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Various versions of the Riemann–Hilbert problem for linear differential equations

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Abstract. A counterexample to Hilbert's 21st problem was found by Bolibrukh in 1988 (and published in 1989). In the further study of this problem he substantially developed the approach using holomorphic vector bundles and meromorphic connections. Here the best-known results of the past that were obtained by using this approach (both for Hilbert's 21st problem and for certain generalizations) are presented.

Contents

Introduction

This paper is devoted to Hilbert's 21st problem (the Riemann–Hilbert problem), which, in one form or another, was considered as far back as the middle of the 19th century by Riemann, and to certain generalizations of it that appeared at the end of the 20th century. This problem belongs to the analytic theory of differential equations and consists in constructing a linear differential equation (or a system of equations) of a certain class that has given singular points and given ramification type of solutions at these singular points.

The most substantial achievements in solving the Riemann–Hilbert problem are associated with the name of A. A. Bolibrukh. Before him the problem had long been wrongly regarded as solved in the affirmative. However, Bolibrukh constructed the first counterexample, which stimulated the development of the theory in the new direction outlined in his survey "The Riemann–Hilbert problem" [1].

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In the present paper we focus on the main recent achievements in the study of the Riemann–Hilbert problem and its generalizations, and we indicate some questions that arise naturally in the consideration of these results.

The basic notions of the analytic theory of linear differential equations and the general approach to studying the classical Riemann–Hilbert problem for Fuchsian systems on the Riemann sphere, together with the best-known results, are presented in the first two sections.

In § 3 we consider the Riemann–Hilbert problem for scalar Fuchsian equations and its relation to non-linear differential equations (Painlevé VI equations, Garnier systems).

Possible generalizations of the classical Riemann–Hilbert problem to the case of Fuchsian systems defined on a compact Riemann surface of arbitrary genus are presented in § 4.

The generalized Riemann–Hilbert problem for linear systems with irregular singular points is considered in § 5.

A brief description of some geometric notions and constructions related to the Riemann–Hilbert problem is given in § 6.

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§ 1. Basic definitions and statement of the classical problem

We consider a system of p linear differential equations on the Riemann sphere $\overline{\mathbb{C}}$, written in matrix form

$$\frac{dy}{dz} = B(z)\,y, \qquad y(z) \in \mathbb{C}^p, \tag{1}$$

where $B(z)$ is the coefficient matrix of the system and is meromorphic on the Riemann sphere, with singularities at the points $a_1, \dots, a_n$.

Definition 1. A singular point a_i of the system (1) is said to be *Fuchsian* if the matrix $B(z)$ has a pole of the first order at this point.

A singular point a_i of the system (1) is said to be *regular* if any solution of the system has at most polynomial growth in a neighbourhood of this point. A singular point that is not regular is said to be *irregular*.

From many viewpoints, Fuchsian singularities are the simplest type of singular points of the system (1). According to Sauvage's theorem [2], a Fuchsian singular point of a linear system is always regular (see also [3], Theorem 11.1).

Regular singularities of a linear system are next in complexity after Fuchsian ones. Generally speaking, the coefficient matrix of the system can have a pole of order higher than 1 at a regular singular point.[1] In such a case it turns out to

[1]The notions of regular and Fuchsian singular points coincide only in the case where a system consists of a single equation ($p = 1$). This fact can be verified by a straightforward integration of the equation.

be difficult to find out whether the singularity is regular or not. (In the general case, verification of all existing criteria for regularity of a singular point of a linear system is fairly difficult; one of the first such criteria was obtained by Moser [4], and subsequently other criteria also appeared [5], [6].) However, there is a simple *necessary* condition for the regularity of a singular point obtained by Horn [7], which consists in the following. We write the Laurent series for the coefficient matrix $B(z)$ of the system (1) in a neighbourhood of a singular point $z=a$ in the form

$$B(z)=\frac{B_{-r-1}}{(z-a)^{r+1}}+\cdots+\frac{B_{-1}}{z-a}+B_0+\cdots, \qquad B_{-r-1}\neq 0.$$

(The number r is called the *Poincaré rank* of the system (1) at this point, or the Poincaré rank of the singular point $z=a$. For example, the Poincaré rank of a Fuchsian singularity is equal to zero.)

If $z=a$ is a regular singular point of the system (1) *and $r>0$, then B_{-r-1} is a nilpotent matrix.*

We now give another simple *necessary* condition for the regularity of a singular point of a linear system.

If $z=a$ is a regular singular point of the system (1) *and $r>0$, then* $\operatorname{tr} B_{-r-1}=\cdots=\operatorname{tr} B_{-2}=0$.

This condition follows from the fact that, by the well-known Liouville theorem, the determinant of a fundamental matrix $Y(z)$ of the system (1) (a matrix whose columns form a basis in the solution space of the system) satisfies the equation $\frac{d}{dz}\det Y=\operatorname{tr} B(z)\det Y$. If a singular point $z=a$ of the system (1) is regular, then it is a regular (consequently, Fuchsian) singularity of the latter equation. Therefore, the function $\operatorname{tr} B(z)$ must have a simple pole at this point.

Irregular singular points form the most complicated type of singularities of a linear system.

A system (1) is said to be *Fuchsian* if all its singular points are Fuchsian. If infinity is not among the singularities of a Fuchsian system, then the system can be written in the form

$$\frac{dy}{dz}=\biggl(\sum_{i=1}^{n}\frac{B_i}{z-a_i}\biggr)y, \qquad \sum_{i=1}^{n}B_i=0.$$

(If one of the singularities, say a_n, is situated at infinity, then the coefficient matrix has the form $B(z)=\sum_{i=1}^{n-1}B_i/(z-a_i)$, but the sum of the residues B_i is no longer equal to the zero matrix.)

One of the important characteristics of a linear system is its *monodromy representation* (or *monodromy*), which is defined as follows.

In a neighbourhood of a non-singular point z_0 we consider a fundamental matrix $Y(z)$ of the system (1). The result of the analytic continuation of the matrix $Y(z)$ along a loop γ starting at the point z_0 and contained in $\overline{\mathbb{C}}\setminus\{a_1,\dots,a_n\}$ is, generally speaking, another fundamental matrix $\widetilde{Y}(z)$. Two bases are connected by a non-singular transition matrix G_γ corresponding to the loop γ:

$$Y(z)=\widetilde{Y}(z)G_\gamma.$$

The map $[\gamma] \mapsto G_\gamma$ (which depends only on the homotopy class $[\gamma]$ of the loop γ) defines a representation

$$\chi\colon \pi_1\bigl(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}, z_0\bigr) \to \mathrm{GL}(p, \mathbb{C})$$

of the fundamental group of the space $\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$ into the space of non-singular complex $p \times p$ matrices. This representation is called the monodromy of the system (1).

The *monodromy matrix* of the system (1) at a singular point a_i (with respect to a fundamental matrix $Y(z)$) is defined as the matrix G_i corresponding to a simple loop γ_i around the point a_i, that is, $G_i = \chi([\gamma_i])$. The matrices $G_1, \dots, G_n$ are generators of the *monodromy group* — the image of the map χ. By the condition $\gamma_1 \cdots \gamma_n = e$ in the fundamental group, these matrices are connected by the relation $G_1 \cdots G_n = I$ (henceforth, I denotes the identity matrix).

If initially another fundamental matrix $Y'(z) = Y(z)C,\ C \in \mathrm{GL}(p, \mathbb{C})$, is considered instead of the fundamental matrix $Y(z)$, then the corresponding monodromy matrices have the form $G_i' = C^{-1} G_i C$. The dependence of the matrices G_i on the choice of the initial point z_0 is of similar nature. Thus, the monodromy of a linear system is determined up to conjugation by a constant non-singular matrix and, more precisely, is an element of the space

$$\mathscr{M}_a = \mathrm{Hom}\bigl(\pi_1\bigl(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}\bigr), \mathrm{GL}(p, \mathbb{C})\bigr)/\mathrm{GL}(p, \mathbb{C})$$

of conjugacy classes of representations of the group $\pi_1\bigl(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}\bigr)$.

The number of parameters on which the monodromy depends can be calculated by considering only irreducible representations (since for $n > 2$ the reducible ones form a subspace of some positive codimension). In such a case, the dimension of the conjugacy class

$$\{(S^{-1} G_1 S, \dots, S^{-1} G_n S) \mid S \in \mathrm{GL}(p, \mathbb{C})\} \cong \mathrm{GL}(p, \mathbb{C})/\mathrm{st}(G_1, \dots, G_n)$$

of the element $(G_1, \dots, G_n)$ is equal to $\dim \mathrm{GL}(p, \mathbb{C}) - \dim \mathrm{st}(G_1, \dots, G_n) = p^2 - 1$. (According to Schur's lemma, if a matrix S commutes with all the matrices G_i, then it is a scalar matrix; therefore, $\dim \mathrm{st}(G_1, \dots, G_n) = 1$.) Consequently, the dimension of the space

$$\mathscr{M}_a \cong \{(G_1, \dots, G_n) \mid G_1 \cdots G_n = I\}/\mathrm{GL}(p, \mathbb{C})$$

is equal to $(n-1)p^2 - (p^2 - 1) = (n-2)p^2 + 1$.

The classical Riemann–Hilbert problem is stated as follows.

Is it possible to realize a given set of singular points $a_1, \dots, a_n$ and a given representation

$$\chi\colon \pi_1\bigl(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}, z_0\bigr) \to \mathrm{GL}(p, \mathbb{C}) \tag{2}$$

by a Fuchsian system? (*that is*, *is it possible to construct a Fuchsian system with given singularities and monodromy*?)

Thus, the Riemann–Hilbert problem is a question about the surjectivity of the *monodromy map* $\mu_a\colon \mathscr{M}_a^* \to \mathscr{M}_a$ from the space

$$\mathscr{M}_a^* \cong \{(B_1, \dots, B_n) \mid B_i \in \mathrm{Mat}(p, \mathbb{C}),\ B_1 + \dots + B_n = 0\}/\mathrm{GL}(p, \mathbb{C})$$

of Fuchsian systems with fixed singularities $a_1, \dots, a_n$ (considered up to equivalence $B_i \sim S^{-1}B_iS,\ i = 1, \dots, n$) into the space $\mathscr{M}_a$. Although the dimensions of these spaces are the same, the map μ_a is not surjective, and the problem has a negative solution in the general case. The first counterexample appears in dimension $p = 3$ for the number of singular points $n = 4$ (Bolibrukh [8]; see also [9], Ch. 2). We shall discuss the monodromy map in more detail in § 6.

There exist numerous sufficient conditions for a positive solution of the classical Riemann–Hilbert problem. We present the best-known of them and also focus on some analogues of them in the analysis of various generalizations of the classical problem (the simplest proofs of most of these sufficient conditions are presented in [10]).

1) *If one of the generators $G_1, \dots, G_n$ of the representation* (2) *is diagonalizable, then the Riemann–Hilbert problem has a positive solution*[2] (Plemelj [12]).

2) *If the representation* (2) *is two-dimensional* ($p=2$), *then the Riemann–Hilbert problem has a positive solution* (Dekkers [13]).

3) *If the representation* (2) *is irreducible, then the Riemann–Hilbert problem has a positive solution* (Bolibrukh [14], Kostov [15]).

4) *If the representation* (2) *is the monodromy of some scalar linear differential equation of order p with regular singularities $a_1, \dots, a_n$, then the Riemann–Hilbert problem has a positive solution* (Bolibrukh [16]).

We make several remarks about the last sufficient condition. The monodromy of a linear differential equation

$$\frac{d^p y}{dz^p} + b_1(z)\frac{d^{p-1}y}{dz^{p-1}} + \dots + b_p(z)y = 0 \tag{3}$$

of order p with singular points $a_1, \dots, a_n$ (poles of the coefficients) is defined in the same way as for the system (1), only instead of a fundamental matrix $Y(z)$ one must consider a row $(y_1, \dots, y_p)$ whose elements form a basis in the solution space of the equation.

In contrast to a system, for a scalar equation there exists a simple criterion for the regularity of a singular point of this equation, obtained by Fuchs [17] (see also [3], Theorem 12.1): *a singularity a_i of equation* (3) *is regular if and only if the coefficient $b_j(z)$ has at this point a pole of order at most j* $(j = 1, \dots, p)$. Scalar differential equations with regular singular points are said to be *Fuchsian*.

Initially, Riemann [18] stated the problem about constructing precisely a *Fuchsian differential equation* with given singular points and monodromy. However, Poincaré [19] showed that, unlike a Fuchsian system, the number of parameters on which a Fuchsian equation depends is less than the dimension of the space $\mathscr{M}_a$ of monodromy representations.[3] After that, Hilbert [20] included in his list of "Mathematical problems" the problem of constructing a *Fuchsian system* with given

[2] This condition was improved by Kostov [11]: *If one of the matrices $G_1, \dots, G_n$ in its Jordan form has at most one block of size* 2, *while the other blocks are of size* 1, *then the Riemann–Hilbert problem has a positive solution.* Further improvement of this condition in terms of the Jordan form of one of the monodromy matrices is impossible, since there exist counterexamples to the Riemann–Hilbert problem in which the Jordan forms of the monodromy matrices contain one block of size 3 or two blocks of size 2.

[3] Therefore, in the construction of a Fuchsian equation with a given monodromy in the general case there necessarily emerge additional (apart from $a_1, \dots, a_n$) *apparent* singular points at

singularities and monodromy, which is what became known as the Riemann–Hilbert problem (details of the history of studies of the Riemann–Hilbert problem and its definitive solution can be found in [21], [22]).

§ 2. Method of solution

In the study of problems related to the Riemann–Hilbert problem, a very useful tool is provided by linear gauge transformations of the form

$$y' = \Gamma(z)\, y \tag{4}$$

of the unknown function $y(z)$. The transformation (4) is said to be *holomorphically* (*meromorphically*) *invertible* at some point z_0 if the matrix $\Gamma(z)$ is holomorphic (meromorphic) at this point and $\det\Gamma(z_0)\neq 0$ $(\det\Gamma(z)\not\equiv 0)$. This transformation transforms the system (1) into the system

$$\frac{dy'}{dz} = B'(z)\, y', \qquad B'(z) = \frac{d\Gamma}{dz}\,\Gamma^{-1} + \Gamma B(z)\Gamma^{-1}, \tag{5}$$

which is said to be, respectively, *holomorphically* or *meromorphically equivalent* to the original system in a neighbourhood of the point z_0.

An important property of meromorphic gauge transformations is the fact that they preserve the monodromy (being meromorphic, the matrix $\Gamma(z)$ is single-valued on the punctured Riemann sphere; therefore the ramification of the fundamental matrix $\Gamma(z)Y(z)$ of the new system coincides with the ramification of the matrix $Y(z)$).

A transformation that is holomorphically invertible in a neighbourhood of a singular point a_i of the system (1) does not change the Poincaré rank of this singularity, whereas a meromorphically invertible transformation may increase or decrease this rank.

Locally, in a neighbourhood of each point a_k, it is easy to produce a system for which a_k is a Fuchsian singularity and the monodromy matrix at this point coincides with the corresponding generator $G_k = \chi([\gamma_k])$ of the representation (2). This system is

$$\frac{dy}{dz} = \frac{E_k}{z-a_k}\, y, \qquad E_k = \frac{1}{2\pi i}\log G_k, \tag{6}$$

with fundamental matrix $(z-a_k)^{E_k} := e^{E_k\log(z-a_k)}$ (the branch of the logarithm of the matrix G_k is chosen so that the eigenvalues ρ_k^j of the matrix E_k satisfy the condition $0\leqslant \operatorname{Re}\rho_k^j < 1$). Indeed,

$$\frac{d}{dz}(z-a_k)^{E_k} = \frac{E_k}{z-a_k}\,(z-a_k)^{E_k},$$

and a single circuit around the point a_k counterclockwise transforms the matrix $(z-a_k)^{E_k}$ into the matrix

$$e^{E_k(\log(z-a_k)+2\pi i)} = e^{E_k\log(z-a_k)}e^{2\pi i E_k} = (z-a_k)^{E_k} G_k.$$

which the coefficients of the equation have singularities but the solutions are single-valued meromorphic functions, and hence the monodromy matrices at these points are identity matrices. In what follows, by additional singular points of an equation or a system we mean precisely such singularities.

We note that in the case when the representation (2) is commutative (that is, when the matrices $G_1, \dots, G_n$ commute pairwise) the Riemann–Hilbert problem has a positive solution. The above arguments must be applied to the fundamental matrix $Y(z) = (z-a_1)^{E_1} \cdots (z-a_{n-1})^{E_{n-1}} (z-a_n)^{E_n - D}$, $D = \sum_{i=1}^n E_i$, of the global Fuchsian system

$$\frac{dy}{dz} = \biggl(\sum_{i=1}^{n-1} \frac{E_i}{z-a_i} + \frac{E_n - D}{z-a_n}\biggr) y$$

with singularities $a_1, \dots, a_n \in \mathbb{C}$. (Here one makes essential use of the relations $[G_i, G_j] = 0$ and the consequent relations $[G_i, (z-a_j)^{E_j}] = [G_i, (z-a_n)^{E_n - D}] = 0$; the commutativity also implies that $\log G_1 \cdots G_n = \log G_1 + \dots + \log G_n$ and, in view of the condition $G_1 \cdots G_n = I$, ensures that D is a diagonal integer-valued matrix that does not affect the monodromy at the point a_n.)

It is interesting that a positive solution of the Riemann–Hilbert problem when the representation (2) is commutative was first obtained by Lappo-Danilevskii [23]. This sufficient condition had not been noted before him (he mentioned this fact at his dissertation defense in 1929).

Of course, not every system with the Fuchsian singularity a_k and the local monodromy matrix G_k is holomorphically equivalent to the system (6) in a neighbourhood of this point.

Let $\Lambda_k = \operatorname{diag}(\lambda_k^1, \dots, \lambda_k^p)$ be a diagonal integer-valued matrix whose elements λ_k^j form a non-increasing sequence, and S_k a non-singular matrix reducing the matrix E_k to an upper-triangular form $E_k' = S_k E_k S_k^{-1}$. Then according to (5) the transformation

$$y' = \Gamma(z)\, y, \qquad \Gamma(z) = (z-a_k)^{\Lambda_k} S_k,$$

transforms the system (6) into the system

$$\frac{dy'}{dz} = \biggl(\frac{\Lambda_k}{z-a_k} + (z-a_k)^{\Lambda_k} \frac{E_k'}{z-a_k} (z-a_k)^{-\Lambda_k}\biggr) y', \tag{7}$$

for which the point a_k is also a Fuchsian singularity (it follows from the form of the matrices Λ_k and E_k' that the matrix $(z-a_k)^{\Lambda_k} E_k' (z-a_k)^{-\Lambda_k}$ is holomorphic) and the matrix G_k is the monodromy matrix.

According to Levelt's theorem [24], in a neighbourhood of a singular point a_k any Fuchsian system is *holomorphically* equivalent to a system of the form (7). At the same time, in a neighbourhood of a regular (in particular, Fuchsian) singular point a_k the *meromorphic* equivalence class of the system is uniquely determined by its local monodromy matrix G_k, since such a system is meromorphically equivalent to a system of the form (6).

We call a set $\{\Lambda_1, \dots, \Lambda_n, S_1, \dots, S_n\}$ of matrices having the properties described above, a set of *admissible matrices.*

The Riemann–Hilbert problem has a positive solution if it is possible to pass from the local systems (7) to a global Fuchsian system defined on the whole Riemann sphere. The use of holomorphic vector bundles and meromorphic connections proves to be effective in the study of this question.

We briefly recall the basic notions of the theory of holomorphic vector bundles (a detailed exposition of scope quite sufficient for applications to linear differential equations in the complex domain is contained in [10], as well as in [25], Ch. 3).

A holomorphic vector bundle $\pi\colon E \to B$ of rank p over a (one-dimensional) complex manifold B has the following properties:

a) for any point $z \in B$ the fibre $\pi^{-1}(z)$ is a p-dimensional vector space and there exists a neighbourhood U of z such that the inverse image $\pi^{-1}(U)$ is biholomorphically equivalent to $U \times \mathbb{C}^p$ (furthermore, the biholomorphic equivalence maps $\pi^{-1}(z)$ onto $\{z\} \times \mathbb{C}^p$ isomorphically as vector spaces);

b) for any neighbourhoods U_α, U_β with non-empty intersection, the local charts $U_\alpha \times \mathbb{C}^p$, $U_\beta \times \mathbb{C}^p$ are compatible:

$$\begin{gathered}\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^p, \qquad \varphi_\beta\colon \pi^{-1}(U_\beta) \to U_\beta \times \mathbb{C}^p;\\ \varphi_\alpha \circ \varphi_\beta^{-1}\colon (U_\alpha \cap U_\beta) \times \mathbb{C}^p \to (U_\alpha \cap U_\beta) \times \mathbb{C}^p,\\ (z, y) \mapsto \bigl(z, g_{\alpha\beta}(z)y\bigr),\end{gathered}$$

where $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(p, \mathbb{C})$ is a holomorphic map.

A set $\{g_{\alpha\beta}(z)\}$ of holomorphically invertible matrix functions satisfying the conditions

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}, \qquad g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = I \quad (\text{for } U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing),$$

is called a *gluing cocycle* corresponding to a covering $\{U_\alpha\}$ of the manifold B.

Bundles E and F over B are (holomorphically) equivalent if there exists a set $\{h_\alpha(z)\}$ of holomorphic maps $h_\alpha\colon U_\alpha \to \mathrm{GL}(p, \mathbb{C})$ such that

$$h_\alpha g_{\alpha\beta} = f_{\alpha\beta}h_\beta \tag{8}$$

for gluing cocycles $\{g_{\alpha\beta}(z)\}$, $\{f_{\alpha\beta}(z)\}$ of these bundles.

A bundle E is holomorphically trivial if it is equivalent to the direct product $B \times \mathbb{C}^p$, that is, if the relations (8) hold for a cocycle $\{g_{\alpha\beta}(z)\}$ with $f_{\alpha\beta}(z) \equiv I$ for all α, β.

A subbundle $E' \subset E$ of rank q is characterized by the condition that for any point $z \in B$ the set $\pi^{-1}(z) \cap E'$ is a q-dimensional vector subspace of $\pi^{-1}(z)$. Then the cocycle $\{g_{\alpha\beta}(z)\}$ can be chosen to be block-upper-triangular:

$$g_{\alpha\beta} = \begin{pmatrix} g_{\alpha\beta}^1 & * \\ 0 & g_{\alpha\beta}^2 \end{pmatrix},$$

where the $g_{\alpha\beta}^1$ are $q \times q$ matrices forming a cocycle of the bundle E'.

A *section* of a bundle E is defined to be a map $s\colon B \to E$ such that $\pi \circ s \equiv \mathrm{id}$. In local charts $U_\alpha \times \mathbb{C}^p$ a holomorphic (meromorphic) section is given by a set $\{s_\alpha(z)\}$ of holomorphic (meromorphic) functions $s_\alpha\colon U_\alpha \to \mathbb{C}^p$ such that $\varphi_\alpha(s(z)) = (z, s_\alpha(z))$ and satisfying the conditions $s_\alpha(z) = g_{\alpha\beta}(z)s_\beta(z)$ on the intersections $U_\alpha \cap U_\beta \neq \varnothing$.

A *holomorphic connection* $\nabla\colon \Gamma(E) \to \Gamma(\tau_B^* \otimes E)$ is a linear map of the space $\Gamma(E)$ of holomorphic sections of the bundle E into the space of holomorphic sections of the bundle $\tau_B^* \otimes E$, where τ_B^* is the cotangent bundle over B. Sections of the bundle $\tau_B^* \otimes E$ are E-valued differential 1-forms on B. Such sections are given by a set $\{\Omega_\alpha\}$ of (vector) differential 1-forms defined in corresponding neighbourhoods U_α and satisfying the conditions $\Omega_\alpha = g_{\alpha\beta}(z)\Omega_\beta$ on the intersections $U_\alpha \cap U_\beta \neq \varnothing$.

In local coordinates of the sets $U_\alpha \times \mathbb{C}^p$, a connection ∇ is given by a set $\{\omega_\alpha\}$ of matrix holomorphic differential 1-forms defined in the corresponding neighbourhoods U_α. The coordinate action of the connection ∇ on the functions $s_\alpha(z)$ defining a section s has the form

$$s_\alpha \mapsto \Omega_\alpha = ds_\alpha - \omega_\alpha s_\alpha.$$

Then the compatibility conditions for $\{s_\alpha\}$ and $\{\Omega_\alpha\}$ are rewritten for $\{\omega_\alpha\}$ as

$$\omega_\alpha = (dg_{\alpha\beta})g_{\alpha\beta}^{-1} + g_{\alpha\beta}\omega_\beta g_{\alpha\beta}^{-1} \qquad (\text{for } U_\alpha \cap U_\beta \neq \varnothing). \tag{9}$$

Similarly, a *meromorphic connection* $\nabla\colon M(E) \to M(\tau_B^* \otimes E)$ is a linear map of the corresponding spaces of meromorphic sections and is given by a set of matrix meromorphic differential 1-forms. A meromorphic connection is said to be *logarithmic* (*Fuchsian*) if all the singular points of these 1-forms are poles of the first order.

A section s is said to be *horizontal* (with respect to a connection ∇) if $\nabla(s) = 0$ or, in coordinates, $ds_\alpha = \omega_\alpha s_\alpha$. Thus, horizontal sections of a holomorphic vector bundle with a meromorphic connection are determined by solutions of a system of linear differential equations. At the same time, every set of p linearly independent sections can be regarded as a basis in the space of horizontal sections with respect to some meromorphic connection.

The *monodromy of a connection* characterizes the ramification of horizontal sections under analytic continuation along closed paths in B avoiding singular points of the connection, and its definition is similar to that of the monodromy of the system (1).

The approach to solving the Riemann–Hilbert problem based on using holomorphic vector bundles emerged in the papers of Röhrl [26], Levelt [24], Deligne [27]. It was developed by Bolibrukh and enabled him to obtain various sufficient conditions for a positive solution of the problem (some of them were given above, and one more will be given in what follows). We now briefly present this approach (see details in [9], [10]).

1. First, from the representation (2) over the punctured Riemann sphere $B = \overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$, we construct a holomorphic vector bundle F of rank p with a holomorphic connection ∇ that has the given monodromy (2). The bundle F over B is obtained from the holomorphically trivial bundle $\widetilde{B} \times \mathbb{C}^p$ over the universal covering $\widetilde{B}$ of the punctured Riemann sphere after identifications of the form $(\tilde z, y) \sim (\sigma \tilde z, \chi(\sigma) y)$, where $\tilde z \in \widetilde{B}$, $y \in \mathbb{C}^p$, and σ is an element of the group of covering transformations of $\widetilde{B}$ which is identified with the fundamental group $\pi_1(B)$. Thus, $F = \widetilde{B} \times \mathbb{C}^p/{\sim}$ and $\pi\colon F \to B$ is the natural projection.

We show that a gluing cocycle $\{g_{\alpha\beta}\}$ of the bundle F is constant. Consider a set $\{U_\alpha\}$ of small neighbourhoods covering B, and a set $\tilde z_\alpha\colon U_\alpha \to \nu^{-1}(U_\alpha)$ of local holomorphic sections of the universal covering $\nu\colon \widetilde{B} \to B$. On the non-empty intersections $U_\alpha \cap U_\beta$ the functions $\tilde z_\alpha(z)$, $\tilde z_\beta(z)$ are connected by the relation

$$\tilde z_\alpha(z) = \delta_{\alpha\beta} \tilde z_\beta(z), \qquad \delta_{\alpha\beta} \in \pi_1(B)$$

(the maps $\delta_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \pi_1(B)$ are locally constant; therefore, we can even consider them to be constant, for example, when all the $U_\alpha \cap U_\beta$ are connected). Local maps $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^p$ are given by the relations

$$\varphi_\alpha\colon [\tilde{z}_\alpha(z), y] \mapsto (z, y),$$

where $[\tilde{z}_\alpha(z), y]$ is the equivalence class of an element $(\tilde{z}_\alpha(z), y) \in \widetilde{B}\times\mathbb{C}^p$. Therefore, for $z \in U_\alpha \cap U_\beta$ we have

$$\begin{aligned}\varphi_\alpha \circ \varphi_\beta^{-1}(z,y) &= \varphi_\alpha\bigl([\tilde{z}_\beta(z), y]\bigr) = \varphi_\alpha\bigl([\delta_{\alpha\beta}^{-1}\tilde{z}_\alpha(z), y]\bigr) \\ &= \varphi_\alpha\bigl([\tilde{z}_\alpha(z), \chi(\delta_{\alpha\beta})y]\bigr) = \bigl(z, \chi(\delta_{\alpha\beta})y\bigr),\end{aligned}$$

that is, $g_{\alpha\beta}(z) = \chi(\delta_{\alpha\beta}) = \text{const}$.

The connection ∇ can now be given by the set $\{\omega_\alpha\}$ of matrix differential 1-forms $\omega_\alpha \equiv 0$, which obviously satisfy the gluing conditions (9) on the intersections $U_\alpha \cap U_\beta \neq \varnothing$. Furthermore, it follows from the construction of the bundle F that the monodromy of the connection ∇ coincides with χ.

2. Next, the pair (F, ∇) is extended to a bundle F^0 with a logarithmic connection ∇^0 over the whole Riemann sphere. For this, the set $\{U_\alpha\}$ should be supplemented by small neighbourhoods $O_1, \dots, O_n$ of the points $a_1, \dots, a_n$, respectively. An extension of the bundle F to each point a_i looks as follows. For some non-empty intersection $O_i \cap U_\alpha$ we set $g_{i\alpha}(z) = (z - a_i)^{E_i}$ in this intersection. For any other neighbourhood U_β that intersects O_i we define $g_{i\beta}(z)$ as the analytic continuation of the matrix function $g_{i\alpha}(z)$ into $O_i \cap U_\beta$ along a suitable path (so that the set $\{g_{\alpha\beta}, g_{i\alpha}(z)\}$ defines a cocycle for the covering $\{U_\alpha, O_i\}$ of the Riemann sphere). An extension of the connection ∇ to each point a_i is given by the matrix differential 1-form $\omega_i = \dfrac{E_i}{z - a_i}\,dz$, which has a simple pole at this point. Then the set $\{\omega_\alpha, \omega_i\}$ defines a logarithmic connection ∇^0 in the bundle F^0, since along with conditions (9) for non-empty $U_\alpha \cap U_\beta$ the conditions

$$(dg_{i\alpha})g_{i\alpha}^{-1} + g_{i\alpha}\omega_\alpha g_{i\alpha}^{-1} = \frac{E_i}{z - a_i}\,dz = \omega_i, \qquad O_i \cap U_\alpha \neq \varnothing,$$

also hold (see (6)). The pair (F^0, ∇^0) is called the *canonical extension* of the pair (F, ∇).

3. In a way similar to that for the construction of the pair (F^0, ∇^0), we can construct the family $\mathscr{F}$ of bundles F^Λ with logarithmic connections ∇^Λ having given singularities and monodromy. For this the matrices $g_{i\alpha}(z)$ in the construction of the pair (F^0, ∇^0) must be replaced by the matrices

$$g_{i\alpha}^\Lambda(z) = (z - a_i)^{\Lambda_i} S_i (z - a_i)^{E_i},$$

and the forms ω_i by the forms

$$\omega_i^\Lambda = \bigl(\Lambda_i + (z - a_i)^{\Lambda_i} E_i'(z - a_i)^{-\Lambda_i}\bigr)\frac{dz}{z - a_i}\,,$$

where $\{\Lambda_1, \dots, \Lambda_n, S_1, \dots, S_n\}$ are all possible sets of admissible matrices. Then the conditions

$$\bigl(dg_{i\alpha}^\Lambda\bigr)\bigl(g_{i\alpha}^\Lambda\bigr)^{-1} + g_{i\alpha}^\Lambda\omega_\alpha\bigl(g_{i\alpha}^\Lambda\bigr)^{-1} = \omega_i^\Lambda \tag{10}$$

again hold on the non-empty intersections $O_i \cap U_\alpha$ (see (7)).

Strictly speaking, the bundle F^Λ also depends on the set $S=\{S_1,\dots,S_n\}$ of matrices S_i reducing the monodromy matrices G_i to upper-triangular form. In view of this dependence the bundles in the family $\mathscr{F}$ should therefore be denoted by $F^{\Lambda,S}$, but in what follows we shall only need the dependence on the set $\Lambda=\{\Lambda_1,\dots,\Lambda_n\}$, and by F^Λ we shall mean the bundle constructed with respect to a given set Λ and some set S. (This does not apply to the canonical extension F^0 of the bundle F, which is independent of the choice of the matrices S_i.)

Definition 2. The eigenvalues $\beta_i^j=\lambda_i^j+\rho_i^j$ of the matrix Λ_i+E_i' are called *exponents* of the logarithmic connection ∇^Λ at the point $z=a_i$. (It follows from the structure of the forms ω_i^Λ that the exponents at the point $z=a_i$ are the eigenvalues of the residue matrix $\operatorname{res}_{a_i}\omega_i^\Lambda$.)

The above-mentioned dependence of the bundle F^Λ on the sets $S=\{S_1,\dots,S_n\}$ is essential only in the case where at least one of the singular points a_i of the connection ∇^Λ is a *resonant* singularity (that is, where among the exponents of the connection at this point there exist two that differ by a positive integer). This observation follows from the fact that the holomorphic equivalence class of a Fuchsian system of the form (7) in a neighbourhood of a non-resonant singular point a_i is uniquely determined by the local monodromy of the system at this point and an admissible matrix Λ_i (see, for example, [10], Exercise 14.5).

If some bundle F^Λ in $\mathscr{F}$ is holomorphically trivial, then the corresponding connection ∇^Λ determines a global system (1) that solves the Riemann–Hilbert problem. Indeed, the triviality of the bundle F^Λ means that for the covering $\{U_\alpha,O_i\}$ of the Riemann sphere there exists a corresponding set $\{h_\alpha(z),h_i(z)\}$ of holomorphically invertible matrices such that

$$h_\alpha(z)g_{\alpha\beta}=h_\beta(z)\qquad\text{and}\qquad h_i(z)g_{i\alpha}^\Lambda(z)=h_\alpha(z)\tag{11}$$

in $U_\alpha\cap U_\beta\neq\varnothing$ and $O_i\cap U_\alpha\neq\varnothing$, respectively. These relations, along with conditions (10), imply that the forms

$$\omega_i'=(dh_i)h_i^{-1}+h_i\omega_i^\Lambda h_i^{-1},\qquad \omega_\alpha'=(dh_\alpha)h_\alpha^{-1}+h_\alpha\omega_\alpha h_\alpha^{-1}\tag{12}$$

coincide on the corresponding non-empty intersections, which fact defines a global form $\omega=B(z)\,dz$ on the whole Riemann sphere. By construction, the global system $dy=\omega y$ has Fuchsian singularities $a_1,\dots,a_n$ and the given monodromy (2).

On the other hand, in view of Levelt's theorem mentioned above, the existence of a global Fuchsian system with the given singular points $a_1,\dots,a_n$ and monodromy (2) implies the triviality of some bundle of the family $\mathscr{F}$.

Thus, *the Riemann–Hilbert problem is soluble if and only if at least one of the bundles of the family $\mathscr{F}$ is holomorphically trivial.*

According to the Birkhoff–Grothendieck theorem, every holomorphic vector bundle E of rank p over the Riemann sphere is equivalent to a direct sum

$$E\cong\mathscr{O}(k_1)\oplus\dots\oplus\mathscr{O}(k_p)$$

of line bundles that has a coordinate description of the form

$$\bigl(U_1=\mathbb{C},\ U_\infty=\overline{\mathbb{C}}\setminus\{a_1\},\ g_{1\infty}=(z-a_1)^K\bigr),\qquad K=\operatorname{diag}(k_1,\dots,k_p),$$

where $k_1 \geqslant \dots \geqslant k_p$ is a tuple of integers, which is called the *splitting type* of the bundle E.

This implies that for a cocycle $\{g_{\alpha\beta}, g^{\Lambda}_{i\alpha}(z)\}$ defining the bundle F^{Λ} in the family $\mathscr{F}$ there exists a set $\{h_\alpha(z), h'_i(z)\}$ of holomorphically invertible matrices such that

$$h_\alpha(z)g_{\alpha\beta} = h_\beta(z), \qquad h'_1(z)g^{\Lambda}_{1\alpha}(z) = (z-a_1)^K h_\alpha(z),$$
$$h'_i(z)g^{\Lambda}_{i\alpha}(z) = h_\alpha(z), \quad i \neq 1,$$

in the corresponding non-empty intersections (where the matrix K defines the splitting type of the bundle F^{Λ}). These relations can be written in the form (11), where all the matrices $h_i(z)$, except $h_1(z)$, are holomorphically invertible in corresponding neighbourhoods O_i and the matrix $h_1(z)$ has the form $h_1(z) = (z-a_1)^{-K}h'_1(z)$.

Consequently, there always exists a global system $dy = \omega y$ with the given singularities $a_1, \dots, a_n$ and monodromy (2), except that its singular point a_1 may be non-Fuchsian, since

$$\omega = -\frac{K}{z-a_1}\,dz + (z-a_1)^{-K}\omega'_1(z-a_1)^K \tag{13}$$

in O_1, where $\operatorname{ord}_{a_1}\omega'_1 = -1$. This result is called Plemelj's theorem.

Theorem 1 (Plemelj [12]). *A given set of points $a_1, \dots, a_n$ and a representation (2) can always be realized by a system (1) that is Fuchsian at all but possibly one point, at which the system is regular.*

We note that the system in Plemelj's theorem can always be chosen so that its Poincaré rank at the regular singular point is at most $(n-1)(p-1)$ (see [28]).

The notion of stability (introduced by Mumford) turns out to be useful in dealing with vector bundles. Recently this notion made it possible to obtain a number of important results, in particular, in the study of the Riemann–Hilbert problem. We now give the requisite preliminary definitions.

Every holomorphic vector bundle over a Riemann surface X has a global meromorphic section that is not identically equal to zero (see [29], Corollary 29.17 and Proposition 30.4). The *degree* of a holomorphic *linear* bundle E_1 (that is, a bundle of rank 1) is defined as the sum of the orders of the zeros minus the sum of the orders of the poles of a global meromorphic section of this bundle. (It is easy to show that this number is independent of the choice of the section if X is compact, which is assumed in what follows.)

If a global meromorphic section of the bundle E_1 is given by a set of meromorphic functions $s_\alpha\colon U_\alpha \to \mathbb{C}$ corresponding to a covering $\{U_\alpha\}$ of the surface X, then the set $\{ds_\alpha/s_\alpha\}$ of differential 1-forms determines a meromorphic connection in the bundle E_1. Thus, the degree of the bundle E_1 coincides with the sum of residues of this connection. It is also easy to show that this sum is the same for any meromorphic connection in E_1.

Definition 3. The *degree* $\deg E$ of a bundle E (over a compact Riemann surface X) given by a cocycle $\{g_{\alpha\beta}(z)\}$ is the degree of the *determinant* bundle $\det E$ given by the cocycle $\{\det g_{\alpha\beta}(z)\}$.

If in a bundle E a connection ∇ is determined by a set $\{\omega_\alpha\}$ of matrix differential 1-forms, then in the bundle $\det E$ one can define the connection $\operatorname{tr}\nabla$ by the set of differential 1-forms $\operatorname{tr}\omega_\alpha$. In this case the degree of the bundle E also coincides with the sum of residues of the connection $\operatorname{tr}\nabla$.

The number $\varkappa(E) = \deg E/p$ is called the *slope* of a bundle E of rank p.

For the bundle F^Λ (according to Definition 2) we have

$$\deg F^\Lambda = \sum_{i=1}^{n} \operatorname{res}_{a_i} \operatorname{tr}\omega_i^\Lambda = \sum_{i=1}^{n}\sum_{j=1}^{p} \beta_i^j.$$

The degree of a holomorphic vector bundle E over the Riemann sphere coincides with the sum of the coefficients k_i of the splitting type of this bundle. Indeed, if $E \cong \mathscr{O}(k_1) \oplus \cdots \oplus \mathscr{O}(k_p)$, then the bundle $\det E$ has a coordinate description of the form

$$\bigl(U_0 = \mathbb{C},\ U_\infty = \overline{\mathbb{C}} \setminus \{0\},\ g_{0\infty} = z^{\operatorname{tr} K}\bigr), \qquad K = \operatorname{diag}(k_1, \dots, k_p).$$

A global meromorphic section of this bundle can be given by the functions $s_0(z) = z^{\operatorname{tr} K}$, $s_\infty(z) = 1$.

Definition 4. A vector bundle E is said to be *stable* (*semistable*) if the slope $\varkappa(E')$ of any proper subbundle E' of it is less than (respectively, less than or equal to) the slope $\varkappa(E)$.

It is easy to verify that over the Riemann sphere there are no stable vector bundles (the slope k_1 of the subbundle $\mathscr{O}(k_1)$ of a bundle $E \cong \mathscr{O}(k_1) \oplus \cdots \oplus \mathscr{O}(k_p)$ is not less than the slope $(k_1 + \cdots + k_p)/p$ of the bundle E), and only bundles with splitting type $(k, \dots, k)$ are semistable. Thus, holomorphically trivial bundles over the Riemann sphere are the semistable bundles of degree zero, and only they.

As is evident, the notions of stability and semistability of a vector bundle over the Riemann sphere are not very deep in meaning, but they will prove to be meaningful and useful for generalizing the problem to the case of a compact Riemann surface of positive genus. For the present, we recall that in solving the Riemann–Hilbert problem it is not just bundles that are investigated, but bundles together with logarithmic connections on them. The introduction of the notion of a *stable pair* (E, ∇) consisting of a vector bundle and a connection has proved effective.

Definition 5. We say that a subbundle F' of the bundle F^Λ *is stabilized* by the connection ∇^Λ if $\nabla^\Lambda(M(F')) \subset M(\tau^*_{\overline{\mathbb{C}}} \otimes F')$. (Generally speaking, the action of the map ∇^Λ takes a section of the subbundle F' to a section of the bundle $\tau^*_{\overline{\mathbb{C}}} \otimes F^\Lambda$.)

This definition is equivalent to the condition that every horizontal section of F^Λ passing through a point $s_0 \in F'$ remains in F' under analytic continuations.

Practically, the existence of a subbundle $F' \subset F^\Lambda$ that is stabilized by the connection means that the pair $(F^\Lambda, \nabla^\Lambda)$ admits a coordinate description $\{g_{\alpha\beta}, g_{i\alpha}^\Lambda\}$, $\{\omega_\alpha, \omega_i^\Lambda\}$ of the following block-upper-triangular form:

$$g_{\alpha\beta}\ (g_{i\alpha}^\Lambda) = \begin{pmatrix} g_{\alpha\beta}^1\ (g_{i\alpha}^1) & * \\ 0 & g_{\alpha\beta}^2\ (g_{i\alpha}^2) \end{pmatrix}, \qquad \omega_\alpha\ (\omega_i^\Lambda) = \begin{pmatrix} \omega_\alpha^1\ (\omega_i^1) & * \\ 0 & \omega_\alpha^2\ (\omega_i^2) \end{pmatrix},$$

where the sizes of all the blocks $g_{\alpha\beta}^1$, $g_{i\alpha}^1$ and ω_α^1, ω_i^1 are the same (the cocycles $\{g_{\alpha\beta}^1, g_{i\alpha}^1\}$ determine a coordinate description of the subbundle F', and the forms ω_α^1, ω_i^1 determine the restriction ∇' to F' of the connection ∇^Λ).

Definition 6. A pair $(F^\Lambda,\nabla^\Lambda)\in\mathscr{F}$ is said to be *stable* (respectively, *semistable*) if the inequality $\varkappa(F')<\varkappa(F^\Lambda)$ (respectively, $\varkappa(F')\leqslant\varkappa(F^\Lambda)$) holds for any proper subbundle $F'\subset F^\Lambda$ that is stabilized by the connection ∇^Λ.

We now give another sufficient condition for a positive solution of the Riemann–Hilbert problem.

5) *If among the elements of the family $\mathscr{F}$ of bundles with logarithmic connections having given singularities $a_1,\dots,a_n$ and monodromy* (2) *there exists at least one stable pair $(F^\Lambda,\nabla^\Lambda)$, then the Riemann–Hilbert problem has a positive solution* (Bolibrukh [30]).

Thus, for studying the Riemann–Hilbert problem by using the notions of stability and semistability of pairs of the family $\mathscr{F}$, one can consider the subset $\mathscr{F}^0\subset\mathscr{F}$ consisting of semistable pairs.

If $\mathscr{F}^0$ is empty, then the Riemann–Hilbert problem has a negative solution (no semistable pairs $\Longrightarrow$ no semistable bundles $\Longrightarrow$ no trivial bundles).

If $\mathscr{F}^0$ is not empty but contains no stable pairs, then the question of solubility of the Riemann–Hilbert problem remains open.

If $\mathscr{F}^0$ contains at least one stable pair, then the Riemann–Hilbert problem has a positive solution.

It should be noted that for finding stable or semistable pairs one does not have to go through all the elements of the family $\mathscr{F}$ but can confine oneself to the pairs $(F^\Lambda,\nabla^\Lambda)$ constructed with respect to sets $\Lambda=\{\Lambda_1,\dots,\Lambda_n\}$ of matrices whose elements do not exceed a certain constant $N(n,p)$ depending only on the dimension p of the representation (2) and the number n of singular points. In some special cases this enables one to produce an effective algorithm for verifying the existence of a stable or semistable pair among elements of the family $\mathscr{F}$ (see [31]).

Bolibrukh's first counterexample to the Riemann–Hilbert problem ($p=3,\ n=4$) is fairly complicated technically. Subsequently he obtained simpler counterexamples for representations of special type — *B-representations*. These are reducible representations whose generators G_i have Jordan form consisting of exactly one block. The counterexamples are based on the following theorem (see [10], Theorem 11.2 and Corollary 11.2).

Theorem 2 (Bolibrukh). *If a B-representation can be realized as the monodromy representation of some Fuchsian system, then the slope of the canonical extension F^0 of the bundle F constructed from this representation is an integer.*

Example 1 (Bolibrukh). Consider the matrices

$$G_1=\begin{pmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix},\qquad G_2=\begin{pmatrix}3&1&1&-1\\-4&-1&1&2\\0&0&3&1\\0&0&-4&-1\end{pmatrix},$$

$$G_3=\begin{pmatrix}-1&0&2&-1\\4&-1&0&1\\0&0&-1&0\\0&0&4&-1\end{pmatrix}.$$

and an arbitrary set of points a_1, a_2, a_3. The reducible representation χ with singular points a_1, a_2, a_3 and generators G_1, G_2, G_3 corresponding to loops around these points cannot be realized as the monodromy representation of any Fuchsian system.

We note that $G_1G_2G_3 = I$, the matrix G_2 can be transformed into the matrix G_1, and the matrix G_3 can be transformed into a Jordan block with eigenvalue -1:

$$S_2^{-1}G_2S_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S_2 = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 & 0 \\ -6 & 3 & -3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 3 \end{pmatrix},$$

$$S_3^{-1}G_3S_3 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad S_3 = \frac{1}{64}\begin{pmatrix} 0 & 16 & 4 & 3 \\ 64 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & -16 & -12 \end{pmatrix}.$$

The eigenvalues ρ_k of the matrix $E_k = (2\pi i)^{-1} \log G_k$ are

$$\rho_1 = \rho_2 = 0, \qquad \rho_3 = \frac{1}{2}\,.$$

According to Definition 2, they coincide with the exponents β_k of the logarithmic connection ∇^0 at the points a_k. According to Definition 3, the degree of the canonical extension F^0 is equal to

$$\deg F^0 = 4\beta_1 + 4\beta_2 + 4\beta_3 = 2,$$

and consequently $\varkappa(F^0) = 1/2 \notin \mathbb{Z}$. Therefore, the B-representation χ cannot be realized as the monodromy representation of any Fuchsian system (by Theorem 2).

§ 3. The Riemann–Hilbert problem for scalar Fuchsian equations

As mentioned earlier, the problem of constructing a Fuchsian differential equation (3) with given singularities $a_1, \dots, a_n$ and the monodromy (2) has a negative solution in the general case, since the number of parameters on which such an equation depends is less than the number of parameters on which the set of conjugacy classes of representations (2) depends. (We recall that the latter is equal to $(n-2)p^2+1$, while the former does not exceed $p+(n-2)p(p+1)/2$; see [10], Proposition 7.1.) Therefore, questions arise about estimating the number of additional singular points of a Fuchsian equation with a given monodromy, as well as about finding conditions under which the construction of an equation without additional singularities is nevertheless possible.

We consider the family $\mathscr{F}$ of holomorphic vector bundles F^Λ with the logarithmic connections ∇^Λ constructed from the representation (2). The *Fuchsian weight* of the bundle F^Λ is defined as the quantity

$$\gamma(F^\Lambda) = \sum_{i=1}^{p} (k_1 - k_i),$$

where $(k_1, \dots, k_p)$ is the splitting type of F^Λ.

If the pair $(F^\Lambda, \nabla^\Lambda) \in \mathscr{F}$ is stable, then the splitting type of the bundle F^Λ satisfies the inequalities

$$k_i - k_{i+1} \leqslant n-2, \qquad i = 1, \ldots, p-1 \tag{14}$$

(see [10], Theorem 11.1). Since in the case of an irreducible representation (2) the family $\mathscr{F}$ consists only of stable pairs (the bundle F^Λ has no subbundles stabilized by the connection ∇^Λ if the representation is irreducible), the quantity

$$\gamma_{\max}(\chi) = \max_{F^\Lambda \in \mathscr{F}} \gamma(F^\Lambda) \leqslant \frac{(n-2)p(p-1)}{2}$$

is defined for such a representation, and is called the *maximal Fuchsian weight* of the irreducible representation χ.

Bolibrukh obtained an expression for the minimal possible number m_0 of additional singular points emerging in the construction of a Fuchsian equation (3) from an *irreducible* representation (2) [32]:

$$m_0 = \frac{(n-2)p(p-1)}{2} - \gamma_{\max}(\chi). \tag{15}$$

In the case of an arbitrary representation there is the following estimate for the number m_0 of additional singularities (see [28]):

$$m_0 \leqslant \frac{(n+1)p(p-1)}{2} + 1.$$

In particular, it follows from formula (15) that *a set of singular points* a_1, a_2, a_3 $(n = 3)$ *and an irreducible two-dimensional representation* $(p = 2)$ *can always be realized by a Fuchsian differential equation of second order*, since in this case $\gamma(F^\Lambda) = 1$ for any bundle F^Λ of odd degree. As shown in [33] (see also [34]), among reducible two-dimensional representations (with three generators G_1, G_2, G_3 and the relation $G_1G_2G_3 = I$) there are two types that cannot be realized by a Fuchsian differential equation of second order with three singular points:

$$G_1 = \begin{pmatrix} c_1 & d_1 \\ 0 & c_1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} c_2 & d_2 \\ 0 & c_2 \end{pmatrix}, \quad G_3 = \begin{pmatrix} c_3 & d_3 \\ 0 & c_3 \end{pmatrix}, \qquad d_i \neq 0; \tag{16}$$

$$G_1 = \begin{pmatrix} c_1 & 0 \\ 0 & d_1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} c_2 & 0 \\ 0 & d_2 \end{pmatrix}, \quad G_3 = \begin{pmatrix} c_3 & 0 \\ 0 & d_3 \end{pmatrix}, \qquad c_i \neq d_i. \tag{17}$$

It also follows from formula (15) that *a set of singular points* $a_1, \ldots, a_n$ *and an irreducible representation* (2) *can be realized by a Fuchsian differential equation* (3) *if and only if among the elements of the family* $\mathscr{F}$ *there exists a bundle with splitting type* $((p-1)(n-2), (p-2)(n-2), \ldots, n-2, 0)$.

Since in the case of an irreducible representation (2) the family $\mathscr{F}$ coincides with the family $\mathscr{F}^{\mathrm{st}} \subset \mathscr{F}$ whose elements are only stable pairs, the following theorem is a generalization of the last assertion to the case of an arbitrary representation.

Theorem 3 (V'yugin [35]). *A set of singular points* $a_1, \ldots, a_n$ *and a representation* (2) *can be realized by a Fuchsian differential equation* (3) *if and only if among the elements of the family* $\mathscr{F}$ *there exists a stable pair* $(F^\Lambda, \nabla^\Lambda)$ *such that the splitting type of the bundle* F^Λ *is equal to* $\big((p-1)(n-2), (p-2)(n-2), \ldots, n-2, 0\big)$.

The following question now becomes natural: $\boxed{1}$ *Is it possible to generalize formula* (15) *to the case of an arbitrary representation and assert that*

$$m_0 = \frac{(n-2)p(p-1)}{2} - \max_{F^\Lambda \in \mathscr{F}^{\mathrm{st}}} \gamma(F^\Lambda)\ ?$$

Example 2 (V'yugin). Using Theorem 3 one can easily show that two-dimensional representations with three generators of the form (16) or (17) indeed cannot be realized by a Fuchsian differential equation of second order with three singular points. This follows from the fact that the family $\mathscr{F}$ constructed from each of these representations does not contain stable pairs.

Indeed, a representation with generators of the form (16) is a B-representation, and a stable pair cannot be constructed from such a representation (this follows from the proof of Theorem 11.2 in [10]).

As for a representation with generators of the form (17), it is the direct sum $\chi = \chi_1 \oplus \chi_2$ of two (one-dimensional) representations with generators G_i^1, G_i^2. Since the spectra of the matrices G_i^1 and G_i^2 are disjoint ($c_i \neq d_i$) for each $i = 1, 2, 3$, the degree of any bundle F^Λ in the family $\mathscr{F}$ satisfies the relation

$$\deg F^\Lambda = \deg F_1 + \deg F_2,$$

where F_1 and F_2 are the (one-dimensional) subbundles corresponding to the subrepresentations χ_1 and χ_2, respectively, that is, are stabilized by the connection ∇^Λ (this equality follows from the fact that in this case the set of exponents of the connection ∇^Λ is the union of the sets of exponents of its restrictions to F_1 and F_2). Consequently, the inequalities $\deg F_1 < \deg F^\Lambda/2$ and $\deg F_2 < \deg F^\Lambda/2$, which are necessary for the stability of the pair $(F^\Lambda, \nabla^\Lambda)$, cannot both hold.

By using Theorem 3 one can also prove (we do not do this here) that the remaining types of reducible two-dimensional representations with three generators can be realized by a Fuchsian differential equation of second order with three singular points.

We mention the interesting connection between the Riemann–Hilbert problem for scalar Fuchsian equations and the *Painlevé* VI *equation* ($\mathrm{P_{VI}}$) — the non-linear differential equation of second order

$$\frac{d^2u}{dt^2} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\left(\frac{du}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)\frac{du}{dt} + \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha + \beta\,\frac{t}{u^2} + \gamma\,\frac{t-1}{(u-1)^2} + \delta\,\frac{t(t-1)}{(u-t)^2}\right) \qquad (18)$$

with respect to the unknown function $u(t)$. Only poles can be *movable singularities* of solutions of this equation (whose locations depend on the initial conditions). In such a case the equation is said to have the *Painlevé property*.[4] Painlevé himself [37] initially studied the special case of equation (18) corresponding to the parameter

[4]Apparently, the idea of using this type of restriction on solutions first appeared in Kovalevskaya's paper [36] in a study of the problem of integration of a spinning top.

values $\alpha = \beta = \gamma = 0$, $\delta = 1/2$. In the general form (18), the equation P_{VI} was first written by R. Fuchs [38] (a son of L. Fuchs) and was added by Gambier [39], a student of Painlevé, to the list of equations now known as the Painlevé I–VI equations. Among the non-linear differential equations of second order that have the Painlevé property, the equations of this list are distinguished by the fact that in the general case their solutions cannot be expressed in terms of elementary or classical special functions (it is assumed that the right-hand sides of the equations are rational in $\dfrac{du}{dt}$ and meromorphic in u, t). R. Fuchs proposed two methods for obtaining the equation P_{VI}. The first method, on which we focus here, is related to isomonodromic deformations of linear differential equations. The second, more geometric, approach uses elliptic integrals.

Let us consider the four points t, 0, 1, ∞ (here $t \in D(t^*)$, where $D(t^*) \subset \mathbb{C} \setminus \{0,1\}$ is a disc of a small radius with centre at a point t^*) and the *irreducible* $\mathrm{SL}(2,\mathbb{C})$-representation

$$\chi^*\colon \pi_1(\mathbb{C} \setminus \{t,0,1\}) \to \mathrm{SL}(2,\mathbb{C})$$

generated by matrices G_1, G_2, G_3 corresponding to the points t, 0, 1.

Depending on the location of the point t, there are two possible cases.

1) *Every vector bundle F^Λ in the family $\mathscr{F}$ constructed with respect to the given four points and the representation χ^* such that* $\deg F^\Lambda = 0$ *is holomorphically trivial* (this is the case for almost all values $t \in D(t^*)$; see [10], Exercise 16.4).

2) *Among the elements of the family $\mathscr{F}$ there exists a holomorphically non-trivial bundle F^Λ of degree zero.*

It follows from the inequality (14) that $\gamma_{\max}(\chi^*) \leqslant 2$; therefore in the first case the splitting types of holomorphically non-trivial bundles F^Λ (of non-zero degree) can only be $(k, k-1)$ or (k, k). The case $(k+1, k-1)$ is impossible, since then the bundle constructed with respect to the set of matrices $\Lambda_1 - kI$, Λ_2, Λ_3, Λ_4 has degree zero, that is, is holomorphically trivial, but at the same time its splitting type is equal to $(1, -1)$. Consequently, $\gamma_{\max}(\chi^*) = 1$ in the first case.

In the second case the splitting type of the holomorphically non-trivial bundle of degree zero is equal to $(1, -1)$, and $\gamma_{\max}(\chi^*) = 2$ in this case.

Thus, in view of formula (15), for almost all values $t \in D(t^*)$ the set of points t, 0, 1, ∞ and the irreducible two-dimensional representation χ^* are realized by a Fuchsian differential equation of second order with one additional singularity. We denote this singularity by $u(t)$ (regarding it as a function of the parameter t). It turns out that the function $u(t)$ satisfies equation (18) for some values of the constants α, β, γ, δ. (The equation P_{VI} was obtained by R. Fuchs precisely as a differential equation that is satisfied by the additional (fifth) singularity $\lambda(t)$ of some Fuchsian equation of second order with singular points 0, 1, t, ∞ and with monodromy independent of the parameter t.) This interesting fact can be explained by using *isomonodromic deformations of Fuchsian systems*.

We choose a value $t = t^0$ for which there exists a Fuchsian system

$$\frac{dy}{dz} = \left(\frac{B_1}{z - t^0} + \frac{B_2}{z} + \frac{B_3}{z - 1}\right) y$$

with singular points t^0, 0, 1, ∞ and with monodromy representation χ^* such that $\operatorname{tr} B_i = 0$ and the matrix $B_\infty = -B_1 - B_2 - B_3$ is diagonal. We denote by $\pm\theta_i/2$ the eigenvalues of the matrices B_i.

It is not difficult to show that such a system exists. It suffices to produce a set $\Lambda = \{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_\infty\}$ of admissible matrices such that $\operatorname{tr}(\Lambda_i + E_i) = 0$, $i = 1, 2, 3, \infty$, and the eigenvalues of the matrix $\Lambda_\infty + E_\infty$ are non-zero (then for almost all values of t^0 the corresponding bundle F^Λ is holomorphically trivial, and the logarithmic connection ∇^Λ determines a Fuchsian system that has the required properties). Since $\det G_i = 1$, the sum $\rho_i^1 + \rho_i^2$ of eigenvalues of the matrix E_i is an integer, equal to 0 or 1 in view of the condition $0 \leqslant \operatorname{Re} \rho_i^j < 1$. In the first case it remains to set $\Lambda_i = \operatorname{diag}(1, -1)$, and in the second $\Lambda_i = \operatorname{diag}(0, -1)$.

This system can be embedded in Schlesinger's [40] *isomonodromic family*[5]

$$\frac{dy}{dz} = \left(\frac{B_1(t)}{z-t} + \frac{B_2(t)}{z} + \frac{B_3(t)}{z-1}\right) y, \qquad B_i(t^0) = B_i, \tag{19}$$

of Fuchsian systems with singularities $t, 0, 1, \infty$ which depends holomorphically on the parameter $t \in D(t^0)$, where $D(t^0)$ is a disc of small radius with centre at the point t^0. Here $B_1(t) + B_2(t) + B_3(t) = -B_\infty = \operatorname{diag}(-\theta_\infty/2, \theta_\infty/2)$. Malgrange [41] showed that the matrix functions $B_i(t)$ can be extended as meromorphic functions to the universal covering T of the space $\mathbb{C} \setminus \{0, 1\}$. The set $\Theta \subset T$ of their poles is called the *Malgrange Θ-divisor*.

We denote by $B(z,t) = (b_{ij}(z,t))$ the coefficient matrix of the family (19). Since the upper-right element of the matrix $B_1(t) + B_2(t) + B_3(t) = -B_\infty$ is equal to zero, for each fixed t the same element of the matrix $z(z-1)(z-t)B(z,t)$ is a first-degree polynomial in z. We define $\tilde{u}(t)$ as the unique root of this polynomial. Next we use the following theorem in [42] (see also [10], Theorem 18.1).

Theorem 4. *The function $\tilde{u}(t)$ satisfies the equation* P_{VI} (18), *where the constants α, β, γ, δ are connected with the parameters θ_1, θ_2, θ_3, θ_∞ by the relations*

$$\alpha = \frac{(\theta_\infty - 1)^2}{2}, \qquad \beta = -\frac{\theta_2^2}{2}, \qquad \gamma = \frac{\theta_3^2}{2}, \qquad \delta = \frac{1 - \theta_1^2}{2}.$$

Let us consider the row vectors

$$q_0 = (1, 0), \qquad q_1(z,t) = \frac{dq_0}{dz} + q_0 B(z,t) = (b_{11}, b_{12})$$

and the matrix composed from them,

$$\Gamma(z,t) = \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_{11} & b_{12} \end{pmatrix},$$

which is meromorphically invertible on $\overline{\mathbb{C}} \times D(t^*)$, since $\det \Gamma(z,t) = b_{12} \not\equiv 0$ by the irreducibility of the representation χ^*. We define meromorphic functions $b_1(z,t)$

[5]Isomonodromic means that the monodromy of systems of this family is independent of the value of the parameter t. Moreover, the eigenvalues of the matrices $B_i(t)$ are also independent of t and coincide with the eigenvalues $\pm\theta_i/2$ of the matrices $B_i(t^0) = B_i$ (see details in [10], Lectures 13, 14).

and $b_2(z,t)$ on $\overline{\mathbb{C}} \times D(t^*)$ so that the relation

$$q_2 := \frac{dq_1}{dz} + q_1 B(z,t) = (-b_2, -b_1)\Gamma(z,t)$$

holds. Then

$$\frac{d\Gamma}{dz} = \frac{d}{dz}\begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} - \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} B(z,t) = \begin{pmatrix} 0 & 1 \\ -b_2 & -b_1 \end{pmatrix}\Gamma - \Gamma B(z,t),$$

whence,

$$\begin{pmatrix} 0 & 1 \\ -b_2 & -b_1 \end{pmatrix} = \frac{d\Gamma}{dz}\Gamma^{-1} + \Gamma B \Gamma^{-1}.$$

The latter means that for each fixed $t \in D(t^0)$ the gauge transformation $y' = \Gamma(z,t)y$ transforms the corresponding system of the family (19) into the system

$$\frac{dy'}{dz} = \begin{pmatrix} 0 & 1 \\ -b_2 & -b_1 \end{pmatrix} y',$$

the first coordinate of whose solution is, as usual, a solution of the scalar equation

$$\frac{d^2 w}{dz^2} + b_1(z,t)\frac{dw}{dz} + b_2(z,t)w = 0.$$

This (Fuchsian) equation has singular points t, 0, 1, ∞ and monodromy χ^*, but it also has an additional singularity $u(t)$ — this is a zero of the function $\det \Gamma(z,t) = b_{12}(z,t)$, as follows from the construction of the functions $b_1(z,t)$, $b_2(z,t)$. By Theorem 4 the function $u(t)$ satisfies the equation $\mathrm{P_{VI}}$. (We remark that $u(t) \neq t, 0, 1, \infty$ if $t \in D(t^*) \setminus \widetilde{\Theta}$, where $\widetilde{\Theta}$ is the countable set consisting of the values of t such that the corresponding family $\mathscr{F}$ constructed with respect to the singularities $t, 0, 1, \infty$ and the representation χ^* contains a non-trivial bundle of degree zero.)

Thus, the *set of points* $t, 0, 1, \infty$ *and the irreducible* $\mathrm{SL}(2,\mathbb{C})$*-representation* χ^* *are realized by a scalar Fuchsian equation with additional singularity* $u(t)$ *which (as a function of the parameter* $t \in D(t^*)$*) satisfies the equation* $\mathrm{P_{VI}}$*. The singular points of the function* $u(t)$ *extended to* T *are poles, and the set* $\widetilde{\Theta} \supset \{t \in D(t^*) \mid u(t) = t,\ 0,\ 1,$ *or* $\infty\}$ *is a countable set of parameter values for which the Riemann–Hilbert problem under consideration is soluble without additional singularities.*

Without dwelling on this here, we point out that the arguments given above can be extended to the general case of $n+3$ singular points $a_1, \dots, a_n$, $a_{n+1} = 0$, $a_{n+2} = 1$, $a_{n+3} = \infty$ and an *irreducible* $\mathrm{SL}(2,\mathbb{C})$-representation. Then the additional singularities $u_1(a), \dots, u_n(a)$ of the scalar Fuchsian equation realizing them (as functions of the parameter $a = (a_1, \dots, a_n)$) satisfy a Garnier system [43]: a system of non-linear partial differential equations of second order (coinciding for $n = 1$ with the equation $\mathrm{P_{VI}}$).

Both the equations P_{VI} and their multi-dimensional generalizations — Garnier systems — can be written in Hamiltonian form. For example, using the second of the aforementioned approaches of R. Fuchs, Painlevé [44] derived an equivalent form of the equation P_{VI} using the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{(l,m)\in\mathbb{Z}^2\setminus\{0\}} \left(\frac{1}{(z+l+m\tau)^2} - \frac{1}{(l+m\tau)^2} \right)$$

with periods 1 and τ $(\operatorname{Im}\tau > 0)$.

After the change of coordinates

$$u = \frac{\wp(q) - e_1}{e_2 - e_1}, \qquad t = \frac{e_3 - e_1}{e_2 - e_1},$$

where the $e_j = \wp(\omega_j)$ are the values of the $\wp$-function at the corresponding half-periods ω_j, $(\omega_0, \omega_1, \omega_2, \omega_3) = (0,\ 1/2,\ (1+\tau)/2,\ \tau/2)$, equation (18) takes the form

$$\frac{d^2 q}{d\tau^2} = \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \wp'(q + \omega_j)$$

in view of some classical formulae in the theory of elliptic functions. Here, $\alpha_0 = \alpha$, $\alpha_1 = -\beta$, $\alpha_2 = \gamma$, $\alpha_3 = -\delta + 1/2$, and $\wp'$ is the derivative of the $\wp$-function.

It is easy to see that in this form the equation looks much more simple and symmetric than in the classical form. This form is especially convenient for working with the Hamiltonian form of the equation. In these variables the Hamiltonian of the corresponding Hamiltonian system

$$2\pi i \, \frac{dq}{d\tau} = \frac{\partial H}{\partial p}, \qquad 2\pi i \, \frac{dp}{d\tau} = -\frac{\partial H}{\partial q}$$

has a simple and short form:

$$H(p, q, \tau) = \frac{p^2}{2} - \sum_{j=0}^{3} \alpha_j \wp(q + \omega_j).$$

The system is non-autonomous in view of the dependence of the $\wp$-function in the potential on the 'time' τ.

There also exist other Hamiltonian forms of the equation P_{VI}. In general, the Hamiltonian approach is widely used for studying the Painlevé equations, Garnier systems, and isomonodromic deformations of Fuchsian systems (see, for example, [45]). There are also interesting geometric interpretations of the equations P_{VI} (in particular, see [46]–[49]) which enable one to find explicitly their algebraic solutions. An important consequence of this fact is the possibility of an effective solution of the Riemann–Hilbert problem for certain two-dimensional representations with four singular points (see [50]).

§ 4. The Riemann–Hilbert problem on a compact Riemann surface

We pass to considering *Pfaffian* systems

$$dy = \omega y, \qquad y \in \mathbb{C}^p, \tag{20}$$

of Fuchsian type on a compact Riemann surface X of genus g, where ω is a matrix differential 1-form meromorphic on X whose singular points $a_1, \dots, a_n$ are poles of first order. If we formulate the Riemann–Hilbert problem in a way analogous to the case of the Riemann sphere, that is, pose the question of realization of a given set of points $a_1, \dots, a_n \in X$ and a representation

$$\chi\colon \pi_1(X \setminus \{a_1, \dots, a_n\}) \to \mathrm{GL}(p, \mathbb{C}) \tag{21}$$

by a Fuchsian system (20), then the answer proves to be negative in the general case, as follows from the calculation of dimensions given below.

The fundamental group of the space $X \setminus \{a_1, \dots, a_n\}$ is a group with $n + 2g$ generators and one relation. (We recall that the group $\pi_1(X \setminus \{a_1, \dots, a_n\})$ is generated by simple loops $\gamma_1, \dots, \gamma_n$ around the points $a_1, \dots, a_n$, respectively, together with loops $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ generating the group $\pi_1(X)$ and satisfying the relation $\gamma_1 \cdots \gamma_n = \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}$.) Consequently, the dimension of the space

$$\mathscr{M}_a = \mathrm{Hom}\big(\pi_1(X \setminus \{a_1, \dots, a_n\}),\, \mathrm{GL}(p, \mathbb{C})\big)/\mathrm{GL}(p, \mathbb{C})$$

of conjugacy classes of representations (21) is equal to $(n - 1 + 2g)p^2 - (p^2 - 1) = (n - 2 + 2g)p^2 + 1$ (see § 1).

At the same time, according to one of the statements of the Riemann–Roch theorem (see [29], Theorem 16.9, Remark 17.10), we have

$$\dim H^0(X, \mathscr{O}_{-D}) - \dim H^0(X, \Omega_D) = 1 - g - \deg D,$$

where

a) $D\colon X \to \mathbb{Z}$ is a divisor on X of degree $\deg D = \sum_{x \in X} D(x)$ (we recall that by definition, $D(x) \neq 0$ for only finitely many points x on a compact Riemann surface),

b) $H^0(X, \mathscr{O}_{-D})$ is the space of functions f meromorphic on X such that $\mathrm{ord}_x f \geqslant D(x)$,

c) $H^0(X, \Omega_D)$ is the space of differential 1-forms ω meromorphic on X such that $\mathrm{ord}_x \omega \geqslant -D(x)$.

In our case $D(a_i) = 1$ for $i = 1, \dots, n$. Hence, $\deg D = n$ and $\dim H^0(X, \Omega_D) = n - 1 + g$ (we note that $H^0(X, \mathscr{O}_{-D}) = 0$, since $f \equiv 0$ is the only function that is holomorphic on the compact Riemann surface and vanishes at the points a_i). Therefore the dimension of the space $\mathscr{M}_a^*$ of Fuchsian systems (20) with singular points $a_1, \dots, a_n$, considered up to the equivalence $\omega \sim S^{-1}\omega S$, $S \in \mathrm{GL}(p, \mathbb{C})$, is equal to $(n - 1 + g)p^2 - (p^2 - 1) = (n - 2 + g)p^2 + 1$.

Thus, the difference of the dimensions of the spaces $\mathscr{M}_a$ and $\mathscr{M}_a^*$ is equal to gp^2. In the case $g > 0$ this difference turns out to be positive, and therefore for constructing a system (20) with given singularities and monodromy it is necessary to introduce additional singular points (at which the solutions do not ramify, but the coefficient matrix has poles). More precisely, the *problem of realization* *of the representation* (21) *is soluble in the class of Pfaffian systems with regular singularities if additional singular points are allowed* (Röhrl [26]).

A more detailed study of the orders of the poles of the coefficient form of the system and of the number of additional singularities is the subject of one of the possible generalizations of the classical Riemann–Hilbert problem to the case of a compact Riemann surface of positive genus. It is known that *every two-dimensional irreducible representation can be realized by a system of the form* (20) *with at most* $3g-1$ *additional singularities*, *and all the singular points of this system are Fuchsian except for one* (*regular*) *singular point*, *which can be chosen among the additional ones*, *and its Poincaré rank is at most* $2g-1$ (Bolibrukh [30]).

An interesting problem is $\boxed{2}$ *to obtain estimates of the number of additional singular points* (*and the Poincaré ranks of the system*) *in the case of representations of arbitrary dimension.*

We consider in more detail a different way of generalizing the classical Riemann–Hilbert problem, proposed by Esnault and Viehweg [51]. We construct the family $\mathscr{F}$ of holomorphic vector bundles F^Λ over X with logarithmic connections ∇^Λ having given singularities and monodromy. (The construction is similar to that for the Riemann sphere in § 2: first, a holomorphic vector bundle with a holomorphic connection is constructed from the representation (21) over $X \setminus \{a_1, \dots, a_n\}$, and then this bundle is extended to the singular points $a_1, \dots, a_n$; the extensions are of a local nature, so they are realized in coordinate neighbourhoods of each point a_i just as in the case of the Riemann sphere.) As mentioned earlier (see § 2), a holomorphically trivial bundle over the Riemann sphere is the same thing as a semistable bundle of degree zero. Therefore, in the case of a compact Riemann surface of positive genus it is natural to consider the following problem.

Is it possible to construct a semistable holomorphic vector bundle of degree zero with a logarithmic connection that has given singularities and monodromy (*that is*, *does there exist a semistable bundle of degree zero among the elements of the family* $\mathscr{F}$)?

The problem thus stated (like its classical analogue) has a negative solution in the general case. A counterexample can again be obtained by using B-representations. (In the case of a compact Riemann surface, a representation (21) is called a *B-representation* if it is reducible and the Jordan forms of the monodromy matrices G_i corresponding to circuits around the singularities a_i consist of exactly one block.) In a way similar to that for Theorem 2 it is proved that *if a B-representation* (21) *is realized by a pair* $(F^\Lambda, \nabla^\Lambda)$, *where* F^Λ *is a semistable holomorphic vector bundle of degree zero*, *then the slope of the canonical extension* F^0 *is an integer.*

The representation (21) with three arbitrary singular points that is given by the local monodromy matrices G_1, G_2, G_3 in Example 1 and the identity matrices $H_1, \dots, H_{2g}$ corresponding to generators of the fundamental group $\pi_1(X)$ is a B-representation but does not have the property mentioned above; therefore, it cannot be realized as the monodromy representation of a logarithmic connection in a semistable vector bundle of degree zero.

Esnault and Hertling [52] showed that in the case of a surface of positive genus it is possible to construct counterexamples with only one singular point if the dimension p is greater than 4.

First of all it would be interesting to find out the following: $\boxed{3}$ *Is the commutativity of the representation* (21) *a sufficient condition for a positive solution of the Riemann–Hilbert problem on a compact Riemann surface*?

We now consider analogues of the sufficient conditions for a positive solution of the classical Riemann–Hilbert problem.

1) *If one of the generators $G_1, \dots, G_n$ of the representation* (21) *corresponding to loops around the singularities $a_1, \dots, a_n$ is diagonalizable*, *then the Riemann–Hilbert problem has a positive solution*[6] (Bolibrukh [30]).

2) *If the representation* (21) *is two-dimensional* ($p = 2$), *then the Riemann–Hilbert problem has a positive solution* (Bolibrukh [30]).

3) *If the representation* (21) *is irreducible*, *then the Riemann–Hilbert problem has a positive solution* (Esnault and Viehweg [51]).

4) $\boxed{4}$ *Suppose that the representation* (21) *is the monodromy of some scalar Fuchsian differential equation of order p on X with singularities $a_1, \dots, a_n$. Does the Riemann–Hilbert problem have a positive solution in this case*?

5) *If among the elements of the family $\mathscr{F}$ of bundles with logarithmic connections having given singularities $a_1, \dots, a_n$ and monodromy* (21) *there exists at least one stable pair* $(F^\Lambda, \nabla^\Lambda)$, *then the Riemann–Hilbert problem has a positive solution* (Bolibrukh [30]).

The fourth sufficient condition in the case of a compact Riemann surface X of positive genus is stated as a question, the answer to which is so far not known. By a Fuchsian differential equation of order p on X with singularities $a_1, \dots, a_n$ we mean a set of local Fuchsian differential equations

$$\frac{d^p y}{dz^p} + b_1(z)\,\frac{d^{p-1} y}{dz^{p-1}} + \dots + b_p(z)y = 0$$

(where z is a local coordinate on X) that are compatible with each other, that is, the solutions of the equations coincide on intersections of charts.

As in the case of the Riemann sphere, in solving the problem of constructing a Fuchsian differential equation on X with given singular points and monodromy, there necessarily emerge additional singularities. Yoshida [53] showed that from an *irreducible* representation (21) one can construct a Fuchsian equation with at most

$$\frac{p(p-1)}{2}\,(n + 2g - 2) + (p-1)(g-1)$$

additional singularities. This estimate improves by g the estimate obtained earlier by Ohtsuki [54]. (In the theorems of Ohtsuki and Yoshida one more condition is imposed: that one of the monodromy matrices $G_1, \dots, G_n$ is *diagonalizable*. This condition can be avoided by using the construction of the family $\mathscr{F}$ based on the results of Levelt; see § 2.)

[6] We remark that here we have in mind the matrices corresponding to circuits around the singularities rather than to traverses along generators of the group $\pi_1(X)$.

The following natural question arises: $\boxed{5}$ *Is it possible to obtain a generalization of the known estimates of the number of additional singularities of a Fuchsian equation to the case of an arbitrary representation* (21)?

§ 5. The Riemann–Hilbert problem for systems with irregular singularities

In this section we present a generalization of the Riemann–Hilbert problem to the case of a system (1) with *irregular* singular points $a_1, \dots, a_n \in \overline{\mathbb{C}}$. This generalization was proposed in [55].

Definition 7. The *minimal Poincaré rank* of the system (1) at a singular point a_i is the smallest of the Poincaré ranks of systems

$$\frac{dy'}{dz} = B'(z)y', \qquad B'(z) = \frac{d\Gamma}{dz}\,\Gamma^{-1} + \Gamma B(z)\Gamma^{-1},$$

that are meromorphically equivalent to the system (1) in a neighbourhood O_i of the point a_i.

For example, the minimal Poincaré rank of a regular singular point is equal to zero, and the minimal Poincaré rank of an irregular singularity is positive.

In contrast to systems with regular singular points, the meromorphic equivalence class of a system in a neighbourhood of an irregular singularity a_i is not uniquely determined by the local monodromy matrix G_i. Taking this into account, we can state the generalized Riemann–Hilbert problem for systems with irregular singular points as follows.

For each $i = 1, \dots, n$ *consider a local system*

$$\frac{dy}{dz} = B_i(z)y, \qquad B_i(z) = \frac{B^i_{-r_i-1}}{(z-a_i)^{r_i+1}} + \dots + \frac{B^i_{-1}}{z-a_i} + B^i_0 + \cdots, \tag{22}$$

in a neighbourhood O_i *of the* (*irregular*) *singular point* a_i *of minimal Poincaré rank* r_i *such that the monodromy matrix of this system coincides with the generator* G_i *of the representation* (2). *Does there exist a global system* (1) *with singularities* $a_1, \dots, a_n$ *of Poincaré ranks* $r_1, \dots, r_n$ *and with given monodromy* (2), *that is meromorphically equivalent to the systems* (22) *in the corresponding neighbourhoods* O_i?

We note that the classical Riemann–Hilbert problem can be stated in the same form. In the Fuchsian case the additional (compared with the classical statement) requirements of meromorphic equivalence of the desired system to the fixed set of local systems are satisfied automatically. Indeed, in the neighbourhood O_k every local Fuchsian system with monodromy G_k is meromorphically equivalent to the system

$$\frac{dy}{dz} = \frac{E_k}{z-a_k}\,y, \qquad E_k = \frac{1}{2\pi i}\log G_k.$$

Thus, in this case the systems (22) are uniquely determined by the monodromy representation (2) and can be omitted.

We call the representation (2) together with the local systems (22) the *generalized monodromy data*.

The generalized monodromy data are said to be *reducible* if the representation (2) and the local systems (22) are reducible. The reducibility of the latter means that they can be reduced by meromorphic transformations to systems with coefficient matrices of the same block-upper-triangular form. In the opposite case the generalized monodromy data are said to be *irreducible*.

It is known (see, for example, [56]) that in a neighbourhood of an irregular singularity $a = a_i$ of Poincaré rank $r = r_i$ the system (22) has a formal fundamental matrix $\widehat{Y}(z)$ of the form

$$\widehat{Y}(z) = \widehat{F}(z)(z-a)^{\widehat{E}}e^{Q(z)}, \tag{23}$$

where:

a) $\widehat{F}(z)$ is a formal (matrix) Laurent series in powers of $z-a$ with finite principal part and with $\det \widehat{F}(z) \not\equiv 0$;

b) $Q(z) = \operatorname{diag}(Q^1, \dots, Q^N)$, where the diagonal matrices $Q^j(z)$ are polynomials P^j in $(z-a)^{-1/s}$ of degree at most rs without constant terms, and each block $Q^j(z)$ is closed with respect to analytic continuation around the singular point $z = a$ (that is, the matrices $Q^j(a + ze^{2\pi i})$ and $Q^j(a + z)$ differ only by some permutation of the diagonal elements);

c) $\widehat{E} = (2\pi i)^{-1} \log \widehat{G}$, where $\widehat{G} = \operatorname{diag}(\widehat{G}^1, \dots, \widehat{G}^N)$ is the *formal monodromy* matrix (of block-diagonal form corresponding to the form of the matrix Q) determined by the relation

$$\widehat{Y}(a + ze^{2\pi i}) = \widehat{Y}(a + z)\widehat{G},$$

and the eigenvalues ρ of the matrix $\widehat{E}$ satisfy the condition $0 \leqslant \operatorname{Re}\rho < 1$.

It is also known that each diagonal element $q(z)$ of the matrix $Q(z)$ has the form

$$q(z) = -\frac{\lambda}{r}(z-a)^{-r} + o(|z-a|^{-r}), \qquad z \to a,$$

where λ is some eigenvalue of the matrix $B^i_{-r_i-1}$ (here, to different $q(z)$ there correspond different eigenvalues of $B^i_{-r_i-1}$).

Definition 8. The *Katz index* of a singular point $z = a$ is the number $(\deg P)/s$, where $P = \operatorname{diag}(P^1, \dots, P^N)$. (We recall that $Q(z) = P((z-a)^{-1/s})$.)

Since the matrix $Q(z)$ is a meromorphic invariant of the system (1), it follows from the properties of this matrix that the Katz index does not exceed the minimal Poincaré rank of the singularity. Moreover, the minimal Poincaré rank is the smallest integer that is greater than or equal to the Katz index of the singularity.

Definition 9. An irregular singularity of the system (22) is said to be *formally unramified* if the diagonal elements of the matrix $Q(z)$ in the expansion (23) are linear combinations of integer powers of $z-a$, that is, if $s = 1$. In the opposite case, the singularity is said to be *formally ramified*. (It is also natural to say that a Fuchsian singularity is unramified.)

In the case of a formally unramified singularity, each block $Q^j(z)$ of the matrix $Q(z)$ in the expansion (23) is a scalar matrix, and the matrix $\widehat{E}$ is in Jordan form.

For systems with irregular singular points, the method for solving the generalized Riemann–Hilbert problem is similar to the method for solving the classical problem. After the construction, over the punctured Riemann sphere $\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$, of a holomorphic vector bundle F of rank p with a holomorphic connection ∇ having a given monodromy (2), the pair (F, ∇) is extended to a bundle F^0 with a meromorphic connection ∇^0, both defined on the whole Riemann sphere (and called the canonical extension of the pair (F, ∇)). To this end, in contrast to the classical case (see § 2), instead of the gluing functions $g_{i\alpha}(z) = (z - a_i)^{E_i}$ which are fundamental matrices of systems of the form (6), we must consider the functions $g_{i\alpha}(z) = Y_i(z)$ which are fundamental matrices of the corresponding systems (22), and instead of the matrix differential 1-forms $\omega_i = E_i\,dz/(z - a_i)$ determining the logarithmic connection ∇^0 in neighbourhoods O_i we must consider the coefficient forms $\omega_i = B_i(z)\,dz = (dY_i)Y_i^{-1}$ of the systems (22).

Next we can construct the family $\mathscr{F}$ of extensions of the pair (F, ∇) by replacing the matrices $g_{i\alpha}(z)$ in the construction of the pair (F^0, ∇^0) by the matrices

$$g'_{i\alpha}(z) = \Gamma_i(z) g_{i\alpha}(z), \tag{24}$$

and the forms ω_i by the forms

$$\omega'_i = (d\Gamma_i)\Gamma_i^{-1} + \Gamma_i \omega_i \Gamma_i^{-1}, \tag{25}$$

where the $y' = \Gamma_i(z)y$ are all possible meromorphic transformations of the system (22) that do not increase its Poincaré rank r_i, $i = 1, \dots, n$.

As in the classical case, the *generalized Riemann–Hilbert problem for systems with irregular singular points is soluble if and only if at least one of the bundles of the family* $\mathscr{F}$ *constructed from the generalized monodromy data* (2), (22) *is holomorphically trivial.*

We now consider the subset $\mathscr{E} \subset \mathscr{F}$ of $\mathscr{F}$ constructed by using meromorphic transformations with the matrices $\Gamma_i(z)$ in (24), (25) of a special form. For this we shall need the following definition of an admissible matrix.

Definition 10. Consider a system (22) with an (irregular) singular point $a = a_i$ and a formal fundamental matrix $\widehat{Y}(z)$ of the form (23). An *admissible matrix* for this system is a diagonal integer-valued matrix $\Lambda_i = \operatorname{diag}(\Lambda_i^1, \dots, \Lambda_i^N)$ divided into blocks in the same way as the matrix $Q(z)$ and such that:

a) the diagonal elements of the block Λ_i^j form a non-increasing sequence if the block $Q^j(z)$ is not ramified;

b) Λ_i^j is a scalar matrix if the block $Q^j(z)$ is ramified.

We represent the matrix $\widehat{Y}(z)$ in the following form:

$$\widehat{Y}(z) = \widehat{F}(z)(z - a)^{-\Lambda_i}(z - a)^{\Lambda_i}(z - a)^{\widehat{E}} e^{Q(z)}. \tag{26}$$

An analogue of Sauvage's lemma (see [3], Lemma 11.2) for formal matrix series implies the existence of a matrix $\Gamma'_i(z)$ holomorphically invertible in O_i and such that

$$\Gamma'_i(z)\widehat{F}(z)(z - a)^{-\Lambda_i} = (z - a)^D \widehat{F}_0(z), \tag{27}$$

where D is a diagonal integer-valued matrix and $\widehat{F}_0(z)$ is an invertible formal (matrix) Taylor series in $z-a$.

We now define the requisite meromorphic transformation for each irregular singular point $a=a_i$ by the matrix $\Gamma^{\Lambda_i}(z)=(z-a)^{-D}\Gamma'_i(z)$, which depends on the admissible matrix Λ_i (since $\Gamma'_i(z)$ depends on Λ_i). It follows from (26), (27) that the transformation $y'=\Gamma^{\Lambda_i}(z)y$ transforms the system (22) into a system with the formal fundamental matrix

$$\widehat{Y}'(z)=\widehat{F}_0(z)(z-a)^{\Lambda_i}(z-a)^{\widehat{E}}e^{Q(z)}.$$

As shown in [55], such a transformation does not increase the Poincaré rank r_i of the system (22). Thus, the family $\mathscr{E}$ of extensions $(F^\Lambda,\nabla^\Lambda)$ of the pair (F,∇) to the whole Riemann sphere obtained by using all possible sets $\Lambda=\{\Lambda_1,\dots,\Lambda_n\}$ of admissible matrices for the singularities $a_1,\dots,a_n$ is a subset of the family $\mathscr{F}$.

We note that the holomorphic triviality of one of the bundles of the family $\mathscr{E}$ implies a positive solution of the Riemann–Hilbert problem (since $\mathscr{E}\subset\mathscr{F}$), but the absence of holomorphically trivial bundles in the family $\mathscr{E}$ does not yet imply a negative solution of the problem.

If the Poincaré rank of one of the singularities of a global system (1) that is required to be constructed from the generalized monodromy data (2), (22) is allowed not to be minimal, then the problem has a positive solution. Namely, the following analogue of Plemelj's theorem holds (see [57]).

Theorem 5. *The generalized monodromy data* (2), (22) *can be realized by a system* (1) *that has minimal Poincaré ranks at all the singular points except possibly for one of them, say a_1, at which the Poincaré rank of the system does not exceed the number $r_1+(p-1)(n+R-1)$, where $R=\sum_{i=1}^n r_i$.*

Analogues of the sufficient conditions for a positive solution of the classical Riemann–Hilbert problem are also known for the problem under consideration. They are stated under the assumption that for at least one local system (22), its singular point a_i is *formally unramified*.

1) *If the formal monodromy matrix $\widehat{G}_i$ (corresponding to the formally unramified singularity a_i) is diagonalizable, then the Riemann–Hilbert problem has a positive solution*[7] (see Theorem 6 below).

2) *If the generalized monodromy data* (2), (22) *are two-dimensional* ($p=2$), *then the Riemann–Hilbert problem has a positive solution* (Malek [55]).

3) *If the generalized monodromy data* (2), (22) *are irreducible, then the Riemann–Hilbert problem has a positive solution* (Bolibrukh [55]).

4) [6] *Let* (2), (22) *be the generalized monodromy data of some scalar linear differential equation of order p with singularities $a_1,\dots,a_n$. Does the Riemann–Hilbert problem have a positive solution in this case*?

5) *If among the elements of the subfamily $\mathscr{E}\subset\mathscr{F}$ there exists at least one stable pair $(F^\Lambda,\nabla^\Lambda)$, then the Riemann–Hilbert problem has a positive solution* (Bolibrukh [55]).

[7] If the singular point a_i is Fuchsian, then we should require that the corresponding monodromy matrix G_i be diagonalizable.

We explain the fourth sufficient condition, stated in the form of a question. By the generalized monodromy data of a scalar differential equation

$$\frac{d^p y}{dz^p} + b_1(z)\,\frac{d^{p-1} y}{dz^{p-1}} + \dots + b_p(z)y = 0$$

of order p with singularities $a_1, \dots, a_n$ we mean its monodromy representation and the set of local systems (22) whose Poincaré ranks are minimal and which are meromorphically equivalent in neighbourhoods of the singular points to systems with coefficient matrix of the form

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -b_p & \dots & \dots & -b_1 \end{pmatrix}. \tag{28}$$

The Katz index K_i of the scalar equation at a singular point a_i can be defined as the corresponding index of a system with coefficient matrix of the form (28). Thus, the problem of realization of the generalized monodromy data of a scalar linear differential equation stipulates the construction of a global system with singularities a_i of Poincaré ranks $r_i = -[-K_i]$, where $[x]$ denotes the integer part of a number x.

Theorem 6. *If for one of the local systems (22) its singular point a_i is formally unramified and the formal monodromy matrix $\widehat{G}_i$ is diagonalizable, then the Riemann–Hilbert problem for the generalized monodromy data (2), (22) has a positive solution.*

Proof. Consider an arbitrary pair $(F^\Lambda, \nabla^\Lambda)$ in the subfamily $\mathscr{E}$ of holomorphic vector bundles with connections constructed from the generalized monodromy data (2), (22) satisfying the hypothesis of the theorem. We can assume without loss of generality that $a_i = a_1$.

As follows from the Birkhoff–Grothendieck theorem (see the explanations before Theorem 1), the connection ∇^Λ determines a global system (1) with singularities $a_1, \dots, a_n$ and generalized monodromy data (2), (22). Furthermore, the Poincaré ranks of the singular points $a_2, \dots, a_n$ of this system are equal to $r_2, \dots, r_n$, respectively (that is, are minimal), and in a neighbourhood of the (formally unramified) singular point a_1 the system has a formal fundamental matrix $\widehat{Y}(z)$ of the form

$$\widehat{Y}(z) = (z-a_1)^{-K}\widehat{F}_0(z)(z-a_1)^{\Lambda_1}(z-a_1)^{\widehat{E}_1}e^{Q(z)},$$

where $K = \operatorname{diag}(k_1, \dots, k_p)$, $k_1 \geqslant \dots \geqslant k_p$, is the splitting type of the bundle F^Λ, $\widehat{F}_0(z)$ is an invertible formal (matrix) Taylor series in $z - a_1$, and $Q(z)$ is a (diagonal) matrix polynomial in $1/(z-a_1)$ of degree r_1.

Since the singular point a_1 is formally unramified, the matrix $\widehat{E}_1 = (2\pi i)^{-1}\log\widehat{G}_1$ is in Jordan form, that is, is diagonal (by the hypothesis of the theorem).

An analogue of Bolibrukh's permutation lemma (see [10], Lemma 10.2) for formal matrix series implies the existence of a holomorphic matrix $\Gamma(z)$ invertible away from a_1 and such that

$$\Gamma(z)(z-a_1)^{-K}\widehat{F}_0(z) = \widehat{H}_0(z)(z-a_1)^{K'},$$

where $\widehat{H}_0(z)$ is an invertible formal (matrix) Taylor series in $z-a_1$, and K' is the diagonal matrix obtained from the matrix $-K$ by some permutation of its diagonal elements.

Thus, the global meromorphic transformation $y' = \Gamma(z)y$ transforms the system under consideration into a system with the same singularities and generalized monodromy data and does not change the Poincaré ranks of the singular points $a_2, \dots, a_n$ (since the matrix $\Gamma(z)$ is holomorphically invertible at these points). It remains to show that the Poincaré rank of the transformed system at the singular point a_1 is equal to r_1. This follows from the form of the formal fundamental matrix $\widehat{Y}'(z)$ of this system:

$$\widehat{Y}'(z) = \Gamma(z)\widehat{Y}(z) = \widehat{H}_0(z)(z-a_1)^D e^{Q(z)},$$

where $D = K' + \Lambda_1 + \widehat{E}_1$ is a diagonal matrix; therefore, the coefficient matrix

$$B'(z) = \frac{d\widehat{Y}'}{dz}\,\widehat{Y}'^{-1} = \frac{d\widehat{H}_0}{dz}\,\widehat{H}_0^{-1} + \widehat{H}_0\biggl(\frac{D}{z-a_1} + \frac{dQ}{dz}\biggr)\widehat{H}_0^{-1}$$

of the transformed system has a pole of order r_1+1 at a_1 (recall that $Q(z)$ is a matrix polynomial in $1/(z-a_1)$ of degree r_1). The theorem is proved.

Apart from the classical Riemann–Hilbert problem, another special case of the problem under consideration is the problem of the Birkhoff standard form, which corresponds to two singular points ($a_1 = 0$, $a_2 = \infty$) — an irregular one and a Fuchsian one. In this case the generalized monodromy data consist of the local system

$$\frac{dy}{dz} = B(z)\,y, \qquad B(z) = \frac{B_{-r-1}}{z^{r+1}} + \dots + \frac{B_{-1}}{z} + B_0 + \cdots, \tag{29}$$

in a neighbourhood of the irregular singular point $z=0$ of minimal Poincaré rank r and the local Fuchsian system

$$\frac{dy}{dz} = \frac{E}{z}\,y, \qquad E = \frac{1}{2\pi i}\log G,$$

in a neighbourhood of infinity, where G is the monodromy matrix of the system (29).

The problem reduces to constructing a global system of the form

$$\frac{dy}{dz} = \biggl(\frac{B'_{-r-1}}{z^{r+1}} + \dots + \frac{B'_{-1}}{z}\biggr)y \tag{30}$$

that is meromorphically equivalent to the system (29) in a neighbourhood of zero.

The system (30) is called a *Birkhoff standard form* of the system (29). The question of whether every system can be transformed (by a meromorphic transformation) to a Birkhoff standard form remains open. It is known that the answer to this question is affirmative in dimensions $p \leqslant 3$, as well as in the case when the system (29) is *irreducible* or its monodromy matrix is *diagonalizable*. Historically, the latter two sufficient conditions had been obtained earlier (by Bolibrukh and Birkhoff, respectively; see [10], Lecture 12, and Balser's survey [58]), but in essence they are special cases of the sufficient conditions **3), 1)** for a positive solution of the Riemann–Hilbert problem for systems with irregular singular points.

Turrittin [59] showed that if all the eigenvalues of the matrix B_{-r-1} in (29) are distinct, then the system can be transformed to a Birkhoff standard form. The following generalization of Turrittin's theorem is a consequence of Theorem 6.

Corollary 1. *If for one of the local systems* (22) *all the eigenvalues of the matrix* $B^i_{-r_i-1}$ *are distinct, then the Riemann–Hilbert problem for the generalized monodromy data* (2), (22) *has a positive solution.*

Proof. By Theorem 6 it is sufficient to show that the singular point a_i is formally unramified and the formal monodromy matrix $\widehat{G}_i$ is diagonalizable.

Consider the expansion of the form (23) for a formal fundamental matrix of the corresponding system (22) in a neighbourhood of the singular point a_i. If this singularity were formally ramified, then the analytic continuation around it of some diagonal element $q^j(z)$ of the matrix $Q(z)$ would coincide with another diagonal element $q^k(z)$ of this matrix, which is impossible, since

$$q^j(z) = -\frac{\lambda_j}{r_i}\,(z-a_i)^{-r_i} + o(|z-a_i|^{-r_i}),$$
$$q^k(z) = -\frac{\lambda_k}{r_i}\,(z-a_i)^{-r_i} + o(|z-a_i|^{-r_i}),$$

where $\lambda_j \neq \lambda_k$ are eigenvalues of the matrix $B^i_{-r_i-1}$.

The diagonalizability of the matrix $\widehat{G}_i = \operatorname{diag}(\widehat{G}^1_i,\dots,\widehat{G}^N_i)$ follows from the fact that the singular point $z=a_i$ is formally unramified and all the diagonal elements of the matrix $Q(z)=\operatorname{diag}(Q^1,\dots,Q^N)$ are distinct (therefore each block $Q^j(z)$, being a scalar matrix, consists of a single element, as does each block $\widehat{G}^j_i$). The corollary is proved.

We can also consider the generalized Riemann–Hilbert problem for scalar differential equations with irregular singular points. In this case the generalized monodromy data are defined to be the representation (2) with generators $G_1,\dots,G_n$ and the set of local equations

$$\frac{d^p y}{dz^p} + b^i_1(z)\,\frac{d^{p-1}y}{dz^{p-1}} + \dots + b^i_p(z)y = 0 \tag{31}$$

defined in neighbourhoods of the corresponding singularities a_i, $i=1,\dots,n$ (the local monodromy matrix of equation (31) coincides with G_i).

The construction of a global equation with given monodromy that is meromorphically equivalent in a neighbourhood of each singular point a_i to the corresponding local equation (31) is accompanied in the general case by the appearance of additional singularities. (Two linear differential equations are said to be *meromorphically equivalent* in a neighbourhood of a singular point if the corresponding linear systems with coefficient matrices of the form (28) are meromorphically equivalent.) The number m of additional singularities satisfies the inequality

$$m \leqslant \frac{(K+n+1)p(p-1)}{2} + 1,$$

where $K = -\sum_{i=1}^n[-K_i]$ with K_i being the Katz index of the local equation (31) (see [57]). Note that in the case when all the local equations (31) are Fuchsian, the problem becomes the classical Riemann–Hilbert problem for scalar Fuchsian equations.

In conclusion of this section we mention a possible simultaneous generalization of the classical Riemann–Hilbert problem to the case of Pfaffian systems with irregular singularities on a compact Riemann surface X of genus g. For the generalized monodromy data we consider the representation (21) with generators $G_1,\dots,G_n$, $H_1,\dots,H_{2g}$ and the local systems

$$dy=\omega_i\,y,\qquad y\in\mathbb{C}^p, \tag{32}$$

in neighbourhoods O_i of the (irregular) singular points a_i of minimal Poincaré ranks r_i (here the local monodromy matrix of each system (32) coincides with the corresponding G_i). From the generalized monodromy data (21), (32) we construct the family $\mathscr{F}$ of holomorphic vector bundles with meromorphic connections having the given singularities $a_1,\dots,a_n$ of Poincaré ranks $r_1,\dots,r_n$, respectively, and the given generalized monodromy data (the construction of the family $\mathscr{F}$ is analogous to the case of the Riemann sphere considered in this section). The problem can now be stated as follows.

[7] *Is it possible to construct a semistable holomorphic vector bundle of degree zero with a connection that has given singularities $a_1,\dots,a_n$ of Poincaré ranks $r_1,\dots,r_n$ and the generalized monodromy data* (21), (32) (*that is, does there exist a semistable bundle of degree zero among the elements of the family $\mathscr{F}$*)?

§ 6. Some geometric properties of the monodromy map

In this section we give a geometric description of certain important notions and constructions related to the Riemann–Hilbert problem. We remark that, although we consider the most 'classical' case — Fuchsian systems and the classical Riemann–Hilbert problem — all the constructions given below also have analogues for various generalizations of the classical problem.

Recall that the monodromy map

$$\mu_a\colon \mathscr{M}_a^*\to\mathscr{M}_a$$

introduced in § 1 is a map from the space $\mathscr{M}_a^*$ of Fuchsian systems with singular points $a_1,\dots,a_n$ into the space $\mathscr{M}_a$ of representations

$$\chi\colon \pi_1(\overline{\mathbb{C}}\setminus\{a_1,\dots,a_n\})\to \mathrm{GL}(p,\mathbb{C})$$

of a group with n generators and one relation. In the coordinate description:

$$\begin{array}{ccc}
\mathscr{M}_a^* & \cong & \mathscr{O}=\Big\{(B_1,\dots,B_n)\,\Big|\,B_i\in\mathrm{Mat}(p,\mathbb{C}),\ \displaystyle\sum_{i=1}^n B_i=0\Big\}\big/\mathrm{GL}(p,\mathbb{C})\\
\Big\downarrow{\scriptstyle\mu_a} & & \Big\downarrow{\scriptstyle\mu_a}\\
\mathscr{M}_a & \cong & \Big\{(G_1,\dots,G_n)\,\Big|\,G_i\in\mathrm{GL}(p,\mathbb{C}),\ \displaystyle\prod_{i=1}^n G_i=I\Big\}\big/\mathrm{GL}(p,\mathbb{C})\ .
\end{array}$$

The question naturally arises of the existence, well-definedness, and properties of the inverse map

$$\mathrm{RH}_a\colon \mathscr{M}_a\to\mathscr{M}_a^*.$$

Such a map is called the *Riemann–Hilbert map*. It should be noted that in the definition of the inverse map of μ_a there arises the difficulty that substantially different systems may have the same monodromy. Nevertheless, it can be shown that the space $\mathscr{M}_a^*$ foliates naturally into leaves for each of which the map RH_a is well defined (although not on the entire space $\mathscr{M}_a$; see details in [60]).

The next step is to pass from Fuchsian systems with n fixed singular points to all possible Fuchsian systems with n arbitrary singularities. We extend the maps considered above to the space of all Fuchsian systems. Let

$$P = \mathbb{C}^n \setminus \bigcup_{i\neq j}\{a_i = a_j\}$$

denote the space of all possible n-tuples of pairwise distinct points of $\mathbb{C}$. Then

$$\mathscr{M}^* = P \times \mathscr{O}$$

is the *moduli space* of Fuchsian systems. A point $(a, \mathbf{B}) \in \mathscr{M}^*$ with $a = (a_1, \dots, a_n)$ and $\mathbf{B} = (B_1, \dots, B_n)$ represents the Fuchsian system

$$\frac{dy}{dz} = \biggl(\sum_{i=1}^{n} \frac{B_i}{z - a_i}\biggr)y, \qquad \sum_{i=1}^{n} B_i = 0.$$

It is also necessary to extend the representation space:

$$\mathscr{M} = \{(a, \chi) \mid a \in P,\ \chi \in \mathscr{M}_a\}.$$

We can now naturally define maps

$$\mu\colon \mathscr{M}^* \to \mathscr{M}, \qquad \mathrm{RH}\colon \mathscr{M} \to \mathscr{M}^*$$

(a point of the base space P — a tuple $a = (a_1, \dots, a_n)$ of poles — goes to itself under both maps, and the maps between fibres are realized by using μ_a and RH_a).

In terms of the notions defined above, the classical Riemann–Hilbert problem is stated as follows: *given a point in the space $\mathscr{M}$, find out whether there exists an inverse image of it with respect to the map μ.*

The maps μ and RH have a number of important and useful properties. Without going into details, we mention some of them.

1. It is natural to regard the map μ, with respect to a number of its properties, as a non-commutative generalization of the exponential map. For example, a set of matrices with zero sum is taken to a set of matrices with product equal to the identity matrix. Furthermore, the spectrum of each monodromy matrix G_k is exactly equal to the spectrum of the exponent $\exp(2\pi i B_k)$ of the corresponding residue matrix B_k, and in the *non-resonance* case (when no two eigenvalues of the matrix B_k differ by a positive integer) not only do the spectra of the matrices G_k and $\exp(2\pi i B_k)$ coincide, but so do the conjugacy classes of these matrices (see [10], Corollary 6.1, Proposition 6.1).

2. The map μ is extremely transcendental. In the general case the set of residues of the system and its monodromy can be expressed in terms of each other by 'new'

special functions. (We remark that this is also true for such a special case as the equation P_{VI}.)

3. The map μ is locally almost everywhere injective (in the sense of the aforementioned foliation of the space $\mathscr{M}_a^*$ into leaves).

4. Regarding the spaces $\mathscr{M}$ and $\mathscr{M}^*$ as bundles over the base space P, one can see that the maps μ and RH are compatible with the natural connections in these bundles: horizontal sections of one bundle are taken to horizontal sections of the other. Whereas the local horizontal sections of the bundle $\mathscr{M}$ have trivial structure (over each point they pass through one and the same representation), the structure of the local horizontal sections of the bundle $\mathscr{M}^*$ is much more interesting: all the Fuchsian systems corresponding to a local horizontal section $\mathbf{B}(a)\colon D(a^0)\to\mathscr{M}^*$, where $D(a^0)\subset P$ is a ball of small radius with centre at a point a^0, have one and the same monodromy; one can verify that the exponents of these systems (the eigenvalues of the matrices $B_i(a)$) also coincide. Thus, every horizontal section of the bundle $\mathscr{M}^*$ is none other than the set of solutions of some extended Riemann–Hilbert problem,[8] parametrized by the location of the poles of the Fuchsian system. This object is also known as an *isomonodromic family*. The set of all local horizontal sections of the bundle $\mathscr{M}^*$ that are taken under the action of μ to some fixed horizontal section of the bundle $\mathscr{M}$, restricted to any fibre $\mathscr{M}_a^*$, gives the set of all solutions of a classical Riemann–Hilbert problem.

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