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ON NON–AUTONOMOUS SINE–GORDON TYPE EQUATIONS WITH A SIMPLE GLOBAL ATTRACTOR AND SOME AVERAGING

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Abstract. We study the global attractors for the dissipative sine–Gordon type wave equation with time dependent external force $g(x,t)$. We assume that the function $g(x,t)$ is translationary compact in $L_2^{\mathrm{loc}}(\mathbb{R}, L_2(\Omega))$ and the nonlinear function $f(u)$ is bounded and satisfies a global Lipschitz condition. If the Lipschitz constant K is smaller than the first eigenvalue of the Laplacian with homogeneous Dirichlet conditions and the dissipation coefficient is large, then the global attractor has a simple structure: it is the closure of all the values of the unique bounded complete trajectory of the wave equation. Moreover, the attractor attracts all the solutions of the equation with exponential rate.

We also consider the wave equation with rapidly oscillating external force $g^\varepsilon(x,t) = g(x,t,t/\varepsilon)$ having the average $g^0(x,t)$ as $\varepsilon \to 0+$. We assume that the function $g(x,t,\zeta) - g^0(x,t)$ has a bounded primitive with respect to ζ. Then we prove that the Hausdorff distance between the global attractor $\mathcal{A}_\varepsilon$ of the original equation and the global attractor $\mathcal{A}_0$ of the averaged equation is less than $O(\varepsilon^{1/2})$.

Introduction. During the last decades the interest in the long–time behaviour of processes in mathematical physics has substantially grown and corresponding mathematical analyses can be found in monographs as, e.g., in [1], [12], [16] and [5]. More recently, within these topics, the influence of oscillatory perturbations in the modeling partial differential equations on the corresponding attractors has been studied in relation with the behaviour of the averaged system and its attractors, see [5, Chap. XVIII], [17] and the references therein.

In the first part of this paper we consider scalar, dissipative, non–autonomous hyperbolic equations of sine–Gordon type which provide a unique, global and exponentially attracting solution $\{z(t)$ for $t \in \mathbb{R}\}$, if the forcing term depends on the time and is a translationary compact function. In the second part we study forcing terms which oscillate rapidly in time and find estimates for the distance between the global attractors of the oscillatory and the averaged equation. These

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results generalize some of those in [13] where the forcing term is an almost periodic function oscillating in time since our assumptions are more general and we give quantitative estimates.

The attractors of dissipative hyperbolic equations with rapidly oscillating terms in the spatial variables were studied in [9], [17], [5].

For autonomous and non–autonomous parabolic equations and the 2D Navier–Stokes system analogous problems were studied in [8], [6], [7] and [17]. In [10] and [11], integral manifolds and their structures are investigated for non–autonomous parabolic equations. In [7] parabolic problems were treated which have coefficients oscillatory in time as well as in space but which generate smooth processes and have regular attractors.

Our paper is organized as follows. In Section 1 we deal with an equation that has a translationary compact forcing term $g(x,t)$ for $t \in \mathbb{R}$ in $L_2^{\mathrm{loc}}\big(\mathbb{R}, L_2(\Omega)\big)$, which means that the corresponding family of translations $\{g(x,t+h)$ for all $h \in \mathbb{R}\}$ forms a precompact subset of $L_2\big([T_1,T_2]\,,\,L_2(\Omega)\big)$, where $[T_1,T_2]$ is an arbitrarily chosen, fixed real time interval. For the interaction function $f(u)$ we assume that it satisfies a global Lipschitz condition. In particular, the function $\sin u$ in the sine–Gordon equation belongs to this class. Finally, we assume that the dissipation coefficient $\gamma > 0$ is larger than an explicitly given number depending on the Lipschitz constant of f as well as on the first eigenvalue of the Laplacian in Ω subjected to the homogeneous Dirichlet boundary condition.

Under these conditions it is shown that the hyperbolic equation has in the energetic space E a unique, global and bounded solution $\{z(t)$ for $t \in \mathbb{R}\}$. This solution $z(t)$ then attracts exponentially any other solution of the hyperbolic equation as $t \to \infty$. We prove that the global attractor $\mathcal{A}$ of the hyperbolic equation coincides with the closure in E of all the values of the function $z(t)$ if t traces $\mathbb{R}$, i.e.,

$$\mathcal{A} = [z(\bullet,t) \,|\, t \in \mathbb{R}]_E\,.$$

In Section 2 we then study the case that the external forcing term in the hyperbolic equation is rapidly oscillating and is of the form

$$g^\varepsilon(x,t) := g\big(x,t,\tfrac{t}{\varepsilon}\big)\,, \quad \text{where } 0 < \varepsilon \le \varepsilon_0\,.$$

We assume that $g^\varepsilon(x,t)$ admits a uniform average $g^0(x,t)$ if $\varepsilon \to 0$. The crucial assumption for our analysis is that there exists a function $J(x,t,\zeta)$ such that

$$g(x,t,\zeta) - g^0(x,t) = \tfrac{\partial}{\partial\zeta} J(x,t,\zeta)$$

and that all the derivatives of J as functions of x have uniformly bounded norms in $L_2(\Omega)$ for all $t,\zeta \in \mathbb{R}$, and that $J(\bullet,t,\zeta) \in H_0^1(\Omega)$ for all $t,\zeta \in \mathbb{R}$. Under these suppositions we prove that for the corresponding attracting global solutions the following estimate holds:

$$\|\big(z^\varepsilon(\bullet,t) - z^0(\bullet,t)\big)\,,\, \partial_t z^\varepsilon(\bullet,t) - \partial_t z^0(\bullet,t)\big)\|_E \le C\varepsilon^{\frac{1}{2}}\,.$$

Here $z^\varepsilon(x,t)$ and $z^0(x,t)$ for $t \in \mathbb{R}$ are the respective exponentially attracting global, unique solutions of the hyperbolic equation corresponding to the rapidly oscillating and the averaged forcing term $g^\varepsilon(x,t)$ and $g^0(x,t)$, respectively.

Then an analogous estimate follows for the Hausdorff distance between the global attractors $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$ of the corresponding equations.

1. **Non–autonomous wave equation of sine–Gordon type.** Here, we consider the dissipative wave equation

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - f(u) + g^0(x,t) \text{ with } u|_{\partial\Omega} = 0 \tag{1.1}$$

in a precompact spatial domain $\Omega \subset \mathbb{R}^n$ where γ is a positive constant, f is continuous on $\mathbb{R}$ and $g^0(x,t)$ for $t \in \mathbb{R}$, $x \in \Omega$ is locally square integrable in time, i.e., $g^0 \in L_2^{\mathrm{loc}}\big(\mathbb{R}, L_2(\Omega)\big)$.

We suppose for the nonlinearity f the inequalities

$$|f(v_1)| \quad \leq \quad C \tag{1.2}$$

and

$$|f(v_1) - f(v_2)| \quad \leq \quad K|v_1 - v_2| \text{ for all } v_1, v_2 \in \mathbb{R}\,. \tag{1.3}$$

In the special case $f(v) = K \sin v$, the equation in (1.1) is the sine–Gordon equation.

We assume that the function $g^0(\bullet, t) =: g^0(t)$ is **translationary compact** in the space $L_2^{\mathrm{loc}}(\mathbb{R}, H)$ where the Hilbert space is here $H = L_2(\Omega)$; that is, the family of translations $\{g^0(t+h)\,,\, h \in \mathbb{R}\}$ forms a precompact set in $L_2([T_1, T_2]\,;\, H)$, where $[T_1, T_2]$ is an arbitrary interval of the time axis $\mathbb{R}$. This property implies that

$$\|g^0\|^2_{L_2^b} := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g^0(\bullet, s)\|^2_{L_2(\Omega)} ds < \infty\,. \tag{1.4}$$

For equation (1.1), we consider the Cauchy problem with initial conditions

$$\begin{aligned} u(\tau, x) &= u_\tau(x) \qquad \text{with } u_\tau \in H_0^1(\Omega) \text{ and} \\ \partial_t u(\tau, x) &= p_\tau(x) \qquad \text{with } p_\tau \in L_2(\Omega)\,. \end{aligned} \tag{1.5}$$

For any given initial values (u_τ, p_τ), this problem has a unique solution $u \in \mathcal{C}\big(\mathbb{R}_\tau, H_0^1(\Omega)\big)\,,\ \partial_t u \in \mathcal{C}\big(\mathbb{R}_\tau, L_2(\Omega)\big)\,,\ \partial_t^2 u \in L_2^{\mathrm{loc}}\big(\mathbb{R}_\tau, H^{-1}(\Omega)\big)$, where $\mathbb{R}_\tau := \{t \in \mathbb{R} \,|\, \tau \leq t\}$, see e.g. [14], [15], [16], [5]. Here and in what follows $H_0^1(\Omega)$ denotes the Sobolev space of order 1 of functions having zero boundary trace on $\partial\Omega$, and $H^{-1}(\Omega)$ is its dual space with respect to the L_2–duality.

Now we will write (1.1), (1.5) as an evolutionary system by introducing $y(t) := \big(u(t)\,,\, p(t)\big)$ and $y_\tau := (u_\tau, p_\tau)$. The vector–valued functions $y(t)$ should belong to the space $E := H_0^1(\Omega) \times L_2(\Omega)$ equipped with the norm

$$\|y\|_E := \{\|\nabla u\|^2_{L_2(\Omega)} + |p|^2\}^{\frac{1}{2}} \text{ where } |p| := \|p\|_{L_2(\Omega)}\,.$$

Then (1.1), (1.5) has the form of an evolutionary system,

$$\begin{aligned} \partial_t u &= p\,, \\ \partial_t p &= -\gamma p + \Delta u - f(u) + g^0(x,t) \text{ for } t \geq \tau\,, \\ u|_{\partial\Omega} &= 0\,;\, u|_{t=\tau} = u_\tau\,,\, p|_{t=\tau} = p_\tau\,. \end{aligned} \tag{1.6}$$

This problem has the so–called time symbol $g^0(\bullet, t)$ with values in $L_2(\Omega)$ and finite L_2^b–norm (1.4). Since (1.6) has a unique solution $y(t) \in \mathcal{C}(\mathbb{R}_\tau, E)$, it defines via $y(t) = U_{g^0}(t, \tau) y_\tau$ a process $\{U_{g^0}(t,\tau) \text{ for } t \geq \tau\}$ corresponding to (1.6), with the mapping property $U_{g^0}(t,\tau) : E \to E$ continuously for every $t \geq \tau$.

It is the aim of this paper to investigate the global attractor $\mathcal{A}$ of this process.

Proposition 1.1. *Under the conditions* (1.2)–(1.4)*, the problem* (1.6) *has a global attractor* $\mathcal{A}$ *which is compact in* E*.*

The proof can be found in [5]. Also note that the process $U_{g^0}(t,\tau)$ is asymptotically compact, i.e., it possesses a compact, attracting set.

As $g^0(x,t)$ is translationary compact in $L_2^{\rm loc}(\mathbb{R},H)$, the hull

$$\mathcal{H}(g^0) := [g^0(x,t+h)\,|\,h\in\mathbb{R}]_{L_2^{\rm loc}(\mathbb{R},H)} \tag{1.7}$$

is compact in $L_2^{\rm loc}(\mathbb{R},H)$ where $[\cdot]_{L_2^{\rm loc}(\mathbb{R},H)}$ denotes the closure in $L_2^{\rm loc}$.

Now, for any $g(x,t)\in\mathcal{H}(g^0)$, the problem (1.6) with g instead of g^0 possesses a corresponding process $\{U_g(t,\tau)\}$ acting on E. As is proved in [5], the family $\{U_g(t,\tau)\,|\,g\in\mathcal{H}(g^0)\}$ of processes is $\big(E\times\mathcal{H}(g^0),E\big)$–continuous.

Let

$$\mathcal{K}_g := \{y_g(x,t) \ \text{ for } t\in\mathbb{R}\,|\,y_g(x,t) \ \text{ is solution of (1.6) satisfying} \\ \|y_g(\bullet,t)\|_E \le M_g \ \text{ for all } t\in\mathbb{R}\}$$

be the so–called kernel of the process $\{U_g(t,\tau)\}$.

Proposition 1.2. *Let $g^0(x,t)$ be translationary compact in $L_2^{\rm loc}\big(\mathbb{R},L_2(\Omega)\big)$ and let the conditions* (1.2) *and* (1.3) *be satisfied. Then the global attractor $\mathcal{A}$ of the process $\{U_{g^0}(t,\tau)\}$ can be represented by the formula*

$$\mathcal{A} = \bigcup_{g\in\mathcal{H}(g^0)} \mathcal{K}_g(0)\,.$$

where $\mathcal{K}_g(0)$ is the section of the kernel $\mathcal{K}_g$ at the time $t=0$:

$$\mathcal{K}_g(0) = \{y_g(\bullet,0)\,|\,\{y_g(x,t)\}\in\mathcal{K}_g\}\,.$$

The proof is given in [5].

Let us specify the case when the global attractor $\mathcal{A}$ has a simple structure and is exponentially attracting. Denote by λ the first eigenvalue of the Laplacian on $H_0^1(\Omega)$, i.e.

$$-\Delta u_\lambda = \lambda u_\lambda \ \text{ in } \Omega \ \text{ and } u_\lambda|_{\partial\Omega} = 0\,.$$

Then we have the following theorem.

Theorem 1.3. *Let the Lipschitz constant K in* (1.3) *satisfy the inequality*

$$K<\lambda \tag{1.8}$$

and let the constant γ in (1.1) *satisfy*

$$\gamma^2 > \gamma_0^2 := 2(\lambda-\sqrt{\lambda^2-K^2})\,. \tag{1.9}$$

For any function $g\in\mathcal{H}(g^0)$, the system (1.6) *with the forcing term g has a unique bounded solution $z(t)$ in E for all $t\in\mathbb{R}$. For every solution $y(t)=U_g(t,\tau)y_\tau$ for $t\ge\tau$ to* (1.6)*, there holds the inequality*

$$\|y(t)-z(t)\|_E \le C\|y_\tau - z(\tau)\|_E e^{-\beta(t-\tau)}\,, \tag{1.10}$$

where the constants $C>0$ and $\beta>0$ are independent of y_τ.

Proof. In what follows, all the relations can be justified by the use of Galerkin's method with standard arguments, see [5], [14], [16]. Let $u_1(x,t)$ and $u_2(x,t)$ for $t\ge\tau$ both be solutions of equation (1.1) with forcing term $g\in\mathcal{H}(g^0)$. Then their difference $w(x,t):=u_1(x,t)-u_2(x,t)$ solves

$$\partial_t^2 w + \gamma\partial_t w = \Delta w - \big(f(u_1)-f(u_2)\big) \ \text{ in } \Omega \ \text{ and } w|_{\partial\Omega}=0\,.$$

This equation can also be written as

$$\partial_t(\partial_t w+\alpha w) + (\gamma-\alpha)(\partial_t w+\alpha w) - \Delta w - \alpha(\gamma-\alpha)w = -\big(f(u_1)-f(u_2)\big)\,, \tag{1.11}$$

where $\alpha > 0$ is a suitable constant to be chosen later on. Multiplying (1.11) by $v = \partial_t w + \alpha w$ and integrating over Ω we obtain, after employing integration by parts and inequality (1.3), the inequality

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\left(|v|^2 + |\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2\right) + (\gamma-\alpha)|v|^2 \\ &+\alpha\left(|\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2\right) \le K|w|\,|v|\,. \end{aligned} \tag{1.12}$$

Now choose $\alpha > 0$ satisfying

$$\alpha(\gamma-\alpha) < \lambda\,. \tag{1.13}$$

Then, with $\lambda|w|^2 \le |\nabla w|^2$, one obtains the inequality

$$\lambda|w|^2 - \alpha(\gamma-\alpha)|w|^2 \le |\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2$$

and

$$|w|^2 \le \frac{|\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2}{\lambda - \alpha(\gamma-\alpha)}\,. \tag{1.14}$$

Hence, the right–hand side in (1.14) is non–negative. With

$$X^2 := |v|^2 \ \text{ and } Y^2 := |\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2 \ge 0,$$

then (1.12) can with (1.14) be written as

$$\frac{1}{2}\frac{d}{dt}(X^2+Y^2) + \left\{(\gamma-\alpha)X^2 + \alpha Y^2 - \frac{K}{\sqrt{\lambda-\alpha(\gamma-\alpha)}}XY\right\} < 0\,. \tag{1.15}$$

The quadratic form $\{\dots\}$ is positive definite provided $\alpha > 0$, $(\gamma-\alpha) > 0$ and its determinant satisfies

$$\alpha(\gamma-\alpha) - \frac{K^2}{4\big(\lambda - \alpha(\gamma-\alpha)\big)} > 0\,. \tag{1.16}$$

The inequality (1.16) is equivalent to

$$\varrho^2 - \lambda\varrho + K^2/4 < 0\,. \tag{1.17}$$

where $\varrho = \alpha(\gamma-\alpha)$. As we assume $K < \lambda$ in (1.8), the quadratic inequality (1.17) is satisfied for every ϱ with

$$\frac{\lambda - \sqrt{\lambda^2-K^2}}{2} < \varrho < \frac{\lambda + \sqrt{\lambda^2-K^2}}{2}\,. \tag{1.18}$$

If ϱ fulfills both inequalities in (1.18) then $\varrho < \lambda$ holds automatically. In this case, we choose $\alpha > 0$ in such a way that ϱ as function of α becomes maximal, i.e., $\alpha = \frac{\gamma}{2}$ and $\varrho = \frac{\gamma^2}{4}$, and the left inequality in (1.18) is just (1.9).

If, however, $\frac{\gamma^2}{4} \ge \frac{1}{2}(\lambda + \sqrt{\lambda^2-K^2})$ then we choose, for instance, $\varrho = \frac{\lambda}{2}$ and $\alpha = \frac{1}{2}(\gamma + \sqrt{\gamma^2 - 2\lambda})$, and (1.13) as well as (1.18) and (1.16) are satisfied.

So, in both cases, (1.16) is satisfied and the quadratic form $\{\dots\}$ in (1.15) is positive definite, i.e., there exists $\beta > 0$ depending on γ, λ and K such that

$$(\gamma-\alpha)X^2 + \alpha Y^2 - \frac{K}{\sqrt{\lambda-\alpha(\gamma-\alpha)}}XY \ge \beta(X^2+Y^2)\,.$$

Then (1.15) becomes

$$\tfrac{1}{2}\tfrac{d}{dt}(X^2+Y^2) \le -\beta(X^2+Y^2)$$

and Gronwall's inequality yields

$$X^2(t) + Y^2(t) \le \big(X^2(\tau) + Y^2(\tau)\big)e^{-2\beta(t-\tau)}\,. \tag{1.19}$$

With our abbreviations we see that

$$X^2 + Y^2 = |\partial_t w + \alpha w|^2 + |\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2$$

is equivalent to $|\partial_t w|^2 + |\nabla w|^2 = \|y_1 - y_2\|_E^2$ because of $\lambda|w|^2 \le |\nabla w|^2$ and (1.14) with (1.13). Hence, (1.19) implies the inequality

$$\|y_1(t) - y_2(t)\|_E^2 \le C^2 \|y_1(\tau) - y_2(\tau)\|_E^2 e^{-2\beta(t-\tau)} \quad \text{for } t \ge \tau \tag{1.20}$$

with some constant $C = C(\gamma, \lambda, \alpha)$.

Since equation (1.1) has a global attractor $\mathcal{A}$ in E, the kernel $\mathcal{K}_g$ of the system (1.6) is not empty (see [5]) and so, there exists for every $g \in \mathcal{H}(g^0)$ a solution $z(t) = z_g(t)$ of this system for all $t \in \mathbb{R}$ which is bounded in E.

If we substitute this solution $z(t)$ into (1.20) then we obtain for any other solution $y(t) = U_g(t,\tau)y_\tau$ the estimate

$$\|y(t) - z(t)\|_E \le C\|y(\tau) - z(\tau)\|_E e^{-\beta(t-\tau)} \tag{1.21}$$

for all $t \ge \tau$. Clearly, this inequality also implies that $z(t)$ is the unique, bounded and complete trajectory of the process $U_g(t,\tau)$ corresponding to (1.1), (1.5). $\square$

Now let us formulate consequences of Theorem 1.3 which can be proved in an analogous manner as for the corresponding propositions for the $2D$–Navier–Stokes system in [4].

The estimate (1.21) shows that the closed set $\mathcal{A} := [\{z_{g^0}(t) \,|\, t \in \mathbb{R}\}]_E$ is the attractor for the process $\{U_{g^0}(t,\tau)\,,\, t \ge \tau\}$ since $\mathcal{A}$ is closed and is attracting and, moreover, is minimal. Note that with the hull $\mathcal{H}(g^0)$ in (1.7) we also have

$$\mathcal{A} = \bigcup_{g \in \mathcal{H}(g^0)} \{z_g(0)\}, \tag{1.22}$$

where $z_g(t)$ is the unique, for all $t \in \mathbb{R}$ bounded solution of the hyperbolic equation (1.1), now with the forcing term $g(x,t)$ where $g \in \mathcal{H}(g^0)$. In fact, one can find examples of functions $g^0(x,t)$ for which the set $\{z_{g^0}(t) \,|\, t \in \mathbb{R}\}$ is not closed in E.

On the other hand, the equation (1.22) provides us with the whole global attractor $\mathcal{A}$ and, moreover, reveals its structure. Also note that inequality (1.21) implies that the global attractor $\mathcal{A}$ is exponential, i.e., under the conditions (1.1), (1.3) and (1.8), (1.9), for any bounded set $\mathcal{B} \subset E$ there holds

$$\text{dist}_E\left(U_{g^0}(t,\tau)\mathcal{B}, \mathcal{A}\right) \le C\|\mathcal{B}\|_E e^{-\beta(t-\tau)} \quad \text{for all } t \ge \tau\,.$$

By using the results and same arguments as in [3] and [4], and under all the previous assumptions for the process governed by (1.1), we can also claim the following propositions.

Proposition 1.4. *If the function g^0 is periodic in t with the period p then $z_{g^0}(t)$ also is periodic with the same period p.*

The proof is an immediate consequence of the uniqueness of the solution $z_{g^0}(t)$.

Proposition 1.5. *If $g^0(t)$ is almost periodic then $z_{g^0}(t)$ is almost periodic, too.*

The proof is analogous to the one for [4, Corollary 2.2].

Proposition 1.6. *If $g^0(t)$ is a quasiperiodic Lipschitzian function with the frequencies $\alpha_1, \dots, \alpha_k$, i.e.,*

$$g^0(\bullet, t) = \varphi(\bullet, \alpha_1 t, \dots, \alpha_k t) = \varphi(\bullet, \overline{\alpha} t)$$

where $\varphi(\bullet, \overline{\omega}) \in \mathcal{C}^{Lip}\big(\mathbb{T}^k, L_2(\Omega)\big)$ is a Lipschitz continuous function on the k–dimensional torus $\mathbb{T}^k$ with $\overline{\omega} \in \mathbb{T}^k$, then $z_{g^0}(t)$ is quasiperiodic with the same frequencies, i.e., there exists $\Phi \in \mathcal{C}^{Lip}\big(\mathbb{T}^k, L_2(\Omega)\big)$ and

$$z_{g^0}(0,t) = \Phi(\bullet, \alpha_1 t, \dots, \alpha_k t)\,.$$

Moreover, the global attractor $\mathcal{A}$ is a Lipschitz continuous image of the k–dimensional torus $\mathbb{T}^k$, i.e.,

$$\mathcal{A} = \Phi(\mathbb{T}^k)\,.$$

2. **Rapidly oscillating forcing term.** In what follows, we now consider also a family of problems depending on $\varepsilon > 0$ where u^ε denotes a solution of the hyperbolic equation of sine–Gordon type,

$$\partial_t^2 u^\varepsilon + \gamma \partial_t u^\varepsilon = \Delta u^\varepsilon - f(u^\varepsilon) + g\left(x, t, \tfrac{t}{\varepsilon}\right) \text{ in } \Omega \text{ with } u^\varepsilon|_{\partial\Omega} = 0 \tag{2.1}$$

with the rapidly oscillating forcing term

$$g^\varepsilon(x,t) := g(x, t, \frac{t}{\varepsilon}) \text{ where } 0 < \varepsilon \le \varepsilon_0\,.$$

For the function $f(v)$ we require that the conditions (1.2), (1.3) together with (1.8) and (1.9) are satisfied. We assume that the function $g(x,t,\zeta)$ with $\zeta = \frac{t}{\varepsilon}$, for $(x,t,\zeta) \in (\Omega \times \mathbb{R}^2)$ provides properties which will enable us to obtain the appropriate estimates in what follows.

For instance, g is a continuous function of (t,ζ) with values in H. Then the function $g(x,t,t/\varepsilon)$ is well defined. Supplementary assumptions for the function $g(x,t,\zeta)$ will be given below.

In any case we assume that

i) $g^\varepsilon(x,t)$ is translationary compact in $L_2^{\mathrm{loc}}(\mathbb{R}, H)$ for every ε with $0 < \varepsilon \le \varepsilon_0$. Then it follows that the following integrals are bounded uniformly:

$$\|g^\varepsilon\|^2_{L_2^b} := \sup_{t\in\mathbb{R}} \int_t^{t+1} |g^\varepsilon(\bullet, \tau)|^2 d\tau \le M^2 < \infty\,,$$

where we assume that M is independent of ε. Further we require that

ii) $g^\varepsilon(x,t)$ has the uniform average $g^0(x,t)$ as ε tends to zero, i.e., for every $T > 0$ and every function $\varphi \in L_2\big([-T,T], L_2(\Omega)\big)$, there holds

$$\lim_{\varepsilon\to 0} \int_{-T}^{T} \langle g^\varepsilon(\bullet, s+h)\,,\, \varphi(\bullet, s)\rangle_{L_2(\Omega)} ds = \int_{-T}^{T} \langle g^0(\bullet, s+h)\,,\, \varphi(\bullet, s)\rangle_{L_2(\Omega)} ds$$

uniformly with respect to $h \in \mathbb{R}$ (see [5], [17]).

We moreover assume that $g^0(x,t)$ is also translationary compact in $L_2^{\mathrm{loc}}(\mathbb{R}, H)$.

The equation (2.1) generates the process $U_{g^\varepsilon}(t,\tau)$ for $t \ge \tau \in \mathbb{R}$ with $y(t) = \big(u^\varepsilon(t)\,,\, \partial_t u^\varepsilon(t)\big) = U_{g^\varepsilon}(t,\tau) y(\tau)$, where $u^\varepsilon(t)$ is the solution of (2.1) in the class $u^\varepsilon \in \mathcal{C}([\tau,t], H_0^1)\,,\, \partial_t u^\varepsilon \in \mathcal{C}([\tau,t], H)$.

As is proved in [5], then $y(t)$ satisfies the estimate

$$\begin{aligned} \|y(t)\|_E^2 &\le c_1 \|y(\tau)\|_E^2 e^{-\varrho(t-\tau)} + c_2(1 + \|g^\varepsilon\|^2_{L_2^b}) \\ &\le c_1 \|y(\tau)\|_E^2 e^{-\varrho(t-\tau)} + c_2(1 + M^2) \qquad \text{for all } t \ge \tau \end{aligned} \tag{2.2}$$

where $\varrho > 0$ and c_1 and c_2 do not depend on ε for $0 < \varepsilon \le \varepsilon_0$.

Thus, the process $U_{g^\varepsilon}(t,\tau)$ has a uniformly absorbing set $\mathcal{B}_0$ which is bounded in E:

$$\|\mathcal{B}_0\|_E^2 \le 2c_2(1 + M^2) =: c_3\,. \tag{2.3}$$

Now, let u^0 be the solution of the averaged equation

$$\partial_t^2 u^0 + \gamma \partial_t u^0 = \Delta u^0 - f(u^0) + g^0(x,t) \text{ in } \Omega \text{ with } u^0|_{\partial\Omega} = 0\,. \tag{2.4}$$

As in [5] and [17] we deduce that the equations (2.1) and (2.4) or the corresponding processes $\{U_{g^\varepsilon}(t,\tau)\}$ and $\{U_{g^0}(t,\tau)\}$, for every fixed $\varepsilon \in (0,\varepsilon_0]$ have global attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$, correspondingly, which are uniformly bounded and are subsets of $\mathcal{B}_0$:

$$\mathcal{A}_\varepsilon \subset \mathcal{B}_0 \text{ and } \|\mathcal{A}_\varepsilon\|_E \le \|\mathcal{B}_0\|_E \le c_3^{\frac{1}{2}} =: C_3\,. \tag{2.5}$$

Clearly, also the averaged equation (2.4) has a global attractor $\mathcal{A}_0$ in $\mathcal{B}_0$, i.e., (2.5) also holds with $\varepsilon = 0$.

For what follows, we assume the following crucial, supplementary condition on

$$\widetilde{g}(x,t,\zeta) := g(x,t,\zeta) - g^0(x,t)\,.$$

Assumption:

There exists a function $J(x,t,\zeta) \in C_b^1(\mathbb{R}^2_{t,\zeta}, H) \cap C_b(\mathbb{R}^2_{t,\zeta}, H_0^1)$ *such that*

$$\widetilde{g}(x,t,\zeta) = \tfrac{\partial J}{\partial \zeta}(x,t,\zeta),$$

where $H = L_2(\Omega)$ *and* $H_0^1 = H_0^1(\Omega)$. *Then evidently, there holds*

$$\widetilde{g}(x,t,\zeta) = \tfrac{\partial J}{\partial \zeta}(x,t,\zeta) = \varepsilon \tfrac{\partial}{\partial \tau}\big(J(x,t,\tfrac{\tau}{\varepsilon})\big), \text{ if we set } \zeta = \tfrac{\tau}{\varepsilon}, \tag{2.6}$$

and with some sufficiently large constant M*:*

$$|J(\bullet,t,\zeta)| + |\tfrac{\partial}{\partial t} J(\bullet,t,\zeta)| + \|J(\bullet,t,\zeta)\|_{H_0^1} \le M \text{ for all } t,\zeta \in \mathbb{R}\,. \tag{2.7}$$

Lemma 2.1. *Assume that the conditions* (1.2), (1.3), **i), ii)** *and* (2.7) *with* (2.6) *are fulfilled. Let the external force in* (2.1) *be* $g^\varepsilon(x,t)$ *and let* $u^\varepsilon(x,t)$ *be the solution of* (2.1) *satisfying the initial conditions*

$$u^\varepsilon(x,t)|_{t=\tau} = u_\tau(x)\,,\ \partial_t u^\varepsilon(x,t)|_{t=\tau} = p_\tau(x)\,.$$

Let $u^0(x,t)$ *be the solution of* (2.4) *with the same initial conditions:*

$$u^0(x,t)|_{t=\tau} = u_\tau(x)\,,\ \partial_t u^0|_{t=\tau} = p_\tau(x)\,.$$

Then the difference $w(x,t) = u^\varepsilon(x,t) - u^0(x,t)$ *satisfies the inequality*

$$\|(w,\partial_t w)\|_E = \|y(t)\|_E \le C\varepsilon^{\frac{1}{2}} \text{ for all } t \ge \tau\,, \tag{2.8}$$

where $y(t) = \big(w(t)\,,\,\partial_t w(t)\big)$ *and* $C = C(M,R)$ *with* $R = \|(u_\tau\,,\,\partial_\tau u_\tau)\|_E$.

Proof. For ease of reading let us suppose that $\tau = 0$. As in Section 1 we introduce

$$X^2 := |\partial_t w + \alpha w|^2\,,\ Y^2 := |\nabla w|^2 - (\gamma-\alpha)\alpha |w|^2\,,$$

and recall the conditions

$$\alpha(\gamma-\alpha) < \lambda \quad\text{and}\quad Y^2 \ge \delta|\nabla w|^2 \quad\text{with some appropriate } \alpha \quad\text{and}\quad \delta > 0\,.$$

In the same manner as in Section 1 we find that

$$\sigma(t) := X^2 + Y^2 \text{ with } \sigma(0) = 0$$

satisfies the inequality

$$\tfrac12 \partial_t\sigma + \beta_0\sigma \le \langle g^\varepsilon(\bullet,t) - g^0(\bullet,t)\,,\,\partial_t w + \alpha w\rangle =: \tfrac12\varphi(t)\,.$$

Then Gronwall's lemma implies the estimate

$$\sigma(t) \le \int\limits_0^t e^{-\beta(t-s)}\varphi(s)ds \text{ with } \beta = 2\beta_0\,,$$

which takes with (2.7) the form

$$\sigma(t) \le 2\varepsilon \int_0^t e^{-\beta(t-s)} \langle \big(\tfrac{\partial}{\partial s} J(x,\eta,\tfrac{s}{\varepsilon})\big)|_{\eta=s}\, ,\, \partial_s w + \alpha w\rangle_{L_2(\Omega)} ds\,.$$

Now, integration by parts of the right–hand side gives

$$\begin{aligned}\sigma(t) \le\ & 2\varepsilon\langle J(\bullet,t,\tfrac{t}{\varepsilon})\, ,\, \partial_t w(t) + \alpha w(t)\rangle_{L_2(\Omega)} \\ & - 2\varepsilon \int_0^t \langle \big(\tfrac{\partial}{\partial \eta} J(\bullet,\eta,\tfrac{s}{\varepsilon})\big)|_{\eta=s}\, ,\, \partial_s w(s) + \alpha w(s)\rangle_{L_2(\Omega)} e^{-\beta(t-s)} ds \\ & - 2\beta\varepsilon \int_0^t \langle J(\bullet,s,\tfrac{s}{\varepsilon})\, ,\, \partial_s w(s) + \alpha w(s)\rangle_{L_2(\Omega)} e^{-\beta(t-s)} ds \\ & - 2\varepsilon \int_0^t \langle J(\bullet,s,\tfrac{s}{\varepsilon})\, ,\, \partial_s^2 w(s) + \alpha \partial_s w\rangle_{L_2(\Omega)} e^{-\beta(t-s)} ds\,.\end{aligned}$$

Since $w = u - u^0 \in H_0^1(\Omega)$ and $\partial_t w \in H$ it follows from equations (2.1) and (2.4) that

$$\partial_s^2 w = \partial_s^2 u - \partial_s^2 u^0 = \Delta w - \big(f(u) - f(u^0)\big) + (g - g^0) \in H^{-1}(\Omega)\,.$$

Hence, the following estimate holds:

$$\left| \int_0^t \big\langle J(\cdot,s,s/\varepsilon), \partial_s^2 w(s)\big\rangle\, e^{-\beta(t-s)} ds \right| \le \int_0^t \|J(\cdot,s,s/\varepsilon)\|_{H_0^1} \left\|\partial_s^2 w(s)\right\|_{H^{-1}} e^{-\beta(t-s)} ds.$$

Using this inequality we obtain the estimate

$$\begin{aligned}\sigma(t) \le\ & 2\varepsilon \int_0^t \|J(\cdot,s,s/\varepsilon)\|_{H_0^1} \left\|\partial_s^2 w(s)\right\|_{H^{-1}} e^{-\beta(t-s)} ds \\ & + 2\varepsilon |J(\bullet,t,\tfrac{t}{\varepsilon})| \big(|\partial_t w(t)| + \alpha |w(t)|\big) \\ & + 2\varepsilon \int_0^t \Big\{ |\big(\tfrac{\partial}{\partial \eta} J(\bullet,\eta,\tfrac{s}{\varepsilon})\big)|_{\eta=s}| + (\beta+\alpha)|J(\bullet,s,\tfrac{s}{\varepsilon})| \Big\} \times \\ & \qquad (|\partial_s w(s)| + \alpha |w(s)|) e^{-\beta(t-s)} ds\,.\end{aligned}$$

Inserting (2.7), we finally obtain

$$\begin{aligned}\sigma(t) &= X^2(t) + Y^2(t) \\ &\le CM\varepsilon \int_0^t e^{-\beta(t-s)} ds \Big\{ \|\partial_s^2 w(s)\|_{H^{-1}(\Omega)} + |\partial_s w(s)| + |w(s)| \Big\} ds \\ &\quad + CM\varepsilon (|\partial_t w| + \alpha |w(t)|)\,.\end{aligned} \tag{2.9}$$

If the initial data $y_0 = (u_0, p_0)$ (now for $\tau = 0$) are bounded in E by the constant R, i.e., $\|y_0\|_E \le R$, then we obtain from (2.2) that

$$\|y^\varepsilon(t)\|_E \le C(R) \ \text{ for all } t \ge 0 \ \text{ and all } \varepsilon \in (0, \varepsilon_0]\,.$$

Analogously, $\|y^0(t)\|_E \le C(R)$. From (2.9) it follows that

$$X^2(t) + Y^2(t) \le c_3 C(R)\varepsilon M + CM\varepsilon \int_0^t e^{-\beta(t-s)} \|\partial_s^2 w(s)\|_{H^{-1}(\Omega)} ds\,. \qquad (2.10)$$

As $X^2(t)+Y^2(t)$ is equivalent to $\|y(t)\|_E^2$, to prove (2.3), it remains to estimate the last integral in (2.10).

We begin with the estimate of $\|\partial_t^2 w(t)\|_{H^{-1}(\Omega)}$. We take the difference between the equations (2.1) and (2.4) and deduce that $w = u^\varepsilon(x,t) - u^0(x,t)$ satisfies the inequality

$$\begin{aligned}
&\|\partial_t^2 w(t)\|_{H^{-1}(\Omega)} \\
&\le \gamma \|\partial_t w(t)\|_{H^{-1}(\Omega)} + c_4 \|w(t)\|_{H_0^1} + c_5\big(|u^\varepsilon(t)| + |u^0(t)| + 1\big) + |g^\varepsilon(t) - g^0(t)| \\
&\le c_6\big(C(R)+1\big) + c_7 |\widetilde{g}(t)|\,,
\end{aligned}$$

see [15, Chap.3 Remark 8.2].

Consequently, the last integral in (2.10) can be estimated by

$$CM\varepsilon \int_0^t e^{-(t-s)} \|\partial_t^2 w(s)\|_{H^{-1}(\Omega)} ds \le c_8(R)\varepsilon\,. \qquad (2.11)$$

From (2.10), (2.11) and the equivalence of $X^2(t)+Y^2(t)$ to $\|y(t)\|_E^2$, there follows

$$\|y(t)\|_E^2 = \|\big(w(t)\,,\, \partial_t w(t)\big)\|_E^2 \le c_9(R,M)\varepsilon$$

and (2.8) is proved. □

In the same manner as in Section 1 we establish that the equations (2.1) have one global bounded solution $z^\varepsilon(t)$ each for all $t \in \mathbb{R}$ which exponentially attract other solutions of (2.1) (see (1.21)). As $\{z^\varepsilon(t)\} \in A_\varepsilon$ then (2.5) implies

$$\|z^\varepsilon(t)\|_E \le c_2 \;\text{ where } c_2 \text{ does not depend on } \varepsilon \;\text{ and } 0 < \varepsilon \le \varepsilon_0\,. \qquad (2.12)$$

Clearly, also the averaged equation (2.4) has one bounded global solution $z^0(t)$ which exponentially attracts other solutions of the averaged equation (2.4) and satisfies the inequality (2.12) with $\varepsilon = 0$.

Now we are in the position to estimate the distance between these global, bounded, exponentially attracting solutions $z^\varepsilon(t)$ and $z^0(t)$, which is our main result in this paper.

Theorem 2.2. *Under the assumptions* (1.2), (1.3), **i)** *and* **ii)** *and* (2.7) *with* (2.6), *the solutions* $z^\varepsilon(t)$ *and* $z^0(t)$ *satisfy the estimate*

$$\|z^\varepsilon(t) - z^0(t)\|_E \le C\varepsilon^{\frac{1}{2}} \;\; \textit{for all}\; t \in \mathbb{R}\,. \qquad (2.13)$$

Proof. We choose one point $\{z^\varepsilon(T)$ with $T \in \mathbb{R}\}$ on the unique bounded solution $\{z^\varepsilon(t)\,,\, t \in \mathbb{R}\}$; for simplicity, let $T = 0$. On $\{z^\varepsilon(t)\}$ we take a point $z^\varepsilon(-\tau)$, where τ will be specified later on. Then let $y^0(t) = \big(u^0(t)\,,\, \partial_t u^0(t)\big)$ be the solution of (2.4) with initial data on $z^\varepsilon(-\tau)$, namely

$$y^0(-\tau) = \big(z^\varepsilon(-\tau)\,,\, \partial_t z^\varepsilon(-\tau)\big)\,. \qquad (2.14)$$

Let $\{(z^0(t), \partial_t z^0(t))$ for $t \in \mathbb{R}\}$ be in E the unique, globally bounded solution of (2.4). Then, as was shown in Theorem 1.3, $\big(z^0(t)\,,\, \partial_t z^0(t)\big)$ exponentially attracts

$y^0(t)$:

$$\|y^0(-\tau+t) - \big(z^0(-\tau+t)\,,\,\partial_t z^0(-\tau+t)\big)\|_E \le c_1\|y^0(-\tau) - \big(z^0(-\tau)\,,\,\partial_t z^0(-\tau)\big)\|_E e^{-\beta t}\,. \tag{2.15}$$

As $y^0(-\tau+t)$ and $\big(z^\varepsilon(-\tau+t)\,,\,\partial_t z^\varepsilon(-\tau+t)\big)$ are solutions of equations (2.4) and (2.1), respectively, with equal initial data (2.14) at the moment $t=0$, there holds

$$\|\big(z^\varepsilon(-\tau+t)\,,\,z_t^\varepsilon(-\tau+t)\big) - y^0(-\tau+t)\|_E \le c\varepsilon^{\frac12} \text{ for all } t\ge 0 \tag{2.16}$$

as was proven in Lemma 2.1. Hence, using (2.15) and (2.16) for $t=\tau$, we get

$$\begin{aligned}
&\|\big(z^\varepsilon(0)\,,\,\partial_t z^\varepsilon(0)\big) - \big(z^0(0)\,,\,\partial_t z^0(0)\big)\|_E \\
&\le \|\big(z^\varepsilon(-\tau+\tau)\,,\,\partial_t z^\varepsilon(-\tau+\tau)\big) - y^0(-\tau+\tau)\|_E \\
&+ \|y^0(-\tau+\tau) - \big(z^0(-\tau+\tau)\,,\,\partial_t z^0(-\tau+\tau)\big)\|_E \\
&\le c\varepsilon^{\frac12} + c_1 e^{-\beta\tau}\|y^0(-\tau) - \big(z^0(-\tau)\,,\,\partial_t z^0(-\tau)\big)\|_E\,.
\end{aligned} \tag{2.17}$$

As $y^0(-\tau) = \big(z^\varepsilon(-\tau)\,,\,\partial_t z^\varepsilon(-\tau)\big) \in \mathcal{A}^\varepsilon$ and $\big(z^0(-\tau)\,,\,\partial_t z^0(-\tau)\big) \in \mathcal{A}^0$ and $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$ are uniformly bounded with respect to ε for $0\le\varepsilon\le\varepsilon_0$ (cf. (2.5)) we deduce from (2.17) and (2.14) that

$$\|\big(z^\varepsilon(0)\,,\,z_t^\varepsilon(0)\big) - \big(z^0(0)\,,\,z_t^0(0)\big)\|_E \le c\varepsilon^{\frac12} + c_3 e^{-\beta\tau} \le c_4(\varepsilon^{\frac12} + e^{-\beta\tau})\,. \tag{2.18}$$

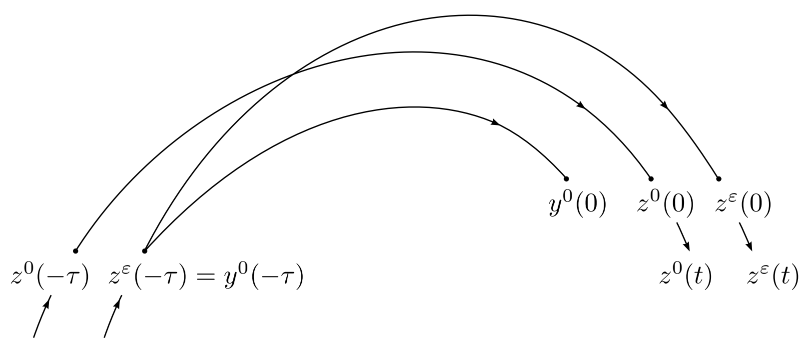
Now we pose $\tau = \frac{1}{2\beta}\log(\frac{1}{\varepsilon})$. Then $e^{-\beta\tau} = \varepsilon^{\frac12}$ and inequality (2.18) becomes

$$\|\big(z^\varepsilon(0)\,,\,\partial_t z^\varepsilon(0)\big) - \big(z^0(0)\,,\,\partial_t z^0(0)\big)\| \le c_4\varepsilon^{\frac12}\,,$$

i.e., the proposed estimate (2.13). □

Analogously it can be proved that

$$\mathrm{dist}_E(\mathcal{A}_\varepsilon, \mathcal{A}_0) \le c\varepsilon^{\frac12}\,.$$



The global, attracting solutions z^0 and z^ε

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