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Optimal Hysteresis Control for *BMAP/SM/*1 Queue with *MAP*-Input of Disasters

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Abstract: A *BMAP/SM/*1 queueing system with two modes of operation is considered. Operational mode is changed according to a hysteresis strategy. The system has input of disasters causing all customers to leave the system unserved. The stationary state distribution of the embedded Markov chain and performance characteristics are obtained. The problem of hysteresis control optimization is solved numerically.

Keywords: BMAP/SM/1 queue, disaster, hysteresis control, mode of operation.

1. Introduction

Queueing systems with dynamic control allow to redistribute system resources during system operation. In particular, such systems have variable parameters of input flow, service and other processes characterizing the queueing system. Each group of parameters used simultaneously is referred to as operational mode. At the decision making epochs certain operational mode is selected for customer processing in accordance with some control strategy (rule of mode selection). Queueing models with controllable mode of operation can be effectively applied for modelling telecommunication network fragments. That is the resource of telecommunication network which hands a mixture of flows having different requirements to the response time; systems which combine transmission of several types of information having different requirements to the quality of service and having possibility to distribute the bandwidth dynamically, etc. We refer reader to [2], [4], [7], [8], [17], [18], [19], [23] for the papers dealing with controllable queueing systems.

During operation of real-life queueing systems the loss of several or all customers is possible due to some disturbances. Such a loss can be described as a negative customer arrival causing one customer or several ones to leave the system unserved. The theory of queues and queueing networks with negative arrivals was originated by E. Gelenbe [10]. Detailed review of achieved results is presented in [1], [3]. Disaster is the special kind of a negative arrival causing immediate departure of all customers from the queue. General model of *BMAP/SM/*1 queueing system with disasters was investigated in [5], [6].

In this paper the investigation of *BMAP/SM/*1 queueing system with two operational modes and disasters is presented. For some basic cases of *BMAP/SM/*1 queue it has been proved [12], [13], [20] that optimal strategy in the class of Markov homogeneous strategies is hysteresis. The partial case of hysteresis strategy is a threshold one. In the previous work [21] class of threshold control strategies has been considered to optimize the system operational cost per time unit. In the present work we continue investigation of *BMAP/SM/1* queue with two operational modes and disasters considering the class of hysteresis control strategies. Note that *BMAP/G/1* queueing system with hysteresis control (without disasters) was investigated in [7], paper [8] involves analysis of *BMAP/SM/1/N* queueing system.

2. The Model

We consider a single-server queue with unlimited waiting space having two modes of operation and additional input of disasters. The r th mode of system operation is described as follows. Customers arrive to the system according to Batch Markovian Arrival Process (BMAP) which is governed by a stochastic process $\nu_t,\, t \geq 0$ having state space $\{0,1,...,W\}$. Transitions of the process $\nu_t,\, t \geq 0$ and arrivals of customers are performed with a matrix generation function $D^{(r)}(z) = \sum_{m=0}^{\infty} D_m^{(r)} z^m,\ |z| \leq 1$. The matrix $D_0^{(r)}$ governs the transitions corresponding to no arrivals, and $D_m^{(r)}$ governs the transitions accompanied by arrival of a batch of m customers, $m \geq 1$.

Denote by $\vec{\varphi}^{(r)}$ the stationary probability row vector of the Markov chain ν_t. It is defined by the system of equations: $\vec{\varphi}^{(r)} D^{(r)}(1) = \vec{0},\ \vec{\varphi}^{(r)} \vec{1} = 1.$ Here and below $\vec{0}$ is a null column-vector of appropriate size, $\vec{1}$ is a row-vector consisted of ones. The intensity $\lambda^{(r)}$ of BMAP is determined as

$$\lambda^{(r)} = \vec{\varphi}^{(r)} \left. \frac{dD^{(r)}(z)}{dz} \right|_{z=1} \vec{1}.$$

The detailed description of BMAP can be found in the paper [14] of D. Lucantoni.

Service process is of SM-type (Semi-Markovian). It means that successful service times of customers are the sojourn time of the process $m_t,\, t \geq 0$ in its states $\{0,1,...,M\}$. During the r th mode of operation it behaves as a Semi-Markovian process with kernel $B^{(r)}(t)$ which is a matrix with entries $B_{m,m'}^{(r)}(x)$, $m,m' = \overline{1,M}$. The function $B_{m,m'}^{(r)}(x)$ is the conditional distribution function of the sojourn time of the process $m_t, t \geq 0$ in the state m given that the next state is m'.

We use the same assumptions about the kernel $B^{(r)}(x)$ as M. Neuts [16] and D. Lucantoni and M. Neuts [15]. Denote by $P^{(r)} = B^{(r)}(\infty)$ the transition matrix of the embedded Markov chain for the process $m_t,\, t \geq 0$. It is assumed that the states of the process change according to the matrix $P^{(r)}$ at service completion epochs regardless of whether service is completed successfully or is interrupted by a disaster appearance. The mean customer service time which is not interrupted by a disaster arrival is determined as

$$b^{(r)} = \vec{\delta}^{(r)} \int_0^{\infty} t dB^{(r)}(t) \vec{1},$$

where $\vec{\delta}^{(r)}$ is solution of the system $\vec{\delta}^{(r)} P^{(r)} = \vec{\delta}^{(r)},\quad \vec{\delta}^{(r)} \vec{1} = 1.$

Disaster arrival process is MAP (Markovian Arrival Process) which is the partial case of BMAP allowing the single arrivals only. We suppose that input of disasters is governed by the process $\eta_t,\, t \geq 0$ having state space $\{0,1,...,N\}$ and the matrix generating function $F^{(r)}(z) = F_0^{(r)} + F_1^{(r)} z,\ |z| \leq 1,\ r = \overline{1,2}$.

Following G. Jain and K. Sigman [11] we suppose that disaster arrival to the busy system interrupts the service and causes all customers to leave the system instantaneously.

If disaster arrives to the empty system it is ignored.

The operation mode can be switched at the service completion epochs only. Time to switch from the r th mode to another one has a distribution function $G_r(t)$ with mean $g_r = \int_0^\infty t dG_r(t)$, $r = \overline{1,2}$. During mode changeover time the server does not process customers, arriving customers are accumulated in the queue, disasters are ignored, input of customers is governed as in the r th mode, $r = \overline{1,2}$.

The quality of system steady-state operation is evaluated by the cost criterion

$$C = a\Lambda L + c_1\Phi_1 + c_2\Phi_2 + dV + hK , \quad (1)$$

where Λ^{-1} is the average interdeparture time; L is the average queue length at customer departure epochs; Φ_r is the average fraction of time, when the r th mode is in use, $r = \overline{1,2}$; V is the average number of customers lost per time unit; K is the average number of mode changeovers per time unit; a, c_1, c_2, d and h are the corresponding cost coefficients.

The operational modes are switched to minimize the cost criterion value. We find optimal switching strategy in the class of hysteresis strategies. These strategies are determined as follows. Two non-negative integers j and k referred to as thresholds are fixed. Let i be the queue length at given service completion epoch. If $i \le j$ the system will operate in the first mode. If $i > k$ the system will operate in the second mode. Otherwise it keeps the current operation mode.

3. Stationary Queue Length Distribution

Let the thresholds (j,k), $0 \le j \le k$ be fixed and t_n be the n th epoch of service completion, $n \ge 1$. Consider the process $\xi_n = \{i_n, \omega_n, \nu_n, \eta_n, m_n\}$ where i_n is a queue length at the epoch $t_n + 0, i_n \ge 0$; ω_n is the number of mode used at the epoch $t_n - 0$, $\omega_n = \overline{1,2}$; ν_n is the state of arrival directing process $\nu_t, t \ge 0$ at the epoch $t_n, \nu_n = \overline{0,W}$; $\eta_n, t \ge 0$ is the state of disasters directing process η_t at the epoch $t_n, \eta_n = \overline{0,N}$ and m_n is the state of service directing process $m_t, t \ge 0$ at the epoch $t_n + 0, m_n = \overline{1,M}$.

The process $\xi_n, n \ge 1$ is the five-dimensional embedded Markov chain. Enumerate the states of the chain in lexicographic order and introduce into consideration the matrices of one-step transition probabilities

$$P_{i,l}^{(r,r')} = (P\{i_{n+1} = l, \omega_{n+1} = r', \nu_{n+1} = \nu', \eta_{n+1} = \eta', m_{n+1} = m' \mid i_n = i,$$

$$\omega_n = r, \nu_n = \nu, \eta_n = \eta, m_n = m\})_{\nu,\nu'=\overline{0,W}, \eta,\eta'=\overline{0,N}, m,m'=\overline{1,M}}, i,l \ge 0, r,r' = \overline{1,2}.$$

Lemma 1. The nonzero matrices $P_{i,l}^{(r,r')}, i,l \ge 0, r,r' = \overline{1,2}$ of one-step transition probabilities of the Markov chain $\xi_n, n \ge 1$ are determined as

$$P_{0,0}^{(1,1)} = \Psi_1\Omega_0^{(1)} + \Psi(1)S^{(1)}; \quad P_{0,0}^{(2,1)} = \Gamma_0^{(2)}(\Psi_1\Omega_0^{(1)} + \Psi(1)S^{(1)}) + \sum_{m=1}^{\infty}\Gamma_m^{(2)}S^{(1)} + \Gamma_1^{(2)}\Omega_0^{(1)};$$

$$P_{0,l}^{(1,1)} = \sum_{n=1}^{l+1}\Psi_n\Omega_{l-n+1}^{(1)}, l > 0; \quad P_{0,l}^{(2,1)} = \Gamma_0^{(2)}\sum_{n=1}^{l+1}\Psi_n\Omega_{l-n+1}^{(1)} + \sum_{n=1}^{l+1}\Gamma_n^{(2)}\Omega_{l-n+1}^{(1)}, l > 0 ;$$

$$P_{1,0}^{(1,1)} = \Omega_0^{(1)} + S^{(1)}, j > 0 \text{ or } j = 0, k > 0; \quad P_{1,0}^{(1,2)} = \Gamma_0\Omega_0^{(2)} + \Gamma^{(1)}(1)S^{(2)}, j = 0, k > 0 ;$$

$$P_{1,0}^{(2,1)} = \Gamma^{(2)}(1)S^{(1)} + \Gamma_0^{(2)}\Omega_0^{(1)},\ j = 1;\quad P_{1,0}^{(2,2)} = \Omega_0^{(2)} + S^{(2)},\ j = 0\ ;$$

$$P_{i,0}^{(r,r)} = S^{(r)},\ r = \begin{cases} 1, & 2 \le i \le k, \\ 2, & i \ge j+1, i \ge 2, \end{cases}$$

$$P_{i,0}^{(1,2)} = \Gamma^{(1)}(1)S^{(2)},\ i \ge max\{2, k+1\}$$

$$P_{j,0}^{(2,1)} = \Gamma^{(2)}(1)S^{(1)},\ j \ge 2\ ;$$

$$P_{i,l}^{(r,r)} = \Omega_{l-i+1}^{(r,r)},\ r = \begin{cases} 1, & 0 < i \le k, l \ge i-1, l \ne 0, \\ 2, & i \ge j+1, l \ge i-1, l \ne 0, \end{cases}$$

$$P_{i,l}^{(1,2)} = \sum_{m=i}^{l+1} \Gamma_{m-i}^{(1)} \Omega_{l-m+1}^{(2)},\ i \ge k+1, l \ge i-1, l \ne 0\ ;$$

$$P_{j,l}^{(2,1)} = \sum_{m=j}^{l+1} \Gamma_{m-j}^{(2)} \Omega_{l-m+1}^{(1)},\ l \ge j-1, l \ne 0,$$

where matrices $\Omega_l^{(r)}$ and $\Gamma_l^{(r)}, l \ge 0, r = 1,2$ are the coefficients of the matrix expansions

$$\beta^{(r)}(z) = \sum_{l=0}^{\infty} \Omega_l^{(r)} z^l = \int_0^{\infty} e^{D^{(r)}(z)t} \otimes e^{F_0^{(r)}t} \otimes dB^{(r)}(t), r = \overline{1,2},$$

$$\Gamma^{(r)}(z) = \sum_{l=0}^{\infty} \Gamma_l^{(r)} z^l = \int_0^{\infty} e^{D^{(r)}(z)t} \otimes e^{F^{(r)}(1)t} dG_r(t) \otimes I_M,$$

$$\Psi_l = \int_0^{\infty} (e^{D_0^{(1)}t} D_l^{(1)}) \otimes e^{F^{(1)}(1)t} dt \otimes I_M = -[(D_0^{(1)} \oplus F^{(1)}(1))^{-1}(D_l^{(1)} \otimes I_{N+1})] \otimes I_M, l \ge 1,$$

$$S^{(r)} = S^{(r)}(1),\quad S^{(r)}(z) = \int_0^{\infty} e^{D^{(r)}(z)t} \otimes (e^{F_0^{(r)}t} F_1^{(r)}) \otimes (P^{(r)} - B^{(r)}(t))\, dt.$$

Here $\otimes$ and $\oplus$ are the symbols of Kronecker product and sum, I_m denotes the identity matrix of size m, O is a zero matrix.

The entries of the matrix $\Omega_l^{(r)}$ are transition probabilities of the process $\{\nu_n, \eta_n, m_n\}, n \ge 1$ during the service time in the r th mode when l customers arrive and no disaster arrives to the queue. The entries of the matrix $\Gamma_l^{(r)}$ are the transition probabilities of the process during changeover time from the r th mode to another one when l customers arrive to the queue, $l \ge 0$. The entries of the matrix $S^{(r)}$ are transition probabilities of the process during uncompleted service time in the r th mode when disaster arrives to the system, $r = \overline{1,2}$. And the entries of the matrix Ψ_l are transition probabilities of the process during the idle period that finishes with arrival of batch of l customers, $l \ge 1$.

Markov chain $\xi_n, n \ge 1$ is ergodic under any parameters of input and disaster flows, service and mode switching processes.

Introduce into consideration the stationary state probabilities of the chain $\xi_n, n \ge 1$

$$\pi(i,\nu,\eta,m)=\lim_{n\to\infty}P\{i_n=i,\omega_n=1,\nu_n=\nu,\eta_n=\eta,m_n=m\},$$

$$\chi(i,\nu,\eta,m)=\lim_{n\to\infty}P\{i_n=i,\omega_n=2,\nu_n=\nu,\eta_n=\eta,m_n=m\},$$

$$i\geq 0,\nu=\overline{0,W},\eta=\overline{0,N},m=\overline{1,M}$$

and consider vectors $\vec{\pi}_i,\ \vec{\chi}_i,\ i\geq 0$ consisted of these probabilities listed in lexicographic order of the components ν,η,m increasing. Note, that $\vec{\chi}_i=\vec{0}$ for $i=\overline{1,j-1}$. Introduce also generating functions

$$\vec{\Pi}_1(z)=\sum_{i=0}^{k}\vec{\pi}_i z^i,\quad \vec{\Pi}_2(z)=\sum_{i=k+1}^{\infty}\vec{\pi}_i z^i,\quad \vec{\chi}(z)=\sum_{i=j}^{\infty}\vec{\chi}_i z^i,\quad |z|\leq 1.$$

Having known the generating functions and the vector $\vec{\chi}_0$ we can obtain the stationary state embedded Markov chain distribution in the form of vectors $\vec{\pi}_i,\ \vec{\chi}_i,\ i\geq 0$.

Theorem 1. The generating functions $\vec{\Pi}_1(z)$, $\vec{\Pi}_2(z)$ and $\vec{\chi}(z)$ satisfy the following equations:

$$\vec{\Pi}_1(z)(zI-\beta^{(1)}(z))+\vec{\Pi}_2(z)z=\vec{\tau}(\Psi(z)-I)\beta^{(1)}(z)+(\vec{\chi}_0+\vec{\chi}_j z^j)\Gamma^{(2)}(z)\beta^{(1)}(z)+\vec{\theta}z, \quad (2)$$

$$\vec{\chi}_0=(\vec{\chi}(1)-\vec{\chi}_j+\vec{\Pi}_2(1)\Gamma^{(1)}(1))S^{(2)}, \quad (3)$$

$$\vec{\chi}(z)(zI-\beta^{(2)}(z))=(\vec{\Pi}_2(z)\Gamma^{(1)}(z)-\vec{\chi}_j z^j)\beta^{(2)}(z), \quad (4)$$

where $\Psi(z)=\sum_{i=1}^{\infty}\Psi_i z^i$, $\vec{\tau}=\vec{\pi}_0+\vec{\chi}_0\Gamma_0^{(2)}$, $\vec{\theta}=[\vec{\tau}(\Psi(1)-I)+\vec{\Pi}_1(1)+(\vec{\chi}_0+\vec{\chi}_j)\Gamma^{(2)}(1)]S^{(1)}$.

Proof. Using the formula of total probability and Lemma 1 the following balance equations can be obtained:

$$\vec{\pi}_0=\vec{\pi}_0\Psi_1\Omega_0^{(1)}+\vec{\pi}_0\Psi(1)S^{(1)}+\vec{\pi}_1\Omega_0^{(1)}+\sum_{i=1}^{k}\vec{\pi}_i S^{(1)}+\vec{\chi}_0\Gamma_1^{(2)}\Omega_0^{(1)}+\vec{\chi}_j\Gamma^{(2)}(1)S^{(1)}$$
$$+\vec{\chi}_0\sum_{i=1}^{\infty}\Gamma_i^{(2)}S^{(1)}+\vec{\chi}_0\Gamma_0^{(2)}\Psi_1\Omega_0^{(1)}+\vec{\chi}_0\Gamma_0^{(2)}\Psi(1)S^{(1)},$$

$$\vec{\pi}_l=(\vec{\pi}_0+\vec{\chi}_0\Gamma_0^{(2)})\sum_{i=1}^{l+1}\Psi_i\Omega_{l-i+1}^{(1)}+\sum_{i=1}^{l+1}\vec{\pi}_i\Omega_{l-i+1}^{(1)}+\vec{\chi}_0\sum_{i=1}^{l+1}\Gamma_i^{(2)}\Omega_{l-i+1}^{(1)},\quad l=\overline{1,j-2},$$

$$\vec{\pi}_l=(\vec{\pi}_0+\vec{\chi}_0\Gamma_0^{(2)})\sum_{i=1}^{l+1}\Psi_i\Omega_{l-i+1}^{(1)}+\sum_{i=1}^{l+1}\vec{\pi}_i\Omega_{l-i+1}^{(1)}+\vec{\chi}_0\sum_{i=1}^{l+1}\Gamma_i^{(2)}\Omega_{l-i+1}^{(1)}+\vec{\chi}_j\sum_{m=j}^{l+1}\Gamma_{m-j}^{(2)}\Omega_{l-m+1}^{(1)},$$
$$l=\overline{j-1,k-1},$$

$$\vec{\pi}_l=(\vec{\pi}_0+\vec{\chi}_0\Gamma_0^{(2)})\sum_{i=1}^{l+1}\Psi_i\Omega_{l-i+1}^{(1)}+\sum_{i=1}^{k}\vec{\pi}_i\Omega_{l-i+1}^{(1)}+\vec{\chi}_0\sum_{i=1}^{l+1}\Gamma_i^{(2)}\Omega_{l-i+1}^{(1)}++\vec{\chi}_j\sum_{m=j}^{l+1}\Gamma_{m-j}^{(2)}\Omega_{l-m+1}^{(1)},$$
$$l\geq k,$$

$$\vec{\chi}_0=\sum_{i=j+1}^{\infty}\vec{\chi}_i S^{(2)}+\sum_{i=k+1}^{\infty}\vec{\pi}_i\Gamma^{(1)}(1)S^{(2)},\quad \vec{\chi}_l=\sum_{i=j+1}^{l+1}\vec{\chi}_i\Omega_{l-i+1}^{(2)},\quad l=\overline{j,k-1},$$

$$\vec{\chi}_l=\sum_{i=j+1}^{l+1}\vec{\chi}_i\Omega_{l-i+1}^{(2)}+\sum_{i=k+1}^{l+1}\vec{\pi}_i\sum_{m=i}^{i+1}\Gamma_{m-i}^{(1)}\Omega_{l-m+1}^{(2)},\quad l\geq k.$$

Multiplying these equations by the corresponding degrees of z and summing them up the theorem statement is derived.

By expanding the both sides of (2) in series of z the following Corollary can be proved.

Corollary 1. Generating function $\vec{\Pi}_1(z)$ is determined as

$$\vec{\Pi}_1(z) = \vec{\tau} Y(z) + \vec{\chi}_0 Q(z) + \vec{\chi}_j R(z) + \vec{\theta} N(z), \tag{6}$$

where $Y(z) = \sum_{i=0}^{k} Y_i z^i$, $Q(z) = \sum_{i=0}^{k} Q_i z^i$, $R(z) = \sum_{i=0}^{k} R_i z^i$, $N(z) = \sum_{i=0}^{k} N_i z^i$,

$$Y_0 = I, \quad Y_{i+1} = \left(Y_i - \sum_{l=1}^{i+1} \Psi_l \Omega_{i-l+1}^{(1)} - \sum_{l=1}^{i} Y_l \Omega_{i-l+1}^{(1)} \right) \left(\Omega_0^{(1)} \right)^{-1}, i = \overline{0,k},$$

$$Q_0 = -\Gamma_0^{(2)}, \quad Q_{i+1} = \left(Q_i - \sum_{l=1}^{i+1} \Gamma_l^{(2)} \Omega_{i-l+1}^{(1)} - \sum_{l=1}^{i} Q_l \Omega_{i-l+1}^{(1)} \right) \left(\Omega_0^{(1)} \right)^{-1}, i = \overline{0,k},$$

$$R_i = O, i = \overline{0, j-1}, \quad R_i = Q_{i-j}, \ j \le i \le k,$$

$$N_0 = O, N_1 = -\left(\Omega_0^{(1)} \right)^{-1}, \quad N_{i+1} = \left(N_i - \sum_{l=1}^{i} N_l \Omega_{i-l+1}^{(1)} \right), i = \overline{1,k}.$$

Relation (6) involves four unknown vectors $\vec{\tau}$, $\vec{\chi}_0$, $\vec{\chi}_j$ and $\vec{\theta}$. Exploiting equation (5) for $\vec{\pi}_k$ we obtain the dependence

$$\vec{\chi}_j = \left(\vec{\tau} Y_{k+1} + \vec{\chi}_0 Q_{k+1} + \vec{\theta} N_{k+1} \right) \left(-Q_{k-j+1} \right)^{-1}.$$

So the relation (6) can be rewritten as

$$\vec{\Pi}_1(z) = \vec{\tau} \tilde{Y}(z) + \vec{\chi}_0 \tilde{Q}(z) + \vec{\theta} \tilde{N}(z), \tag{7}$$

where $\tilde{\Delta}(z) = \Delta(z) + \Delta_{k+1} \left(-Q_{k-j+1} \right)^{-1} R(z)$, Δ is in the set of symbols $\{Y, Q, N\}$. Further using the equations (7) and (2) we get the dependence of function $\vec{\Pi}_2(z)$ on vectors $\vec{\tau}$, $\vec{\chi}_0$ and $\vec{\theta}$

$$\vec{\Pi}_2(z) = \vec{\tau} \tilde{Y}_2(z) z^{-1} + \vec{\chi}_0 \tilde{Q}_2(z) z^{-1} + \vec{\theta} \tilde{N}_2(z) z^{-1}, \tag{8}$$

where

$$\tilde{Y}_2(z) = [(\Psi(z) - I)\beta^{(1)}(z) + Y_{k+1}(-Q_{k-j+1})^{-1} z^j \Gamma^{(2)}(z)\beta^{(1)}(z) + \tilde{Y}(z)(\beta^{(1)}(z) - zI)] z^{-1},$$

$$\tilde{Q}_2(z) = [\Gamma^{(2)}(z)\beta^{(1)}(z) + Q_{k+1}(-Q_{k-j+1})^{-1} \Gamma^{(2)}(z)\beta^{(1)}(z) + \tilde{Q}(z)(\beta^{(1)}(z) - zI)] z^{-1},$$

$$\tilde{N}_2(z) = [N_{k+1}(-Q_{k-j+1})^{-1} z^j \Gamma^{(2)} \beta^{(1)}(z) + zI + \tilde{N}(z)(\beta^{(1)}(z) - zI)] z^{-1}.$$

To obtain relation between vectors $\vec{\chi}_0$, $\vec{\tau}$ and $\vec{\theta}$ from (3) the value of $\vec{\chi}(1)$ needs to be known. To do this we substitute $z = 1$ in (2)-(4), sum the equations up and use the technique of M. Neuts, see for e.g. [6], [16]. Finally we obtain

$$\vec{\chi}(1) = \vec{\rho} + \vec{\tau}\hat{Y} + \vec{\chi}_0\hat{Q} + \vec{\theta}\hat{N}, \qquad (9)$$

$$\hat{Y} = (\Psi(1) - I)A_1 Z + \tilde{Y}(1)(A_1 - I - 1\vec{\rho})Z + Y_{k+1}(-Q_{k-j+1})^{-1}(\Gamma^{(2)}A_1 - I - 1\vec{\rho})Z$$
$$+ \tilde{Y}_2(1)(\Gamma^{(1)}Z - Z - \Gamma^{(1)}(1)) + Y_{k+1}(-Q_{k-j+1})^{-1},$$

$$\hat{Q} = \tilde{Q}(1)(A_1 - I - 1\vec{\rho})Z + (I + Q_{k+1}(-Q_{k-j+1})^{-1})(\Gamma^{(2)}(1)A_1 - I - 1\vec{\rho})Z$$
$$+ \tilde{Q}_2(1)(\Gamma^{(1)}(1)Z - Z - \Gamma^{(1)}(1)) + Q_{k+1}(-Q_{k-j+1})^{-1},$$

$$\hat{N} = \tilde{N}(1)(A_1 - I - 1\vec{\rho})Z + N_{k+1}(-Q_{k-j+1})^{-1}(\Gamma^{(2)}A_1 - I - 1\vec{\rho})Z$$
$$+ \tilde{N}_2(1)(\Gamma^{(1)}(1)Z - Z - \Gamma^{(1)}(1)) + N_{k+1}(-Q_{k-j+1})^{-1},$$

$$Z = (I - A_2 + 1\vec{\rho})^{-1}, \quad A_i = \beta^{(i)}(1) + S^{(i)}, i = \overline{1,2},$$

$\vec{\rho}$ is the row eigenvector of the matrix A_2 corresponding to the eigenvalue 1. Using (3), (8) and (9) we get the relation

$$\vec{\chi}_0 = \vec{\rho}A^* + \vec{\tau}(\tilde{Y}_2(1)\Gamma^{(1)}(1) + \hat{Y})A^* + \vec{\theta}(\tilde{N}_2(1)\Gamma^{(1)}(1) + \hat{N}(1))A^*,$$

where $A^* = S^{(2)}(I - \tilde{Q}_2(1)\Gamma^{(1)}(1)S^{(2)} - \tilde{Q}_3 S^{(2)})^{-1}$. Then susbtituting $z = 1$ into (8) the dependence $\vec{\theta} = \vec{\rho}U + \vec{\tau}T$ is derived, where

$$U = A^*[\tilde{Q}(1) + (I + Q_{k+1}(-Q_{k-j+1})^{-1})\Gamma^{(2)}(1)]\tilde{A},$$

$$T = [\Psi(1) - I + \tilde{Y}(1) + (\tilde{Y}_2(1)\Gamma^{(1)}(1) + \hat{Y})A^*(\tilde{Q}(1) + (I + Q_{k+1}(-Q_{k-j+1})^{-1})\Gamma^{(2)}(1))$$
$$+ Y_{k+1}(-Q_{k-j+1})^{-1}\Gamma^{(2)}(1)]\tilde{A},$$

$$\tilde{A} = S^{(1)}[I - (\tilde{N}_2(1)\Gamma^{(1)}(1) + \hat{N})A^*(\tilde{Q}(1) + (I + Q_{k+1}(-Q_{k-j+1})^{-1})\Gamma^{(2)}(1))S^{(1)}$$
$$- N_{k+1}(-Q_{k-j+1})^{-1}\Gamma^{(2)}(1)S^{(1)} - \tilde{N}(1)S^{(1)}]^{-1}.$$

Finally to calculate unknown vector $\vec{\pi}_0$ we use the functional equation (2) and the property of the equation $det(zI - \beta^{(2)}(z)) = 0$ to have exactly $J = (W+1)(N+1)M$ roots inside a unit disc $|z| < 1$ of the complex plane (Theorem 3 in [9]). Denote these roots as z_k with corresponding multiplicities $n_k, k = \overline{1, H}$, $\sum_{k=1}^{H} n_k = J$, where H is the number of different root. Exploiting analyticity property of the function $\vec{\Pi}_2(z)$ inside the disc $|z| < 1$ we obtain the following system of equations for the vector $\vec{\tau}$ entries:

$$\vec{\tau}\frac{d^n}{dz^n}L_1(z)(zI - \beta^{(2)}(z))\Big|_{z=z_k} = \vec{\rho}\frac{d^n}{dz^n}L_2(z)(zI - \beta^{(2)}(z))\Big|_{z=z_k}, \; n = \overline{0, n_k - 1}, k = \overline{1, H},$$

where

$$\vec{L}_1(z) = [\tilde{Y}_2(z) + (\tilde{Y}_2(1)\Gamma^{(1)}(1) + \hat{Y})A^*\tilde{Q}_2(z) + T(\tilde{N}_2(1)\Gamma^{(1)}(1) + \hat{N})A^*\tilde{Q}_2(z)$$

$$+ T\tilde{N}_2(z)]\Gamma^{(1)}(z)\beta^{(2)}(z) - [Y_{k+1} + (\tilde{Y}_2(1)\Gamma^{(1)}(1) + \hat{Y} + T(\tilde{N}_2(1)\Gamma^{(1)}(1) + \hat{N}))A^*Q_{k+1} + TN_{k+1}](-Q_{k-j+1})^{-1}z^{j+1},$$

$$\vec{L}_2(z) = [A^*\tilde{Q}_2(z) + U(\tilde{N}_2(1)\Gamma^{(1)}(1) + \hat{N})A^*\tilde{Q}_2(z) + U\tilde{N}_2(z)]\Gamma^{(1)}(z)\beta^{(2)}(z) - [A^*Q_{k+1} + U(\tilde{N}_2(1)\Gamma^{(1)}(1) + \hat{N})A^*Q_{k+1} + UN_{k+1}](-Q_{k-j+1})^{-1}z^{j+1}.$$

Thus having known the value of $\vec{\tau}$ and having calculated the vectors $\vec{\chi}_0$, $\vec{\chi}_j$, $\vec{\theta}$ and $\vec{\chi}(1)$ we can obtain the partial generating function $\vec{\Pi}_1(z)$, $\vec{\Pi}_2(z)$ and $\vec{\chi}(z)$ of the stationary state distribution of the embedded Markov chain for the fixed thresholds (j,k).

4. Performance Characteristics and the Cost Criterion Value

As some customers may leave the system without being served, important performance characteristics is the probability of arbitrary customer successful service. Denote this probability as P_+. Using the ergodic theorems for functionals defined on the Markov chains [22] the formula for P_+ can be obtained

$$P_+ = \frac{\vec{u}\vec{1}}{\vec{u}\vec{1} + \vec{\gamma}'(1)\vec{1}},$$

where $\vec{u} = \vec{\Pi}_1(1) + \vec{\Pi}_2(1) - \vec{\theta}(I - S^{(1)})^{-1} + \vec{\chi}(1)$,

$$\vec{\gamma}(z) = [(\vec{\pi}_0 + \vec{\theta}(S^{(1)} - I)\Gamma_0^{(2)})(\Psi(z) - I) + \vec{\Pi}_1(z) + (\vec{\chi}_j z^j + \vec{\chi}_0)\Gamma^{(2)}(z)]S^{(1)}(z) + [\vec{\chi}(z) - \vec{\chi}_j z^j + \vec{\Pi}_2(z)\Gamma^{(1)}(z)]S^{(2)}(z).$$

Using theory of renewal processes we derive the formula for the mean interdeparture time

$$\Lambda = \frac{\lambda^{(1)}}{\vec{u}\vec{1} + \vec{\gamma}'(1)\vec{1} + (\lambda^{(1)} - \lambda^{(2)})(\vec{\chi}_j g_2 + \vec{\chi}_0 g_2 + (\vec{\chi}(1) - \vec{\chi}_j))\overline{B}\vec{1}},$$

where

$$\overline{B} = \int_0^\infty e^{D^{(2)}(1)t} \otimes e^{F_0^{(2)}t} \otimes t\,dB^{(2)}(t) + \int_0^\infty e^{D^{(2)}(1)t} \otimes (te^{F_0^{(2)}t}F_1^{(2)}) \otimes (P^{(2)} - B^{(2)}(t))\,dt\,.$$

The mean queue length L at service completion epochs is $(\vec{\Pi}_1(z) + \vec{\Pi}_2(z) + \vec{\chi}'(1))\vec{1}$. Other performance characteristics involved to the cost criterion (1) are calculated as

$$\Phi^{(2)} = \Lambda(\vec{\chi}_j g_2 + \vec{\chi}_0 g_2 + (\vec{\chi}(1) - \vec{\chi}_j)\overline{B})\vec{1}, \quad \Phi^{(1)} = 1 - \Phi^{(2)},$$

$$V = (1 - P_+)(\lambda^{(1)}\Phi^{(1)} + \lambda^{(2)}\Phi^{(2)}), \quad K = 2\Lambda[\vec{\chi}_0 + \vec{\chi}_j]\vec{1}.$$

Substituting the values of the cost criterion components in (1) we get the value of the cost criterion under the fixed thresholds (*j*, *k*). This allows us to find optimal thresholds (*j**, *k**) minimizing the cost criterion value.

5. Numerical Example

To illustrate obtained results we present below simple numerical example. BMAP-input of customers is defined by the matrices

$$D_0^{(1)} = \begin{pmatrix} -4.4 & 2.4 \\ 2.8 & 7.8 \end{pmatrix}, \ D_1^{(1)} = D_2^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 2.5 \end{pmatrix},$$

$$D_0^{(2)} = \begin{pmatrix} -2.2 & 1.2 \\ 4.8 & 7.8 \end{pmatrix}, \ D_1^{(2)} = \begin{pmatrix} 0.75 & 0 \\ 0 & 2.25 \end{pmatrix}, \ D_2^{(2)} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.75 \end{pmatrix}.$$

The intensities of input flow are equal to $\lambda^{(1)} = 5.08$ and $\lambda^{(1)} = 1.75$. The Semi-Markovian kernel characterizing the service process has the form

$$B^{(r)}(t) = \begin{pmatrix} 0.3B_1^{(1)}(t) & 0.7B_2^{(1)}(t) \\ 0.8B_1^{(2)}(t) & 0.2B_2^{(2)}(t) \end{pmatrix},$$

where

$$B_i^{(r)}(t) = \int_0^t \frac{b_i^{(r)}(b_i^{(r)}\tau)^{k_i^{(r)}-1}}{(k_i^{(r)}-1)!} e^{-b_i^{(r)}\tau} d\tau, \ b_1^{(1)} = 4, \ b_2^{(1)} = 8, \ b_1^{(2)} = 7, \ b_2^{(2)} = 14,$$

$$k_1^{(1)} = 1, \ k_2^{(1)} = k_1^{(2)} = 2, \ k_2^{(2)} = 4.$$

The mean service times are equal $b^{(1)} = 0.25$ and $b^{(2)} = 0.285$.

MAP -input of disasters is defined by the matrices

$$F_0^{(1)} = \begin{pmatrix} -0.25 & 0.15 \\ 0.24 & -0.33 \end{pmatrix}, \ F_1^{(1)} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.09 \end{pmatrix},$$

$$F_0^{(2)} = \begin{pmatrix} -0.21 & 0.16 \\ 0.27 & -0.35 \end{pmatrix}, \ F_1^{(2)} = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.08 \end{pmatrix}.$$

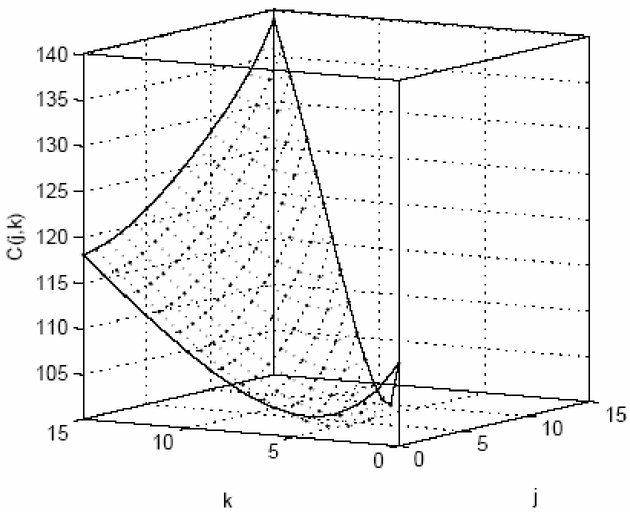

Figure 1. The dependence of the cost criterion value on the thresholds.

The intensity of disaster arrivals is equal to 0.096 in the first mode and 0.061 in the second mode. The time to switch from the r th mode to another one has exponential distribution with rate $1/g_r$, $g_1 = 0.2$, $g_2 = 0.1$. Cost coefficients involved in (1) have the values $a = 2$, $c_1 = 10$, $c_2 = 115$, $d = 40$, $h = 5$. For this system the following results are obtained. Figure 1 illustrates dependence of the cost criterion on the thresholds (j,k) in the set of their values $\{(j,k): 0 \le j \le k \le 15\}$. Optimal cost criterion value is $C^* = 101.17$, optimal thresholds are $j^* = 1$, $k^* = 4$. Using the results of [6] the value of the cost criterion when the system uses only one of the modes has been obtained, that is $C_1 = 124.87$ when the first mode is used only and $C_2 = 122.151$ otherwise. So in our case the optimal hysteresis control allows to reduce the system operation cost more than 17%.

6. Conclusion

Queueing system $BMAP/SM/1$ with two modes of operation and hysteresis control has been considered. The system also has input of disasters destroying the entire queue. The stationary state distribution of the embedded Markov chain and performance characteristics are obtained. The problem of hysteresis strategy optimization is solved numerically.

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