DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **10**, Numbers **1& 2**, January & March **2004** pp. 117–135

ON THE FRACTAL DIMENSION OF INVARIANT SETS; APPLICATIONS TO NAVIER–STOKES EQUATIONS

V.V. Chepyzhov and A.A. Ilyin

Institute for Information Transmission Problems
Bol'shoĭ Karetnyĭ 19, Moscow 101447, Russia

Keldysh Institute of Applied Mathematics
Miusskaya Sq. 4, Moscow 125047, Russia

Abstract. A semigroup S_t of continuous operators in a Hilbert space H is considered. It is shown that the fractal dimension of a compact strictly invariant set X ($X \Subset H$, $S_tX = X$) admits the same estimate as the Hausdorff dimension, namely, both are bounded from above by the Lyapunov dimension calculated in terms of the global Lyapunov exponents. Applications of the results so obtained to the two–dimensional Navier–Stokes equations are given.

1. Introduction. Let X be a compact set in a Hilbert space H: $X \Subset H$. We recall the following definitions (see [1]).

Definition 1.1. The Hausdorff dimension of X in H is the number

$$\dim_H X = \inf\{d |\, \mu_H(X,d) = 0\},$$

where $\mu_H(X,d) = \lim_{\varepsilon\to 0+} \mu_H(X,d,\varepsilon)$, $\mu_H(X,d,\varepsilon) = \inf_{X\subset U} V_d(U)$. Here the infimum is taken over all coverings U of the set X by balls $B(x_i,r_i)$ with centres at x_i and radii $r_i \le \varepsilon$, and

$$V_d(U) = \sum r_i^d.$$

Definition 1.2. The fractal dimension of X in H is the number

$$\dim_F X = \limsup_{\varepsilon\to 0+} \frac{\log_2(N_X(\varepsilon))}{\log_2(1/\varepsilon)},$$

where $N_X(\varepsilon)$ is the minimum number of balls of radius ε which is necessary to cover X.

The following definition of the fractal dimension is similar to Definition 1.1.

Definition 1.3. The fractal dimension of X in H is the number

$$\dim_F X = \inf\{d |\, \mu_F(X,d) = 0\},$$

where

$$\mu_F(X,d) = \limsup_{\varepsilon\to 0+} \varepsilon^d N_X(\varepsilon).$$

AMS 1991 Subject Classification: Primary 34G20, 35Q30.
Key words and phrases: Fractal dimension, Attractors, Navier–Stokes equations.
This work was carried out with the financial support from INTAS, grant no. 899, US Civilian Research & Development Foundation (CRDF), grant no. RM1-2343-MO-02, and the Russian Foundation for Fundamental Research, grants no. 02–01–00227, 00–01–00387, and 02–01–00801

As easily seen, Definitions 1.2 and 1.3 are equivalent. Furthermore, if the covering U consists of $N_X(\varepsilon)$ balls of the same radius ε, then

$$\varepsilon^d N_X(\varepsilon) = V_d(U).$$

It is well known (and follows from Definitions 1.1 and 1.3) that $\dim_H(X) \le \dim_F(X)$.

Suppose that a continuous map S acts in H and let a compact set X be strictly invariant: $SX = X$, $X \Subset H$.

It was shown in [2],[1] that if the differential DS uniformly contracts d-dimensional volumes on X (the corresponding definitions are given in §2), then

$$\dim_H X \le d.$$

As for the fractal dimension, the following estimate was obtained in [1], [3], [4] (for earlier work on the estimates of the fractal dimension see also [5], [6]):

$$\dim_F X \le cd, \qquad c = \text{const} > 1.$$

Under an additional concavity condition (that is satisfied for the majority of the maps $S = S_t$ defined by evolution equations) the estimate $\dim_F X \le d$ was proved in [7]. Finally, for a diffeomorphism S the estimate $\dim_F X \le k$ was obtained for an integer k in [8] (see also [9]) under the condition that the differential DS contracts k-dimensional volumes.

The purpose of this work is to prove the estimate $\dim_F X \le d$ in the general case. Our approach is similar to that of [8], however, our proof is aimed at and is adjusted for the estimates of the fractal dimension of attractors of partial differential equations.

The main estimate proved in the abstract setting is then applied to the two-dimensional Navier–Stokes system. We obtain new estimates for the fractal dimension of the attractor that significantly improve (as far as the numerical values of the constants are concerned) the estimates from [7] and [15]:

$$\dim_F \mathcal{A} \le \frac{1}{\sqrt{2}\,\pi} (\lambda_1|\Omega|)^{1/2} \frac{\|f\|}{\lambda_1\nu^2} \le \frac{1}{2\pi^{3/2}} \frac{\|f\||\Omega|}{\nu^2}.$$

The improvement of the constants is due to the new recent results of [17], [18] combined with a more precise account of the non-divergence condition.

2. Main estimate. Suppose that the map S is uniformly quasidifferentiable on X, that is, for any $u \in X$ there exists a linear operator $DS(u)$ such that

$$\|S(u) - S(v) - DS(u)(u - v)\| \le h(r)\|u - v\|, \tag{2.1}$$

for all $u, v \in X$ such that $\|u - v\| \le r$, where $h(r) \to 0$ as $r \to 0$.

We also assume that the operator $L(u) = DS(u)$ is compact (the non-compact case reduces to the compact case by means of Proposition V.1.1. from [1]). Then the unit ball $B(0,1)$ in H is mapped by L into the ellipsoid $\mathcal{E} = \mathcal{E}(u) = L(u)B(0,1)$ with semiaxes $\alpha_1(u) \ge \alpha_2(u) \ge \dots$. The $\alpha_i(u)$ are the eigenvalues of the self-adjoint positive operator $(L^*L)^{1/2}$ and are called the s-numbers of the operator L: $\alpha_i(u) = s_i(L(u))$.

For each k we define the numbers $\omega_k(u) = \omega_k(\mathcal{E}(u))$:

$$\begin{aligned}\omega_0(u) &= 1,\\ \omega_k(u) &= \alpha_1(u)\alpha_2(u)\cdots\alpha_k(u),\\ \bar{\omega}_k &= \sup_{u\in X}\omega_k(u).\end{aligned}$$

For d of the form $d = k + s$, $0 < s \le 1$ we set

$$\omega_d(u) = \omega_k(u)^{1-s}\omega_{k+1}(u)^s = \omega_k(u)\alpha_{k+1}(u)^s.$$

We note that $\omega_d(u) \le \omega_1(u)^d = \alpha_1(u)^d = \|L(u)\|^d_{\mathcal{L}(H,H)}$. Furthermore,

$$\bar{\omega}_d = \sup_{u\in X}\omega_k(u)^{1-s}\omega_{k+1}(u)^s \le \bar{\omega}_k^{1-s}\bar{\omega}_{k+1}^s. \tag{2.2}$$

Lemma 2.1. *Let $\mathcal{E}$ be an ellipsoid in H with semiaxes $\alpha_1 \ge \alpha_2 \ge \dots$. If $r \ge \alpha_{n+1}$, then the minimum number of balls of radius $r\sqrt{n+1}$ which is necessary to cover $\mathcal{E}$ satisfies the estimate*

$$N_{\mathcal{E}}(r\sqrt{n+1}) \le 2^n\frac{\omega_k(\mathcal{E})}{r^k}, \tag{2.3}$$

where $\alpha_{k+1} \le r \le \alpha_k$ and $\omega_k(\mathcal{E}) = \alpha_1 \dots \alpha_k$. In particular, if $r = \alpha_{n+1}$, then

$$N_{\mathcal{E}}(\alpha_{n+1}\sqrt{n+1}) \le 2^n\frac{\omega_n(\mathcal{E})}{\alpha_{n+1}^n}, \tag{2.4}$$

Furthermore, for any $\eta > 0$ the following estimate holds:

$$N_{\mathcal{E}+B(0,\eta)}(r\sqrt{n+1} + \eta) \le N_{\mathcal{E}}(r\sqrt{n+1}) \le 2^n\frac{\omega_k(\mathcal{E})}{r^k}. \tag{2.5}$$

Proof. Estimate (2.3) is the well-known covering lemma [1], [2]. Estimate (2.5) follows from the fact that if N balls of radius ε cover $\mathcal{E}$, then N concentric ball of radius $\varepsilon + \eta$ cover $\mathcal{E} + B(0,\eta)$. The lemma is proved.

Lemma 2.2. *Let $d = n + s$, $0 < s \le 1$. Then in the notation of the previous lemma there exists a covering U of the set $\mathcal{E} + B(0, \sqrt{n+1}\,\alpha_{n+1})$ by balls of radius $2\,\alpha_{n+1}\sqrt{n+1}$ such that*

$$V_d(U) \le \beta_d\omega_d(\mathcal{E}), \quad \beta_d = 2^{n+d}(n+1)^{d/2}. \tag{2.6}$$

Proof. Using estimate (2.5) with $\eta = \sqrt{n+1}\,\alpha_{n+1}$ and $r = \alpha_{n+1}$ we obtain for the covering U from (2.5) the following inequality:

$$\begin{aligned}&V_d(U) = N_{\mathcal{E}+B(0,\sqrt{n+1}\,\alpha_{n+1})}(\sqrt{n+1}\,\alpha_{n+1} + \sqrt{n+1}\,\alpha_{n+1})(2\sqrt{n+1}\,\alpha_{n+1})^d \le\\ &2^n\frac{\omega_n(\mathcal{E})}{\alpha_{n+1}^n}\,2^d(n+1)^{d/2}\,\alpha_{n+1}^{n+s} = 2^{n+d}(n+1)^{d/2}\omega_d(\mathcal{E}).\end{aligned}$$

The lemma is proved.

Theorem 2.1. *Suppose that the map S is uniformly quasidifferentiable on X and $SX = X$. Let $d = n+s$, $0 < s \le 1$. Suppose that $L(u) = DS(u)$ is norm-continuous with respect to $u \in X$:*

$$\|L(u) - L(u_0)\|_{\mathcal{L}(H,H)} \to 0, \quad as \quad \|u - u_0\| \to 0, \quad u, u_0 \in X.$$

Suppose further that the quasidifferential $DS(u)$ contracts d-dimensional volumes uniformly for $u \in X$, that is, the following inequality holds:

$$\bar{\omega}_d = \sup_{u\in X} \omega_d(u) < 1.$$

Then

$$\dim_F X \le d.$$

Proof. Replacing S by S^m, where m is sufficiently large, we can assume that the number $\bar{\omega}_d$ is arbitrarily small (see [1]).

Since X is compact and $L(u)$ is norm-continuous, we have

$$\|L(u_1) - L(u_2)\|_{\mathcal{L}(H,H)} \le \delta(\|u_1 - u_2\|), \quad \delta(r) \to 0,\ r \to 0.$$

In view of the inequality

$$|s_j(A) - s_j(B)| \le \|A - B\|_{\mathcal{L}(H,H)}, \quad j = 1, \dots ,$$

see [21], Ch.II, Corollary 2.3, the numbers $\alpha_j(u) = s_j(L(u))$ are uniformly continuous on X and the following inequality holds:

$$|\alpha_j(u_1) - \alpha_j(u_2)| \le \delta(\|u_1 - u_2\|), \quad \delta(r) \to 0,\ r \to 0.$$

Hence for each $\varepsilon > 0$ we have

$$\frac{\alpha_j(u_1) + \varepsilon}{\alpha_j(u_2) + \varepsilon} \le 1 + \delta(\|u_1 - u_2\|)\varepsilon^{-1}, \quad \delta(r)\varepsilon^{-1} \to 0,\ r \to 0. \tag{2.7}$$

Since the numbers $\alpha_j(u)$ are non-negative and monotone decreasing, we have

$$\alpha_{n+1}(u) \le \omega_{n+1}(u)^{1/(n+1)} \le \omega_d(u)^{1/d} \le \bar{\omega}_d^{1/d}.$$

Similarly, all the $\alpha_j(u)$ are bounded uniformly for $u \in X$:

$$\alpha_j(u) \le \bar{\omega}_j^{1/j} \le \bar{\alpha}_1 = \sup_{u\in X} \|L(u)\|_{\mathcal{L}(H,H)} < \infty.$$

(Recall that X is compact and $L(u)$ is norm-continuous). Therefore by the mean value formula we have for small $\varepsilon > 0$ the inequality:

$$|(\alpha_1(u) + \varepsilon)\dots(\alpha_n(u) + \varepsilon)(\alpha_{n+1}(u) + \varepsilon)^s - \alpha_1(u)\dots\alpha_n(u)\alpha_{n+1}(u)^s| < C\varepsilon \tag{2.8}$$

holding uniformly for $u \in X$ with $C = C(\bar{\alpha}_1)$ for $0 < \varepsilon < 1$. Therefore for $\varepsilon > 0$ we have

$$L(u)B(0,1) \subset \mathcal{E}',$$

where $\mathcal{E}'$ is the ellipsoid with semiaxes $\alpha'_j(u) = \alpha_j(u) + \varepsilon$, $j = 1, \dots$. Taking m in S^m sufficiently large and then fixing ε sufficiently small we see from (2.8) that the corresponding number $\bar\omega'_d = \sup_{u\in X} \alpha'_1(u)\cdots\alpha'_n(u)\alpha'_{n+1}(u)^s$, along with $\bar\omega_d$, can be chosen arbitrarily small. Precise conditions are as follows: $2^{d+1}\beta_d\bar\omega'_d \le 1$ and $4\sqrt{n+1}\,(\bar\omega'_d)^{1/d} \le 1$ (see (2.11) and (2.12)). In addition, the $(n+1)$th semiaxis $\alpha'_{n+1}(u)$ is bounded from above and, thanks to ε, is bounded away from zero uniformly for $u \in X$. Omitting the primes we rewrite this property and (2.7) in the form

$$\begin{aligned} &0 < a \le 4\sqrt{n+1}\,\alpha_{n+1}(u) \le b, \quad u \in X,\\ &\frac{\alpha_{n+1}(u_1)}{\alpha_{n+1}(u_2)} \le 1 + \delta_1(\|u_1 - u_2\|), \quad \delta_1(r) \to 0,\ r \to 0, \end{aligned} \tag{2.9}$$

where $\delta_1(r) = \delta(r)/\varepsilon$ and b can be arbitrary small because $\alpha_{n+1}(u) \le (\bar\omega_d)^{1/d}$.

We now proceed with the proof. There exists a finite covering U_0 of the set X by balls of radius r_0 and without loss of generality we can assume that their centres belong to X:

$$X = \bigcup_{i_0=1}^{N_0} B(u^0_{i_0}, r_0) \cap X, \quad u^0_{i_0} \in X, \quad N_0 = N_X(r_0).$$

Then

$$SX = X = \bigcup_{i_0=1}^{N_0} S(B(u^0_{i_0}, r_0) \cap X). \tag{2.10}$$

By quasidifferentiability for any $v \in B(u^0_{i_0}, r_0) \cap X$ we have

$$\|S(v) - S(u^0_{i_0}) - L(u^0_{i_0})(v - u^0_{i_0})\| \le h(r_0)\|v - u^0_{i_0}\|.$$

Hence, if r_0 is so small that $h(r_0) \le a/4 \le \sqrt{n+1}\alpha_{n+1}(u)$ for all $u \in X$, then

$$\begin{aligned} S(B(u^0_{i_0}, r_0) \cap X) = S((u^0_{i_0} + B(0, r_0)) \cap X) \subset S(u^0_{i_0}) + r_0(\mathcal{E}_{i_0} + B(0, h(r_0))) \subset \\ S(u^0_{i_0}) + r_0(\mathcal{E}_{i_0} + B(0, \sqrt{n+1}\,\alpha_{n+1}(u^0_{i_0}))), \end{aligned}$$

where $\mathcal{E}_{i_0} = L(u^0_{i_0})B(0,1)$.

In view of Lemma 2.2 for each set $r_0(\mathcal{E}_{i_0} + B(0, \sqrt{n+1}\,\alpha_{n+1}(u^0_{i_0})))$ there exists a covering $U_1^{i_0}$ of this set made up of balls of radius $r_0 2\sqrt{n+1}\,\alpha_{n+1}(u^0_{i_0})))$ such that

$$V_d(U_1^{i_0}) \le r_0^d \beta_d \omega_d(\mathcal{E}_{i_0}) \le r_0^d \beta_d \bar\omega_d.$$

The set $S(u^0_{i_0}) + r_0(\mathcal{E}_{i_0} + B(0, \sqrt{n+1}\,\alpha_{n+1}(u^0_{i_0})))$ is covered by the family $\tilde U_0^{i_0} = S(u^0_{i_0}) + U_1^{i_0}$, and clearly

$$V_d(\tilde U_1^{i_0}) = V_d(U_1^{i_0}).$$

We now throw out from $\tilde U_1^{i_0}$ the balls which do not intersect X. The remaining balls containing some points $u^1_{i_1} \in X$ we replace by balls of twice the radius (that is, of radius $r_0\,4\sqrt{n+1}\,\alpha_{n+1}(u^0_{i_0}))$ with centres at these points. As a result we obtain the covering $\hat U_1^{i_0}$ of the set $S(B(u^0_{i_0}, r_0) \cap X) = X \cap S(B(u^0_{i_0}, r_0) \cap X)$ containing the same number of balls or fewer of twice the radius, whose centres belong to X. Then we see that

$$V_d(\hat{U}_1^{i_0}) \le 2^d V_d(\tilde{U}_1^{i_0}) \le r_0^d 2^d \beta_d \bar{\omega}_d \le \frac{1}{2} r_0^d,$$

provided that $\bar{\omega}_d$ has been chosen so that following inequality holds:

$$2^{d+1}\beta_d\bar{\omega}_d \le 1. \tag{2.11}$$

We denote by U_1 the covering of the set X which is the union of the coverings $\hat{U}_1^{i_0}$ for $i = 1, \dots, N_0$ and write this in the form

$$X \subset \bigcup_{i_1=1}^{N_1} B(u_{i_1}^1, r_{i_1}^1), \quad u_{i_1}^1 \in X,$$

where the radii $r_{i_1}^1$ are of the form $r_{i_1}^1 = r_0\, 4\sqrt{n+1}\alpha_{n+1}(u_{i_0}^0)$ for some points $u_{i_0}^0$, each $u_{i_0}^0 \in X$ being the centre of a ball from the previous covering U_0. It follows from (2.9) that

$$r_0 a \le r_{i_1}^1 \le r_0 b.$$

Then we find that

$$V_d(U_1) \le \frac{1}{2} N_0 r_0^d = \frac{1}{2} V_d(U_0).$$

We now return to step (2.10):

$$SX = X = \bigcup_{i_1=1}^{N_1} S(B(u_{i_1}^1, r_{i_1}^1) \cap X).$$

If $\bar{\omega}_d$ is so small that

$$4\sqrt{n+1}\,\bar{\omega}_d^{1/d} \le 1, \tag{2.12}$$

(in fact, (2.12) follows from (2.11)), then $4\sqrt{n+1}\,\alpha_{n+1}(v) \le 1$ uniformly for $v \in X$ and we can repeat our construction and as a result obtain the covering U_2. After k steps we obtain the covering U_k:

$$X \subset \bigcup_{i_k=1}^{N_k} B(u_{i_k}^k, r_{i_k}^k), \;\; r_{i_k}^k = r_0(4\sqrt{n+1})^k \alpha_{n+1}(u_{i_0}^0) \dots \alpha_{n+1}(u_{i_{k-1}}^{k-1}),$$

where $u_{i_j}^j \in X$, $j = 0, \dots, k$ are the centres of balls from the coverings $U_0, \dots, U_k$. The following inequality holds for U_k:

$$V_d(U_k) \le 2^{-k} V_d(U_0). \tag{2.13}$$

Moreover, in view of the first inequality in (2.9) for an arbitrary collection of k points $u_0, \dots, u_{k-1} \in X$ we have

$$r_0 a^k \le r_0(4\sqrt{n+1})^k \alpha_{n+1}(u_0) \dots \alpha_{n+1}(u_{k-1}) \le r_0 b^k. \tag{2.14}$$

In particular, the radii $r^k_{i_k}$ of the balls of the covering U_k satisfy the above inequality:

$$r_0 a^k \le r^k_{i_k} \le r_0 b^k.$$

We fix an arbitrary small η, $0 < \eta \ll 1$ and construct a covering $U(\eta)$. We fix a point $u \in X$. By the strict invariance of X there exists a sequence $u(i) \in X$ such that

$$Su(1) = u, \quad Su(i) = u(i-1), \quad i = 1, \dots .$$

We define the functions

$$R_k = R_k(u(1), \dots, u(k)) = r_0(4\sqrt{n+1})^k \alpha_{n+1}(u(1)) \dots \alpha_{n+1}(u(k)).$$

Since $a \le R_k/R_{k-1} \le b$ (see the first inequality in (2.9)), it follows that there exists a number $k = k(u)$ such that R_k gets into the interval $r_0 a\eta \le r \le r_0\eta$:

$$r_0 a\eta \le R_k \le r_0\eta. \tag{2.15}$$

For a given u we fix such a $k = k(u)$. In view of (2.14) the number k so obtained cannot satisfy the inequality $\eta \le a^k$, that is, the inequality $k \le \log_2 \eta / \log_2 a$. Analogously, such a k cannot satisfy the inequality $b^k \le a\eta$, that is, the inequality $k \ge (\log_2 a + \log_2 \eta)/\log_2 b$. Thus, all such numbers k satisfy the inequality

$$K_2(\eta) := \frac{\log_2 \eta}{\log_2 a} \le k \le \frac{\log_2 a + \log_2 \eta}{\log_2 b} =: K_1(\eta). \tag{2.16}$$

We introduce the notation

$$u^0 = u(k), \dots, u^i = u(k-i), \dots, u^{k-1} = u(1),\ u^k = u.$$

Accordingly,

$$Su^0 = u^1, \dots, Su^i = u^{i+1}, \dots, Su^{k-1} = u^k = u.$$

The point u^0 belongs to a ball from the covering U_0:

$$u^0 \in B(u^0_{i_0}, r_0), \quad \|u^0 - u^0_{i_0}\| \le r_0.$$

Next, the point u^1 belongs to a ball from the covering U_1. This ball is a member of the subcovering $\hat{U}_1^{i_0}$ that covers the set $S(B(u^0_{i_0}, r_0) \cap X)$. Since $u^0 \in B(u^0_{i_0}, r_0)$, it follows that $u^1 = Su^0 \in S(B(u^0_{i_0}, r_0) \cap X)$ and belongs to a ball from U_1 of the form

$$u^1 \in B(u^1_{i_1}, r_0 4\sqrt{n+1}\alpha_{n+1}(u^0_{i_0})), \quad \|u^1 - u^1_{i_1}\| \le r_0 4\sqrt{n+1}\alpha_{n+1}(u^0_{i_0})) \le r_0.$$

Here we have used (2.12).

In a similar way we see that u^j is covered by a ball from U_j of the form

$$u^j \in B(u^j_{i_j}, r_0(4\sqrt{n+1})^j \alpha_{n+1}(u^0_{i_0})) \dots \alpha_{n+1}(u^{j-1}_{i_{j-1}})), \quad \|u^j - u^j_{i_j}\| \le r_0.$$

Finally, u^k belongs to a ball from U_k:

$$u = u^k \in B(u^k_{i_k}, r_k),\ r_k = r_k(u) = r_0(4\sqrt{n+1})^k \alpha_{n+1}(u^0_{i_0}) \dots \alpha_{n+1}(u^{k-1}_{i_{k-1}}).$$

We include this ball in the covering $U(\eta)$. Since we can carry out this procedure for each point $u \in X$ including every time only new balls and since all the balls being included in the family $U(\eta)$ belong to the union of $K_1(\eta) - K_2(\eta) < K_1(\eta)$ (finite) coverings U_k, it follows that $U(\eta)$ is a finite covering of X. In view of (2.13) and (2.16) for each covering U_k, whose balls are included in $U(\eta)$, we have

$$V_d(U_k) \le 2^{-k} V_d(U_0) \le 2^{-K_2(\eta)} V_d(U_0).$$

Therefore

$$V_d(U(\eta)) \le K_1(\eta) 2^{-K_2(\eta)} V_d(U_0). \tag{2.17}$$

We now estimate the radii $r_k = r_k(u)$ of the balls of $U(\eta)$ by comparing them with $R_k = R_k(u^0, \dots, u^{k-1})$. In view of the inequalities $\|u^j - u^j_{i_j}\| \le r_0$, $j = 0, \dots, k-1$ we can use the second inequality in (2.9). With (2.15) and (2.16) taken into account this gives

$$\frac{r_k(u)}{R_k} \le (1 + \delta_1(r_0))^k \le (1 + \delta_1(r_0))^{K_1(\eta)}, \quad \max_{u \in X}(r_k(u)) \le r_0 \eta (1 + \delta_1(r_0))^{K_1(\eta)}$$

and

$$\frac{R_k}{r_k(u)} \le (1 + \delta_1(r_0))^k \le (1 + \delta_1(r_0))^{K_1(\eta)}, \quad \min_{u \in X}(r_k(u)) \ge r_0 \eta a (1 + \delta_1(r_0))^{-K_1(\eta)}.$$

Hence, we have

$$\frac{\max_{u \in X}(r_k(u))}{\min_{u \in X}(r_k(u))} \le a^{-1}(1 + \delta_1(r_0))^{2K_1(\eta)}.$$

We now replace each ball in $U(\eta)$ by a concentric ball of radius

$$r_\eta = \max_{u \in X}(r_k(u))$$

and denote the covering so obtained by $\tilde{U}(\eta)$. Then by the above inequality we see that

$$V_d(\tilde{U}(\eta)) \le a^{-d}(1 + \delta_1(r_0))^{2d\,K_1(\eta)} V_d(U(\eta)).$$

All the balls of the covering $\tilde{U}(\eta)$ have the same radius r_η. With (2.17) taken into account this gives

$$V_d(\tilde{U}(\eta)) \le a^{-d}(1 + \delta_1(r_0))^{2d\,K_1(\eta)} K_1(\eta) 2^{-K_2(\eta)} V_d(U_0). \tag{2.18}$$

We now recall that

$$K_1(\eta) = \frac{\log_2(1/(a\eta))}{\log_2(1/b)} \le 2\frac{\log_2(1/\eta)}{\log_2(1/b)} \text{ if } \eta \le a$$

and that $\delta_1(r_0) \to 0$ as $r_0 \to 0$. Hence

$$\begin{gathered}(1 + \delta_1(r_0))^{2d\,K_1(\eta)} \le 2^{4d \log_2(1+\delta_1(r_0)) \log_2(1/\eta)/\log_2(1/b)} = \\ \left(\frac{1}{\eta}\right)^{4d \log(1+\delta_1(r_0))/\log(1/b)} \le \left(\frac{1}{\eta}\right)^{4d\delta_1(r_0)/\log(1/b)} = \left(\frac{1}{\eta}\right)^{\delta_2(r_0)},\end{gathered} \tag{2.19}$$

where $\delta_2(r_0) = 4d\delta_1(r_0)/\log(1/b) \to 0$ as $r_0 \to 0$.

Finally, $K_2 = \log_2(1/\eta)/\log_2(1/a)$ and

$$2^{-K_2(\eta)} = \eta^{1/\log_2(1/a)}.$$

Therefore for $\eta \le a$ we obtain from (2.18) the inequality

$$V_d(\tilde{U}(\eta)) \le \frac{2a^{-d}}{\log_2(1/b)} V_d(U_0)\log_2(1/\eta)\eta^{\varkappa}, \tag{2.20}$$

where $\varkappa = 1/\log_2(1/a) - \delta_2(r_0)$.

If we now take (and fix) the radius r_0 of the balls of the initial covering U_0 so small that $\delta_2(r_0) < 1/\log_2(1/a)$, i.e., $\varkappa > 0$, then

$$V_d(\tilde{U}(\eta)) \to 0 \text{ as } \eta \to 0.$$

Recall that the covering $\tilde{U}(\eta)$ consists of the balls having the same radius r_η, which clearly tends to zero as $\eta \to 0$. Therefore (see Definition 1.3) $\mu_F(X, d) = 0$ and, hence, $\dim_F(X) \le d$. The theorem is proved.

Remark 2.1. We have proved a slightly stronger result than Theorem 2.1. Notice that the radius r_η of the covering $\tilde{U}(\eta)$ satisfies the inequality

$$r_0 a\eta(1+\delta_1(r_0))^{-K_1(\eta)} \le r_\eta \le r_0\eta(1+\delta_1(r_0))^{K_1(\eta)}.$$

Using inequality (2.19) we find that

$$r_0 a\eta^{1+\delta_2/(2d)} \le r_\eta \le r_0\eta^{1-\delta_2/(2d)},\ \delta_2 = \delta_2(r_0).$$

It follows from (2.20) that

$$N_X(r_\eta) \le CV_d(U_0) r_\eta^{-d}\log_2(1/\eta)\eta^{1/\log_2(1/a)-\delta_2},$$

where $C = 2a^{-d}/\log_2(1/b)$. Therefore

$$N_X\left(r_0\eta^{1-\delta_2/(2d)}\right) \le N_X(r_\eta) \le C_1 N_X(r_0)\log_2(1/\eta)\eta^{-d+1/\log_2(1/a)-3\delta_2/2},$$

where $C_1 = 2a^{-2d}/\log_2(1/b)$. This gives that

$$\dim_F X \le \frac{d - 1/\log_2(1/a) + 3\delta_2/2}{1-\delta_2/(2d)}.$$

Since $\delta_2(r_0) \to 0$ as $r_0 \to 0$ we obtain that

$$\dim_F X \le d - 1/\log_2(1/a) < d.$$

3. Applications to semigroups and differential equations. We now consider applications of the results of §2 to semigroups of continuous operators S_t acting in a Hilbert space H. Let X be a compact strictly invariant set for S_t: $S_t X = X$, $X \Subset H$. We assume that the map S_t is uniformly quasidifferentiable on X for each

t. In other words, (2.1) holds, where S is replaced by S_t. We assume, in addition, that for a fixed t the operator $L = L(t, u) = DS_t(u)$ is norm-continuous with respect to $u \in X$,

The eigenvalues of the self-adjoint positive (compact) operator$(L^*L)^{1/2}$ are denoted by $\alpha_1(t, u) \geq \alpha_2(t, u) \geq \dots$, and similarly to §2 we set

$$\begin{aligned}\omega_0(t, u) &= 1,\\ \omega_k(t, u) &= \alpha_1(t, u)\alpha_2(t, u)\cdots\alpha_k(t, u),\\ \bar{\omega}_k(t) &= \sup_{u\in X}\omega_k(t, u).\end{aligned}$$

Then there exists a limit $\lim_{t\to\infty} t^{-1}\ln\bar{\omega}_k(t) = q(k)$ (see[1], Section V.2.3) and hence for any $\varepsilon > 0$ we have for t large enough

$$\bar{\omega}_k(t) \leq e^{(q(k)+\varepsilon)t}, \tag{3.1}$$

The numbers $q(k)$ are called the sums of the first k global Lyapunov exponents.

For an arbitrary $d = k + s$, $0 < s \leq 1$, we set as in §2

$$\omega_d(t, u) = \omega_k(t, u)^{1-s}\omega_{k+1}(t, u)^s.$$

By (3.1) we have the estimate

$$\bar{\omega}_d(t) = \sup_{u\in X}\omega_k(t, u)^{1-s}\omega_{k+1}(t, u)^s \leq e^{(q(d)+\varepsilon)t} \tag{3.2}$$

for large t, where

$$q(d) = q(k + s) = (1 - s)q(k) + sq(k + 1).$$

Theorem 3.1. *Suppose that for an integer $n > 0$ the inequalities $q(n) \geq 0$ and $q(n + 1) < 0$ hold. Then*

$$\dim_F X \leq d_0 = n + \frac{q(n)}{q(n) - q(n + 1)}\,. \tag{3.3}$$

Proof. If $d > d_0$, then $q(d) < 0$. Therefore inequality (3.2) gives that $\omega_d(t) \to 0$ as $t \to \infty$. Applying Theorem 2.1 to $S = S_t$, where t is sufficiently large, we find that $\dim_F X \leq d$. The proof is complete.

Remark 3.1. The number d_0 is called the Lyapunov dimension of X, $d_0 = \dim_L X$. It has a clear geometrical meaning. It is the point of intersection of the straight line joining the points $(n, q(n))$ and $(n + 1, q(n + 1))$ with the horizontal axis.

Remark 3.2. The estimate (3.3) was proved in [7] under the following additional condition. It is required that the graph of the piecewise linear function $q(d)$ lies below the straight line described in the previous remark.

In conclusion we give a result useful for practical applications (especially for partial differential equations).

Corollary 3.1. *Suppose that $q(m) \le f(m)$, where $f(d)$ is a (continuous) function of the continuous variable d, and let $f(d_*) = 0$. Then if f is concave (at least in the interval $d_* - 1 < d < d_* + 1$), then*

$$\dim_F X \le d_*. \tag{3.4}$$

In the general case

$$\dim_F X \le d_* + 1. \tag{3.5}$$

4. Navier–Stokes system. We illustrate the above results using the two-dimensional Navier–Stokes system:

$$\begin{aligned} &\partial_t u + \sum_{i=1}^{2} u^i \partial_i u = \nu \Delta u - \nabla p + f, \\ &\operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \quad u(0) = u_0, \qquad \Omega \Subset \mathbb{R}^2. \end{aligned}$$

We denote by P the orthogonal projection in $L_2(\Omega)^2$ onto the Hilbert space H which is the closure in $L_2(\Omega)^2$ of the set of smooth solenoidal vector functions with compact supports in Ω. Applying P we obtain

$$\partial_t u + \nu A u + B(u,u) = f, \quad u(0) = u_0, \tag{4.1}$$

where $A = -P\Delta$ and $B(u,v) = P(\sum_{i=1}^2 u^i \partial_i v)$. We denote by $\lambda_1 \le \lambda_2 \le \dots$ the eigenvalues of the Stokes operator A.

The equation (4.1) generates the semigroup $S_t : H \to H$, $S_t u_0 = u(t)$, which is uniformly differentiable in H and has a compact global attractor $\mathcal{A} \Subset H$ (see, for instance, [1], [11]). The attractor $\mathcal{A}$ is the maximal strictly invariant compact set.

Theorem 4.1. *The fractal dimension of $\mathcal{A}$ satisfies the following estimate in terms of dimensionless numbers $G = \|f\|/(\lambda_1\nu^2)$ and $G' = \|f\||\Omega|/\nu^2$ (where $|\Omega|$ denotes the area of Ω):*

$$\dim_F \mathcal{A} \le \frac{1}{\sqrt{2}\,\pi}\,(\lambda_1|\Omega|)^{1/2}\,\frac{\|f\|}{\lambda_1\nu^2} \le \frac{1}{2\pi^{3/2}}\,\frac{\|f\||\Omega|}{\nu^2}\,. \tag{4.2}$$

In addition, $\dim_F \mathcal{A} = 0$ *if*

$$\frac{\|f\|}{\lambda_1\nu^2} < (3/2)^{3/2}\pi^{1/2} = 3.2562\ldots \quad \text{or} \quad \frac{\|f\||\Omega|}{\nu^2} < (3\pi)^{3/2}2^{-1/2} = 20.4593\ldots. \tag{4.3}$$

Proof. We estimate the numbers $q(m)$. Taking the scalar product of (4.1) with u and integrating in t we obtain the well-known estimate

$$\limsup_{t\to\infty}\ \sup_{u_0\in\mathcal{A}} \frac{1}{t}\int_0^t \|\operatorname{rot} u(\tau)\|^2 d\tau \le (\lambda_1\nu^2)^{-1}\|f\|^2. \tag{4.4}$$

The semigroup S_t is uniformly differentiable in H and the differential is the linear operator $L(t,u_0) : \xi \in H \to U(t) \in H$, where $U(t)$ is the solution of the first variation equation (see [10], where it is also shown that $L(t,u_0)$ is Hölder continuous with respect to the initial point u_0):

$$\partial_t U = -\nu AU - B(U,u(t)) - B(u(t),U) =: \mathcal{L}(t,u_0)U, \quad U(0)=\xi. \tag{4.5}$$

Following [1],[4] we have for $q(m)$ the estimate

$$q(m) \le \limsup_{t\to\infty} \sup_{u_0\in\mathcal{A}} \sup_{\substack{\xi_i\in H\\ i=1,\dots m}} \left(\frac{1}{t}\int_0^t \operatorname{Tr}\mathcal{L}(\tau,u_0)\circ Q_m(\tau)d\tau\right), \tag{4.6}$$

where $Q_m(\tau)$ is the orthogonal projection in H onto $\operatorname{Span}(U_1(\tau),\dots,U_m(\tau))$ and U_i is the solution of the problem (4.5) with $U_k(0)=\xi_k$. Suppose that vector functions $v_1(t)\ \dots, v_m(t) \in H\cap H_0^1(\Omega)^2$ make up an orthonormal basis in $\operatorname{Span}\{U_1(t),\dots,U_m(t)\} = Q_m(t)H$. Then using the well-known orthogonality relation $(B(u,v),v)=0$ and Lemma 4.1 below we obtain

$$\begin{aligned}
&\operatorname{Tr}\mathcal{L}(t,u_0)\circ Q_m(t) \;=\; \sum_{j=1}^m(\mathcal{L}(t,u_0)v_j,v_j) \;=\\
&-\sum_{j=1}^m(\nu(A\,v_j,v_j) \;+\; (B(v_j,u(t)),v_j) \;+\; (B(u(t),v_j),v_j)) \;=\\
&-\nu\sum_{j=1}^m\|\operatorname{rot}v_j\|^2 \;-\; \int\sum_{j=1}^m\sum_{k,i=1}^2 v_j^k\partial_k u^i v_j^i dx \;\le\\
&-\nu\sum_{j=1}^m\|\operatorname{rot}v_j\|^2 \;+\; 2^{-1/2}\int\rho(x)|\nabla u(x)|\,dx \;\le\\
&-\nu\sum_{j=1}^m\|\operatorname{rot}v_j\|^2 \;+\; 2^{-1/2}\|\rho\|\|\nabla u\| \;=\; -\nu\sum_{j=1}^m\|\operatorname{rot}v_j\|^2 \;+\; 2^{-1/2}\|\rho\|\|\operatorname{rot}u\|,
\end{aligned}$$

where $\rho(x) = \sum_{j=1}^m|v_j(x)|^2$, $|\nabla u(x)|^2 = \sum_{i,k=1}^2(\partial_k u^i(x))^2$. Using the following lower bound for the spectrum of the Stokes operator (see [15])

$$\sum_{j=1}^m\|\operatorname{rot}v_j\|^2 \;\ge\; \lambda_1 \;+\dots+\; \lambda_m \;\ge\; \frac{\pi m^2}{|\Omega|}$$

and the Lieb–Thirring inequality (see Appendix)

$$\|\rho\|^2 \;=\; \int\Big(\sum_{j=1}^m|v_j(x)|^2\Big)^2dx \;\le\; \frac{1}{\pi}\sum_{j=1}^m\|\operatorname{rot}v_j\|^2, \tag{4.7}$$

we find that

$$\begin{aligned}
&\operatorname{Tr}\,\mathcal{L}(t,u_0)\circ Q_m(t) \;\le\\
&\nu\sum_{j=1}^m\|\operatorname{rot}v_j\|^2 \;+\; 2^{-1/2}\left(\frac{1}{\pi}\sum_{j=1}^m\|\operatorname{rot}v_j\|^2\right)^{1/2}\|\operatorname{rot}u(t)\| \;\le\\
&-\frac{\nu}{2}\sum_{j=1}^m\|\operatorname{rot}v_j\|^2 \;+\; \frac{1}{4\pi\nu}\|\operatorname{rot}u(t)\|^2 \;\le\; -\frac{\nu\pi m^2}{2|\Omega|} \;+\; \frac{1}{4\pi\nu}\|\operatorname{rot}u(t)\|^2.
\end{aligned}$$

Using in (4.6) the last estimate and (4.4), we finally obtain

$$q(m) \le -\frac{\nu\pi m^2}{2|\Omega|} + \frac{\|f\|^2}{4\pi\lambda_1\nu^3}. \tag{4.8}$$

Using Corollary 3.1 (estimate (3.4)) we find that

$$\dim_F \mathcal{A} \le \frac{1}{\sqrt{2}\,\pi}(\lambda_1|\Omega|)^{1/2}\frac{\|f\|}{\lambda_1\nu^2}.$$

The second inequality in (4.2) follows from the estimate $\lambda_1 \ge 2\pi/|\Omega|$ (see [15]).

It was shown in [15] that $\dim_F \mathcal{A} = 0$ if $\frac{\|f\|}{\lambda_1\nu^2} < \frac{1}{c}$, where c is the constant in the estimate $|(B(u,u),v)| \le c\|u\|\|\operatorname{rot} u\|\|\operatorname{rot} v\|$. In the following Lemma 4.2 we prove that we can take in this inequality $c = (3/2)^{-3/2}\pi^{-1/2}$. This proves that if the first inequality in (4.3) holds, then the dynamics is trivial. The second inequality in (4.3) implies the first since $\lambda_1 \ge 2\pi/|\Omega|$. The theorem is proved.

Lemma 4.1. *If* $\operatorname{div} u(x) = 0$*, then the following inequality holds:*

$$\left|\sum\nolimits_{k,i=1}^{2} v^k(x)\partial_k u^i(x) v^i(x)\right| \le \frac{1}{\sqrt{2}}|\nabla u(x)||v(x)|^2,$$

where $|\nabla u| = \left(\sum_{k,i=1}^{2}(\partial_k u^i)^2\right)^{1/2}$.

Proof. We have

$$\left|\sum\nolimits_{k,i=1}^{2} v^k\partial_k u^i v^i\right| = |\nabla u\, v\cdot v| = \left|\frac{1}{2}(\nabla u + \nabla u^*)v\cdot v\right| \le |\lambda||v|^2,$$

where

$$\nabla u = \begin{pmatrix}\partial_1 u^1 & \partial_1 u^2\\ \partial_2 u^1 & \partial_2 u^2\end{pmatrix}, \quad \nabla u^* = \begin{pmatrix}\partial_1 u^1 & \partial_2 u^1\\ \partial_1 u^2 & \partial_2 u^2\end{pmatrix},$$

and λ is the maximum (in absolute value) eigenvalue of the matrix $\frac{1}{2}(\nabla u + \nabla u^*)$. From the characteristic polynomial with the condition $\operatorname{div} u = 0$ taken into account we see that the eigenvalues of this matrix are $\lambda > 0$ and $-\lambda$, where

$$\lambda^2 = (\partial_1 u^1)^2 + \frac{1}{4}(\partial_1 u^2 + \partial_2 u^1)^2 \le \frac{1}{2}\,|\nabla u|^2.$$

The lemma is proved.

This lemma makes it possible to estimate the non-linear term $B(v,v)$ with a smaller constant than previously known.

Lemma 4.2. *For* $u, v \in H_0^1(\Omega)^2 \cap H$ *the following estimate holds:*

$$|(B(v,v),u)| \le \left(\frac{8}{27\,\pi}\right)^{1/2}\|v\|\|\operatorname{rot} v\|\|\operatorname{rot} u\|. \tag{4.9}$$

Proof. We have

$$\begin{gathered}|(B(v,v),u)| = |(B(v,u),v)| = \left|\int\sum\nolimits_{k,i=1}^{2} v^k\partial_k u^i v^i\,dx\right| \le \\ \frac{1}{\sqrt{2}}\int|\nabla u||v|^2dx \le \frac{1}{\sqrt{2}}\|\nabla u\|\|v\|_{L_4}^2 \le \frac{\vec{c}_4^{\,2}}{\sqrt{2}}\|v\|\|\operatorname{rot} v\|\|\operatorname{rot} u\|,\end{gathered} \tag{4.10}$$

where $\vec{c}_4$ is the (sharp) constant in the vector Gagliardo–Nirenberg inequality

$$\|v\|_{L_4} \le \vec{c}_4\|v\|^{1/2}\|\nabla v\|^{1/2}, \quad v \in H_0^1(\Omega)^2, \quad \Omega \subseteq \mathbb{R}^2.$$

In fact, $\vec{c}_q = c_q$ (in other words, the constants in multiplicative inequalities do not increase in going over from scalars to vectors). To see this we use the scalar Gagliardo–Nirenberg inequality

$$\|\varphi\|_{L_q} \le c_q\|\varphi\|^{2/q}\|\nabla\varphi\|^{1-2/q}, \quad \varphi \in H_0^1(\Omega), \quad \Omega \subseteq \mathbb{R}^2, \quad q \ge 2$$

and Young's inequality with parameter $\varepsilon > 0$ in the form

$$a^{2\theta}b^{2-2\theta} \le \varepsilon\theta a^2 + (1-\theta)\varepsilon^{\theta/(\theta-1)}b^2, \quad 0 \le \theta \le 1.$$

Then we have

$$\begin{aligned}
&\|v\|_{L_q}^2 = \|(v^1)^2 + (v^2)^2\|_{L_{q/2}} \le \|(v^1)^2\|_{L_{q/2}} + \|(v^2)^2\|_{L_{q/2}} = \|v^1\|_{L_q}^2 + \|v^2\|_{L_q}^2 \le \\
&c_q^2\left(\|v^1\|^{2\theta}\|\nabla v^1\|^{2-2\theta} + \|v^2\|^{2\theta}\|\nabla v^2\|^{2-2\theta}\right) \le \\
&c_q^2\left(\varepsilon\theta(\|v^1\|^2 + \|v^2\|^2) + (1-\theta)\varepsilon^{\theta/(\theta-1)}(\|\nabla v^1\|^2 + \|\nabla v^2\|^2)\right) = \\
&c_q^2\left(\varepsilon\theta\|v\|^2 + (1-\theta)\varepsilon^{\theta/(\theta-1)}\|\nabla v\|^2\right), \quad \theta = 2/q.
\end{aligned}$$

Minimizing the right-hand side in ε we obtain

$$\|v\|_{L_q} \le c_q\|v\|^{2/q}\|\nabla v\|^{1-2/q}.$$

This shows that $\vec{c}_q \le c_q$. Since clearly $\vec{c}_q \ge c_q$, we have $\vec{c}_q = c_q$.

To complete the proof of (4.9) we recall the best to date closed form estimate of the constant c_4 from [22]:

$$c_4 \le \left(\frac{16}{27\pi}\right)^{1/4} = 0.6590\ldots.$$

Remark 4.1. The sharp value of the constant c_4 was obtained numerically in [23]:

$$c_4 = (\pi \cdot 1.8622\ldots)^{-1/4} = 0.6429\ldots.$$

Using this in Lemma 4.2 we can slightly improve (4.3). Namely, $\dim_F \mathcal{A} = 0$ if

$$\frac{\|f\|}{\lambda_1\nu^2} < 3.4206\ldots \quad \text{or} \quad \frac{\|f\||\Omega|}{\nu^2} < 21.4925\ldots.$$

Remark 4.2. Important contributions to the construction of the set which is now called the global attractor of the Navier–Stokes system have been made in [12], [10], [13]. The estimates of the Hausdorff and fractal dimension of the attractor of the Navier–Stokes system of the form

$$\dim_H \mathcal{A} \le c(\Omega)\frac{\|f\|}{\lambda_1\nu^2}, \qquad \dim_F \mathcal{A} \le 2c(\Omega)\frac{\|f\|}{\lambda_1\nu^2}$$

were obtained for in [4], [19] (see also [1]). Here $c(\Omega)$ is a dimensionless constant depending on the shape of the domain Ω: $c(\lambda\Omega) = c(\Omega)$. The Lieb-Thirring inequalities [16], [20] were essential in the proof.

Remark 4.3. Using the results of [14] one can show as in [15] that Theorem 4.1 holds for an arbitrary open domain Ω with finite area.

Remark 4.4. The function $q(m)$ is concave (see (4.8)). In this case the estimate $\dim_F \mathcal{A} \le d_*$, where $q(d_*) = 0$, also follows from [7].

5. Appendix. Lieb–Thirring inequalities for solenoidal vector functions. We consider a Schrödinger operator in $L_2(\mathbb{R}^n)$

$$-\Delta - V, \tag{5.1}$$

where $V \ge 0$ is a scalar function (which is sufficiently smooth and sufficiently rapidly decays at infinity). Then this operator is self-adjoint and bounded from below in $L_2(\mathbb{R}^n)$. We denote by $\mu_j = \mu_j(V)$ the negative eigenvalues of the operator (5.1) (each negative eigenvalue repeated according to its multiplicity). The following estimates were obtained in [16]:

$$\sum_{\mu_j<0} |\mu_j|^\gamma \le \mathrm{L}_{\gamma,n} \int_{\mathbb{R}^n} V^{\gamma+n/2} dx. \tag{5.2}$$

Here $\gamma > \max(0, 1-n/2)$. Furthermore, by notational definition $\mathrm{L}_{\gamma,n}$ is the best constant in (5.2). Explicit majorants for the Lieb–Thirring constants $\mathrm{L}_{\gamma,n}$ were found in [16] and it was also shown that

$$\mathrm{L}_{\gamma,n} \ge \mathrm{L}^{\mathrm{cl}}_{\gamma,n} := \frac{\Gamma(\gamma+1)}{(4\pi)^{n/2}\Gamma(\gamma+n/2+1)}\,. \tag{5.3}$$

Having in mind applications to the two-dimensional Navier–Stokes system we set $n = 2$ and consider the operator (5.1) acting on vector functions $u = (u^1, u^2)^T$ as follows:

$$-\Delta u - Vu = -\begin{pmatrix}\Delta u^1\\ \Delta u^2\end{pmatrix} - \begin{pmatrix}Vu^1\\ Vu^2\end{pmatrix}. \tag{5.4}$$

For an eigenvalue μ of the operator (5.1) with eigenfunction φ there clearly corresponds the repeated eigenvalue μ with eigenfunctions $(\varphi, 0)^T$ and $(0, \varphi)^T$. Therefore the following estimate holds for negative eigenvalues ν_j of the operator (5.4):

$$\sum_{\nu_j<0} |\nu_j|^\gamma \le \mathrm{L}^{\mathrm{vec}}_{\gamma,2} \int_{\mathbb{R}^2} V^{\gamma+1} dx, \qquad \gamma > 0, \tag{5.5}$$

where the best constant $\mathrm{L}^{\mathrm{vec}}_{\gamma,2}$ here satisfies the equality

$$\mathrm{L}^{\mathrm{vec}}_{\gamma,2} = 2\,\mathrm{L}_{\gamma,2}\,. \tag{5.6}$$

The following theorem on estimates for orthonormal families of functions is proved in [16]. For reader's convenience we reproduce the proof from [16] for the vector case.

Theorem 5.1. *Suppose that vector functions $u_1,\dots,u_m \in H_0^1(\Omega)^2$ make up an orthonormal family in $L_2(\Omega)^2$, $\Omega\subseteq\mathbb{R}^2$:*

$$\int_\Omega u_i(x)\cdot u_j(x)\,dx \;=\; \delta_{ij}.$$

Then the following inequality holds:

$$\int_\Omega \rho(x)^2dx \le \mathrm{k}_2\sum_{j=1}^m\|\nabla u_j\|^2 = \mathrm{k}_2\sum_{j=1}^m\bigl(\|\operatorname{rot}u_j\|^2+\|\operatorname{div}u_j\|^2\bigr), \tag{5.7}$$

where $\rho(x)=\sum_{j=1}^m|u_j(x)|^2$ and the sharp constant k_2 satisfies the equality

$$\mathrm{k}_2 = 4\,\mathrm{L}_{1,2}^{\mathrm{vec}} = 8\,\mathrm{L}_{1,2}\,.$$

Proof. Extending the vector functions u_j by zero outside Ω we can assume that $u_j\in H_0^1(\mathbb{R}^2)^2$. We furthermore suppose that $u_j\in C_0^\infty(\mathbb{R}^2)^2$. Having proved (5.7) for smooth functions we then apply the standard closure procedure. We consider the vector Schrödinger operator (5.4) with potential $V(x)=\alpha\rho(x)$, where $\alpha>0$ is a positive parameter. We denote this operator by A:

$$A\,u = -\Delta\begin{pmatrix}u^1\\u^2\end{pmatrix} - V\begin{pmatrix}u^1\\u^2\end{pmatrix}. \tag{5.8}$$

We set $H=L_2(\mathbb{R}^n)^2$ and denote by $\bigwedge^m H$ the m exterior product of H which is a Hilbert space whose elements are linear combinations of the products $v_1\wedge\dots\wedge v_m$. The scalar product of $v_1\wedge\dots\wedge v_m$ and $w_1\wedge\dots\wedge w_m$ is defined as follows

$$(w_1\wedge\dots\wedge w_m, v_1\wedge\dots\wedge v_m) \;=\; \det\{(w_i,v_j)\},\quad 1\le i,j\le m$$

and then is extended to $\bigwedge^m H$ by linearity. We define the operator $A_m:\bigwedge^m H\to\bigwedge^m H$ by the equality

$$A_m(v_1\wedge\dots\wedge v_m) \;=\; (A\,v_1\wedge\dots\wedge v_m+\dots+v_1\wedge\dots\wedge A\,v_m).$$

Corresponding to A_m is the quadratic form

$$a_m(v_1\wedge\dots\wedge v_m, v_1\wedge\dots\wedge v_m) \;=\; (A_m(v_1\wedge\dots\wedge v_m), v_1\wedge\dots\wedge v_m).$$

If the family $v_1,\dots,v_m$ is orthonormal, then the following equality holds:

$$a_m(v_1\wedge\dots\wedge v_m, v_1\wedge\dots\wedge v_m) = \sum_{j=1}^m\|\nabla v_j\|^2 \;-\; \sum_{j=1}^m\alpha\int\rho(x)|v_j(x)|^2dx. \tag{5.9}$$

We set $E \;=\; \inf\sigma(A_m)$, where $\sigma(A_m)$ is the spectrum of A_m. Two cases are possible.

1. The operator A has $k\ge1$ negative eigenvalues. Then $E=\sum_{i=1}^m\nu_i$ if $m\le k$, and $E=\sum_{i=1}^k\nu_i$ if $m\ge k$. In any case we have

$$E \;\ge\; \sum_{\nu_j(f)\le0}\nu_j(f) \;\ge\; -\,\mathrm{L}_{1,2}^{\mathrm{vec}}\,\alpha^2\int\rho(x)^2dx, \tag{5.10}$$

where in the second inequality we used (5.5).
2. The operator A has no negative eigenvalues. Then since $u_j \in C_0^\infty(\mathbb{R}^2)^2$, it follows that $\rho \in C_0^\infty(\mathbb{R}^2)$. Therefore $\sigma(A) = \sigma_e(A) = [0,\infty)$, where $\sigma_e(\cdot)$ denotes the continuous spectrum. Hence, $\sigma(A_m) = \sigma_e(A) = [0,\infty)$ and $E = 0$. This shows that (5.10) also formally holds.

On the other hand, by the variational principle and (5.9)

$$E \le a_m(u_1 \wedge \dots \wedge u_m, u_1 \wedge \dots \wedge u_m) = \sum_{j=1}^m \|\nabla u_j\|^2 - \alpha \int \rho(x)^2 dx. \tag{5.11}$$

Combining (5.10) and (5.11) and setting $\alpha = (2\,\mathrm{L}_{1,2}^{\mathrm{vec}})^{-1}$ in the resulting inequality we obtain

$$\int \rho(x)^2 dx \le 4\,\mathrm{L}_{1,2}^{\mathrm{vec}} \sum_{j=1}^m \|\nabla u_j\|^2, \tag{5.12}$$

which gives (5.7) with $\mathrm{k}_2 \le 4\,\mathrm{L}_{1,2}^{\mathrm{vec}}$. Let us prove the reverse inequality.

We consider the operator (5.8) with some non-negative potential $V \in C_0^\infty(\mathbb{R}^2)$. Let ν_j, $j = 1,\dots,N$ be the negative eigenvalues of it with the corresponding orthonormal eigenfunctions v_j. Then

$$\nu_i = \int |\nabla v_j(x)|^2 dx - \int V(x)|v_j(x)|^2 dx.$$

Therefore setting $\rho(x) = \sum_{j=1}^N |v_j(x)|^2$ and using (5.7), we obtain

$$\sum_{\nu_j<0} |\nu_j| = \int V(x)\rho(x)dx - \sum_{j=1}^N \|\nabla v_j\|^2 \le \|V\|\|\rho\| - \sum_{j=1}^N \|\nabla v_j\|^2 \le$$
$$\|V\|\|\rho\| - (\mathrm{k}_2)^{-1}\|\rho\|^2 \le \max_{y>0}(\|V\|\,y - (\mathrm{k}_2)^{-1}y^2) = \frac{\mathrm{k}_2}{4}\int V^2 dx.$$

Combining this with (5.5) we find that $\mathrm{k}_2/4 \ge \mathrm{L}_{1,2}^{\mathrm{vec}}$. The theorem is proved.

We now observe that in the case of Navier–Stokes system we are dealing with families of orthonormal systems of solenoidal vector functions. If we take this into account, we can reduce the constant k_2 in Theorem 5.1 at least by a factor of two. Namely, the following theorem holds.

Theorem 5.2. *Suppose that vector functions $u_1,\dots,u_m \in H_0^1(\Omega)^2$ make up an orthonormal family in $L_2(\Omega)^2$, $\Omega \subseteq \mathbb{R}^2$:*

$$\int_\Omega u_i(x)\cdot u_j(x)\,dx = \delta_{ij}.$$

Suppose that $\operatorname{div} u_j = 0$ (*or* $\operatorname{rot} u_j = 0$) *for* $j = 1,\dots,m$. *Then the following inequalities hold:*

$$\begin{aligned} &\int \rho(x)^2 dx \le \mathrm{k}_2^{\mathrm{sol}} \sum_{j=1}^m \|\operatorname{rot} u_j\|^2, \qquad \operatorname{div} u_j = 0,\\ &\int \rho(x)^2 dx \le \mathrm{k}_2^{\mathrm{pot}} \sum_{j=1}^m \|\operatorname{div} u_j\|^2, \qquad \operatorname{rot} u_j = 0, \end{aligned} \tag{5.13}$$

where $\rho(x)=\sum_{j=1}^m |u_j(x)|^2$ *and the best constants* $\mathrm{k}_2^{\mathrm{sol}}$ *and* $\mathrm{k}_2^{\mathrm{pot}}$ *satisfy the relation*

$$\mathrm{k}_2^{\mathrm{sol}}=\mathrm{k}_2^{\mathrm{pot}}\le\frac{\mathrm{k}_2}{2}=4\,\mathrm{L}_{1,2}\,. \tag{5.14}$$

Proof. Obviously, we have to prove only (5.14). We use here the method specific to the two-dimensional case. Given a vector function $u(x)=(u^1(x),u^2(x))^T$ we consider the vector function $\hat u(x)$: $\hat u(x)=(-u^2(x),u^1(x))^T$. It easy to see that

$$|u(x)|=|\hat u(x)|,\quad \operatorname{div}u(x)=\operatorname{rot}\hat u(x),\quad \operatorname{rot}u(x)=-\operatorname{div}\hat u(x).$$

Furthermore, if $u_1,\dots,u_m$ are orthonormal in $L_2(\Omega)^2$, then $\hat u_1,\dots,\hat u_m$ are orthonormal and vice versa. This immediately shows that $\mathrm{k}_2^{\mathrm{sol}}=\mathrm{k}_2^{\mathrm{pot}}$. Let us prove the inequality contained in (5.14). Suppose that the family $u_1,\dots,u_m$ is orthonormal in $L_2(\Omega)^2$ and let $\operatorname{div}u_j=0$, $j=1,\dots,m$. We set $\rho(x)=\sum_{j=1}^m|u_j(x)|^2$ and consider the family of $2m$ vector functions $u_1,\dots,u_m,\hat u_1,\dots,\hat u_m$. Since $\operatorname{rot}\hat u_j=0$, $j=1,\dots m$, it follows that $(u_i,\hat u_j)=0$ for $1\le i,j\le m$ and this family is orthonormal. Applying Theorem 5.1 to this family of $2m$ functions we obtain

$$4\int\rho(x)^2dx=\int\left(\sum_{j=1}^m(|u_j(x)|^2+|\hat u_j(x)|^2)\right)^2dx\le$$
$$\mathrm{k}_2\sum_{j=1}^m(\|\operatorname{rot}u_j\|^2+\|\operatorname{div}\hat u_j\|^2)=2\,\mathrm{k}_2\sum_{j=1}^m\|\operatorname{rot}u_j\|^2.$$

Therefore $\mathrm{k}_2^{\mathrm{sol}}\le\mathrm{k}_2/2$. The proof is complete.

Important results in finding the constants $\mathrm{L}_{\gamma,n}$ have been obtained recently. It was shown in [17] that $\mathrm{L}_{\gamma,n}=\mathrm{L}_{\gamma,n}^{\mathrm{cl}}$ (see (5.3)) for $\gamma\ge\frac32$ and all $n\ge1$. On this basis the best to date estimates of the Lieb–Thirring constants for $\gamma<\frac32$ were found in [18]. In particular, it was shown that

$$\mathrm{L}_{\gamma,n}\;\le\;2\,\mathrm{L}_{\gamma,n}^{\mathrm{cl}}\quad\text{for}\quad 1\le\gamma<\frac32,\quad\text{and}\quad n\ge1.$$

Thus,

$$\mathrm{L}_{1,2}\;\le\;2\,\mathrm{L}_{1,2}^{\mathrm{cl}}\;=\;\frac{1}{4\pi}\,.$$

In view of (5.14), and using the lower bound for $\mathrm{k}_2^{\mathrm{sol}}$ from [15], we finally obtain

$$\frac{1}{2\pi}\;\le\;\mathrm{k}_2^{\mathrm{sol}}\;=\;\mathrm{k}_2^{\mathrm{pot}}\;\le\;4\,\mathrm{L}_{1,2}\;\le\;\frac1\pi\,.$$

This completes the proof of the estimate (4.7).

REFERENCES

[1] R. Temam, "Infinite dimensional dynamical systems in mechanics and physics", 2nd ed., Springer-Verlag, New York, 1997.

[2] A. Douady and J. Oesterlé, *Dimension de Hausdorff des attracteurs*, C. R. Acad. Sci. Paris, Sér. A, 290 (1980), 1135–1138.

[3] P. Constantin and C. Foias, *Global Lyapunov exponents, Kaplan–Yorke formulas and the dimension of the attractors for the 2D Navier–Stokes equations*, Comm. Pure Appl. Math., 38 (1985), 1–27.

[4] P. Constantin, C. Foias, and R. Temam, "Attractors representing turbulent flows", Memoirs of Amer. Math. Soc., vol. 53, Providence RI, 1985.

[5] J. Mallet–Paret, *Negatively invariant sets of compact maps and extention of a theorem by Cartwright*, J. Diff. Equations, 22 (1976), 331–348.

[6] R. Mãné, *On the dimension of the compact invariant sets of certain nonlinear maps*, Lecture Notes in Mathematics, vol. 899, Springer-Verlag, New York, 1981.

[7] V.V. Chepyzhov and A.A. Ilyin, *A note on the fractal dimension of attractors of dissipative dynamical systems*, Nonlinear Anal. Theory, Methods & Applications, 44 (2001), 811-819.

[8] M.A. Blinchevskaya and Yu.S. Ilyashenko, *Estimate for the entropy dimension of the maximal attractor for k-contracting systems in an infinite-dimensional space*, Russian Jour. of Math. Phys., 6 (1999), N 1, 20–26.

[9] B.R. Hunt, *Maximum local Lyapunov dimension bounds the box dimension of chaotic attractors*, Nonlinearity, 9 (1996), 845–852

[10] A.V. Babin and M.I.Vishik, *Attractors of partial differential equations and estimates of their dimension*, Uspekhi Mat. Nauk, 38 (1983), 133–187; English transl. in Russian Math. Surveys, 38 (1983).

[11] A.V. Babin and M.I. Vishik "Attractors of evolution equations", Nauka, Moscow, 1989; English transl., North-Holland, Amsterdam, 1992.

[12] O.A. Ladyzhenskaya, *On the dynamical system generated by the Navier–Stokes equations*, Zap. Nauchn. Sem. LOMI, Leningrad, 27 (1972), 90–114; English transl. in J. of Soviet Math., 3 (1975), 458–479.

[13] Foias and R.Temam, *Some analytic and geometric properties of the solutions of the evolution Navier–Stokes equations*, J. Math. Pures Appl., 58 (1979), 339–368.

[14] O.A. Ladyzhenskaya, *First boundary value problem for Navier–Stokes equations in domain with non smooth boundaries*, C. R. Acad. Sc. Paris, 314, serie 1 (1992), 253–258.

[15] A.A. Ilyin, *Attractors for Navier–Stokes equations in domains with finite measure*, Nonlinear Anal. Theory, Methods & Applications, 27 (1996), 605–616.

[16] E. Lieb and W. Thirring, *Inequalities for the moments of the eigenvalues of the Schrödinger hamiltonian and their relation to Sobolev inequalities,* Studies in Mathematical Physics. Essays in honor of Valentine Bargmann, Princeton University Press (1976), 269–303.

[17] A. Laptev and T. Weidl, *Sharp Lieb–Thirring inequalities in high dimensions* Acta Mathematica, 184 (2000), 1, 87–111.

[18] D. Hundertmark, A. Laptev, and T. Weidl, *New bounds on the Lieb–Thirring constants*, Inventiones Mathematicae, 140 (2000), 3, 693–704.

[19] R. Temam. *Attractors for Navier–Stokes equations*, Research Notes in Mathematics, 122 (1985), 272–292.

[20] E. Lieb, *On characteristic exponents in turbulence*, Commun. Math. Phys., 92 (1984), 473–480.

[21] I.Ts. Gohberg and M.G. Krein, "Introduction to the theory of linear non-selfadjoint operators in Hilbert space", Nauka, Moscow, 1965; English transl., Amer. Math. Soc., Providence, RI, 1969.

[22] Sh.M. Nasibov, *On optimal constants in some Sobolev inequalities and their application to a nonlinear Schrödinger equation*, Dokl. Akad. Nauk SSSR, 307 (1989), 538–542; English transl. in Soviet Math. Dokl. 40 (1990).

[23] M. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys., 87 (1983), 567–576.

E-mail address : chep@iitp.ru; ilyin@spp.keldysh.ru

Received February 2002; revised November 2002; final version March 2003.