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FUCHS INEQUALITIES FOR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH REGULAR SINGULAR POINTS

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ABSTRACT. The upper and lower bounds for the sum of all exponents of a system of linear differential equations with regular singular points given on the Riemann sphere are presented.

Introduction. The present work is devoted to one of the problems of the analytic theory of ordinary differential equations. Consider the system

$$\frac{dy}{dz} = B(z)y \tag{1}$$

of p linear differential equations with the matrix $B(z)$ whose entries are meromorphic on the Riemann sphere $\overline{\mathbb{C}}$ and holomorphic outside the set of singular points $a_1, \dots, a_n$. We assume that all singularities of this system are *regular* (a singular point a_i is said to be regular if any solution of the system grows not rapidly a certain power of the quantity $|z - a_i|$ as the argument z approaches the point a_i along any sectorial neighborhood with vertex at the point a_i and of angle less than 2π).

Let the Laurent series expansion of the matrix $B(z)$ have the following form in a neighborhood of a singular point $a_i \neq \infty$:

$$B(z) = \frac{B^i_{-r_i-1}}{(z-a_i)^{r_i+1}} + \dots + \frac{B^i_{-1}}{z-a_i} + B^i_0 + \cdots, \quad B^i_{-r_i-1} \neq 0 \tag{2}$$

(if $a_i = \infty$, then the principal part of the matrix $B(z)$ in a neighborhood of infinity is a polynomial of degree $r_i - 1$). The number r_i is called the *Poincaré rank* of the system (1), (2) at the singular point a_i.

A singularity a_i is said to be *Fuchsian* if its Poincaré rank r_i is equal to zero. A Fuchsian singular point is always regular (Sauvage theorem; see [7, Chap. IV]).

In a neighborhood of a nonsingular point z_0, let us consider a certain *fundamental matrix* $Y(z)$ of the solution space of system (1) (i.e., the matrix whose columns compose a basis in the solution space). Along any loop γ starting from the point z_0 and lying in $\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$, the matrix $Y(z)$ admits an analytic continuation whose result is a (in general, another) fundamental matrix $Y'(z)$. Since the columns of the matrices $Y(z)$ and $Y'(z)$ compose bases in the solution space of system (1) considered in a neighborhood of the point z_0, we obtain the relation

$$Y(z) = Y'(z)G_\gamma, \quad G_\gamma \in GL(p, \mathbb{C}).$$

The correspondence $\gamma \mapsto G_\gamma$ depends only on the homotopy class $[\gamma]$ of the loop γ and defines the homeomorphism

$$\chi : \pi_1\big(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}, z_0\big) \to GL(p, \mathbb{C})$$

of the fundamental group of the space $\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$ into the group of nonsingular complex matrices of order p. This homomorphism is called the *monodromy representation* of system (1), and the group $\operatorname{Im}\chi$ is called the *monodromy group* of this system. Under changes of the fundamental matrices $Y(z)$ (by matrices $Y(z)S$ with all possible $S \in GL(p, \mathbb{C})$), the monodromy matrices G_γ are replaced by matrices $S^{-1}G_\gamma S$. Therefore, the monodromy of system (1) is defined with accuracy up to an equivalence.

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The matrix $G_i = \chi(g_i)$ corresponding to a continuation along a simple loop g_i going around a singular point a_i is called the *monodromy matrix* of system (1) at the singular point a_i. Since the generators $g_1, \dots, g_n$ of the fundamental group $\pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}, z_0)$ are connected by the relation $g_1 \cdots g_n = e$, it follows that

$$G_1 \cdots G_n = I.$$

The structure of the solution space of system (1) in a neighborhood of a regular singular point was described by Levelt in [8] (see also [1, Chap. I] and [2, Lect. 5]). For each of the singularities of such a system, the *exponents* are defined; they are numbers characterizing the rate of power growth of solutions in a neighborhood of the singular point (for the definition of exponents, see Sec. 1).

The goal of this work is to present certain estimates for the sum of exponents of system (1) over all singular points, the so-called *Fuchs inequalities*. These inequalities can be used in studying separate problems of the analytic theory of differential equations (for example, in finding the multiplicities of zeros of components of the solution of the system at an arbitrary point of the Riemann sphere; see [3]). The name of these estimates is connected with the name of the German mathematician L. Fuchs, who obtained [6] the relation for the sum Σ of exponents of the linear differential equation

$$\frac{d^p y}{dz^p} + b_1(z)\frac{d^{p-1} y}{dz^{p-1}} + \cdots + b_p(z) y = 0$$

of order p with regular singular points $a_1, \dots, a_n$ even as far back as in 1866. This relation has the form

$$\Sigma = \frac{(n-2)p(p-1)}{2}$$

and is called the *classical Fuchs relation*.

As for system (1), from then up until recently, it was known that the sum Σ of exponents of this system is an integer not exceeding zero, and, moreover, $\Sigma = 0$ if and only if all singular points are Fuchsian (see [1, Chap. I] and [2, Lect. 7]). And only in 1999 did Corel [4] obtain effective estimates for the quantity Σ depending on the size p of the system and the Poincaré ranks $r_1, \dots, r_n$:

$$-\frac{p(p-1)}{2} \sum_{i=1}^{n} r_i \le \Sigma \le -\sum_{i=1}^{n} r_i.$$

In the case where the ranks of the matrices $B^i_{-r_i-1}$ corresponding to non-Fuchsian singularities a_i of system (1) are maximal, Corel found in [5] the sharp expression for the quantity Σ:

$$\Sigma = -\frac{p(p-1)}{2} \sum_{i=1}^{n} r_i.$$

In the present work, we refine the Fuchs inequalities obtained by Corel. The refinement takes into account the appearance of the dependence on the ranks of the matrices $B^i_{-r_i-1}$, $i = 1, \dots, n$, in the estimate.

1. Structure of the fundamental matrix of the system in a neighborhood of a regular singular point. We first say a few words about the local *gauge transformations* of system (1) used for obtaining the Fuchs inequalities. These are transformations of the form

$$y' = \Gamma(z) y,$$

where $\Gamma(z)$ is a holomorphically (meromorphically) invertible matrix-valued function in a neighborhood of a singular point a_i. The holomorphic invertibility of the matrix $\Gamma(z)$ means that this matrix is holomorphic in a neighborhood of the point a_i and $\det \Gamma(a_i) \ne 0$, and the meromorphic invertibility means that $\Gamma(z)$ is meromorphic at the point a_i and $\det \Gamma(z) \not\equiv 0$.

Remark 1. This transformation transforms system (1) into a system (defined in a neighborhood of the point a_i) with the following matrix of coefficients:

$$B'(z) = \Gamma(z)B(z)\Gamma^{-1}(z) + \frac{d\Gamma(z)}{dz}\Gamma^{-1}(z). \tag{3}$$

A holomorphically invertible transformation does not change the Poincaré rank r_i, replacing the leading coefficient, the matrix $B^i_{-r_i-1}$, by the matrix $\Gamma(a_i)B^i_{-r_i-1}\Gamma^{-1}(a_i)$. A meromorphically invertible transformation can enlarge, as well as reduce, the Poincaré rank.

For convenience (without loss of generality), we assume that the singular point a_1 of system (1) coincides with zero. In the theorem presented below, we indicate the form of the fundamental matrix of the system in a neighborhood of a singular point (an analogous expansion also holds for the corresponding fundamental matrices in neighborhoods of other singular points).

Theorem 1 (Levelt [8])**.** *There exists a fundamental matrix $Y(z)$ of system* (1) (*which is said to be Levelt*) *in a neighborhood of the singular point $z = 0$ that is represented in the form*

$$Y(z) = U(z)z^A z^E, \tag{4}$$

where $U(z)$ is a matrix holomorphic in a neighborhood of zero, $A = \mathrm{diag}(\varphi^1, \ldots, \varphi^p)$ is an integral matrix of valuations of the systems at zero, $\varphi^1 \geq \cdots \geq \varphi^p$, $E = (1/2\pi i)\ln G$, G is the monodromy matrix of the system at zero (*i.e., the matrix by which $Y(z)$ is postmultiplied after the analytic continuation around zero*) *of upper-triangular form, and the eigenvalues ρ^j of the matrix E satisfy the condition*

$$0 \leq \mathrm{Re}\,\rho^j < 1. \tag{5}$$

Moreover, the matrix $U(z)$ is holomorphically invertible at zero if and only if zero is a Fuchsian singular point of system (1).

Definition 1. The eigenvalues $\beta^j = \varphi^j + \rho^j$ of the matrix $A + E$ are called the (Levelt) exponents of system (1) at the singular point $z = 0$.

It is worth mentioning that, although there can exist several Levelt fundamental matrices of the form (4) (with different $U(z)$ and E), the exponents of the system are uniquely defined (for a detailed description of the concept of valuations and for the construction of the Levelt fundamental matrix, see also [1, Chap. I] and [2, Lect. 5]).

2. Fuchs inequalities. For each singular point a_i, denote by $Y_i(z)$ the corresponding Levelt fundamental matrix of the form

$$Y_i(z) = U_i(z)(z - a_i)^{A_i}(z - a_i)^{E_i},$$

where the matrix $U_i(z)$ is holomorphic in a neighborhood of the point a_i, $A_i = \mathrm{diag}(\varphi_i^1, \ldots, \varphi_i^p)$ is the valuation matrix of system (1) at the point a_i, $\varphi_i^1 \geq \cdots \geq \varphi_i^p$, and E_i is an upper-triangular matrix whose eigenvalues ρ_i^j satisfy condition (5).

Denote by $\beta_i^j = \varphi_i^j + \rho_i^j$ the exponents of system (1) at the point a_i $(j = 1, \ldots, p)$. Express the sum

$$\Sigma = \sum_{i=1}^{n}\sum_{j=1}^{p}\beta_i^j = \sum_{i=1}^{n}\mathrm{tr}(A_i + E_i)$$

of all exponents through the orders $\mathrm{ord}_{a_i}\det U_i(z)$ of zeros of the functions $\det U_i(z)$ at the corresponding points a_i.

It follows from the Liouville formula

$$d\ln\det Y_i(z) = \mathrm{tr}\,B(z)dz$$

and the relation

$$\det Y_i(z) = \det U_i(z)(z-a_i)^{\mathrm{tr}(A_i+E_i)}$$

that

$$\mathrm{res}_{a_i} \mathrm{tr}\, B(z)dz = \mathrm{res}_{a_i} d\ln\det Y_i(z) = \mathrm{res}_{a_i} d\ln\det U_i(z)$$
$$+\,\mathrm{res}_{a_i} d\ln(z-a_i)^{\mathrm{tr}(A_i+E_i)} = \mathrm{ord}_{a_i}\det U_i(z) + \mathrm{tr}(A_i+E_i).$$

Applying the theorem on the sum of residues to the form $\mathrm{tr}\, B(z)dz$, we obtain the following formula for the quantity Σ:

$$\Sigma = -\sum_{i=1}^{n} \mathrm{ord}_{a_i}\det U_i(z). \tag{6}$$

Therefore, the sum of exponents of system (1) is an integer not exceeding zero (Theorem 1 implies that $\Sigma = 0$ if and only if all singular points are Fuchsian).

To obtain more precise estimates for the quantity Σ, let us estimate the order of zeros of the functions $\det U_i(z)$ at the corresponding points a_i (for this purpose, it suffices to estimate the order of zero of the function $\det U(z)$ from expansion (4) at the point $a_1 = 0$). Denote by r the Poincaré rank of system (1) at zero. Then the Laurent series expansion of the matrix $B(z)$ at a neighborhood of zero has the form

$$B(z) = \frac{B_{-r-1}}{z^{r+1}} + \cdots + \frac{B_{-1}}{z} + B_0 + \cdots, \quad B_{-r-1} \neq 0. \tag{7}$$

We assume that $r > 0$ (if $r = 0$, then, according to Theorem 1, $\mathrm{ord}_0 \det U(z) = 0$). The estimate for the quantity $\mathrm{ord}_0 \det U(z)$ depends on the size p of the system, the Poincaré rank r, and the rank of the matrix B_{-r-1}. In what follows, we will need the following auxiliary lemma.

Lemma 1. *For any matrix $U(z)$ holomorphic in a neighborhood of zero, there exist a matrix polynomial $P(z)$ and a matrix $V(z)$ holomorphically invertible in a neighborhood of zero such that*

$$U(z) = V(z)P(z).$$

Proof (*following A. A. Bolibruch*). If $\det U(0) \neq 0$, then it suffices to set $V(z) = U(z)$ and $P(z) = I$. If $\det U(0) = 0$ (and $\mathrm{ord}_0 \det U(z) = l > 0$), then there exists a nonsingular upper-triangular matrix S_1 such that the column with number i of the matrix $U(0)S_1$ vanishes (for a certain i). Hence

$$U(z)S_1 = U_1(z)z^{K_1},$$

where K_1 is a matrix all whose entries are equal to zero, except for the entry with subscript (i,i), which is equal to unity, $U_1(z)$ is a matrix holomorphic in a neighborhood of zero, and $\mathrm{ord}_0 \det U_1(z) = l - 1$. Therefore,

$$U(z) = U_1(z)z^{K_1}S_1^{-1} = U_1(z)P_1(z),$$

where $P_1(z) = z^{K_1}S_1^{-1}$ is an upper-triangular polynomial. Repeating analogous arguments for the matrix $U_1(z)$, we obtain the decomposition

$$U_1(z) = U_2(z)P_2(z),$$

where $P_2(z)$ is an upper-triangular polynomial, $U_2(z)$ is a matrix holomorphic in a neighborhood of zero, and $\mathrm{ord}_0 \det U_2(z) = l - 2$. Finally, after l steps of this procedure, we have

$$U(z) = U_l(z)P_l(z)\ldots P_1(z) = V(z)P(z),$$

where $V(z) = U_l(z)$ is a matrix holomorphically invertible in a neighborhood of zero and $P(z) = P_l(z)\ldots P_1(z)$ is an upper-triangular polynomial. ∎

Lemma 1 immediately implies that decomposition (4) simplifies the relation

$$Y(z) = V(z)P(z)z^A z^E$$

(the matrices $V(z)$ and $P(z)$ are the same as in the lemma), and we need to estimate the quantity $\operatorname{ord}_0 \det P(z)$.

As is seen from Remark 1, under a transformation $y = V(z)y'$ holomorphically invertible in a neighborhood of zero, the Poincaré rank r does not change, and the place of the leading coefficient, the matrix B_{-r-1}, is now occupied by the matrix $V^{-1}(0)B_{-r-1}V(0)$, i.e., the rank of the leading coefficient also does not change. Therefore, we can assume that the fundamental matrix $Y(z)$ of system (1) has the following form in a neighborhood of zero:

$$Y(z) = P(z)z^A z^E, \tag{8}$$

and then the matrix $B(z)$ of coefficients of system (1) considered in a neighborhood of zero is upper triangular.

Moreover, it turns out that the matrices $B_{-r-1}, \dots, B_{-2}$ in the Laurent series (7) for $B(z)$ are nilpotent. Indeed,

$$B(z) = \frac{dY}{dz}Y^{-1} = \frac{dP}{dz}P^{-1} + P\frac{A}{z}P^{-1} + Pz^A\frac{E}{z}z^{-A}P^{-1}.$$

Denote by $p_{jj}(z)$ the diagonal entries of the upper-triangular polynomial $P(z)$; then the diagonal entries $\dfrac{dp_{jj}}{dz}(1/p_{jj})$ of the matrix $\dfrac{dP}{dz}P^{-1}$ have no more than a simple pole at zero. The diagonal entries φ^j/z and ρ^j/z of the other two summands of the matrix $B(z)$ also have a pole of first order at zero, and hence this holds for the diagonal entries of the matrix $B(z)$ itself. Therefore, the diagonal entries of the upper-triangular matrices $B_{-r-1}, \dots, B_{-2}$ can only be zero, i.e., these matrices are nilpotent.

Consider a transformation of the form $\tilde{y} = z^C y$ with an integral matrix $C = \operatorname{diag}(c_1, \dots, c_p)$ such that $c_1 > \dots > c_p \geq 0$. Because of formula (3), this transformation transforms system (1) into a system with the following matrix of coefficients:

$$\widetilde{B}(z) = z^C B(z) z^{-C} + \frac{C}{z}.$$

The entries of the first summand of this sum have the form $b_{ij}(z)z^{c_i - c_j}$ ($b_{ij}(z)$ are entries of the matrix $B(z)$). Since the matrix $B(z)$ is upper triangular and its diagonal entries have a simple pole at zero, this transformation reduces the Poincaré rank of the singularity.

Let us reduce the Poincaré rank up to zero using a specially chosen matrix C. Let $\operatorname{rank} B_{-r-1} = p - k$, $1 \leq k < p$. We set

$$C = \operatorname{diag}\big((p-1)r, (p-2)r, \dots, r, 0\big) + \operatorname{diag}(0, \dots, k-1),$$

where the ith diagonal entry in the second summand is equal to the number

$$i - 1 - \operatorname{rank}\langle \mathbf{b}^1_{-r-1}, \dots, \mathbf{b}^i_{-r-1} \rangle, \quad i = 1, \dots, p$$

($\mathbf{b}^j_{-r-1}$ is the jth column of the matrix B_{-r-1}). In other words, the diagonal of the second summand of the matrix C starts from zero, and then the entry with number i is one greater than the previous one only in the case where the adding of the column $\mathbf{b}^i_{-r-1}$ does not increase the rank of the system of vectors $\mathbf{b}^1_{-r-1}, \dots, \mathbf{b}^{i-1}_{-r-1}$.

If, for certain $i < j$, the difference $c_i - c_j$ is minimal, i.e.,

$$c_i - c_j = (j-i)r - (j-i) \geq r - 1,$$

then this means that the adding of the columns $\mathbf{b}^{i+1}_{-r-1}, \dots, \mathbf{b}^j_{-r-1}$ does not increase the rank of the system of vectors $\mathbf{b}^1_{-r-1}, \dots, \mathbf{b}^i_{-r-1}$, i.e., the entries with subscript (i,j) of the matrix B_{-r-1} is equal to zero (because of the upper-triangular form and the nilpotency of this matrix) and $\operatorname{ord}_0 b_{ij}(z) \geq -r$. Therefore, the function $b_{ij}(z)z^{c_i - c_j}$ has no more than a simple pole at zero.

In other cases,

$$c_i - c_j \geq (j-i)r - (j-i) + 1 \geq r,$$

and the corresponding functions $b_{ij}(z)z^{c_i-c_j}$ also have no more than a simple pole at zero.

Therefore, in a neighborhood of zero, the transformation $\tilde{y} = z^C y$ transforms the initial system (1), (7) into a system with the Fuchsian singular point 0 and a certain fundamental matrix $\widetilde{Y}(z)$ of the form (4):

$$\widetilde{Y}(z) = \widetilde{U}(z) z^{\widetilde{A}} z^{\widetilde{E}};$$

moreover, the matrix $\widetilde{U}(z)$ is holomorphically invertible at zero (since the obtained system is Fuchsian at zero), $\widetilde{A}$ is the valuation matrix of this system, the upper-triangular matrix $\widetilde{E}$ is conjugated to the matrix E (and hence $\operatorname{tr}\widetilde{E} = \operatorname{tr} E$), and the matrix $z^{\widetilde{A}}\widetilde{E}z^{-\widetilde{A}}$ is holomorphic at zero. The matrix $z^C Y(z)$ is also fundamental for the obtained system, and, therefore,

$$z^C Y(z) = \widetilde{Y}(z) R,$$

where R is a constant nonsingular matrix. Then

$$Y(z) = z^{-C}\widetilde{U}(z) z^{\widetilde{A}} z^{\widetilde{E}} R.$$

Comparing this decomposition with decomposition (8) for the matrix $Y(z)$ and equating the order of the determinants at zero, we obtain the relation

$$\operatorname{tr} A + \operatorname{ord}_0 \det P(z) = -\operatorname{tr} C + \operatorname{tr}\widetilde{A}.$$

Now to reveal in which way the quantities $\operatorname{tr} A$ and $\operatorname{tr}\widetilde{A}$ are connected with each other, we use the following lemma (see [1, Corollary 1.2.1]).

Lemma 2. *Let a fundamental matrix $Y(z)$ of system (1) with a regular singularity at zero have the form*

$$Y(z) = U(z) z^L z^E,$$

where the matrix $U(z)$ is holomorphic at zero, L is a diagonal integral matrix, and E is an upper-triangular matrix whose eigenvalues satisfy condition (5)*. If the matrix $z^L E z^{-L}$ is holomorphic at zero, then the trace of the matrix L does not exceed the sum* $\operatorname{tr} A$ *of valuations of system* (1) *at zero.*

Let us apply this lemma to the system with the fundamental matrix

$$Y'(z) = z^{(p-1)r} Y(z) R^{-1} = z^{(p-1)rI - C}\widetilde{U}(z) z^{\widetilde{A}} z^{\widetilde{E}}$$

and the valuation matrix $A + (p-1)rI$. The matrices $z^{(p-1)rI-C}\widetilde{U}(z)$ and $z^{\widetilde{A}}\widetilde{E}z^{-\widetilde{A}}$ are holomorphic at zero, and, therefore, according to Lemma 2,

$$\operatorname{tr}\widetilde{A} \leq \operatorname{tr} A + p(p-1)r,$$

which implies

$$\operatorname{ord}_0 \det P(z) = \operatorname{tr}\widetilde{A} - \operatorname{tr} A - \operatorname{tr} C \leq p(p-1)r - \operatorname{tr} C.$$

Since $\operatorname{tr} C \geq p(p-1)r/2 + k(k-1)/2$, we finally obtain the following assertion.

Proposition 1. *The following inequality holds for the quantity* $\operatorname{ord}_0 \det U(z)$ *from decomposition* (4) *of the fundamental matrix of the system* (1), (7) *in a neighborhood of the regular singular point $z = 0$ of Poincaré rank $r > 0$:*

$$\operatorname{ord}_0 \det U(z) \leq \frac{p(p-1)}{2} r - \frac{k(k-1)}{2},$$

where $k = p - \operatorname{rank} B_{-r-1} = \dim\ker B_{-r-1}$.

To obtain a lower estimate for the quantity $\operatorname{ord}_0 \det U(z)$, we use the following lemma from [7], which is usually called the Sauvage lemma in the literature.

Lemma 3. *For any matrix $U(z)$ holomorphic in a neighborhood of zero, there exists a matrix $\Gamma(z)$ holomorphically invertible in a neighborhood of zero such that*

$$\Gamma(z)U(z) = z^D V(z),$$

where $D = \mathrm{diag}(d_1, \ldots, d_p)$ is an integral matrix with the condition $d_1 \geq \cdots \geq d_p \geq 0$ and the matrix $V(z)$ is holomorphically invertible in a neighborhood of zero.

In view of Remark 1, the transformation $y' = \Gamma(z)y$ does not change the Poincaré rank r and the rank of the leading coefficient, the matrix B_{-r-1}, and, therefore, we can assume that the fundamental matrix $Y(z)$ of system (1) has the following form in a neighborhood of zero:

$$Y(z) = z^D V(z) z^A z^E.$$

Then the following relation holds for the matrix $B(z)$ of coefficients:

$$\begin{aligned} B(z) &= \frac{B_{-r-1}}{z^{r+1}} + \cdots = \frac{dY}{dz} Y^{-1} \\ &= \frac{D}{z} + z^D \left(\frac{dV}{dz} V^{-1} + V \left(\frac{A}{z} + z^A \frac{E}{z} z^{-A} \right) V^{-1} \right) z^{-D} = \frac{D}{z} + z^D \frac{B_0(z)}{z} z^{-D}, \end{aligned}$$

where $B_0(z)$ is a matrix holomorphic in a neighborhood of zero. It follows from this formula for $B(z)$ that only those of its entries which lie lower that the principal diagonal can have a pole of order $r+1$ at zero, i.e., B_{-r-1} is a lower-triangular matrix. Denote its entries by b_{ij}. If $b_{ij} \neq 0$ for a certain $i > j$, then $d_i - d_j \leq -r$ and $d_j \geq r$. Since there exist at least $\mathrm{rank}\, B_{-r-1}$ nonzero columns of the matrix B_{-r-1}, it follows that there also exist $\mathrm{rank}\, B_{-r-1}$ nonzero entries b_{ij} (for $i > j$) lying in different columns. Hence, at least for $\mathrm{rank}\, B_{-r-1}$ numbers d_j, the estimate $d_j \geq r$ holds. Therefore, $\mathrm{tr}\, D \geq \mathrm{rank}\, B_{-r-1} r$. Since $\mathrm{ord}_0 \det U(z) = \mathrm{tr}\, D$ (the matrices $\Gamma(z)$ and $V(z)$ are holomorphically invertible at zero), we obtain the following assertion.

Proposition 2. *The following inequality holds for the quantity* $\mathrm{ord}_0 \det U(z)$ *from decomposition* (4) *of the fundamental matrix of the system* (1), (7) *in a neighborhood of a regular singular point $z = 0$ of Poincaré rank $r > 0$:*

$$\mathrm{ord}_0 \det U(z) \geq \mathrm{rank}\, B_{-r-1} r.$$

Remark 2. If the rank of the (nilpotent) matrix B_{-r-1} is maximal, i.e., if it is equal to the number $p-1$, then, as was established by Corel [5],

$$\mathrm{ord}_0 \det U(z) = \frac{p(p-1)}{2} r.$$

To prove this assertion, it suffices to show that in the case of the maximal rank of the matrix B_{-r-1}, the inequality $\mathrm{ord}_0 \det U(z) \geq p(p-1)r/2$ holds.

As in the proof of Proposition 2, we assume that $B_{-r-1} = (b_{ij})$ is a lower-triangular matrix. Since $\mathrm{rank}\, B_{-r-1} = p-1$, it follows that $b_{i+1,i} \neq 0$ for each $i = 1, \ldots, p-1$. Hence the entries of the matrix $B(z)$ with subscripts $(i+1, i)$, $i = 1, \ldots, p-1$, have a pole of order $r+1$ at zero, which implies $d_{i+1} - d_i \leq -r$ for each $i = 1, \ldots, p-1$. Therefore, $\mathrm{ord}_0 \det U(z) = \mathrm{tr}\, D \geq p(p-1)r/2$.

Therefore, Propositions 1 and 2 and formula (6) imply the following assertion on the sum of exponents of system (1) over all its singular points.

Theorem 2. *The following inequalities hold for the sum Σ of exponents of the system* (1), (2) *with regular singular points $a_1, \ldots, a_n$ of Poincaré rank $r_1, \ldots, r_n$, respectively:*

$$-\frac{p(p-1)}{2} \sum_{i=1}^{n} r_i + \sum_{i=1}^{n} \frac{k_i(k_i-1)}{2} \leq \Sigma \leq -\sum_{i=1}^{n} \mathrm{rank}\, B^i_{-r_i-1} r_i,$$

where $k_i = p - \mathrm{rank}\, B^i_{-r_i-1} = \dim \ker B^i_{-r_i-1}$ if $r_i > 0$ and $k_i = 0$ if $r_i = 0$.

We note additionally that the relation

$$\Sigma = -\frac{p(p-1)}{2}\sum_{i=1}^{n} r_i$$

(the minimal possible value of the quantity Σ in the class of systems (1) with fixed Poincaré ranks $r_1, \dots, r_n$) holds if and only if at all non-Fuchsian singular points a_i, the ranks of the corresponding leading coefficients $B^i_{-r_i-1}$ are maximal, i.e., are equal to the number $p-1$ (see Remark 2).

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