}$ to $\mathbb{R}^-$ and $\mathbb{R}^+$, respectively. Given a Banach space $\mathcal{V}$ and a nonnegative nonincreasing function $\mu \in W^{1,1}(0, \infty)$, we consider the Banach space

$$\mathcal{M} = \mathcal{M}[\mu, \mathcal{V}] = \left\{ \eta : (0, \infty) \to \mathcal{V} : \int_0^\infty \mu(s) \|\eta(s)\|_{\mathcal{V}}^2 \, \mathrm{d}s < \infty \right\},$$

endowed with the standard norm. Clearly, if $\mathcal{V}$ is a Hilbert space, then so is $\mathcal{M}$. Besides this, we call T the infinitesimal generator of the right-translation semigroup on $\mathcal{M}$ (cf. [16]), that is, the linear operator $T = -\partial_s$ on $\mathcal{M}$ with domain

$$\mathcal{D}(T) = \left\{ \eta \in \mathcal{M} : \partial_s \eta \in \mathcal{M}, \lim_{s \to 0} \eta(s) = 0 \right\},$$

where $\partial_s \eta$ is the (distributional) derivative of $\eta = \eta(s)$ with respect to s. Finally, given a Banach space $\mathcal{H}$ and an interval $I \subset \mathbb{R}$, we denote by $C_{\mathrm{b}}(I, \mathcal{H})$ and $C_{\mathrm{loc}}(I, \mathcal{H})$ the space of bounded $\mathcal{H}$-valued continuous functions on I and the space of $\mathcal{H}$-valued continuous functions on I endowed with the local uniform convergence topology, respectively. Recall that a sequence u_n converges to u in $C_{\mathrm{loc}}(I, \mathcal{H})$ if, for any $[\alpha, \beta] \subset I$, it holds that

$$\lim_{n \to \infty} \max_{t \in [\alpha, \beta]} \|u_n(t) - u(t)\|_{\mathcal{H}} = 0.$$

2. Evolution equations with memory

Let $\mathcal{H}$ be a Banach space. An evolution equation with memory for the variable $v : \mathbb{R} \to \mathcal{H}$ has the following structure:

$$\partial_t v(t) = \widehat{A}(v(t), v^t(\cdot)), \qquad t > 0, \tag{2.1}$$

where we set $v^t(s) = v(t - s)$, for $s > 0$. Thus, the (possibly nonlinear) operator $\widehat{A}$ acts on $v(t)$ as well as on the *past values* of v up to the time t. In other words, $\widehat{A}$ is a *memory type* operator. The function v

is supposed to be known for all $t \leq 0$, so that the initial datum has the form

$$v(t) = v_0(t), \qquad t \leq 0, \tag{2.2}$$

where $v_0 : \mathbb{R}^- \to \mathcal{H}$ is a given function. It is important to note that the function v need not satisfy Eq. (2.1) for $t \leq 0$.

We look for solutions belonging to some translation-invariant complete metric space $\mathcal{E} \hookrightarrow C_b(\mathbb{R}, \mathcal{H})$. Hence, $v(\cdot) \in \mathcal{E}$ if and only if $v(\cdot + \tau) \in \mathcal{E}$, for every $\tau \in \mathbb{R}$. It is understood that the notion of boundedness in $\mathcal{E}$ is the one provided by the metric of $\mathcal{E}$. The choice of $\mathcal{E}$ will depend on the particular problem under consideration (see the examples below).

Definition 2.1. A function $v \in \mathcal{E}$ is a solution to the Cauchy problem (2.1) and (2.2) in $\mathcal{E}$ if v satisfies (2.1) for $t > 0$, in some weak sense, and the initial condition (2.2) for $t \leq 0$. Clearly, this forces the choice $v_0 \in \mathcal{E}^-$.

Although the above formulation seems to be the natural way to describe evolution equations with memory, in most cases an alternative approach is possible, setting the problem in the so-called *history space*, following an idea of Dafermos [12] (see also [13]). At first glance, this might appear as an unnecessary complication. Nonetheless, as we shall see, this different point of view is very effective.

Assume that we are given a Banach space $\mathcal{M} = \mathcal{M}[\mu, \mathcal{V}]$. Then, for $u = u(t) : \mathbb{R}^+ \to \mathcal{H}$ and $\eta = \eta^t : \mathbb{R}^+ \to \mathcal{M}$, we consider the system

$$\begin{cases} \partial_t u(t) = A(u(t), \eta^t), \\ \partial_t \eta^t = T\eta^t + B(u), \end{cases} \tag{2.3}$$

for $t > 0$. Here, $A : \mathcal{D}(A) \subset \mathcal{H} \times \mathcal{M} \to \mathcal{H}$ and $B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{V}$ are (possibly nonlinear) operators with dense domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$, respectively, whereas T is the linear operator on $\mathcal{M}$ defined above. In that case, the initial conditions read

$$\begin{cases} u(0) = u_0, \\ \eta^0 = \eta_0, \end{cases} \tag{2.4}$$

where $u_0 \in \mathcal{H}$ and $\eta_0 \in \mathcal{M}$.

Definition 2.2. A pair of functions $(u, \eta) \in C(\mathbb{R}^+, \mathcal{H} \times \mathcal{M})$ is a solution to the Cauchy problem (2.3) and (2.4) if (u, η) satisfies (2.3) for $t > 0$, in some weak sense, and the initial conditions (2.4) for $t = 0$.

It is worth mentioning that, if $B(u) \in L^1(0, \tau; \mathcal{V})$ for every $\tau > 0$, then the solution η has the explicit representation formula (see [16])

$$\eta^t(s) = \begin{cases} \displaystyle\int_0^s B(u(t-y))\,\mathrm{d}y, & 0 < s \leq t, \\ \displaystyle\eta^0(s-t) + \int_0^t B(u(t-y))\,\mathrm{d}y, & s > t. \end{cases} \tag{2.5}$$

In fact, formula (2.5) always holds if (2.3) and (2.4) has a solution.

The possible link between Eq. (2.1) and system (2.3) is detailed in the following.

Definition 2.3. System (2.1) and (2.2) can be translated into the history space setting if there are two operators A and B and a Banach space $\mathcal{M} = \mathcal{M}[\mu, \mathcal{V}]$ such that, given any solution v to (2.1) and (2.2), there exists a corresponding solution (u, η) to (2.3) and (2.4) with

$$u_0 = v_0(0) \qquad \text{and} \qquad \eta_0(s) = \int_0^s B(v_0(-y))\,\mathrm{d}y,$$

such that $u(t) = v(t)$ for all $t \geq 0$. In that case, we call system (2.3) the *translation* of (2.1) into the history space setting.

Remark 2.4. In particular, in the above situation, we can express (2.5) in terms of v as

$$\eta^t(s) = \int_0^s B(v(t-y))\,\mathrm{d}y. \tag{2.6}$$

Before proceeding, some comments are in order. First, in concrete problems, there is a natural way to construct A, B and $\mathcal{M}$. It is then clear that this translation is possible only if $\int_0^s B(v_0(-y))\,\mathrm{d}y \in \mathcal{M}$, whenever $v_0 \in \mathcal{E}^-$. Indeed, the choice of $\mathcal{E}$ is made in such a way that this relation is satisfied. Nonetheless, (2.3) and (2.4) is, in general, solvable for a wider class of initial data. In this respect, provided that the problem can be translated into the history space setting, (2.3) and (2.4) is actually a generalization of the original problem (2.1) and (2.2). Another important fact is that (2.3) and (2.4) generates a flow in the Banach space $\mathcal{H} \times \mathcal{M}$. In particular, if one also has a continuous dependence result, then (2.3) and (2.4) generates a strongly continuous semigroup $\Sigma(t)$ on the phase space $\mathcal{H} \times \mathcal{M}$, so that all the powerful tools of the theory of dynamical systems apply. This is crucial if one is interested in investigating more deeply the asymptotic behavior of the solutions, to prove, for instance, the finite dimensionality of the attractor.

3. Two concrete examples

We now present two paradigmatic examples that illustrate the different approaches presented above. Throughout this section, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. We set $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. Let $k : [0, \infty) \to \mathbb{R}^+$ be such that $\mu(s) = -k'(s)$ is a nonnegative function belonging to $W^{1,1}(0, \infty)$. In addition, we assume that there exists $\delta > 0$ such that

$$\mu'(s) + \delta\mu(s) \leq 0, \qquad \text{a.e. } s > 0.$$

Notice that the above condition implies the exponential decay at infinity of $\mu(s)$. For the sake of simplicity, we will consider a nonlinearity of the form $\phi(r) = r^3 - r$, and an external source $f \in H$ constant in time.

3.1. *The damped wave equation with memory*

For $(v, \partial_t v) : \mathbb{R} \to V \times H$, consider the equation

$$\partial_{tt} v(t) + \partial_t v(t) - k(0)\Delta v(t) - \int_0^\infty k'(s)\Delta v(t-s)\,\mathrm{d}s + \phi(v(t)) + f = 0, \tag{3.1}$$

for $t > 0$. Setting

$$\mathcal{E} = \{(v(t), \partial_t v(t)) : v \in C_\mathrm{b}(\mathbb{R}, V),\ \partial_t v \in C_\mathrm{b}(\mathbb{R}, H)\},$$

with the distance inherited from the norm of $C_{\mathrm{b}}(\mathbb{R}, V \times H)$, given any initial data $(v_0, \partial_t v_0) \in \mathcal{E}^-$, Eq. (3.1) with the initial condition

$$(v(t), \partial_t v(t)) = (v_0(t), \partial_t v_0(t)), \qquad t \leq 0,$$

admits a unique solution in the sense of Definition 2.1 (see [4]). Choosing $\mathcal{M} = \mathcal{M}[\mu, V]$, this problem can be translated into the history space setting, namely, for $t > 0$,

$$\begin{cases} \partial_{tt} u(t) + \partial_t u(t) - k(\infty)\Delta u(t) - \int_0^\infty \mu(s)\Delta\eta^t(s)\,\mathrm{d}s + \phi(u(t)) + f = 0, \\ \partial_t \eta^t = T\eta^t + \partial_t u(t). \end{cases} \tag{3.2}$$

System (3.2) is obtained by integrating by parts the convolution integral appearing in (3.1). According to [11,17], system (3.2) generates a strongly continuous semigroup on the phase space $[V \times H] \times \mathcal{M}$.

3.2. The semilinear heat equation with memory

For $v : \mathbb{R} \to H$, consider the equation

$$\partial_t v(t) - \Delta v(t) - \int_0^\infty k(s)\Delta v(t-s)\,\mathrm{d}s + \phi(v(t)) + f = 0, \tag{3.3}$$

for $t > 0$. We set

$$\mathcal{E} = C_{\mathrm{b}}(\mathbb{R}, H) \cap L^2_{\mathrm{tb}}(\mathbb{R}, V),$$

where

$$L^2_{\mathrm{tb}}(\mathbb{R}, V) = \left\{ w : \mathbb{R} \to V : \|w\|_{L^2_{\mathrm{tb}}(\mathbb{R},V)} = \left(\sup_{t\in\mathbb{R}} \int_t^{t+1} \|w(y)\|_V^2\,\mathrm{d}y \right)^{1/2} < \infty \right\}.$$

Then, Eq. (3.3) admits a unique solution in the sense of Definition 2.1, for any initial datum $v_0 \in \mathcal{E}^-$ (see [5]). The translation into the history space setting (again, obtained by integrating by parts the convolution integral) now reads, for $t > 0$,

$$\begin{cases} \partial_t u(t) - \Delta u(t) - \int_0^\infty \mu(s)\Delta\eta^t(s)\,\mathrm{d}s + \phi(u(t)) + f = 0, \\ \partial_t \eta^t = T\eta^t + u(t). \end{cases} \tag{3.4}$$

It is known from [10,14,15] that system (3.4) generates a strongly continuous semigroup on the phase space $H \times \mathcal{M}$, with $\mathcal{M} = \mathcal{M}[\mu, V]$ as in the previous case.

4. Trajectory and global attractors

In the two recent papers [4,5], Chepyzhov and Miranville introduced the notion of a *trajectory attractor* for problems of the form (2.1) and (2.2). Assuming that (2.1) and (2.2) has a *unique* solution for every initial datum $v_0 \in \mathcal{E}^-$, there is a natural way to construct the solving semigroup $\widehat{S}(h)$ acting on the space $\mathcal{E}^-$ via the formula

$$(\widehat{S}(h)v_0)(t) = v(t+h)|_{t\leq 0}, \qquad h \geq 0,$$

where $v(t)$ is the solution to (2.1) with initial datum $v_0 \in \mathcal{E}^-$. Due to the translation invariance of $\mathcal{E}$, it is apparent that, for every fixed $h \geq 0$, the function $t \mapsto v(t+h)|_{t\leq 0}$ is an element of $\mathcal{E}^-$. The trajectory attractor of problem (2.1) and (2.2) will simply be the global attractor of $\widehat{S}(h)$ on $\mathcal{E}^-$.

Definition 4.1. A set $\widehat{\mathfrak{A}} \subset \mathcal{E}^-$ is the trajectory attractor of problem (2.1) and (2.2) if it is compact in $C_{\mathrm{loc}}(\mathbb{R}^-, \mathcal{H})$, bounded in $\mathcal{E}^-$, strictly invariant under the action of $\widehat{S}(h)$, and attracts any bounded set $\widehat{\mathcal{B}} \subset \mathcal{E}^-$ in the topology of $C_{\mathrm{loc}}(\mathbb{R}^-, \mathcal{H})$; that is, for every $M > 0$, there holds

$$\lim_{h\to\infty} \mathrm{dist}_{C([-M,0],\mathcal{H})}(\widehat{S}(h)\widehat{\mathcal{B}}, \widehat{\mathfrak{A}}) = 0, \tag{4.1}$$

where “dist” denotes the usual Hausdorff semidistance.

The trajectory attractor, if it exists, is unique.

Remark 4.2. The above notion of a trajectory attractor differs from the one given by Chepyzhov and Vishik [6,7]. However, the two objects are somehow close, since the former one consists of all the solutions (trajectories) satisfying the equation with memory for all times.

Papers [4,5] demonstrate the existence of trajectory attractors for the two concrete models presented in the former section. As previously shown, these models can be translated into the history space setting, where they generate strongly continuous semigroups possessing global attractors (see [10,11,14,15,17]). Recall that the global attractor is the unique compact set which is at the same time strictly invariant under the action of the semigroup, and attracts any bounded subset of the phase space with respect to the Hausdorff semidistance.

5. The main result

It is now natural to ask whether there is a relationship between the trajectory attractor of (2.1) and (2.2) and the global attractor of the translated problem (2.3) and (2.4). The positive answer to this question is the main result of this work. Thus, assume that problem (2.1) and (2.2) admits a unique solution, and that the translated problem (2.3) and (2.4) generates a strongly continuous (in space and time) semigroup $\Sigma(t)$ on $\mathcal{H} \times \mathcal{M}$ such that the map $(u, \eta) \mapsto \Sigma(t)(u, \eta)$ is continuous as a function from $\mathcal{H} \times \mathcal{M}$ to $C_{\mathrm{loc}}(\mathbb{R}^+, \mathcal{H} \times \mathcal{M})$. This is true, for instance, if the semigroup is *jointly continuous* in space and time, as happens in most concrete applications. Finally, let $\Sigma(t)$ possess a (unique) global attractor $\mathfrak{A}_0$. It is well known that, for every $\tau \in \mathbb{R}$,

$$\mathfrak{A}_0 = \left\{(u(\tau), \eta^\tau) : (u(t), \eta^t) \in \mathfrak{A}\right\},$$

where $\mathfrak{A}$ is the set of all bounded complete trajectories of $\Sigma(t)$. In particular, $\mathfrak{A}$ is translation invariant. Moreover, due to the representation formula (2.5), if $(u(t), \eta^t) \in \mathfrak{A}$, it follows that

$$\eta^t(s) = \int_0^s B(u(t-y))\,\mathrm{d}y. \tag{5.1}$$

Then we have:

Theorem 5.1. *Suppose that, given any bounded set $\widehat{\mathcal{B}} \subset \mathcal{E}^-$, there holds*

$$\sup_{v_0\in\widehat{\mathcal{B}}} \left\| \int_0^s B(v_0(-y))\,\mathrm{d}y \right\|_{\mathcal{M}} < \infty, \tag{5.2}$$

where B is the operator appearing in system (2.3)*. Then problem* (2.1) *and* (2.2) *has a (connected) trajectory attractor $\widehat{\mathfrak{A}}$ which coincides with $\mathbb{P}\mathfrak{A}^-$.*

Here, the operator $\mathbb{P}$ denotes the projection in $\mathcal{H} \times \mathcal{M}$ onto the first component $\mathcal{H}$, namely, $\mathbb{P}(u, \eta) = u$.

Proof. Observe first that (5.2) actually implies that

$$\sup_{v_0 \in \widehat{\mathcal{B}}} \sup_{M \geq 0} \left\| \int_0^s B(v(-y + M))\,\mathrm{d}y \right\|_{\mathcal{M}} < \infty. \tag{5.3}$$

Here, v is the solution to (2.1) with initial datum v_0. Indeed, (5.3) follows from (2.6) and the existence of the global attractor for $\Sigma(t)$.

It is clear that $\mathbb{P}\mathfrak{A}^-$ is compact in $C_{\text{loc}}(\mathbb{R}^-, \mathcal{H})$ and bounded in $\mathcal{E}^-$. We proceed to discuss the invariance. Let $v_0 \in \mathbb{P}\mathfrak{A}^-$, and let v be the solution to (2.1) with initial datum v_0. Then, there exists $(u, \eta) \in \mathfrak{A}$ (thus, defined for all times) such that $u(t) = v(t)$ for all $t \leq 0$. We reach our goal if we also show that $u(t) = v(t)$ for $t > 0$, since, in that case, we may lean on the translation invariance of $\mathbb{P}\mathfrak{A}$. Hence, let $(\hat{u}, \hat{\eta})$ be the solution to the corresponding translated problem (thus, defined for $t \geq 0$). By definition, $\hat{u}(t) = v(t)$ for all $t \geq 0$, and

$$\hat{\eta}^0(s) = \int_0^s B(v(-y))\,\mathrm{d}y.$$

On the other hand, $u(0) = v(0)$, and, recalling (5.1),

$$\eta^0(s) = \int_0^s B(v(-y))\,\mathrm{d}y.$$

Since we have uniqueness, we conclude that $u(t) = \hat{u}(t)$ for all $t \geq 0$.

To finish the proof, we are left to show that $\mathbb{P}\mathfrak{A}^-$ satisfies the attraction property (4.1). Therefore, given any set $\widehat{\mathcal{B}}$ bounded in $\mathcal{E}^-$, we want to show that, for any fixed $M > 0$, there holds

$$\lim_{h \to \infty} \text{dist}_{C([-M,0],\mathcal{H})}(\widehat{S}(h)\widehat{\mathcal{B}}, \mathbb{P}\mathfrak{A}^-) = 0,$$

which is the same as saying that

$$\lim_{h \to \infty} \text{dist}_{C([-M,0],\mathcal{H})}(\widehat{S}(h)\widehat{\mathcal{B}}, \mathbb{P}\mathfrak{A}) = 0. \tag{5.4}$$

Thus, let $\widehat{\mathcal{B}}$ and M be fixed. Recalling Definition 2.3, we associate with $v_0 \in \widehat{\mathcal{B}}$ a pair $(u_0, \eta_0) \in \mathcal{H} \times \mathcal{M}$ as follows:

$$u_0 = v(M) \qquad \text{and} \qquad \eta_0(s) = \int_0^s B(v(-y + M))\,\mathrm{d}y,$$

where v is the solution to (2.1) and (2.2). In this way, we produce a bounded subset $\mathcal{B}_0 \subset \mathcal{H} \times \mathcal{M}$. Indeed, the bound on the first component is straightforward and the bound on the second one is given by (5.3). Next, we name as $\mathcal{B}$ the set given by the trajectories of $\Sigma(t)$ on $\mathbb{R}^+$ originating from $\mathcal{B}_0$, that is,

$$\mathcal{B} = \left\{ (u(t), \eta^t), t \geq 0 : \ (u(0), \eta^0) = (u_0, \eta_0) \in \mathcal{B}_0 \right\}.$$

Notice that, by construction,

$$u(t) = v(t + M), \qquad \forall t \geq 0.$$

Besides this, due to the dissipativity of $\Sigma(t)$ witnessed by the existence of the global attractor $\mathfrak{A}_0$, the set $\mathcal{B}$ is bounded in $C_b(\mathbb{R}^+, \mathcal{H} \times \mathcal{M})$. The existence of the global attractor $\mathfrak{A}_0$ gives at once the trajectory attractor $\mathfrak{A}^+$ (cf. [6,7]) for the semigroup of (left) translations $S(t)$ acting on the space

$$\mathcal{K}^+ = \left\{(w(t), \xi^t), t \geq 0 : \ (w(0), \xi^0) \in \mathcal{H} \times \mathcal{M}\right\} \subset C_b(\mathbb{R}^+, \mathcal{H} \times \mathcal{M}).$$

That is, $\mathfrak{A}^+$ is bounded in $\mathcal{K}^+$ (endowed with the $C_b(\mathbb{R}^+, \mathcal{H}\times\mathcal{M})$-metric), compact in $C_{loc}(\mathbb{R}^+, \mathcal{H}\times\mathcal{M})$, strictly invariant under the action of $S(t)$, and attracts the bounded subsets of $\mathcal{K}^+$ with respect to the $C_{loc}(\mathbb{R}^+, \mathcal{H} \times \mathcal{M})$-metric. Hence,

$$\lim_{h\to\infty} \mathrm{dist}_{C([0,M],\mathcal{H}\times\mathcal{M})}(S(h)\mathcal{B}, \mathfrak{A}) = 0,$$

which in turn implies

$$\lim_{h\to\infty} \mathrm{dist}_{C([0,M],\mathcal{H})}(\mathbb{P}S(h)\mathcal{B}, \mathbb{P}\mathfrak{A}) = 0.$$

Then, for $h \geq 2M$, exploiting the translation invariance of $\mathfrak{A}$, we have

$$\begin{aligned}
&\mathrm{dist}_{C([0,M],\mathcal{H})}(\mathbb{P}S(h-2M)\mathcal{B}, \mathbb{P}\mathfrak{A}) \\
&\quad = \sup_{u\in\mathbb{P}\mathcal{B}} \inf_{w\in\mathbb{P}\mathfrak{A}} \sup_{t\in[0,M]} \|u(h+t-2M) - w(t)\|_{\mathcal{H}} \\
&\quad = \sup_{u\in\mathbb{P}\mathcal{B}} \inf_{w\in\mathbb{P}\mathfrak{A}} \sup_{t\in[-M,0]} \|u(h+t-M) - w(t)\|_{\mathcal{H}} \\
&\quad = \sup_{v_0\in\widehat{\mathcal{B}}} \inf_{w\in\mathbb{P}\mathfrak{A}} \sup_{t\in[-M,0]} \|v(h+t) - w(t)\|_{\mathcal{H}} \\
&\quad = \sup_{v_0\in\widehat{\mathcal{B}}} \inf_{w\in\mathbb{P}\mathfrak{A}} \sup_{t\in[-M,0]} \|(\widehat{S}(h)v_0)(t) - w(t)\|_{\mathcal{H}} \\
&\quad = \mathrm{dist}_{C([-M,0],\mathcal{H})}(\widehat{S}(h)\widehat{\mathcal{B}}, \mathbb{P}\mathfrak{A}),
\end{aligned}$$

which entails (5.4).

Remark 5.2. With a view to concrete applications, notice that (5.2) is implied by the following (stronger) condition: given any bounded set $\widehat{\mathcal{B}} \subset \mathcal{E}^-$, there exists a positive function $\psi_{\widehat{\mathcal{B}}} \in L^1_\mu(0,\infty)$ such that

$$\sup_{v_0\in\widehat{\mathcal{B}}} \left\| \int_0^s B(v_0(-y))\,\mathrm{d}y \right\|_{\mathcal{V}}^2 \leq \psi_{\widehat{\mathcal{B}}}(s). \tag{5.5}$$

6. Applications of the abstract result

We conclude our work by showing that Theorem 5.1 applies for the two examples presented above. As a consequence, the trajectory attractors inherit the regularity of the corresponding global attractors in the history space setting. This allows us to improve the regularity results of [4,5] for the particular nonlinearity considered above (see, e.g., [10] for the parabolic case; however, note that no growth assumption on the nonlinear term is made in [5]).

Let us first consider the damped wave equation with memory (3.1). Here, we have $B(v_0, \partial_t v_0) = \partial_t v_0$. Thus,

$$\int_0^s B(v_0(-y), \partial_t v_0(-y))\,\mathrm{d}y = v_0(0) - v_0(-s).$$

Due to the boundedness of $\widehat{\mathcal{B}}$, we readily get that

$$\|v_0(0) - v_0(-s)\|_V^2 \le c, \qquad \forall s > 0$$

so that (5.5) is satisfied.

Concerning the semilinear heat equation with memory (3.3), we have $B(v_0) = v_0$. Thus, from the boundedness of $\widehat{\mathcal{B}}$ and the Hölder inequality,

$$\left\| \int_0^s B(v_0(-y))\,\mathrm{d}y \right\|_V^2 \le s \int_0^s \|v_0(-y)\|_V^2\,\mathrm{d}y \le c(1+s^2), \qquad \forall s > 0.$$

Therefore, (5.5) is satisfied, since $(1+s^2) \in L^1_\mu(0,\infty)$, due to the exponential decay at infinity of $\mu(s)$.

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