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TRAJECTORY AND GLOBAL ATTRACTORS OF DISSIPATIVE HYPERBOLIC EQUATIONS WITH MEMORY

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Abstract. We consider in this article a general construction of trajectory attractors and global attractors of evolution equations with memory. In our approach, the corresponding dynamical system acts in the space of initial data of the Cauchy problem under study; we can note that, in previous studies, the so-called history space setting was introduced and the study of global attractors was made in an extended phase space.

As an application, we construct trajectory and global attractors for dissipative hyperbolic equations with linear memory. We also prove the existence of a global Lyapunov function for the dissipative hyperbolic equation with memory. The existence of such a Lyapunov function implies a regular structure for the trajectory and global attractors of the equation under consideration.

1. **Introduction.** In the recent years, many interesting papers which study various models from mathematical physics with memory effects from the theory of dynamical systems and global attractors viewpoint were published (see [1 – 11]). Such models are described by evolution integro-differential equations having terms that depend on the past values of the unknown functions. Usually, these terms have the form of linear time convolutions of the unknown functions with some known functions which are called memory kernels. Such memory kernels are monotone functions that vanish at infinity. This behavior of the kernels reflects the fading in systems with memory.

To construct a dynamical system corresponding to a model with memory, the authors of the above papers use the so-called *history space setting*. This approach suggests to expand the usual phase space of initial data (known from the theory of evolution equations without memory) by some components depending on the past history of the system. This idea of using the past history as a variable of the system was proposed by Dafermos (see [12]). These components belong to a weighted Hilbert space and the weight in this space is determined by the memory kernel of the equation under consideration. The extended phase space is needed to prove (in the autonomous case) the semigroup property of the corresponding dynamical

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system. Then, the well-known techniques from the theory of semigroups in Banach or Hilbert spaces allow to construct and to study the global attractors of these new systems (see [13, 14, 15]). If the considered model is non-autonomous, then the standard construction of the skew-product flow reduces the system to another semigroup and the corresponding extended phase space also includes the hull of all time dependent coefficients and terms of the equation in a suitable topological space (see [15, 16, 17, 18, 19]). In the works [$1-11$], these methods were successfully applied to various systems with memory.

In the present paper, we propose an alternative approach for the study of the longtime behavior of the solutions of evolution equations with memory by constructing a trajectory attractor and a global attractor for an equation with memory. The main feature of the method is the following: the dynamics of the solutions is studied in the space of initial data of the corresponding Cauchy problem. Trajectory attractors were proposed in [20, 19] for the investigation of the limit behavior of some evolution equations of mathematical physics, with an emphasis on equations for which the uniqueness is not known. In [19, 21], the trajectory attractor's approach was then used to construct the global attractors of such equations.

An evolution equation with memory can be written as follows:

$$\partial_t y(t) = A(y(t), y^t(\cdot)),\ t \geq 0. \tag{1.1}$$

The (nonlinear) operator $A(\cdot,\cdot)$ in the right-hand side of (1.1) depends on the value of an unknown function $y(t)$ at time t, as well as all the values of the function $y(t')$ for all $t' \leq t$. In (1.1), $y^t(\cdot)$ denotes the function $y(t')$ for all $t' \leq t$. (In the case of an equation without memory, the operator $A(\cdot)$ depends only on the first variable $y(t)$.) The values of the unknown function $y(t)$ belong to a Banach space E of initial data of equation (1.1). For an equation without memory, this initial data reads

$$y|_{t=0} = z \in E. \tag{1.2}$$

However, when dealing with equations with memory, it is reasonable to assume that the function $y(t)$ is known for all $t \leq 0$ and the initial data has the form

$$y|_{t\leq 0} = z(t),\ t \leq 0. \tag{1.3}$$

We note that the function $y(t)$ does not necessarily satisfy equation (1.1) for $t \leq 0$. We assume that the function $z(\cdot)$ belongs to $\mathcal{E}^- \subseteq C_b(\mathbb{R}_-; E)$, where $\mathbb{R}_- = (-\infty, 0]$ and $\mathcal{E}^-$ is a subspace of $C_b(\mathbb{R}_-; E)$. For simplicity, we set $\mathcal{E}^- = C_b(\mathbb{R}_-; E)$.

We assume that problem (1.1) and (1.3) has a unique solution $y(\cdot) \in C_b(\mathbb{R}; E)$ for every $z \in \mathcal{E}^-$. We construct the semigroup $\{S(h), h \geq 0\}$ acting in $\mathcal{E}^-$ by the formula

$$(S(h)z)(t) = y(t+h),\ t \leq 0,\ h \geq 0,$$

where $y(t)$ is the solution of (1.1) with initial data z (see (1.3)). It is clear that the family of mappings $\{S(h), h \geq 0\}$ forms a semigroup.

The problem is to study the global attractor of this semigroup $\{S(h)\}$ in the space $\mathcal{E}^-$. We call this attractor the *trajectory attractor* since the semigroup $\{S(h)\}$ acts in the space of trajectories (solutions) of equation (1.1). Recall that the trajectory attractor of problem (1.1) and (1.3) is a bounded (in $C_b(\mathbb{R}_-; E)$) and compact (in $C^{\rm loc}(\mathbb{R}_-; E)$) set $\mathfrak{A} \subseteq \mathcal{E}^-$ which is strictly invariant with respect to $\{S(h)\}$ and attracts any bounded (in $C_b(\mathbb{R}_-; E)$) set $B \subset \mathcal{E}^-$ in the topology of $C^{\rm loc}(\mathbb{R}_-; E)$, that is, for any $M > 0$,

$$\mathrm{dist}_{C([-M,0];E)}(S(h)B, \mathfrak{A}) \to 0\ (h \to +\infty). \tag{1.4}$$

Using the trajectory attractor, we define the *global attractor* of problem (1.1) and (1.3) in the space E. A compact set $\mathcal{A} \subset\subset E$ is called the global attractor of (1.1) and (1.3) if, for any bounded set $B \subset \mathcal{E}^-$

$$\operatorname{dist}_E\left(\left(S(h)B\right)(0), \mathcal{A}\right) \to 0 \ (h \to +\infty) \tag{1.5}$$

and $\mathcal{A}$ is the minimal set that satisfies (1.5). Here, for any $B' \subseteq \mathcal{E}^-$, the set $B'(0)$ denotes the following set in $E: \ B'(0) = \{z(0) \mid z \in B'\}$. It follows from (1.4) that $\mathcal{A} = \mathfrak{A}(0)$. Besides, relation (1.5) implies that

$$\operatorname{dist}_E\left(y(t), \mathcal{A}\right) \to 0 \ (t \to +\infty) \tag{1.6}$$

for every solution $y(t)$ of problem (1.1) and (1.3) and this limit holds uniformly with respect to $z \in B$ for every bounded set $B \subset \mathcal{E}^-$. We note that the limit relation (1.6) always holds for the global attractor of evolution equations without memory of the form (1.1) (in that case, $A(\cdot) = A(y)$) with initial data (1.2).

In Section 2, we present the results on the existence and the structure of trajectory and global attractors of a general autonomous equation with memory of the form (1.1). The main theorem states that the trajectory and global attractors exist if the semigroup has a bounded and compact attracting set.

In Section 4, the proposed scheme is applied to the study of the trajectory and global attractors of the following dissipative hyperbolic equation with memory:

$$\begin{aligned} \partial_t^2 u(t) + \gamma \partial_t u(t) &= k(0)\Delta u(t) + \int_0^\infty k'(s)\Delta u(t-s)ds - f(u(t)) + g(x); \\ u|_{\partial\Omega} &= 0; \ x \in \Omega \subset\subset \mathbb{R}^n, \ t \geq 0, \ u(t) = u(x,t). \end{aligned} \tag{1.7}$$

Here, $\gamma > 0, \ g(x) \in L_2(\Omega)$. The nonlinear function $f(v), v \in \mathbb{R}$, satisfies some growth conditions (see (4.3) – (4.5)) which are standard for hyperbolic equations without memory and having a nonlinear term with a moderate growth (see [22, 13, 15, 14]). We set $\mathbb{R}_+ = [0, +\infty)$. We assume that $k(s) \in C^2(\mathbb{R}_+)$, $k(0) > k(+\infty) > 0$, and k satisfies

$$\left.\begin{array}{l} \mu(s) \overset{\text{def}}{\equiv} -k'(s) \geq 0, \\ \mu'(s) = -k''(s) \leq 0, \\ \mu'(s) + \delta\mu(s) \leq 0, \ \ \forall s \geq 0 \ (\delta > 0). \end{array}\right\} \tag{1.8}$$

The condition $k(\cdot) \in C^2(\mathbb{R}_+)$ can be slightly weakened by assuming that $k(\cdot) \in C(\mathbb{R}_+) \cap C^2(0,\infty), \ -k'(\cdot) = \mu(\cdot) \in L_1(\mathbb{R}_+)$, and inequalities (1.8) hold for all $s > 0$. Thus, these conditions allow $\mu(s)$ to have a singularity at $s = 0$, whose order is less than 1, e.g., $\mu(s) = s^{-\rho}e^{-\delta s}, \ 0 \leq \rho < 1$. However, for simplicity, we assume in this paper that $k(\cdot) \in C^2(\mathbb{R}_+)$.

The hyperbolic equations (1.1) with memory kernels satisfying (1.8) were considered in [23, 24, 25, 26]. Such equations arise, for example, in the theory of electromagnetic materials with memory. Similar models describe homogeneous and viscoelastic solids. Another possible application is the equation governing the temperature evolution in a rigid conductor with memory according to the models proposed by Gurtin and Pipkin (see [27, 28, 23, 24, 25, 26] and the references therein for more detailed explanations).

We supplement equation (1.7) with the following initial conditions:

$$u|_{t\leq 0} = v(t), \ \partial_t u|_{t\leq 0} = \partial_t v(t). \tag{1.9}$$

We assume that $v(\cdot) \in C_b(\mathbb{R}_-; H_0^1(\Omega))$ and $\partial_t v(\cdot) \in C_b(\mathbb{R}_-; L_2(\Omega))$. We denote by $E = H_0^1(\Omega) \times L_2(\Omega)$ the usual energy space. It is clear that $z(\cdot) = (v(\cdot), \partial_t v(\cdot)) \in C_b(\mathbb{R}_-; E)$.

In Section 4, we prove that problem (1.7) and (1.9) has a unique solution $u(t), t \in \mathbb{R}$, such that $u(t) \in C_b(\mathbb{R}; E)$. Thus, we can construct a semigroup $\{S(h)\}$ in the space

$$\mathcal{E}^- = \left\{ z(t) = (v(t), \partial_t v(t)) \mid v \in C_b(\mathbb{R}_-; H_0^1(\Omega)),\ \partial_t v \in C_b(\mathbb{R}_-; L_2(\Omega)) \right\}.$$

It is obvious that $\mathcal{E}^-$ is a subspace of $C_b(\mathbb{R}_-; E)$.

We then prove that the semigroup $\{S(h)\}$ acting in the space $\mathcal{E}^-$ has a bounded in $C_b(\mathbb{R}_-; E)$ and compact in $C^{\text{loc}}(\mathbb{R}_-; E)$ attracting set. According to the results of Section 2, this property allows to construct the trajectory and global attractors of equation (1.7) in the spaces $C^{\text{loc}}(\mathbb{R}_-; E)$ and E, respectively.

In Section 3, we consider the questions concerning the structure of trajectory and global attractors of an equation with memory of the form (1.1) when it has a global Lyapunov function.

We denote by $\mathcal{K}$ the kernel of equation (1.1) in the space $C_b(\mathbb{R}; E)$. The kernel consists of all bounded complete solutions of the equation, i.e.

$$\mathcal{K} = \{ y(\cdot) \in C_b(\mathbb{R}; E) \mid y(t) \text{ satisfies (1.1) for all } t \in \mathbb{R} \}.$$

We denote by $\mathcal{N}$ the set of all stationary points of equation (1.1), i.e.

$$\mathcal{N} = \{ w \in E \mid A(w, w) = 0 \}.$$

We also consider the unstable set $M^+(\mathcal{N}) \subset C_b(\mathbb{R}; E)$ issuing from the set $\mathcal{N}$:

$$M^+(\mathcal{N}) = \{ y(\cdot) \in \mathcal{K} \mid y(t) \to \mathcal{N}\ (t \to -\infty) \}.$$

Using the general results from [14, 15], we prove in Section 3 that the trajectory attractor $\mathfrak{A}$ and the global attractor $\mathcal{A}$ of the equation with memory (1.1) constructed in Section 2 have a regular structure, that is,

$$\mathfrak{A} = \Pi_- M^+(\mathcal{N}),\ \mathcal{A} = M^+(\mathcal{N})(0). \tag{1.10}$$

Here, Π_- denotes the restriction operator to the semiaxis $\mathbb{R}_-$. In particular, if the set $\mathcal{N}$ is finite, i.e. $\mathcal{N} = \{ w_i \mid i = 1, \ldots, N \}$, then

$$\mathfrak{A} = \Pi_- \bigcup_{i=1}^{N} M^+(w_i),\ \mathcal{A} = \bigcup_{i=1}^{N} M^+(w_i)(0). \tag{1.11}$$

In Section 5, we prove that the hyperbolic equation (1.7) with memory has the following global Lyapunov function:

$$\begin{aligned} \Phi(z(\cdot)) \quad &= \quad \int_\Omega \left[|\partial_t v(x,0)|^2 + \beta |\nabla v(x,0)|^2 + 2F(v(x,0)) - 2g(x) v(x,0) \right] dx \\ &\quad + \int_0^\infty \mu(s) \int_\Omega |\nabla v(x,0) - \nabla v(x, 0-s)|^2 dx ds, \end{aligned} \tag{1.12}$$

$z(x,t) = (v(x,t), \partial_t v(x,t)),\ x \in \Omega, t \leq 0$. Here, $\beta = k(+\infty)$ and $F(v) = \int_0^v f(w) dw$. Recall that, in the equation without memory, the function $\mu(s) = -k'(s)$ vanishes for all $s \geq 0$, the second integral term in (1.12) vanishes, and $\Phi(z(\cdot))$ reduces to the usual Lyapunov function of the dissipative hyperbolic equation (see [14, 13]). Finally, we establish formulas (1.10) for the hyperbolic equation (1.7). In particular, if the set $\mathcal{N}$ is finite (this holds in the generic case, see [14]), then we obtain (1.11).

Moreover, for any solution $u(\cdot,t)$ of equation (1.7) with initial data $(v,\partial_t v)\in\mathcal{E}^-$, there exists a stationary point $w_i=(q_i(\cdot),0)\in\mathcal{N}$ such that

$$\|u(\cdot,t)-q_i(\cdot)\|_{H_0^1(\Omega)}+\|\partial_t u(\cdot,t)-0\|_{L_2(\Omega)}\to 0\ (t\to+\infty).$$

This property is well-known for the dissipative hyperbolic equation without memory. We conclude that it is also valid for the equation with memory.

2. **Trajectory and Global Attractors; the General Case.** We study autonomous equations with memory of the form

$$\partial_t y(t)=A(y(t),y^t(\cdot)),\ t\geq 0. \tag{2.1}$$

Here, $A(\cdot,\cdot)$ denotes a nonlinear function operator which depends on two function parameters. The first parameter is the value of an unknown function y at time t, while the second parameter is the entire function y for $t'\leq t$. The notation $y^t(\cdot)$ stands for the function $y(t')$ for $t'\leq t$. We assume that the values of the function $y(t)$ for $t\in\mathbb{R}$ belong to a Banach space E. In the next section, we shall consider an example of equation of the form (2.1).

We solve equation (2.1) for $t\geq 0$. It is common to assume that, for $t\leq 0$, the function $y(t)$ is known and it does not necessarily satisfy equation (2.1) for negative t. Thus, for $t\leq 0$, we have an "initial data" for the equation of the form

$$y|_{t\leq 0}=z(t),\ t\leq 0. \tag{2.2}$$

Our task is to find a function $y(t)$ for all $t\in\mathbb{R}$ such that $y(t)=z(t)$ for $t\leq 0$ and $y(t)$ satisfies equation (2.1) for $t\geq 0$. A problem is then to study the behavior of these solutions as $t\to+\infty$.

In this section, we do not discuss in which sense a function $y(\cdot)$ satisfies equation (2.1). This work should be done in each particular case. We only present the main properties of the solutions that we need in order to construct the theory of trajectory and global attractors for such equations.

We denote by $C(\mathbb{R}_-;E)$ the space of continuous functions on $\mathbb{R}_-=(-\infty,0]$ with values in E. We shall also use the space $C_b(\mathbb{R}_-;E)$ of bounded continuous functions. Similarly, we introduce the spaces $C(\mathbb{R};E)$ and $C_b(\mathbb{R};E)$ defined on the entire time axis $\mathbb{R}$.

We consider a subspace $\mathcal{E}^-\subseteq C_b(\mathbb{R}_-;E)$. The case $\mathcal{E}^-=C_b(\mathbb{R}_-;E)$ is not excluded. The subspace $\mathcal{E}^-$ serves as the space of "initial data" for equation (2.1). The elements of the space $\mathcal{E}^-$ are denoted by $z(s), s\leq 0$. Thus, the function $z(s)$ in (2.2) belongs to $\mathcal{E}^-$ (here, the time variable t is replaced by s).

We assume that problem (2.1) and (2.2) has a unique solution $y\in C_b(\mathbb{R};E)$ for any function z in $\mathcal{E}^-$. This property must be checked for each particular equation. Using this property, we can construct a semigroup acting in the space $\mathcal{E}^-$. We fix an arbitrary time $t\geq 0$. Consider the mapping $S(t)$ acting from $\mathcal{E}^-$ into $C_b(\mathbb{R}_-;E)$ by the formula

$$(S(t)z)(s)=y(t+s),\ s\leq 0,$$

where $y(t)$ is the solution of (2.1) and (2.2).

We assume that $(S(t)z)(s)\in\mathcal{E}^-$ for all $t\geq 0$ and for every $z\in\mathcal{E}^-$.

PROPOSITION 2.1. *The family of mappings $\{S(t),t\geq 0\}$ forms a semigroup in $\mathcal{E}^-$, that is, $S(t_1+t_2)=S(t_1)\circ S(t_2)$ for all $t_1,t_2\geq 0$ and $S(0)=\mathrm{Id}$ is the identity operator.*

Proof. It is obvious that $S(0)z = z$ for $z \in \mathcal{E}^-$. Consider the function $y(t_1+t_2+s) = S(t_1+t_2)z(s)$, where $y(t)$ is a solution of (2.1) with initial data $z(s), s \le 0$. Let $y_1(t)$ be a solution of (2.1) with initial data $y(t_1+s), s \le 0$. It is clear that the function $y(t_1+t)$ satisfies equation (2.1) with the same initial data $y(t_1+s), s \le 0$. By assumption, this problem is uniquely solvable, hence $y_1(t) = y(t_1+t)$ for all $t \in \mathbb{R}$. Therefore, $y_1(t_2+s) = y(t_1+t_2+s)$ for $s \le 0$, that is, $S(t_1)S(t_2)z = S(t_1+t_2)z$ for all $z \in \mathcal{E}^-$. □

REMARK 2.2. Fix an arbitrary $t_1 > 0$ and consider the function $z_1(s) = S(t_1)z(s)$ for $s \le 0$, where $z \in \mathcal{E}^-$. It is clear that the function $y_1(t), t \in \mathbb{R}$, where $y_1(t)$ is a solution of equation (2.1) with initial data $z_1(s), s \le 0$, satisfies equation (2.1) on the interval $-t_1 \le t < +\infty$. One of our purposes is to construct complete solutions for equation (2.1) which satisfy (2.1) for all $t \in \mathbb{R}$.

We shall study the global attractor of the semigroup $\{S(t)\} = \{S(t), t \ge 0\}$ corresponding to problem (2.1) and (2.2) and acting in the space $\mathcal{E}^-$. Let us define a topology in $\mathcal{E}^-$. Since $\mathcal{E}^- \subseteq C(\mathbb{R}_-; E)$, we consider the local uniform convergence topology $C^{\mathrm{loc}}(\mathbb{R}_-; E)$ in the space $C(\mathbb{R}_-; E)$. By definition, a sequence of functions $\{f_n\} \subset C(\mathbb{R}_-; E)$ converges to a function $f \in C(\mathbb{R}_-; E)$ in the topology of $C^{\mathrm{loc}}(\mathbb{R}_-; E)$ if, for any $M > 0$,

$$\max_{s\in[-M,0]} \|f_n(s) - f(s)\|_E \to 0 \ (n \to +\infty). \tag{2.3}$$

It easily follows that the topological space $C^{\mathrm{loc}}(\mathbb{R}_-; E)$ is metrizable and the corresponding metric space is complete. Recall that

$$\mathcal{E}^- \subseteq C_b(\mathbb{R}_-; E) \subset C^{\mathrm{loc}}(\mathbb{R}_-; E), \tag{2.4}$$

so that we can use the above convergence to define the topology in $\mathcal{E}^-$.

We also need a notion of a bounded set in $\mathcal{E}^-$. By definition, a set $B \subset \mathcal{E}^-$ is called *bounded* if it is bounded in the norm of $C_b(\mathbb{R}_-; E)$, that is,

$$\sup_{f\in B} \|f\|_{C_b(\mathbb{R}_-;E)} = \sup_{f\in B}\sup_{s\le 0} \|f(s)\|_E < \infty. \tag{2.5}$$

We define in a standard way the notions of an absorbing set and an attracting set of the semigroup $\{S(t)\}$ in $\mathcal{E}^-$.

DEFINITION 2.3. *A set $B_0 \subset \mathcal{E}^-$ is said to be absorbing for the semigroup $\{S(t)\}$ if, for any bounded set $B \subset \mathcal{E}^-$, there exists a number $t_1 = t_1(B)$ such that $S(t)B \subseteq B_0$ for all $t \ge t_1$.*

DEFINITION 2.4. *A set $P \subset \mathcal{E}^-$ is said to be attracting (in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$) for the semigroup $\{S(t)\}$ if, for any bounded set $B \subset \mathcal{E}^-$ and for any $\varepsilon > 0$, there exists a number $t_1 = t_1(B, \varepsilon)$ such that $S(t)B \subseteq \mathcal{O}_\varepsilon(P)$ for all $t \ge t_1$, where $\mathcal{O}_\varepsilon(P)$ denotes the ε-neighbourhood of the set P in a suitable metric generating the topology $C^{\mathrm{loc}}(\mathbb{R}_-; E)$. This property is equivalent to the following: for any $M > 0$,*

$$\mathrm{dist}_{C([-M,0];E)}(S(t)B, P) \to 0 \ (t \to +\infty). \tag{2.6}$$

Here, $\mathrm{dist}_{\mathcal{M}}(A, B)$ denotes the Hausdorff semi-distance from a set A to a set B in a metric space $\mathcal{M}$ with metric $\rho_{\mathcal{M}}(\cdot,\cdot)$, that is,

$$\mathrm{dist}_{\mathcal{M}}(A, B) = \sup_{a\in A}\inf_{b\in B} \rho_{\mathcal{M}}(a, b).$$

If a semigroup $\{S(t)\}$ has a compact absorbing set, it is called *compact.* A semigroup $\{S(t)\}$ having a compact attracting set is called *asymptotically compact.*

We now define the *trajectory attractor* of problem (2.1) and (2.2). We use the term trajectory attractor because the semigroup acts in the trajectory space $\mathcal{E}^- \subseteq C_b(\mathbb{R}_-; E)$ or, to be more precise, in the space of negative semi-trajectories (see also [19, 20]).

DEFINITION 2.5. *A set $\mathfrak{A} \subset \mathcal{E}^-$ is said to be the trajectory $(C_b(\mathbb{R}_-; E), C^{\mathrm{loc}}(\mathbb{R}_-; E))$-attractor of problem (2.1) and (2.2) if*

1. *the set $\mathfrak{A}$ is bounded in $C_b(\mathbb{R}_-; E)$ and compact in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$;*
2. *the set $\mathfrak{A}$ is strictly invariant with respect to $\{S(t)\}$, that is, $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$;*
3. *$\mathfrak{A}$ is an attracting (in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$) set of the semigroup $\{S(t)\}$.*

REMARK 2.6. This definition is analogous to that given in [14] for the notion of the global (F, D)-attractor (see also [19]).

Following [14] and [19], we formulate the main theorem on the existence of the trajectory attractor of problem (2.1) and (2.2).

Recall that the semigroup $\{S(t)\}$ is called *uniformly bounded* in E if, for any bounded set $B \subset \mathcal{E}^-$, the set $\bigcup_{t \geq 0} S(t)B$ is bounded in $C_b(\mathbb{R}_-; E)$.

THEOREM 2.7. *Let the semigroup $\{S(t)\}$ acting in $\mathcal{E}^- \subseteq C_b(\mathbb{R}_-; E)$ and corresponding to problem (2.1) and (2.2) be uniformly bounded and have an attracting set $P \subset \mathcal{E}^-$. We assume that the set P is bounded in $C_b(\mathbb{R}_-; E)$ and compact in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$. Then there exists the trajectory attractor $\mathfrak{A}$ of the semigroup $\{S(t)\}$ and $\mathfrak{A} = \omega(P)$, where $\omega(P)$ is the ω-limit set of P w.r.t. $\{S(t)\}$ in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$.*

Proof. We only sketch the main steps of the reasoning. Together with $C_b(\mathbb{R}_-; E)$ and $C^{\mathrm{loc}}(\mathbb{R}_-; E)$, we consider the spaces $C_b(\mathbb{R}; E)$ and $C^{\mathrm{loc}}(\mathbb{R}; E)$. We denote by $\mathcal{K}^+$ the space of all solutions of equation (2.1) with all possible initial data $z \in \mathcal{E}^-$, that is,

$$\mathcal{K}^+ = \{y(t), t \in \mathbb{R} \mid \begin{cases} y(t) = z(t)\ , & t \leq 0 \\ y(t) \text{ satisfies (2.1)}\ , & t \geq 0 \end{cases}, \ z \in \mathcal{E}^-\}. \tag{2.7}$$

The translation semigroup $\{T(h)\} = \{T(h), h \geq 0\}$ acts on $\mathcal{K}^+$ by the formula

$$T(h)y(t) = y(t+h),\ h \geq 0.$$

It follows easily that the semigroup $\{T(h)\}$ is continuous in the topology $C^{\mathrm{loc}}(\mathbb{R}; E)$. Consider the set $P^+ \subset \mathcal{K}^+$

$$P^+ = \{y(t), t \in \mathbb{R} \mid \begin{cases} y(t) = z(t)\ , & t \leq 0 \\ y(t) \text{ satisfies (2.1)}\ , & t \geq 0 \end{cases}, \ z \in P\}. \tag{2.8}$$

It is clear that the set P^+ is bounded in $C_b(\mathbb{R}; E)$, since the semigroup $\{S(t)\}$ is uniformly bounded and the set P is bounded in $C_b(\mathbb{R}_-; E)$. Besides, the set P^+ is precompact in $C^{\mathrm{loc}}(\mathbb{R}; E)$, since P is compact in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$. Moreover, the set P^+ is attracting for the semigroup $\{T(h)\}$. Therefore, the theorem from [14] on the existence of the global (F, D)-attractor of a continuous semigroup is applicable to the semigroup $\{T(h)\}$ acting in $\mathcal{K}^+$ (see also [19]). We set $F = C_b(\mathbb{R}; E)$ and $D = C^{\mathrm{loc}}(\mathbb{R}; E)$. We denote by $\mathfrak{A}^+ = \omega(P^+)$ the corresponding global (F, D)-attractor. We now restrict the set $\mathfrak{A}^+$ to the negative semiaxis $\mathbb{R}_- = (-\infty, 0]$. We denote by $\mathfrak{A}$ the set so constructed. This set satisfies all the properties of the definition of the trajectory attractor. We leave the details to the reader. $\square$

REMARK 2.8. Note that, in the proof of Theorem 2.7, we do not use the continuity of the semigroup $\{S(t)\}$ itself in the space $C^{\text{loc}}(\mathbb{R}_-; E)$. However, if it is known that the semigroup is continuous, then we can prove that the set $\mathfrak{A}$ is connected.

COROLLARY 2.9. *If $\{S(t)\}$ is continuous on $\mathcal{E}^-$ in the topology $C^{\text{loc}}(\mathbb{R}_-; E)$, then the trajectory attractor $\mathfrak{A}$ is a connected set in $C^{\text{loc}}(\mathbb{R}_-; E)$. In fact, it is sufficient that the semigroup $\{S(t)\}$ is continuous on any bounded (in $C_b(\mathbb{R}_-; E)$) set.*

Proof. Follows from the fact that a continuous image of a connected set is a connected set. (In our case, we may assume that the set P is connected). Then the ω-limit set $\omega(P)$ is also connected (see [13] for a similar reasoning). $\square$

To describe the general structure of the trajectory attractor $\mathfrak{A}$, we have to define the *kernel* of equation (2.1).

A function $y(\cdot) \in C_b(\mathbb{R}; E)$ is said to be a *complete trajectory* of equation (2.1) if $y(t)$ satisfies this equation for all $t \in \mathbb{R}$, that is, speaking formally, the function $y(t+h), t \geq 0$, is a solution of (2.1) for any $h \in \mathbb{R}$. Recall that $y(t)$ is bounded in $C_b(\mathbb{R}; E)$.

DEFINITION 2.10. *The kernel $\mathcal{K}$ in the space $C_b(\mathbb{R}; E)$ is the family of all (bounded) complete trajectories of equation (2.1):*

$$\mathcal{K} = \{y(t), t \in \mathbb{R} \mid y \in C_b(\mathbb{R}; E),\ y(t) \textit{ is a solution of (2.1) for all } t \in \mathbb{R}\}.$$

We denote by Π_- the restriction operator to the semiaxis $\mathbb{R}_-$ which maps a function $y(s), s \in \mathbb{R}$, onto the function $\Pi_- y(s) = y(s), s \leq 0$, with range $\mathbb{R}_- = (-\infty, 0]$.

COROLLARY 2.11. *Under the assumptions of Theorem 2.7, the following identity holds:*

$$\mathfrak{A} = \Pi_- \mathcal{K}, \tag{2.9}$$

where $\mathcal{K}$ is the kernel of equation (2.1). The set $\mathcal{K}$ is bounded in $C_b(\mathbb{R}; E)$ and compact in $C^{\text{loc}}(\mathbb{R}; E)$.

Proof. We note that the kernel $\mathcal{K}$ is invariant with respect to the translation group $\{T(h), h \in \mathbb{R}\}$, that is, if $y(\cdot) \in \mathcal{K}$, then $T(h)y(\cdot) = y(h + \cdot) \in \mathcal{K}$ for all $h \in \mathbb{R}$. Therefore, $\Pi_- \mathcal{K} \subseteq \mathfrak{A}$. The inverse inclusion follows from the invariance of the trajectory attractor $\mathfrak{A}$.

The set $\mathcal{K}$ is bounded in $C_b(\mathbb{R}; E)$ and compact in $C^{\text{loc}}(\mathbb{R}; E)$ since $\mathfrak{A}$ is bounded in $C_b(\mathbb{R}_-; E)$ and compact in $C^{\text{loc}}(\mathbb{R}_-; E)$. $\square$

The following compactness criterion in the space $C^{\text{loc}}(\mathbb{R}_-; E)$ is very useful.

LEMMA 2.12. *A set $B \subset C^{\text{loc}}(\mathbb{R}_-; E)$ is compact in $C^{\text{loc}}(\mathbb{R}_-; E)$ if and only if it is compact in $C([-M, 0]; E)$ for every $M > 0$.*

We also need the

LEMMA 2.13. *Let $E_1 \subset\subset E \subset E'$ be three Banach spaces and let the first embedding be compact. We fix numbers $p > 1$ and $M > 0$. Let a set B be bounded in the space $L_\infty(-M, 0; E_1)$ and let the set $B' = \{\partial_t z \mid z \in B\}$ be bounded in the space $L_p(-M, 0; E')$. Here, $\partial_t z = \partial_t z(t)$ denotes the derivative of the function $z = z(t)$ in the sense of distributions of the space $\mathcal{D}'(-M, 0; E')$ (see [29]). Then the set B is precompact in $C([-M, 0]; E)$.*

For the proof, see, for example, [19].

We now construct the *global attractor* $\mathcal{A}$ in the space E for the autonomous equation with memory of the form (2.1). This attractor plays the role of the global attractor of the equation without memory, where the operator $A(\cdot)$ only depends on the value of an unknown function y at time t (see [13], [14], [19]).

For any set $B \subset \mathcal{E}^- \subseteq C_b(\mathbb{R}_-; E)$, we set

$$B(0) = \{z(0) \mid z \in B\} \subset E.$$

In particular, we may consider the set

$$\mathfrak{A}(0) = \{z(0) \mid z \in \mathfrak{A}\} \subset E.$$

DEFINITION 2.14. *A set $\mathcal{A} \subset E$ is said to be the global (E, E)-attractor of equation (2.1) if*

1. *the set $\mathcal{A}$ is compact in E;*
2. *the set $\mathcal{A}$ attracts the bounded solutions of (2.1), that is, for any bounded set B from the space $\mathcal{E}^-$*
$$\operatorname{dist}_E \left(\left(S(t)B\right)(0), \mathcal{A}\right) \to 0 \ (t \to +\infty); \tag{2.10}$$
3. *$\mathcal{A}$ is the minimal compact attracting set, that is, if a set $\mathcal{A}'$ is compact in E and satisfies (2.10) (with $\mathcal{A}'$ in place of $\mathcal{A}$), then $\mathcal{A} \subseteq \mathcal{A}'$.*

REMARK 2.15. Notice that the global attractor $\mathcal{A}$ is unique if it exists.

It follows from property (2.10) that, if $y(t)$ is a solution of equation (2.1) with initial data $z \in \mathcal{E}^-$, then

$$\operatorname{dist}_E \left(y(t), \mathcal{A}\right) \to 0 \ (t \to +\infty)$$

uniformly w.r.t. $z \in B$, where B is an arbitrary bounded set from $\mathcal{E}^-$. (Recall that, in the above limit relation, we have $y(t) = z(t)$ for all $t \leq 0$.) This is the key property known from the theory of global attractors of equations without memory.

THEOREM 2.16. *Under the assumptions of Theorem 2.7, the set*

$$\mathcal{A} = \mathfrak{A}(0) = \mathcal{K}(0) \tag{2.11}$$

is the global (E, E)-attractor of problem (2.1) and (2.2). Here, $\mathcal{K}$ is the kernel of equation (2.1) in $C_b(\mathbb{R}_-; E)$.

Proof. It follows from Corollary 2.11 that the set $\mathcal{A}$ defined in (2.11) is compact in E, since the kernel $\mathcal{K}$ is compact in $C^{\mathrm{loc}}(\mathbb{R}; E)$. Moreover, for any $M \geq 0$, the set $\mathfrak{A} = \Pi_-\mathcal{K}$ attracts the set $S(t)B$ in $C([-M, 0]; \mathbb{R})$ for any bounded (in $C_b(\mathbb{R}_-; E)$) set $B \subset \mathcal{E}^-$. In particular, this attraction holds for $M = 0$ and we obtain the limit relation (2.10).

It only remains to verify the minimality property *3* from Definition 2.14. We note that the set $\Pi_-\mathcal{K} = \mathfrak{A}$ is strictly invariant with respect to the semigroup $\{S(t)\}$, that is, $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$. Therefore

$$\left(S(t)\mathfrak{A}\right)(0) = \mathfrak{A}(0) \text{ for all } t \geq 0. \tag{2.12}$$

Recall that the set $\mathfrak{A}$ is bounded in $C_b(\mathbb{R}_-; E)$. Let $\mathcal{A}' \subset\subset E$ be a compact attracting set, that is, for any bounded (in $C_b(\mathbb{R}_-; E)$) set B from $\mathcal{E}^-$, we have

$$\operatorname{dist}_E \left(\left(S(t)B\right)(0), \mathcal{A}'\right) \to 0 \ (t \to +\infty).$$

In particular, for the set $B = \mathfrak{A}$

$$\operatorname{dist}_E\left(\left(S(t)\mathfrak{A}\right)(0), \mathcal{A}'\right) \to 0 \ (t \to +\infty).$$

From (2.12), we conclude that

$$\operatorname{dist}_E\left(\mathfrak{A}(0), \mathcal{A}'\right) \to 0 \ (t \to +\infty),$$

hence, $\operatorname{dist}_E\left(\mathfrak{A}(0), \mathcal{A}'\right) = 0$ and $\mathfrak{A}(0) \subseteq \mathcal{A}'$ since $\mathcal{A}'$ is closed. This completes the proof. □

REMARK 2.17. By definition, the global attractor of an equation without memory is invariant with respect to the corresponding semigroup acting in E. By contrast, the global attractor $\mathcal{A}$ of equation (2.1) is not invariant, since we cannot define the semigroup in E and we use the minimality property instead. Such a property is also used in the definition of the global attractor of a non-autonomous equation (see [18, 19]).

We also formulate an analogue of Corollary 2.9.

COROLLARY 2.18. *If the semigroup* $\{S(t)\}$ *is continuous in* $C^{\mathrm{loc}}(\mathbb{R}_-; E)$ *on any bounded set in* $C_b(\mathbb{R}_-; E)$, *then the global attractor* $\mathcal{A}$ *is connected in* E.

The proof follows from formula (2.11) and Corollary 2.9.

3. **On Lyapunov Functions for Equations with Memory.** We need some known facts from the theory of semigroups having global Lyapunov functions. For more details, see, for example, [13], [14], [15]. We give some results with application to equation (2.1). Consider the equation

$$\partial_t y(t) = A(y(t), y^t(\cdot)), \ t \geq 0, \tag{3.1}$$

with initial data

$$y|_{t\leq 0} = z(t), \ t \leq 0. \tag{3.2}$$

As in Section 2, we consider the phase space $\mathcal{E}^- \subseteq C_b(\mathbb{R}_-; E)$, where E is an appropriate Banach space. We assume that problem (3.1) and (3.2) has a unique solution for all $z(\cdot) \in \mathcal{E}^-$. Consider the corresponding semigroup $\{S(t)\}$ acting in $\mathcal{E}^-$ by the formula

$$(S(t)z)(s) = y(t+s), \ t \geq 0, \ s \leq 0, \tag{3.3}$$

where $y(t)$ is the solution of (3.1) and (3.2). We assume that $S(t)$ maps $\mathcal{E}^-$ onto itself for all $t \geq 0$.

DEFINITION 3.1. *A* (*nonlinear*) *functional* $\Phi : \mathcal{E}^- \to \mathbb{R}$ *is called a global Lyapunov function of equation (3.1) if*

$$\Phi\left(S(t)z\right) \leq \Phi\left(z\right) \ \textit{for all } z \in \mathcal{E}^-, \ t \geq 0,$$

and, if $\Phi\left(S(t)z\right) = \Phi\left(z\right)$ *for some* $t > 0$, *then* $z(s)$ *is a stationary solution of (3.1), i.e.* z *is independent of the time* s *and* $A(z, z) = 0$.

Similarly, one defines a global Lyapunov function on an invariant subset $\mathcal{M}$ from $\mathcal{E}^-$, that is, $S(t)\mathcal{M} \subseteq \mathcal{M}$ for all $t \geq 0$.

We denote by $\mathcal{N}$ the set of all stationary points of (3.1), that is, the union of the elements $w \in E$ such that $A(w, w) = 0$.

We consider the kernel $\mathcal{K}$ of equation (3.1) (see Definition 2.10). We denote by $\mathcal{M}^+(\mathcal{N})$ the set

$$\mathcal{M}^+(\mathcal{N}) = \left\{y(\cdot) \in \mathcal{K} \mid \operatorname{dist}_E(y(t), \mathcal{N}) \to 0 \ (t \to -\infty)\right\}, \tag{3.4}$$

that is, $\mathcal{M}^+(\mathcal{N})$ consists of all the bounded complete trajectories of equation (3.1) issuing from the set $\mathcal{N}$. The set $\mathcal{M}^+(\mathcal{N})$ is called the *unstable trajectory set* of the set $\mathcal{N}$. We clearly have the

PROPOSITION 3.2. *If the set $\mathcal{N}$ is finite, $\mathcal{N} = \{w_1, w_2, \ldots, w_N\}$, then*

$$\mathcal{M}^+(\mathcal{N}) = \bigcup_{i=1}^{N} \mathcal{M}^+(w_i), \tag{3.5}$$

where

$$\mathcal{M}^+(w_i) = \{y(\cdot) \in \mathcal{K} \mid \|y(t) - w_i\|_E \to 0 \ (t \to -\infty)\}$$

is the unstable trajectory set of the point w_i.

We conclude from the definition of the set $\mathcal{M}^+(\mathcal{N})$ that

$$\mathcal{M}^+(\mathcal{N}) \subseteq \mathcal{K}. \tag{3.6}$$

Moreover, the set $\mathcal{M}^+(\mathcal{N})$ is strictly invariant with respect to the translation semigroup $\{T(h)\}$, that is,

$$T(h)\mathcal{M}^+(\mathcal{N}) = \mathcal{M}^+(\mathcal{N}), \ \forall h \geq 0.$$

In Section 2, we have studied the case where the sets $\Pi_-\mathcal{K}$ and $\mathcal{K}(0)$ serve as the trajectory and global attractors of equation (3.1) in the spaces $C^{\mathrm{loc}}(\mathbb{R}_-; E)$ and E, respectively. Now the question is: when can we take the equality in (3.6), that is, when are the sets $\Pi_-\mathcal{M}^+(\mathcal{N})$ and $\mathcal{M}^+(\mathcal{N})(0)$ the trajectory and global attractors of equation (3.1), respectively?

THEOREM 3.3. *Assume that the assumptions of Theorem 2.7 hold and that the semigroup $\{S(t)\}$ is continuous (in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$) on any bounded (in $C_b(\mathbb{R}_-; E)$) set from $\mathcal{E}^-$. Let equation (3.1) have the global Lyapunov function Φ on the trajectory attractor $\mathfrak{A}$ and let this function Φ be continuous (in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$) on $\mathfrak{A}$. Then the trajectory attractor $\mathfrak{A}$ and the global attractor $\mathcal{A}$ of equation (3.1) have the forms:*

$$\mathfrak{A} = \Pi_-\mathcal{M}^+(\mathcal{N}), \tag{3.7}$$

$$\mathcal{A} = \mathcal{M}^+(\mathcal{N})(0). \tag{3.8}$$

We omit the proof since it almost repeats the reasoning of a more general theorem from [14].

We recall the following important property of a semigroup $\{S(t)\}$ having a global Lyapunov function.

COROLLARY 3.4. *Under the assumptions of Theorem 3.3, if it is known that Φ is a continuous Lyapunov function on the entire phase space $\mathcal{E}^-$, then, for any function $z \in \mathcal{E}^-$, the corresponding solution $y(t)$ of problem (3.1) and (3.2) satisfies the relation*

$$\mathrm{dist}_E(y(t), \mathcal{N}) \to 0 \ (t \to +\infty). \tag{3.9}$$

In particular, if $\mathcal{N}$ is finite, then there exists $w_j \in \mathcal{N}$ such that

$$\|y(t) - w_j\|_E \to 0 \ (t \to +\infty). \tag{3.10}$$

4. **Dissipative Hyperbolic Equations with Linear Memory; Trajectory and Global Attractors.** We study the following equation:

$$\partial_t^2 u(t) + \gamma \partial_t u(t) \; = \; k(0)\Delta u(t) + \int_0^\infty k'(s)\Delta u(t-s)ds - f(u(t)) + g(x); \tag{4.1}$$

$$u|_{\partial\Omega} \; = \; 0;\; x \in \Omega \subset\subset \mathbb{R}^n,\; t \geq 0,\; u(t) = u(x,t). \tag{4.2}$$

Here, $\gamma > 0,\ \Delta = \Delta_x$ is the laplacian in $\mathbb{R}^n,\ g(x) \in L_2(\Omega)$ and $u(t) = u(x,t)$ is the unknown scalar function depending on x and t. The nonlinear function $f(v), v \in \mathbb{R}$, belongs to the class $C^1(\mathbb{R})$ and satisfies the following inequalities:

$$F(u) \geq -mu^2 - C_m,\ F(u) \quad = \quad \int_0^u f(w)dw, \tag{4.3}$$

$$f(u)u - \gamma_1 F(u) + mu^2 \quad \geq \quad -C_m,\ \forall u \in \mathbb{R}, \tag{4.4}$$

where $\gamma_1 > 0,\ m > 0$, with m fixed and sufficiently small and $C_m > 0$ arbitrarily large. The value of m will be defined later on (see the paragraphs just above (4.33) and (4.38)). Moreover, we assume that

$$|f'(u)| \leq C\left(1 + |u|^\rho\right),\ \text{where} \begin{cases} 0 \leq \rho, & n = 1,2, \\ 0 \leq \rho < 2/(n-2), & n \geq 3. \end{cases} \tag{4.5}$$

Note that conditions (4.3)–(4.5) are standard for the dissipative hyperbolic equations without memory and with a moderate growth of the nonlinear function $f(u)$ (see [13, 14, 15, 18]).

We now consider the integral term which reflects the effects of memory in the model. Concerning the kernel $k(s) \in C^2(\mathbb{R}_+)$, we assume that

$$\left.\begin{array}{l} \mu(s) \stackrel{\text{def}}{\equiv} -k'(s) \geq 0, \\ \mu'(s) = -k''(s) \leq 0, \end{array}\right\} \tag{4.6}$$

$$\mu'(s) + \delta\mu(s) \leq 0,\ \ \forall s \geq 0,\ \delta > 0. \tag{4.7}$$

From (4.6) and (4.7), we conclude that

$$0 \leq \mu(s) \leq \mu(0)e^{-\delta s},\ \forall s \geq 0, \tag{4.8}$$

and, consequently, $k'(s) \in L_1(\mathbb{R}_+)$. Since $k'(s) \leq 0$, we have

$$\beta \stackrel{\text{def}}{\equiv} k(+\infty) = \lim_{s\to+\infty} k(s) < +\infty. \tag{4.9}$$

We assume that

$$\beta = k(+\infty) = \lim_{s\to+\infty} k(s) = \inf_{s\geq 0} k(s) > 0. \tag{4.10}$$

We also assume that

$$k(0) > k(+\infty). \tag{4.11}$$

Otherwise, $k'(s) \equiv 0$ for all $s \geq 0$ and equation (4.1) has no memory and can be treated in a standard way (see [13, 14, 18]). We note that

$$\beta - k(0) = k(+\infty) - k(0) = \int_0^\infty k'(s)ds,$$

that is,

$$k(0) = \beta - \int_0^\infty k'(s)ds,$$

and equation (4.1) can be rewritten in the form

$$\partial_t^2 u(t) + \gamma \partial_t u(t) = \beta \Delta u(t) + \int_0^\infty \mu(s) \left(\Delta u(t) - \Delta u(t-s)\right) ds - f(u(t)) + g(x). \tag{4.12}$$

We now set

$$\theta_u(t,s) = \theta_u(x,t,s) = u(x,t) - u(x,t-s). \tag{4.13}$$

It follows that

$$\begin{aligned} \partial_t \theta_u(t,s) &= \partial_t u(t) - \partial_t u(t-s), \\ \partial_s \theta_u(t,s) &= \partial_t u(t-s). \end{aligned}$$

Therefore

$$\partial_t u(t) = \partial_t \theta_u(t,s) + \partial_s \theta_u(t,s). \tag{4.14}$$

Remark 4.1. This identity will be very important in the sequel. It will help to prove the additional dissipation of the system coming from the memory term of equation (4.1).

Remark 4.2. In [23, 24, 25, 26], the function $\theta_u(\cdot,t,s) = u(\cdot,t) - u(\cdot,t-s)$ plays the role of the additional variable of the past history in the extended phase space.

Using the function $\theta_u(t,s)$, equation (4.12) becomes

$$\partial_t^2 u(t) + \gamma \partial_t u(t) = \beta \Delta u(t) + \int_0^\infty \mu(s) \Delta \theta_u(t,s) ds - f(u(t)) + g(x). \tag{4.15}$$

We now assume that the function $u(t) = u(x,t)$ is known for $t \leq 0$ and is equal to a function $v(t) = v(x,t), t \leq 0$. However, the function $u(t)$ does not necessarily satisfy equation (4.1) for negative t. We are looking for a function $u(t) = u(x,t),\ x \in \Omega,\ t \in \mathbb{R}$, such that $u(t) = v(t)$ for all $t \leq 0$ and $u(t)$ satisfies (4.1) (or (4.15)) for all $t \geq 0$.

We then define a solution of the Cauchy problem for equation (4.1). We associate with (4.1) the initial data

$$u|_{t\leq 0} = v(t),\ \partial_t u|_{t \leq 0} = \partial_t v(t). \tag{4.16}$$

(Note that the first identity implies the second.) We assume that the function $v(\cdot) \in C_b(\mathbb{R}_-; H_0^1(\Omega))$ and $\partial_t v(\cdot) \in C_b(\mathbb{R}_-; L_2(\Omega))$.

Let us be given an arbitrary function $u \in C_b(\mathbb{R}; H_0^1(\Omega))$ with time derivative $\partial_t u \in C_b(\mathbb{R}; L_2(\Omega))$. It follows from (4.5) that

$$|f(u)| \leq C\left(1 + |u|^{\rho+1}\right),\ \forall u \in \mathbb{R}. \tag{4.17}$$

Besides, the Sobolev embedding theorem states that $H_0^1(\Omega) \subset L_{2(\rho+1)}(\Omega)$ (since $2(\rho+1) < 2n/(n-2)$ for $n \geq 3$ and ρ is arbitrary positive for $n = 1, 2$, see (4.5)). Therefore, $f(u(x,t)) \in L_\infty(\mathbb{R}; L_2(\Omega))$. At the same time, $\Delta u \in L_\infty(\mathbb{R}; H^{-1}(\Omega))$, where $H^{-1}(\Omega)$ is the dual of the space $H_0^1(\Omega)$. Using inequalities (4.8), we obtain

$$\xi_u(x,t) = \int_0^\infty \mu(s) \Delta \theta_u(t,s) ds \in L_\infty(\mathbb{R}; H^{-1}(\Omega)).$$

Hence, we conclude that all the terms of equation (4.15) (except $\partial_t^2 u(t)$) belong to the space $L_\infty(\mathbb{R}; H^{-1}(\Omega))$. Thus, the equation has a sense in the space of distributions $\mathcal{D}'(\mathbb{R}_+; H^{-1}(\Omega))$ and $\partial_t^2 u \in C_b(\mathbb{R}_+; H^{-1}(\Omega))$.

Definition 4.3. *A function* $u \in C_b(\mathbb{R}; H_0^1(\Omega))$, $\partial_t u \in C_b(\mathbb{R}; L_2(\Omega))$, *is called a solution of problem (4.15) and (4.16) for* $t \geq 0$ *if* $u(t) = v(t)$ *for all* $t \leq 0$ *(thus,* $\partial_t u(t) = \partial_t v(t)$ *for* $t \leq 0$ *as well) and* $u(t)$ *satisfies (4.15) in the space* $\mathcal{D}'(\mathbb{R}_+; H^{-1}(\Omega))$ *(see* [29], [13]*).*

Theorem 4.4. *Under all the above assumptions, problem (4.15) and (4.16) has a unique solution* $u(t) \in C_b(\mathbb{R}; H_0^1(\Omega))$ *and* $\partial_t u \in C_b(\mathbb{R}; L_2(\Omega))$ *for* $t \geq 0$.

We shall consider the main elements of the proof of Theorem 4.4 later on. It relies on the Faedo–Galerkin approximation method.

We introduce a vector function $y(t) = (u(t), \partial_t u(t)), t \in \mathbb{R}$. Consider the energy space $E = H_0^1(\Omega) \times L_2(\Omega)$ of vector functions $y(x) = (u(x), p(x)), x \in \Omega$, with norm

$$\|y\|_E = \left(|\nabla u|^2 + |p|^2\right)^{1/2} = \left(\int_\Omega |\nabla u(x)|^2 dx + \int_\Omega |p(x)|^2 dx\right)^{1/2}.$$

We denote by $|u| = \left(\int_\Omega |u(x)|^2 dx\right)^{1/2}$ the norm in $L_2(\Omega)$. Then equation (4.15) is equivalent to the following system:

$$\begin{aligned} \partial_t y(t) &= A(y(t), y^t(\cdot)), \\ y|_{t\leq 0} &= z(t),\ t \leq 0, \end{aligned} \tag{4.18}$$

where $z(s) = (v(s), \partial_t v(s)), s \leq 0$, and

$$A(y(t), y^t(\cdot)) = \left(\partial_t u(t), -\gamma \partial_t u(t) + \beta \Delta u(t) + \int_0^\infty \mu(s) \Delta \theta_u(t,s) ds - f(u(t)) + g(x)\right).$$

It is clear that $z(\cdot) \in C_b(\mathbb{R}_-; E)$. The function $y(t) \in C_b(\mathbb{R}; E)$ is called a solution of problem (4.18) for $t \geq 0$, where $u(t)$ is the solution of problem (4.15) and (4.16).

Therefore, equation (4.15) generates a semigroup $\{S(t)\}$ acting in the space

$$\mathcal{E}^- = \left\{z(s) = (v(s), \partial_t v(s)) \mid v \in C_b(\mathbb{R}_-; H_0^1(\Omega)),\ \partial_t v \in C_b(\mathbb{R}_-; L_2(\Omega))\right\}$$

by the formula

$$(S(t)z)(s) = y(t+s),\ s \leq 0,\ \forall t \geq 0.$$

It is obvious that $\mathcal{E}^-$ is a vector subspace of $C_b(\mathbb{R}_-; E)$. We will construct the trajectory and global attractors of the semigroup $\{S(t)\}$ acting in $\mathcal{E}^-$.

Let a function $y(t) = (u(t), \partial_t u(t))$ belong to $C_b(\mathbb{R}; E)$, where $u \in C_b(\mathbb{R}; H_0^1(\Omega))$ and $\partial_t u \in C_b(\mathbb{R}; L_2(\Omega))$. Consider the following functional:

$$\begin{aligned} \psi_\alpha(t) &= |\partial_t u(t) + \alpha u(t)|^2 + \beta |\nabla u(t)|^2 + 2\int_\Omega F(u(x)) dx \\ &\quad + \int_0^\infty \mu(s) |\nabla \theta_u(t,s)|^2 ds,\ t \in \mathbb{R}. \end{aligned} \tag{4.19}$$

Here, α is a fixed positive number. Recall that $\theta_u(t,s) = u(t) - u(t-s),\ t \in \mathbb{R},\ s \geq\ 0$.

Proposition 4.5. *If* $y(t)$ *is a solution of problem (4.18) for* $t \geq 0$, *then*

$$\psi_\alpha(t) \leq \psi_\alpha(0) e^{-\kappa t} + C_1,\ \forall t \geq 0, \tag{4.20}$$

where $\kappa > 0$, α *is sufficiently small and* C_1 *is independent of* y.

Proof. We assume for simplicity that $u \in C^2(\mathbb{R}_+; H_0^1(\Omega))$. Nevertheless, all the transformations remain true for the general case. We shall clarify this point later on. We fix a positive number α. We now rewrite equation (4.15) in the following form:

$$\begin{aligned}&\partial_t(\partial_t u(t) + \alpha u(t)) + (\gamma - \alpha)\,(\partial_t u(t) + \alpha u(t)) - \alpha(\gamma - \alpha)u(t)\\ =\quad &\beta\Delta u(t) + \int_0^\infty \mu(s)\Delta\theta_u(t,s)ds - f(u(t)) + g(x).\end{aligned} \tag{4.21}$$

Taking the scalar product in $L_2(\Omega)$ with $\eta(t) = \partial_t u(t) + \alpha u(t)$, we obtain, using the standard integration by parts

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{|\eta(t)|^2 + \beta|\nabla u(t)|^2 + 2\int_\Omega F(u(x,t))dx\right\} + (\gamma - \alpha)|\eta(t)|^2 + \beta\alpha|\nabla u(t)|^2\\ &-\alpha(\gamma - \alpha)(u(t), \eta(t)) + \alpha(f(u(t)), u(t)) = \int_0^\infty \mu(s)\left(\Delta\theta_u(t,s), \partial_t u(t)\right)ds\\ &+\alpha\int_0^\infty \mu(s)\left(\Delta\theta_u(t,s), u(t)\right)ds + (g, \eta(t)).\end{aligned} \tag{4.22}$$

Here, we have used the equality

$$\int_\Omega f(u(x,t))\partial_t u(x,t)dx = \frac{d}{dt}\int_\Omega F(u(x,t))dx, \tag{4.23}$$

which is easy to verify by a regularization method.

We now consider the first integral in the right-hand side of equality (4.22). Using identity (4.14), we have

$$\begin{aligned}&\int_0^\infty \mu(s)\left(\Delta\theta_u(t,s), \partial_t u(t)\right)ds = \int_0^\infty \mu(s)\left(\Delta\theta_u(t,s), \partial_t\theta_u(t,s) + \partial_s\theta_u(t,s)\right)ds\\ &= \int_0^\infty \mu(s)(\Delta\theta_u, \partial_t\theta_u)ds + \int_0^\infty \mu(s)(\Delta\theta_u, \partial_s\theta_u)ds\\ &= -\frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(s)|\nabla_x\theta_u(t,s)|^2ds - \frac{1}{2}\int_0^\infty \mu(s)\frac{d}{ds}|\nabla_x\theta_u(t,s)|^2ds.\end{aligned} \tag{4.24}$$

Here, we have integrated by parts in x (recall that $\theta_u|_{\partial\Omega} = 0$ and the function θ_u is sufficiently smooth w.r.t. s, $\theta_u \in C_b(\mathbb{R}; H_0^1(\Omega))$).

In the second integral in (4.24), we integrate by parts in s. Recall that $\mu(+\infty) = 0$ (see (4.8)) and $\theta_u(t,0) = u(t) - u(t) = 0$. We obtain from (4.24)

$$\begin{aligned}&\int_0^\infty \mu(s)\left(\Delta\theta_u(t,s), \partial_t u(t)\right)ds\\ =\quad &-\frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(s)|\nabla_x\theta_u(t,s)|^2ds + \frac{1}{2}\int_0^\infty \mu'(s)|\nabla_x\theta_u(t,s)|^2ds.\end{aligned} \tag{4.25}$$

We inject this term into (4.22) and obtain, integrating by parts in the second integral in the right-hand side of (4.22),

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{|\eta(t)|^2 + \beta|\nabla u(t)|^2 + 2\int_\Omega F(u(x,t))dx + \int_0^\infty \mu(s)|\nabla_x\theta_u(t,s)|^2ds\right\}\\ &+(\gamma - \alpha)|\eta(t)|^2 + \alpha\beta|\nabla u(t)|^2 + \alpha(f(u(t)), u(t)) - \frac{1}{2}\int_0^\infty \mu'(s)|\nabla_x\theta_u(t,s)|^2ds\\ &-\alpha(\gamma - \alpha)(u(t), \eta(t)) = -\alpha\int_0^\infty \mu(s)\left(\nabla_x\theta_u(t,s), \nabla_x u(t)\right)ds + (g, \eta(t)).\end{aligned} \tag{4.26}$$

From (4.4), it follows that

$$(f(u),u) \geq \gamma_1 \int_\Omega F(u(x))dx - m|u|^2 - C_m|\Omega|. \tag{4.27}$$

Besides, we have the following elementary inequalities:

$$\alpha(\gamma-\alpha)(u,\eta) \leq \frac{\gamma-\alpha}{4}|\eta|^2 + (\gamma-\alpha)\alpha^2|u|^2, \tag{4.28}$$

$$(g,\eta) \leq \frac{\gamma-\alpha}{4}|\eta|^2 + (\gamma-\alpha)^{-1}|g|^2. \tag{4.29}$$

Finally, we also obtain

$$\begin{aligned} -\alpha \int_0^\infty \mu(s)\left(\nabla_x\theta_u, \nabla_x u(t)\right)ds &\leq \alpha \int_0^\infty \mu(s)|\nabla_x\theta_u|\cdot|\nabla_x u(t)|ds \\ \leq \frac{\alpha}{2}\left(\frac{\mu_0}{\beta}\right)\int_0^\infty \mu(s)|\nabla_x\theta_u|^2ds &+ \frac{\alpha}{2}\left(\frac{\beta}{\mu_0}\right)\int_0^\infty \mu(s)|\nabla_x u(t)|^2 ds \\ = \left(\frac{\alpha\mu_0}{2\beta}\right)\int_0^\infty \mu(s)|\nabla_x\theta_u|^2ds &+ \left(\frac{\alpha\beta}{2}\right)|\nabla_x u(t)|^2, \end{aligned} \tag{4.30}$$

where we have set

$$\mu_0 = \int_0^\infty \mu(s)ds = -\int_0^\infty k'(s)ds = k(0) - k(+\infty) = k(0) - \beta > 0$$

(see (4.11)).

Recall that $-\mu'(s) \geq \delta\mu(s) \geq 0$ for all $s \geq 0$ (see (4.7) and (4.8)). Using this fact, we now inject inequalities (4.27)–(4.30) into (4.26) and we obtain

$$\begin{aligned} &\frac{d}{dt}\left\{|\eta(t)|^2 + \beta|\nabla u(t)|^2 + 2\int_\Omega F(u(x,t))dx + \int_0^\infty \mu(s)|\nabla_x\theta_u(t,s)|^2ds\right\} \\ &+(\gamma-\alpha)|\eta(t)|^2 + \alpha\beta|\nabla u(t)|^2 + 2\gamma_1\alpha\int_\Omega F(u(x,t))dx - 2\alpha m|u(t)|^2 \\ &-2\alpha C_m|\Omega| + \left(\delta - \frac{\alpha\mu_0}{\beta}\right)\int_0^\infty \mu(s)|\nabla_x\theta_u(t,s)|^2ds - 2(\gamma-\alpha)\alpha^2|u(t)|^2 \\ &\leq 2(\gamma-\alpha)^{-1}|g|^2. \end{aligned} \tag{4.31}$$

Using notation (4.19), we rewrite (4.31) as follows:

$$\begin{aligned} &\frac{d}{dt}\psi_\alpha(t) + (\gamma-\alpha)|\eta(t)|^2 + \alpha\beta|\nabla u(t)|^2 \\ &+2\gamma_1\alpha\left\{\int_\Omega F(u(x,t))dx + m|u(t)|^2 + C_m|\Omega|\right\} \\ &+\left(\delta - \frac{\alpha\mu_0}{\beta}\right)\int_0^\infty \mu(s)|\nabla_x\theta_u(t,s)|^2ds - 2\alpha\left(m + (\gamma-\alpha)\alpha + \gamma_1 m\right)|u(t)|^2 \\ &\leq 2(\gamma-\alpha)^{-1}|g|^2 + 2\alpha C_m|\Omega| + 2\gamma_1\alpha C_m|\Omega|. \end{aligned} \tag{4.32}$$

Let λ_1 be the first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions. We now choose the positive numbers m and α small enough so that

$$\kappa = \min\left\{\gamma-\alpha,\ \gamma_1\alpha,\ \delta - \alpha\mu_0\beta^{-1},\ \alpha\lambda_1^{-1}\left[\lambda_1 - 2\beta^{-1}(m + (\gamma-\alpha)\alpha + \gamma_1 m)\right]\right\} > 0.$$

Since

$$|u|^2 \leq \lambda_1^{-1}|\nabla u|^2, \tag{4.33}$$

inequality (4.32) implies that

$$\begin{aligned}\frac{d}{dt}\psi_\alpha(t) &+& \kappa\Big\{|\eta(t)|^2+\beta|\nabla u(t)|^2+2\left(\int_\Omega F(u(x,t))dx+m|u(t)|^2+C_m|\Omega|\right)\\ &+& \int_0^\infty \mu(s)|\nabla_x\theta_u(t,s)|^2ds\Big\}\le C_2, \qquad (4.34)\end{aligned}$$

where

$$C_2=2(\gamma-\alpha)^{-1}|g|^2+2\alpha C_m|\Omega|+2\gamma_1\alpha C_m|\Omega|.$$

We finally obtain

$$\frac{d}{dt}\psi_\alpha(t)+\kappa\psi_\alpha(t)\le C_2. \qquad (4.35)$$

Integrating (4.35) in t, we deduce (4.20). □

PROPOSITION 4.6. *The semigroup $\{S(t)\}$ corresponding to problem (4.18) and acting in the space $\mathcal{E}^-\subset C_b(\mathbb{R}_-;E)$ is uniformly bounded and the estimate*

$$\|y(t)\|_E^2\le C_3\|z\|^{\rho+2}_{C_b(\mathbb{R}_-;E)}e^{-\kappa t}+C_4,\ t\ge 0, \qquad (4.36)$$

holds, where $y(t)=(u(t),\partial_t u(t))$.

Proof. We observe that the norm $\|y\|_E^2=|\nabla u|^2+|p|^2$ in $E=H_0^1(\Omega)\times L_2(\Omega)$ is equivalent to the norm $|y|_E^2=|\nabla u|^2+|p+\alpha u|^2$.

Using (4.3) and (4.33), we obtain

$$2\int_\Omega F(u(x))dx\ge -2m|u|^2-2C_m|\Omega|\ge -2m\lambda_1^{-1}|\nabla u|^2-2C_m|\Omega|. \qquad (4.37)$$

Suppose that $m\le\beta\lambda_1/4$. Taking into account (4.37), we have

$$\begin{aligned}\psi_\alpha(t) &\ge& |\partial_t u(t)+\alpha u(t)|^2+\beta|\nabla u(t)|^2+2\int_\Omega F(u(t,x))dx\\ &\ge& |\partial_t u(t)+\alpha u(t)|^2+\frac{\beta}{2}|\nabla u(t)|^2-2C_m|\Omega|\\ &\ge& \min\{1,\beta/2\}|y(t)|_E^2-2C_m|\Omega|. \qquad (4.38)\end{aligned}$$

We obtain, in view of (4.17)

$$2\int_\Omega F(u(x))dx\le C_5\left(\int_\Omega |u(x)|^{\rho+2}dx+1\right)\le C_6\left(|\nabla u|^{\rho+2}+1\right),$$

since $H_0^1(\Omega)\subset L_{\rho+2}(\Omega)$ for $\rho+2<(2n-2)/(n-2)<2n/(n-2)$. Hence

$$\begin{aligned}\psi_\alpha(0) &=& |\partial_t u(0)+\alpha u(0)|^2+\beta|\nabla u(0)|^2+2\int_\Omega F(u(0,x))dx\\ && +\int_0^\infty \mu(s)|\nabla\theta_u(0,s)|^2ds\le C_7\left(|y(0)|_E^{\rho+2}+1\right)\\ && +\int_0^\infty \mu(s)|\nabla\theta_u(0,s)|^2ds\le C_8\left(|z|^{\rho+2}_{C(\mathbb{R}_-;E)}+1\right), \qquad (4.39)\end{aligned}$$

since

$$\int_0^\infty \mu(s)|\nabla\theta_u(0,s)|^2=\int_0^\infty \mu(s)|\nabla\left(v(0)-v(-s)\right)|^2ds\le C_9|v|^2_{C(\mathbb{R}_-;H_0^1(\Omega))}.$$

Relations (4.38), (4.39), and (4.20) imply that

$$
\begin{aligned}
|y(t)|_E^2 &\le C_{10}\psi_\alpha(t) + C_{11} \le C_{10}\left(\psi_\alpha(0)e^{-\kappa t} + C_1\right) + C_{11} \qquad (4.40)\\
&\le C_{10}\left(C_8\left(|z|^{\rho+2}_{C(\mathbb{R}_-;E)} + 1\right)e^{-\kappa t} + C_1\right) + C_{11} = C_3\|z\|^{\rho+2}_{C_b(\mathbb{R}_-;E)}e^{-\kappa t} + C_4.
\end{aligned}
$$

□

Corollary 4.7. *The semigroup $\{S(t)\}$ generated by problem (4.18) has a bounded (in the space $C_b(\mathbb{R}_-;E)$) attracting (in $C^{\mathrm{loc}}(\mathbb{R}_-;E)$) set.*

It follows from (4.36) that the set

$$
P_0 = \{z(\cdot) \in C_b(\mathbb{R}_-;E) \mid \|z(\cdot)\|_{C_b(\mathbb{R}_-;E)} \le R^*\}, \qquad (4.41)
$$

where $R^* = 2C_4$, is the required bounded attracting set. Note that this set *is not absorbing* for the semigroup $\{S(t)\}$ acting in $\mathcal{E}^- \subset C_b(\mathbb{R}_-;E)$ because the elements $z(\cdot)$ of $\mathcal{E}^-$ have "tails" and the norm (in $C_b(\mathbb{R}_-;E)$) of these tails must be included in the norm of the trajectory $S(t)z(\cdot)$ measured in $C_b(\mathbb{R}_-;E)$. So, we clearly have

$$
\|S(t)z(\cdot)\|_{C_b(\mathbb{R}_-;E)} \ge \|z(\cdot)\|_{C_b(\mathbb{R}_-;E)}
$$

and the semigroup $\{S(t)\}$ has no bounded (in $C_b(\mathbb{R}_-;E)$) absorbing set. However, the weights of the tails are forgotten in the local topology $C^{\mathrm{loc}}(\mathbb{R}_-;E)$ and the trajectories $S(t)z(\cdot)$ tend to P_0 as $t \to +\infty$ uniformly with respect to $z(\cdot) \in P$, where P is any fixed bounded (in $C_b = C_b(\mathbb{R}_-;E)$) set of initial data. This phenomenon reflects the main difference between the dissipative hyperbolic equations with and without memory. The semigroup corresponding to the equation without memory always has bounded (in E) absorbing sets (see [13, 14, 15, 18]).

We now sketch the proof of Theorem 4.4. It can be done similarly to the proof of the corresponding theorem for the hyperbolic equation without memory (see [29, 13, 14, 15]).

We apply the standard Faedo–Galerkin method, using the basis of eigenfunctions $w_j(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ of the Laplace operator $-\Delta$ with zero boundary conditions, $-\Delta w_j(x) = \lambda_j w_j(x),\ w_j|_{\partial\Omega} = 0,\ \lambda_j > 0,\ \lambda_j \nearrow +\infty\ (j \to +\infty)$. For each $m \in \mathbb{N}$, we are looking for a solution $u_m = u_m(x,t)$ of the form

$$
u_m(x,t) = \sum_{j=1}^m a_{jm}(t)w_j(x) \qquad (4.42)
$$

of the system of ordinary differential equations with memory

$$
\begin{aligned}
&\left(\frac{d^2u_m}{dt^2}, w_j\right) + \gamma\left(\frac{du_m}{dt}, w_j\right) = (k(0)\Delta u_m, w_j) \qquad (4.43)\\
&\quad + \int_0^\infty (k'(s)\Delta u_m(t-s), w_j)\,ds - (f(u_m), w_j) + (g(x), w_j)\,,\ j = 1, \dots, m,
\end{aligned}
$$

with initial data

$$
u_m|_{t\le 0} = v_m(t),\ \partial_t u_m|_{t\le 0} = \partial_t v_m(t), \qquad (4.44)
$$

where

$$
(v_m(t), w_j) = (v(t), w_j),\ j = 1, \dots, m,\ t \le 0,
$$

that is,

$$
v_m(x,t) = \sum_{j=1}^m (v(t), w_j)\,w_j(x),\ t \le 0.
$$

We note that

$$\begin{aligned} v_m(t) &\to v(t) \text{ in } H_0^1(\Omega), \\ \partial_t v_m(t) &\to \partial_t v(t) \text{ in } L_2(\Omega),\ \forall t \le 0, \end{aligned}$$

and

$$v_m(t) \to v(t) \text{ strongly in } L_2^{\mathrm{loc}}(\mathbb{R}_-; H_0^1(\Omega)). \tag{4.45}$$

It is easy to prove the existence of a solution $u_m(t)$ of problem (4.43) and (4.44) on an interval $[0, T_m)$ for some maximal $T_m > 0$. In particular,

$$u_m(t) \in C^2([0, T_m); H^2(\Omega) \cap H_0^1(\Omega)).$$

The memory integral in (4.43) does not cause any difficulty. Then we prove that the constructed solution satisfies identity (4.22) on the interval $[0, T_m)$, with $u(x,t)$ replaced by $u_m(x,t)$. Furthermore, the corresponding function $\psi_m(t)$ (see (4.19)) satisfies inequality (4.20) and, therefore, $T_m = \infty$. Besides, the sequence of functions $y_m(t) = (u_m(t), \partial_t u_m(t))$ is bounded in the space $L_\infty(\mathbb{R}; E)$, where $E = H_0^1(\Omega) \times L_2(\Omega)$. Consequently, the sequence $\partial_t^2 u_m(t)$ is bounded in $L_\infty(\mathbb{R}_+; H^{-1}(\Omega))$. Passing to the limit, we may assume that

$$\begin{aligned} u_m(t) &\to u(t)\ *\text{-weakly in } L_\infty(-M, M; H_0^1(\Omega)), \\ \partial_t u_m(t) &\to \partial_t u(t)\ *\text{-weakly in } L_\infty(-M, M; L_2(\Omega)), \text{ and} \\ \partial_t^2 u_m(t) &\to \partial_t^2 u(t)\ *\text{-weakly in } L_\infty(0, M; H^{-1}(\Omega)) \text{ as } m \to \infty, \end{aligned}$$

for any $M > 0$, where the function $y(t) = (u(t), \partial_t u(t)) \in L_\infty(\mathbb{R}; E)$. Then, by a Sobolev embedding theorem,

$$u_m(t) \to u(t) \text{ strongly in } L_2(0, M; H_0^1(\Omega)) \text{ as } m \to \infty$$

and

$$f(u_m(t)) \to f(u(t)) \text{ weakly in } L_2(0, M; H_0^1(\Omega)) \text{ as } m \to \infty.$$

There remains to pass to the limit in equation (4.43) and to prove that $u(x,t)$ satisfies equation (4.1) in the space $D'(\mathbb{R}_+; H^{-1}(\Omega))$. Recall that the convergence (4.45) of the "tails" is used in this reasoning. The passage to the limit in the linear terms of equation (4.43) can be done because the corresponding linear differential operators are continuous in the distribution space $D'(\mathbb{R}_+; H^{-1}(\Omega))$ (see [29]). Note that it follows from the following lemma that $y(t) \in C_w(\mathbb{R}_+; E)$.

Lemma 4.8. *Let X and Y be two Banach spaces such that $X \subset Y$. If a function $\phi \in L^\infty(0, T; X)$ is such that $\phi \in C_w(0, T; Y)$, then $\phi \in C_w(0, T; X)$.*

See [30, 31] for the proof.

To prove the uniqueness result and the strong continuity of the solution $y(t)$ of equation (4.1), we use the following lemma which generalizes Lemma II.4.1 from [13].

Lemma 4.9. *Consider a function $w(t) \in L_2^{\mathrm{loc}}(\mathbb{R}; H_0^1(\Omega))$ such that the derivative $\partial_t w(t) \in L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega))$ and $\partial_t^2 w(t) \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H^{-1}(\Omega))$. We assume that*

$$\partial_t^2 w(t) - \beta \Delta w(t) - \int_0^\infty \mu(s) \Delta \left(w(t) - w(t-s) \right) ds \in L_2^{\mathrm{loc}}(\mathbb{R}_+; L_2(\Omega)).$$

Then the function

$$\varphi(t) = |\partial_t w(t)|^2 + \beta |\nabla w(t)|^2 + \int_0^\infty \mu(s) |\nabla w(t) - \nabla w(t-s)|^2 ds,\ t \ge 0,$$

is absolutely continuous and the following identity holds:

$$\frac{1}{2}\frac{d}{dt}\left\{|\partial_t w(t)|^2 + \beta|\nabla w(t)|^2 + \int_0^\infty \mu(s)|\nabla w(t) - \nabla w(t-s)|^2 ds\right\}$$
$$= \left(\partial_t^2 w(t) - \beta\Delta w(t) - \int_0^\infty \mu(s)\Delta\left(w(t) - w(t-s)\right) ds, \partial_t w(t)\right)$$
$$+\frac{1}{2}\int_0^\infty \mu'(s)|\nabla w(t) - \nabla w(t-s)|^2 ds \quad \text{for } t \geq 0. \tag{4.46}$$

Here, $\mu(\cdot) \in C^1(\mathbb{R}_+;\mathbb{R}_+)$ *and satisfies (4.6) – (4.9).*

Proof. Obviously, (4.46) holds if $w \in C^2(\mathbb{R}; H_0^1(\Omega))$. In the general case, together with the function w, we consider the functions

$$w_\varepsilon(t) = \rho_\varepsilon * w(t), \tag{4.47}$$

where $\rho(t) \in C_0^\infty(]-1,1[)$, $\rho(t) \geq 0$, $\rho(t) = \rho(-t)$, $\int_{-1}^1 \rho(s)ds = 1$ and $\rho_\varepsilon(t) = \varepsilon^{-1}\rho(t/\varepsilon)$. From the regularization theory, it is known that $w_\varepsilon(t) \in C^\infty(\mathbb{R}; H_0^1(\Omega))$ and $\partial_t\left(w_\varepsilon(t)\right) = \left(\partial_t w\right)_\varepsilon(t)$ for all $t \in \mathbb{R}$. Finally, it is easy to prove that

$$\begin{aligned} w_\varepsilon(t) &\to w(t) \text{ strongly in } L_2^{\mathrm{loc}}(\mathbb{R}; H_0^1(\Omega)),\\ \partial_t w_\varepsilon(t) &\to \partial_t w(t) \text{ strongly in } L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)),\end{aligned}$$

and

$$\begin{aligned} \partial_t^2 w_\varepsilon(t) - \beta\Delta w_\varepsilon(t) &- \int_0^\infty \mu(s)\Delta\left(w_\varepsilon(t) - w_\varepsilon(t-s)\right) ds\\ &\to \partial_t^2 w(t) - \beta\Delta w(t) - \int_0^\infty \mu(s)\Delta\left(w(t) - w(t-s)\right) ds\\ &\quad \text{strongly in } L_2^{\mathrm{loc}}(\mathbb{R}_+; L_2(\Omega)).\end{aligned}$$

The function w_ε satisfies identity (4.46) for $t \geq 0$, that is,

$$\frac{1}{2}\frac{d}{dt}\left\{|\partial_t w_\varepsilon(t)|^2 + \beta|\nabla w_\varepsilon(t)|^2 + \int_0^\infty \mu(s)|\nabla w_\varepsilon(t) - \nabla w_\varepsilon(t-s)|^2 ds\right\}$$
$$= \left(\partial_t^2 w_\varepsilon(t) - \beta\Delta w_\varepsilon(t) - \int_0^\infty \mu(s)\Delta\left(w_\varepsilon(t) - w_\varepsilon(t-s)\right) ds, \partial_t w_\varepsilon(t)\right)$$
$$+\frac{1}{2}\int_0^\infty \mu'(s)|\nabla\left(w_\varepsilon(t) - w_\varepsilon(t-s)\right)|^2 ds,$$

in which we can pass to the limit in the distribution space $D'(\mathbb{R}_+)$, according to the above limit relations, and obtain (4.46) for the function $w(t)$ itself. □

Coming back to the proof of Theorem 4.4, we note that the function

$$\psi(t) = |\partial_t u(t)|^2 + \beta|\nabla u(t)|^2 + \int_0^\infty \mu(s)|\nabla u(t) - \nabla u(t-s)|^2 ds$$

is continuous for $t \geq 0$. Since the function $u(t)$ is weakly continuous in $H_0^1(\Omega)$ and $\partial_t u(t)$ is weakly continuous in $L_2(\Omega)$, it follows easily that these functions are strongly continuous for $t \geq 0$ in $H_0^1(\Omega)$ and $L_2(\Omega)$, respectively.

Using Lemma 4.9, we prove the uniqueness of the solution of problem (4.18). Let us be given two solutions $u_1(t)$ and $u_2(t)$ of equation (4.1) such that $u_1(t) = u_2(t)$

for all $t \le 0$. Then the difference $w(t) = u_1(t) - u_2(t)$ satisfies the following equation for $t \ge 0$:

$$\begin{aligned} \partial_t^2 w(t) + \gamma \partial_t w(t) &= \beta \Delta w(t) + \int_0^\infty \mu(s) \left(\Delta w(t) - \Delta w(t-s) \right) ds \\ &\quad - \left(f(u_1(t)) - f(u_2(t)) \right). \end{aligned} \tag{4.48}$$

Recall that $w \in C_b(\mathbb{R}; H_0^1(\Omega))$, $\partial_t w \in C_b(\mathbb{R}; L_2(\Omega))$ and that it follows from (4.3) that $f(u_1(t)) - f(u_2(t)) \in L_\infty(\mathbb{R}; L_2(\Omega))$. Therefore,

$$\partial_t^2 w(t) + \gamma \partial_t w(t) - \beta \Delta w(t) - \int_0^\infty \mu(s) \left(\Delta w(t) - \Delta w(t-s) \right) ds \in L_\infty(\mathbb{R}_+; L_2(\Omega)).$$

We apply Lemma 4.9 and have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\{ |\partial_t w(t)|^2 + \beta |\nabla w(t)|^2 + \int_0^\infty \mu(s) |\nabla w(t) - \nabla w(t-s)|^2 ds \right\} + \gamma |\partial_t w(t)|^2 \\ &- \frac{1}{2} \int_0^\infty \mu'(s) |\nabla w(t) - \nabla w(t-s)|^2 ds = - \left(f(u_1(t)) - f(u_2(t)), \partial_t w \right) \end{aligned} \tag{4.49}$$

for all $t \ge 0$. Besides, all the functions in the left and the right-hand sides of (4.49) belong to the space $L_1^{\mathrm{loc}}(\mathbb{R}_+)$.

It follows from (4.5) that

$$\begin{aligned} |f(u_1) - f(u_2)|^2 &= \int_\Omega |f(u_1(x)) - f(u_2(x))|^2 dx \\ &\le C \int_\Omega \left(1 + |u_1(x)|^{2\rho} + |u_2(x)|^{2\rho} \right) |u_1(x) - u_2(x)|^2 dx \\ &\le C \left(\int_\Omega \left(1 + |u_1(x)|^{\rho n} + |u_2(x)|^{\rho n} \right) dx \right)^{\frac{2}{n}} \left(\int_\Omega |u_1(x) - u_2(x)|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \\ &\le C \left(1 + \|u_1\|_{L_{\rho n}(\Omega)} + \|u_2\|_{L_{\rho n}(\Omega)} \right)^{2\rho} \|u_1 - u_2\|^2_{H_0^1(\Omega)}. \end{aligned} \tag{4.50}$$

Here, we have used the Hölder inequality with $p = n/2$, $q = n/(n-2)$, and the Sobolev embedding theorem ($H_0^1(\Omega) \subset L_{\rho n}(\Omega)$, since $\rho < 2/(n-2)$) for $n \ge 3$ (the cases $n = 1, 2$ are simpler). Recall that u_1 and u_2 are bounded in $L_\infty(\mathbb{R}; H_0^1(\Omega))$. Therefore,

$$|f(u_1) - f(u_2)| \le C_1 |\nabla w|$$

and the right-hand side of (4.49) does not exceed

$$|f(u_1(t)) - f(u_2(t))| \cdot |\partial_t w| \le C_1 |\nabla w| |\partial_t w| \le \frac{\gamma}{2} |\partial_t w|^2 + D |\nabla w|^2,$$

where $D = \frac{C_1^2}{2\gamma}$. We deduce from (4.49) and (4.7) the inequality

$$\begin{aligned} &\frac{d}{dt} \left\{ |\partial_t w(t)|^2 + \beta |\nabla w(t)|^2 + \int_0^\infty \mu(s) |\nabla \left(w(t) - w(t-s) \right)|^2 ds \right\} + \gamma |\partial_t w(t)|^2 \\ &+ \delta \int_0^\infty \mu(s) |\nabla \left(w(t) - w(t-s) \right)|^2 ds \le 2D |\nabla w(t)|^2 \text{ (see also (4.7)).} \end{aligned} \tag{4.51}$$

Integrating (4.51), we obtain the inequality

$$\psi_0(t) \le \psi_0(0) + 2D\beta^{-1} \int_0^t \psi(s) ds, \tag{4.52}$$

where

$$\psi_0(t) = |\partial_t w(t)|^2 + \beta|\nabla w(t)|^2 + \int_0^\infty \mu(s)|\nabla\left(w(t) - w(t-s)\right)|^2 ds.$$

Note that $\psi_0(0) = 0$, since $w(s) = 0$ for all $s \le 0$. We finally conclude from the Gronwall inequality that $\psi_0(t) = 0$ for all $t \ge 0$. The proof of Theorem 4.4 is complete. □

Proposition 4.10. *The semigroup $\{S(t)\}$ acting in the space $\mathcal{E}^-$ is continuous in the topology $C^{\mathrm{loc}}(\mathbb{R}_-; E)$ on any bounded set in $C_b(\mathbb{R}_-; E)$.*

Proof. We apply inequality (4.52), which holds for the difference $w = u_1 - u_2$ of any two solutions u_1 and u_2 with different initial data v_1 and v_2. From the Gronwall lemma, we obtain

$$\psi(t) \le C'\psi(0)e^{2D_1 t},\ D_1 \ge 0,$$

where

$$\psi(t) = \|y_1(t) - y_2(t)\|_E^2 + \int_0^\infty \mu(s)|\nabla\left(w(t) - w(t-s)\right)|^2 ds,$$

$y_i(t) = (u_i(t), \partial_t u_i(t)), i = 1, 2$, and

$$\psi(0) = \|y_1(0) - y_2(0)\|_E^2 + \int_0^\infty \mu(s)|\nabla\left(v_1(0) - v_2(0) - (v_1(-s) - v_2(-s))\right)|^2 ds. \tag{4.53}$$

Recall that $\int_0^\infty \mu(s)ds < \infty$ (see (4.8)), so that the right-hand side of (4.53) tends to zero as

$$\|v_1 - v_2\|_{C([-M,0];H_0^1(\Omega))} + \|\partial_t v_1 - \partial_t v_2\|_{C([-M,0];L_2(\Omega))} \to 0$$

for any $M > 0$, if the initial values $(v_1, \partial_t v_1)$ and $(v_2, \partial_t v_2)$ are taken from a fixed bounded set in $C_b(\mathbb{R}_-; E)$. Then $\|y_1 - y_2\|_{C([-M,M];E)} \to 0$ for any $M > 0$. Thus, the semigroup $\{S(t)\}$ is continuous in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$ on any bounded set in $C_b(\mathbb{R}_-; E)$. □

Proposition 4.11. *The semigroup $\{S(t)\}$ generated by problem (4.18) has an attracting (in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$) set which is bounded in $C_b(\mathbb{R}_-; E)$ and compact in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$.*

Proof. We use the method from [22, 13] which we adjust to the case of equations with memory. We consider the following homogeneous linear equation with memory:

$$\begin{aligned} \partial_t^2 w(t) + \gamma\partial_t w(t) &= \beta\Delta w(t) + \int_0^\infty \mu(s)\left(\Delta w(t) - \Delta w(t-s)\right) ds,\ t \ge 0, \\ w|_{t\le 0} &= v(t),\ \partial_t w|_{t \le 0} = \partial_t v(t). \end{aligned} \tag{4.54}$$

Similarly to Theorem 4.4, we prove (for the case $f \equiv 0, g \equiv 0$) that problem (4.54) has a unique solution $(w(t), \partial_t w(t)) \in C_b(\mathbb{R}; E)$ for any $(v(t), \partial_t v(t)) \in \mathcal{E}^- \subset C_b(\mathbb{R}_-; E)$. Moreover, similarly to (4.36), we prove the following inequality:

$$\|(w(t), \partial_t w(t))\|_E^2 \le C\|(v, \partial_t v)\|_{C_b(\mathbb{R}_-;E)}^2 e^{-\kappa t},\ \forall t \ge 0. \tag{4.55}$$

Let $y(t) = (u(t), \partial_t u(t))$ be a solution of the nonlinear equation (4.18) with the same initial data $u|_{t\le 0} = v,\ \partial_t u|_{t\le 0} = \partial_t v$. We now consider the function

$u_1(t) = u(t) - w(t)$. It satisfies the following nonhomogeneous linear equation for $t \geq 0$:

$$\partial_t^2 u_1(t) + \gamma \partial_t u_1(t) = \beta \Delta u_1(t) + \int_0^\infty \mu(s)\,(\Delta u_1(t) - \Delta u_1(t-s))\,ds - f(u(t)) + g(x),$$
$$u_1|_{t\leq 0} = 0,\ \partial_t u_1|_{t\leq 0} = 0. \tag{4.56}$$

We now differentiate equation (4.56) in t. The function $q(t) = \partial_t u_1(t),\ t \in \mathbb{R}$, satisfies, for $t \geq 0$, the equation

$$\begin{aligned} \partial_t^2 q(t) + \gamma \partial_t q(t) &= \beta \Delta q(t) + \int_0^\infty \mu(s)\,(\Delta q(t) - \Delta q(t-s))\,ds - f'(u(t))\partial_t u(t), \\ q|_{t\leq 0} &= 0,\ \partial_t q|_{t\leq 0} = 0. \end{aligned} \tag{4.57}$$

Consider the most difficult case $n \geq 3$ (the cases $n = 1, 2$ can be treated in a similar way). Note that, since $u \in L_\infty(\mathbb{R}; H_0^1(\Omega))$ and $\partial_t u \in L_\infty(\mathbb{R}; L_2(\Omega))$, the function $f'(u(\cdot))\partial_t u(\cdot) \in L_\infty(\mathbb{R}; H^{-\sigma})$, where $\sigma = \rho(n-2)/2 < 1$ (ρ is taken from inequality (4.5)). This fact follows from the generalization of Lemma 3.3 proved in [13, Chapter IV] (where the case $n = 3$ was considered with $\sigma = \rho/2$).

We assume that

$$\|(u(t), \partial_t u(t))\|_E \leq R \text{ for all } t \geq 0.$$

Then, clearly,

$$\|f'(u(t))\partial_t u(t)\|_{H^{-\sigma}} \leq C(R) \text{ for all } t \geq 0.$$

We claim that equation (4.57) has a unique solution $q(t)$ such that $q \in L_\infty(\mathbb{R}_+; H^{1-\sigma})$, $\partial_t q \in L_\infty(\mathbb{R}_+; H^{-\sigma})$ and

$$\|(q(t), \partial_t q(t))\|_{E_{-\sigma}}^2 = \|q(t)\|_{H^{1-\sigma}}^2 + \|\partial_t q(t)\|_{H^{-\sigma}}^2 \leq C_2(R) \tag{4.58}$$

for all $t \geq 0$. Here, we have set $E_s = H^{1+s}(\Omega) \times H^s(\Omega)$. Inequality (4.58) can be proved as inequality (4.20). In a first step, we multiply the equation by $\partial_t q + \alpha q$, using the scalar product in $H^{-\sigma}$, and carry out the corresponding transformations from Proposition 4.5, taking into account that $q|_{t\leq 0} = 0$ and $\partial_t q|_{t\leq 0} = 0$. We omit the details which are similar to the reasoning from Lemma 3.4 in [13, Chapter IV]. It follows from (4.58) that the function $q(t) = \partial_t u_1(t)$ satisfies

$$\|(\partial_t u_1(t), \partial_t^2 u_1(t))\|_{E_{-\sigma}}^2 = \|\partial_t u_1(t)\|_{H^{1-\sigma}}^2 + \|\partial_t^2 u_1(t)\|_{H^{-\sigma}}^2 \leq C_2(R) \tag{4.59}$$

for all $t \geq 0$.

Note that, in equation (4.56), the sum $-f(u(t)) + g(x) \in L_\infty(\mathbb{R}_+; L_2(\Omega))$. Therefore, from (4.56) and (4.59), we conclude that

$$\left\| \beta \Delta u_1(t) + \int_0^\infty \mu(s)\,(\Delta u_1(t) - \Delta u_1(t-s))\,ds \right\|_{H^{-\sigma}} \leq C_3(R) \tag{4.60}$$

for all $t \geq 0$.

We now need the following

LEMMA 4.12. *Let a function $\varphi(t) \in L_\infty^{\mathrm{loc}}(\mathbb{R}_+; X)$, where X is a Banach space. Let also $\varphi(t)$ satisfy the inequality*

$$\left\| (\beta + \mu_0)\varphi(t) - \int_0^t \mu(s)\varphi(t-s)ds \right\|_X \leq A,\ \forall t \geq 0, \tag{4.61}$$

where $\mu(s) \geq 0$ *for* $s \geq 0$ *and* $\int_0^\infty \mu(s)ds = \mu_0 < \infty$. *Then*

$$\operatorname*{ess\,sup}_{s\in\mathbb{R}_+} \|\varphi(s)\|_X \leq \frac{A}{\beta}. \tag{4.62}$$

Proof. We fix an arbitrary $T \geq 0$ and set $\varphi^* = \operatorname{ess\,sup}_{s\in[0,T]} \|\varphi(s)\|_X$. From (4.61), we obtain

$$\begin{aligned} \|(\beta+\mu_0)\varphi(t)\|_X &\leq \left\|(\beta+\mu_0)\varphi(t) - \int_0^t \mu(s)\varphi(t-s)ds\right\|_X \\ + \int_0^t \mu(s)\|\varphi(t-s)\|_X\, ds &\leq A + \int_0^t \mu(s)ds \left(\operatorname*{ess\,sup}_{s\in[0,t]} \|\varphi(s)\|_X\right) \\ &\leq A + \mu_0\varphi^*,\ \forall t \in [0,T]. \end{aligned} \tag{4.63}$$

Taking the ess sup from both sides of (4.63), we conclude that

$$(\beta+\mu_0)\varphi^* \leq A + \mu_0\varphi^*,$$

so that

$$\operatorname*{ess\,sup}_{s\in[0,T]} \|\varphi(s)\|_X = \varphi^* \leq \frac{A}{\beta},\ \forall T > 0,$$

and (4.62) is proved. □

We continue the proof of Proposition 4.11. It follows from inequality (4.60) that

$$\left\|(\beta+\mu_0)\Delta u_1(t) - \int_0^t \mu(s)\Delta u_1(t-s)ds\right\|_{H^{-\sigma}} \leq C_3(R)$$

for all $t \geq 0$ (recall that $u_1(t) = 0$ for $t \leq 0$). We apply Lemma 4.12 to the function $\varphi(t) = \Delta u_1(t)$ and the space $X = H^{-\sigma}$ and have

$$\|\Delta u_1(t)\|_{H^{-\sigma}} \leq \frac{C_3(R)}{\beta} = C_4(R),$$

so that, finally,

$$\|u_1(t)\|_{H^{2-\sigma}} \leq C_5(R),\ \forall t \geq 0. \tag{4.64}$$

Together with (4.59), this gives the inequality

$$\|(u_1(t), \partial_t u_1(t))\|_{E_{1-\sigma}} = \left(\|u_1(t)\|^2_{H^{2-\sigma}} + \|\partial_t u_1(t)\|^2_{H^{1-\sigma}}\right)^{1/2} \leq C_6(R),\ \forall t \geq 0. \tag{4.65}$$

Recall that $\sigma = \rho(n-2)/n < 1$, that is, $1-\sigma > 0$. From (4.65) and (4.56), we conclude that

$$\left\|\partial_t^2 u_1(t)\right\|_{H^{-\sigma}} \leq C_7(R),\ \forall t \geq 0. \tag{4.66}$$

Consider the set

$$P_1 = \left\{ \begin{array}{l} (w, \partial_t w) \in C_b(\mathbb{R}_-; E) \mid \\ \|(w,\partial_t w)\|_{C_b(\mathbb{R}_-;E_{1-\sigma})} \leq C_6(R^*), \|\partial_t^2 w\|_{C_b(\mathbb{R}_-;H^{-\sigma})} \leq C_7(R^*) \end{array} \right\}, \tag{4.67}$$

where R^* is taken from (4.41). It follows from Lemmas 2.12 and 2.13 that the set P_1 is compact in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$. We claim that the set P_1 is attracting for the semigroup $\{S(t)\}$. To display this fact, we choose an arbitrary bounded set

$$B \subset C_b(\mathbb{R}_-; E),$$

that is, there exists $r > 0$ such that, for every $z = (v, \partial_t v) \in B$,

$$\|v(s)\|^2_{H^1_0} + \|\partial_t v(s)\|^2_{L_2} \leq r^2 \text{ for all } s \leq 0.$$

Consider the solution $y(t) = (u(t), \partial_t u(t))$, $t \geq 0$, of equation (4.18) with initial data $z \in B$.

We fix numbers $M > 0$ and $\varepsilon > 0$ (M is large, while ε is small). It follows from Proposition 4.6 (see (4.36), with $2C_4 = R^*$) that

$$\|(u(t), \partial_t u(t))\|_E = \|(S(t)z)(0)\|_E \leq R^* \text{ for all } t \geq t_1 = t_1(r).$$

We split the solution $u(t)$ into the sum $u(t) = w(t) + u_1(t)$, where $w(t)$ satisfies equation (4.54) and $u_1(t)$ is a solution of (4.56). Then, from (4.65) and (4.66), we conclude that

$$\|(u_1(t), \partial_t u_1(t))\|_{E_{1-\sigma}} \leq C_6(R^*), \tag{4.68}$$

$$\left\|\partial_t^2 u_1(t)\right\|_{H^{-\sigma}} \leq C_7(R^*),\ \forall t \geq t_1. \tag{4.69}$$

Increasing t_1, if necessary, we can assume (see (4.55)) that

$$\|(w(t), \partial_t w(t))\|_E^2 \leq Cr^2 e^{-\kappa t} \leq \varepsilon,\ \forall t \geq t_1(r, \varepsilon). \tag{4.70}$$

From (4.68) – (4.70), we conclude that

$$\operatorname{dist}_{C([-M,0];E)}(S(t)B, P_1) \leq \varepsilon \text{ for all } t \geq t_1 + M.$$

Since ε is arbitrary, the set P_1 attracts $S(t)B$ in $C([-M, 0]; E)$ as $t \to +\infty$ for any fixed $M > 0$. Hence, P_1 attracts $S(t)B$ in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$ as $t \to +\infty$ for any bounded (in $C_b(\mathbb{R}_-; E)$) set B of initial data.

This finishes the proof of Proposition 4.11. □

Summarizing, we have constructed the semigroup $\{S(t)\}$ acting in the space $\mathcal{E}^- \subset C^{\mathrm{loc}}(\mathbb{R}_-; E)$ and corresponding to the equation with memory (2.1). We have proved that this semigroup is continuous (on bounded sets in $C_b(\mathbb{R}_-; E)$) and asymptotically compact, that is, it has a bounded (in $C_b(\mathbb{R}_-; E)$) and compact (in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$) attracting set. Thus, Theorems 2.7, 2.16 and Corollaries 2.9, 2.11, and 2.18 are applicable and we have proved the

THEOREM 4.13. *The semigroup $\{S(t)\}$ of equation (4.1) acting in $\mathcal{E}^- \subset C^{\mathrm{loc}}(\mathbb{R}_-; E)$ has the trajectory attractor $\mathfrak{A}$ and the global attractor $\mathcal{A}$. Furthermore, the following identities hold:*

$$\mathfrak{A} = \Pi_- \mathcal{K}, \tag{4.71}$$

$$\mathcal{A} = \mathcal{K}(0), \tag{4.72}$$

where $\mathcal{K}$ is the kernel of equation (4.1) in $C_b(\mathbb{R}; E)$. The set $\mathfrak{A}$ is connected in $C^{\mathrm{loc}}(\mathbb{R}_-; E)$ and the set $\mathcal{A}$ is connected in E.

We formulate one more result on the smoothness of the attractors.

THEOREM 4.14. *The trajectory attractor $\mathfrak{A}$ is bounded in the space $C_b(\mathbb{R}_-; E_1)$, while the global attractor $\mathcal{A}$ is bounded in E_1. Here, $E_1 = H^2(\Omega) \times H_0^1(\Omega)$.*

The proof is analogous to the proof of Lemma 3.5 from [13] and consists in verifying that the kernel $\mathcal{K}$ is bounded in $C_b(\mathbb{R}; E_1)$. We thus omit the details.

5. **Lyapunov Function for the Hyperbolic Equation with Memory.** We now prove that equation (4.1) has a continuous Lyapunov function. Let a function $z(s) = (v(s), \partial_t v(s)) \in C_b(\mathbb{R}_-; E)$. Consider the following functional:

$$\begin{aligned}\Phi(z(\cdot)) \quad &= \quad |\partial_t v(0)|^2 + \beta|\nabla v(0)|^2 + 2\int_\Omega F(v(x,0))dx \\ &\quad + \int_0^\infty \mu(s)|\nabla v(0) - \nabla v(0-s)|^2 ds - 2\int_\Omega g(x)v(x,0)dx. \qquad (5.1)\end{aligned}$$

PROPOSITION 5.1. *The functional $\Phi(z)$ is a Lyapunov function of the semigroup $\{S(t)\}$ in $C_b(\mathbb{R}_-; E)$ on the trajectory attractor.*

Proof. Let $y(t) = (u(t), \partial_t u(t))$ be the solution of (4.1) for $t \geq 0$. We consider identity (4.26) proved in Proposition 4.5, taking $\alpha = 0$. We obtain the following equality:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{|\partial_t u(t)|^2 + \beta|\nabla u(t)|^2 + 2\int_\Omega F(u(x,t))dx + \int_0^\infty \mu(s)|\nabla u(t) - \nabla u(t-s)|^2 ds\right\} \\ &+\gamma|\partial_t u(t)|^2 - \frac{1}{2}\int_0^\infty \mu'(s)|\nabla u(t) - \nabla u(t-s)|^2 ds = (g, \partial_t u(t)). \qquad (5.2)\end{aligned}$$

(To justify (5.2) rigorously, we apply Lemma 4.9.) Note that

$$(g, \partial_t u(t)) = \frac{d}{dt}\int_\Omega g(x)u(x,t)dx.$$

Then, from (5.2), we have

$$\frac{d}{dt}\Phi(S(t)z) = -2\gamma|\partial_t u(t)|^2 + \int_0^\infty \mu'(s)|\nabla u(t) - \nabla u(t-s)|^2 ds, \qquad (5.3)$$

where the function $\Phi(S(t)z)$ is absolutely continuous for $t \geq 0$. Integrating (5.3) in t, we obtain

$$\Phi(S(t)z) - \Phi(z) = -2\gamma\int_0^t |\partial_t u(\tau)|^2 d\tau + \int_0^t\int_0^\infty \mu'(s)|\nabla u(\tau) - \nabla u(\tau-s)|^2 ds d\tau. \quad (5.4)$$

Recall that $\mu'(s) \leq 0$ (see (4.6)). Therefore, we conclude from (5.4) that

$$\Phi(S(t)z) \leq \Phi(z) \text{ for all } t \geq 0.$$

Finally, if $\Phi(S(t)z) = \Phi(z)$ for some $t \geq 0$, then we observe from (5.4) that

$$\int_0^t |\partial_t u(\tau)|^2 d\tau = 0$$

and, hence, $|\partial_t u(t_1)|^2 = 0$ for all $t_1 \in [0,t]$, that is, $u(t_1) = u(0)$ for $t_1 \in [0,t]$.

The function $\mu(s)$ is positive and non-increasing (see (4.6)). Then there are two possibilities: either $\mu(s) > 0$ for all $s \geq 0$ or $\mu(s) > 0$ for all positive $s < s_0$ and $\mu(s) = 0$ for all $s \geq s_0$, where s_0 is some positive number. In the first case, clearly, $\mu'(s) \leq -\delta\mu(s) < 0$ for all $s \geq 0$ and, therefore, it follows from (5.4) that

$$\int_0^\infty \mu'(s)|\nabla u(\tau) - \nabla u(\tau - s)|^2 ds = 0$$

for all $\tau \in [0,t]$. Hence, $u(s) = u(t)$ for all $s \in]-\infty, t]$, that is, the solution $u(t) = q$ is independent of the time and $z = (q, 0)$ is a stationary point, $z \in \mathcal{N}$. In the second case, clearly,

$$\int_0^{s_0} \mu'(s)|\nabla u(\tau) - \nabla u(\tau - s)|^2 ds = 0,\ \tau \in [0,t],$$

and, therefore, $u(s) = u(t) = q$ for all $s \in]-s_0, t]$, where $(q,0) = w \in \mathcal{N}$. By uniqueness, $u(t) = q$ for all $t \geq 0$.

Note that, in the second (degenerate) case, the solution $u(t)$, which satisfies the equation for $t \geq 0$, can be non-constant for $t \leq s_0$, while $u(t) = q$ for all $t \geq -s_0$. It is easy to construct examples of such solutions. Fortunately, they do not lie on the trajectory attractor. Indeed, if $u(t)$ is a solution of (4.1) *for all* $t \in \mathbb{R}$ and $\Phi(S(t)z) = \Phi(z)$, where $S(t)z = u(t)$, then the above reasoning leads to the equality $u(t) = q$ for all $t \in \mathbb{R}$, for some $z = (q,0) \in \mathcal{N}$. This finishes the proof. □

Proposition 5.1, together with the continuity of the functional Φ, allows to apply Theorem 3.3.

Theorem 5.2. **1)** *The trajectory and global attractors of the hyperbolic equation (4.1) with memory satisfy*

$$\begin{aligned} \mathfrak{A} &= \Pi_- \mathcal{M}^+(\mathcal{N}), \\ \mathcal{A} &= \mathcal{M}^+(\mathcal{N})(0). \end{aligned}$$

In particular, if the set $\mathcal{N} = \{w_1, \ldots, w_N\}$ *is finite, then*

$$\mathfrak{A} = \Pi_- \bigcup_{i=1}^N \mathcal{M}^+(w_i),\ \mathcal{A} = \bigcup_{i=1}^N \mathcal{M}^+(w_i)(0).$$

2) *For any solution* $y(t) = (S(t)z)(0)$

$$\mathrm{dist}_E(y(t), \mathcal{N}) \to 0\ (t \to +\infty)$$

and, if $\mathcal{N}$ *is finite, then*

$$\mathrm{dist}_E(y(t), w_j) \to 0\ (t \to +\infty)$$

for some $w_j \in \mathcal{N}$.

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