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Kolmogorov ε-Entropy in the Problems on Global Attractors for Evolution Equations of Mathematical Physics[1]

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Abstract—We study the Kolmogorov ε-entropy and the fractal dimension of global attractors for autonomous and nonautonomous equations of mathematical physics. We prove upper estimates for the ε-entropy and fractal dimension of the global attractors of nonlinear dissipative wave equations.

Andrey Nikolaevich Kolmogorov discovered applications of notions of information theory in the theory of dynamical systems. In particular, he introduced the notion of ε-entropy $\mathbf{H}_\varepsilon(X)$ of a compact set X in a Banach space E. The well-known paper of Kolmogorov and V.M. Tikhomirov [1] contains many important estimates from above and from below for the ε-entropy $\mathbf{H}_\varepsilon(X)$ of a number of particular function sets X. For example, in the paper, the ε-entropy is studied for the set of real functions $\{u(t),\ t \in \mathbb{R}\}$ that have bounded spectrum, and a variant of the Kotelnikov theorem is proved (see also [2]).

In the last decades, global attractors $\mathcal{A}$ were intensively investigated for basic evolution equations of mathematical physics, for which the initial Cauchy problem is studied deep enough. Recall that a global attractor $\mathcal{A}$ is a compact set in the corresponding Banach or Hilbert space that obeys the invariance property with respect to the corresponding dynamical system and that attracts bounded sets of trajectories as time $t \to +\infty$.

For certainty and brevity, in this paper we study the Kolmogorov ε-entropy and fractal dimension of a global attractor $\mathcal{A}$ of the dissipative wave equation in a bounded domain $\Omega \Subset \mathbb{R}^n$. We consider in more detail the case of the sine-Gordon equation, where the global attractor admits a simple structure.

For the case of an autonomous hyperbolic equation, where all coefficients and the exiting force of the equation do not depend on time, we present in Section 2 an upper estimate for the ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A})$ of its global attractor $\mathcal{A}$ and give an estimate from above for the fractal dimension $\mathbf{d}_F(\mathcal{A})$ of this attractor. Before this, we formulate the general theorem on the estimation of the ε-entropy $\mathbf{H}_\varepsilon(X)$ of an invariant set X of an abstract autonomous dynamical system.

Section 3 is devoted to the construction of the global attractor $\mathcal{A}$ for the nonautonomous wave equation as well as for an abstract nonautonomous evolution equation. In Section 4, we prove an upper estimate for the ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A})$ of the global attractor $\mathcal{A}$ of the nonautonomous wave equation with, for example, exiting force $g(x,t)$, $x \in \Omega$, that depends on time t. We also present in this section some general facts concerning abstract nonautonomous dynamical systems. In the particular case where the exiting force $g(x,t)$ of the wave equation is an almost periodic function of t, the ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A})$ of the global attractor $\mathcal{A}$ for $0 < \varepsilon \le \varepsilon_0$ does not exceed the sum of three terms:

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first, $\mathbf{H}_\beta(\mathcal{H}(g))$, where $\mathcal{H}(g)$ is the hull of the function $g(x,t)$ in the space $C_b(\mathbb{R};L_2(\Omega))$, $\beta > 0$, and we give an explicit expression for the value $\beta = \beta(\varepsilon)$ (see Section 4); second, $D\log_2(1/\varepsilon)$, this term is analogous to those encountered in the upper estimate of the ε-entropy of the attractor $\mathcal{A}$ of an autonomous hyperbolic equation; third, $\mathbf{H}_{\varepsilon_0}(\mathcal{A})$, where ε_0 is fixed. It should be noted that the fractal dimension $\mathbf{d}_F(\mathcal{A})$ of the global attractor $\mathcal{A}$ in the nonautonomous case can be infinite. However, if the function $g(x,t)$ is quasiperiodic in t, that is, $g(x,t) = G(x,\alpha_1 t,\dots,\alpha_k t)$, where $G(x,\omega_1,\dots,\omega_k)$ is a 2π-periodic function in each variable ω_i, $i = 1,\dots,k$, then the fractal dimension $\mathbf{d}_F(\mathcal{A})$ is finite. In this case, $\mathbf{d}_F(\mathcal{A}) \le \mathbf{d}_F(\mathcal{H}(g)) + D \le k + D$ and the ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A}) \lesssim k\log_2\left(\frac{1}{\varepsilon}\right) + D\log_2\left(\frac{1}{\varepsilon}\right)$. Here k is the number of rationally independent frequencies $\{\alpha_i\}$, $i = 1,\dots,k$, of the quasiperiodic function $g(x,t)$. In particular, for $k = 0$, we get an estimate for the autonomous equation.

In conclusion, note that the Kolmogorov ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A})$ of the global attractor $\mathcal{A}$ is always finite because the set $\mathcal{A}$ is compact in the corresponding function space. The behavior of the quantity $\mathbf{H}_\varepsilon(\mathcal{A})$ as a function of ε as $\varepsilon \to 0+$ describes the complexity of the global attractor $\mathcal{A}$ of the dynamical system under consideration.

1. ε-ENTROPY AND FRACTAL DIMENSION OF COMPACT SETS

Let us formulate the definition of the Kolmogorov ε-entropy of a compact set X in a Banach space E. Denote by $N_\varepsilon(X,E) = N_\varepsilon(X)$ the minimum number of open balls in E of radius ε which is necessary to cover X:

$$X \subset \bigcup_{i=1}^{N} B(x_i,\varepsilon), \qquad N_\varepsilon(X) = \min N. \tag{1}$$

Here $B(x_i,\varepsilon) = \{x \in E \mid \|x - x_i\|_E < \varepsilon\}$ is the ball in E with center x_i and radius ε. Since the set X is compact, it is easy to see that $N_\varepsilon(X) < +\infty$ for any $\varepsilon > 0$.

Definition 1. The Kolmogorov ε-entropy of the set X in the space E is the number

$$\mathbf{H}_\varepsilon(X,E) := \mathbf{H}_\varepsilon(X) := \log_2 N_\varepsilon(X). \tag{2}$$

For particular sets X, the problem is to study the asymptotic behavior of the function $\mathbf{H}_\varepsilon(X)$ with respect to ε as $\varepsilon \to 0+$. This characteristic of a compact set was originally introduced by Kolmogorov and was studied in the joint work with Tikhomirov (see [1]). In this paper, ε-entropy of various classes of functions was investigated. Moreover, in the paper, an important notion of the *entropy dimension* of a compact set was also defined. This dimension is now often called the *fractal dimension*.

Definition 2. The (upper) fractal dimension of a compact set X in the space E is the number

$$\mathbf{d}_F(X,E) := \mathbf{d}_F(X) := \overline{\lim_{\varepsilon\to 0+}} \frac{\mathbf{H}_\varepsilon(X)}{\log_2(1/\varepsilon)}. \tag{3}$$

The fractal dimension of a compact set in an infinite-dimensional Banach space can be infinite. However, if it is known that $0 < \mathbf{d}_F(X) < +\infty$, then $\mathbf{H}_\varepsilon(X) \approx \mathbf{d}_F(X)\log_2\left(\frac{1}{\varepsilon}\right)$; therefore, in this case, one needs $N_\varepsilon(X) \approx \left(\frac{1}{\varepsilon}\right)^{\mathbf{d}_F(X)}$ points to approximate the set X with precision ε. In [1], sets from various function spaces were considered for which $\mathbf{H}_\varepsilon(X) \approx D\log_2\left(\frac{1}{\varepsilon}\right)^a$, where $a > 1$, and even $\mathbf{H}_\varepsilon(X) \approx D\left(\frac{1}{\varepsilon}\right)^a$. For such sets, evidently, their fractal dimension $\mathbf{d}_F(X) = +\infty$; thus, ε-entropy becomes an important characteristic of sets in infinite-dimensional spaces.

There is another useful characteristic of a compact set, namely, the Hausdorff dimension

$$\mathbf{d}_H(X) = \inf\{d \mid \mu(X,d) = 0\},$$

where $\mu(X,d) = \inf \sum r_i^d$ and the infimum is taken over all the coverings of X by balls $B(x_i, r_i)$ of radii $r_i \le \varepsilon$. It is easy to see that $\mathbf{d}_H(X) \le \mathbf{d}_F(X)$. There exist many examples of sets such that $\mathbf{d}_H(X) = 0$ but $\mathbf{d}_F(X) = +\infty$. In the present paper, we shall only consider the fractal dimension of a compact set, because this dimension is closely related to the ε-entropy of the set.

Note that the fractal dimension is very useful in the study of the structure of various "nonsmooth" sets in finite-dimensional spaces, for example, self-similar sets or the fractals. The simplest example of such a set is the Cantor set K on the segment $[0,1]$, for which $\mathbf{d}_F(K) = \log_3 2 < 1$. The fractal dimension of a compact smooth manifold is equal to its usual dimension, i.e., is an integer. However, the example of the Cantor set shows that the fractal dimension can be noninteger.

Another important application of the ε-entropy and the fractal dimension arises in the study of global attractors of dynamical systems that describe the so-called deterministic chaos, which was originally introduced in the works of Lorenz (see [3]). Global attractors will be discussed in more detail in Section 3. Here we recall that the global attractor of the dynamical system is a compact strictly invariant set $\mathcal{A}$ of the phase space E that attracts all the trajectories of this dynamical system. For the Lorenz system, the phase space is $\mathbb{R}^3$. The Lorenz system is the three-mode Galerkin approximation of the Boussinesq system describing the convection of heated fluid. The system has the form

$$\begin{cases} \dfrac{dx}{dt} = -\sigma x + \sigma y, \\ \dfrac{dy}{dt} = rx - y - xz, \\ \dfrac{dz}{dt} = -bz + xy, \end{cases}$$

where σ, r, and b are positive parameters. The original Lorenz parameters are $\sigma = 10$, $b = 8/3$, and $r = 28$. Recently, a new upper estimate for the fractal dimension of the Lorenz attractor was proved:

$$\mathbf{d}_F(\mathcal{A}) \le \mathbf{d}_L(\mathcal{A}) = 2.401\ldots. \tag{4}$$

Here, $\mathbf{d}_L(\mathcal{A})$ denotes the Lyapunov dimension of the attractor $\mathcal{A}$ (this dimension will be discussed below). The Lyapunov dimension is always less than the fractal dimension of a compact set. It is interesting that there exists a simple explicit formula for the Lyapunov dimension of the Lorenz attractor (see [4]). However, the question on nontrivial lower bounds for the fractal dimension of the Lorenz attractor remains open. It is only known that $\mathbf{d}_F(\mathcal{A}) \ge 2$.

In the next section, we consider infinite-dimensional dynamical systems and their global attractors.

2. ε-ENTROPY AND FRACTAL DIMENSION OF GLOBAL ATTRACTORS OF AUTONOMOUS EQUATIONS OF MATHEMATICAL PHYSICS

In this section, we study the ε-entropy and fractal dimension of global attractors of autonomous dynamical systems in infinite-dimensional spaces. Such systems are generated by autonomous evolution equations that can be written in the following abstract form:

$$\partial_t y = A(y), \quad y|_{t=0} = y_0(x) \in E, \quad t \ge 0. \tag{5}$$

Here, $y = y(x,t)$ is a solution of equation (5), x is the spatial variable, and t is time. The right-hand side $A(y)$ of equation (5) is a (nonlinear) operator, which depends on the function y and its partial derivatives in x. The function $y_0(x)$ in (5) determines the initial state of the dynamical system described by this equation, that is, $y(x,0) = y_0(x)$. The initial condition $y_0(x)$ belongs to some infinite-dimensional Banach space E, called the phase space of problem (5). The phase space E is chosen based on the physical sense of the problem. For instance, E can be a Sobolev space. The value $y_0(x)$ can be taken from this space arbitrarily. We assume that, for any function $y_0(x)$ from E, problem (5) has a unique solution $y(x,t)$, $t \geq 0$, in some function space, and $y(\cdot\,,t) \in E$ for all $t \geq 0$. Then problem (5) generates a family of nonlinear operators $\{S(t),\ t \geq 0\}$, $S(t)\colon E \to E$, acting by the formula

$$y_0(x) \longmapsto S(t)y_0(x) = y(x,t),$$

where $y(x,t)$ is a solution of (5) with initial condition $y_0(x)$. The operators $\{S(t)\} = \{S(t),\ t \geq 0\}$ form a semigroup; that is, $S(0) = \mathrm{Id}$ is the identity operator and $S(t_1 + t_2) = S(t_1) \circ S(t_2)$ for all $t_1, t_2 \geq 0$.

A vast variety of partial differential equations of mathematical physics of the form (5) and the corresponding semigroups $\{S(t)\}$ can be found in [5–7]. As an example of problem (5), we consider the following nonlinear wave equation with dissipation:

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - f(u) + g(x), \tag{6}$$

$$u|_{\partial\Omega} = 0, \quad x \in \Omega \Subset \mathbb{R}^n. \tag{7}$$

Here $u = u(x,t)$ is an unknown scalar function of variables x and t. The equation is considered in a bounded domain Ω of the space $\mathbb{R}^n$. For simplicity, we assume that the boundary $\partial\Omega$ of the domain Ω belongs to the class C^1. The boundary condition (7) means that $u(x,t) = 0$ for all $x \in \partial\Omega$ and $t \geq 0$. In equation (6), the symbol Δ denotes the Laplas operator in x: $\Delta u(x) = \sum\limits_{i=1}^{n} \dfrac{\partial^2 u}{\partial x_i^2}(x)$. Also, we use the notations $\partial_t u = \dfrac{\partial u}{\partial t}$ and $\partial_t^2 u = \dfrac{\partial^2 u}{\partial t^2}$. The equation contains the dissipation term $\gamma\partial_t u$, where γ is a positive number. The nonlinear function $f(u)$ belongs to the class C^1 and satisfies the following inequalities:

$$F(u) \geq -mu^2 - C_m, \qquad F(u) = \int\limits_0^u f(v)\,dv, \tag{8}$$

$$f(u)u \geq \gamma_1 F(u) - mu^2 - C_m, \quad \forall u \in \mathbb{R}, \tag{9}$$

where $m > 0$, $\gamma_1 > 0$, and the number m is sufficiently small ($m < \lambda_1$, where λ_1 is the first eigenvalue of the operator $-\Delta$ with zero boundary conditions). We also assume that

$$|f'(u)| \leq C_0(1 + |u|^\rho), \quad \forall u \in \mathbb{R}, \tag{10}$$

where ρ is an arbitrary positive number for $n = 1, 2$, and $\rho < 2/(n-2)$ for $n \geq 3$. Equation (6) with boundary conditions (7) was considered in many works (see [5, 7–11]). The limit case $\rho = 2/(n-2)$ was also studied under some extra conditions on the function $f(u)$. Here we restrict ourselves to the case $\rho < 2/(n-2)$.

Remark 1. Wave equations of the form (6) appears in many problems of mathematical physics. For example, the model sine-Gordon equation is used in the study of the Josephson junction with nonlinear function

$$f(u) = \beta \sin(u). \tag{11}$$

It easily follows that in this case conditions (8)–(10) are valid and $\rho = 0$.

Another model equations of the form (6), encountered in relativistic quantum mechanics, has the nonlinear term

$$f(u) = |u|^\rho u. \tag{12}$$

In this case, $F(u) = |u|^{\rho+2}/(\rho+2)$ and inequalities (8)–(10) take place with $\gamma_1 = 1/(\rho+2)$ (see [7, 8, 10]).

Inequality (10) implies that

$$|f(u)| \le C_1(1 + |u|^{\rho+1}), \quad \forall u \in \mathbb{R}. \tag{13}$$

Note that, owing to the Sobolev embedding theorem, we have

$$H_0^1(\Omega) \subset L_{2(\rho+1)}(\Omega). \tag{14}$$

Indeed, for $n = 1, 2$, this embedding holds for any $\rho \ge 0$; for $n \ge 3$, we have $2(\rho+1) < 2n/(n-2)$ due to the assumptions imposed on ρ, while the number $2n/(n-2)$ is the critical exponent in the Sobolev theorem. Thus, if $u(x) \in H_0^1(\Omega)$, then by (13) and (14) we observe that $f(u(x)) \in L_2(\Omega)$. If a function $u(x,t) \in L_\infty(\mathbb{R}_+; H_0^1(\Omega))$ is given such that $\partial_t u(x,t) \in L_\infty(\mathbb{R}_+; L_2(\Omega))$, then $\Delta u(x,t) \in L_\infty(\mathbb{R}_+; H^{-1}(\Omega))$, where $H^{-1}(\Omega)$ is the dual space for $H_0^1(\Omega)$. Furthermore, $f(u(x,t)) \in L_\infty(\mathbb{R}_+; L_2(\Omega))$. Consequently, $-\gamma\partial_t u + \Delta u - f(u) + g(x) \in L_\infty(\mathbb{R}_+; H^{-1}(\Omega))$, and equation (6) can be considered in the space $D'(\mathbb{R}_+; H^{-1}(\Omega))$ of distributions with values in $H^{-1}(\Omega)$ (see [8]). In particular, if a function $u(x,t)$ is a solution of equation (6), then $\partial_t^2 u(x,t) \in L_\infty(\mathbb{R}_+; H^{-1}(\Omega))$.

Equation (6) and boundary conditions (7) are supplemented with the initial conditions

$$u|_{t=0} = u_0(x), \qquad \partial_t u|_{t=0} = p_0(x). \tag{15}$$

Proposition 1. *If $u_0(x) \in H_0^1(\Omega)$ and $p_0(x) \in L_2(\Omega)$, then, under conditions* (8)–(10), *problem* (6), (7), (15) *has a unique solution $u(x,t) \in C_b(\mathbb{R}_+; H_0^1(\Omega))$, $\partial_t u(x,t) \in C_b(\mathbb{R}_+; L_2(\Omega))$ in the space $D'(\mathbb{R}_+; H^{-1}(\Omega))$. Moreover, $\partial_t^2 u(x,t) \in L_\infty(\mathbb{R}_+; H^{-1}(\Omega))$.*

The proof of this assertion can be found in [5–8].

Denote for brevity $y(x,t) = (u(x,t), \partial_t u(x,t)) = (u(t), p(t))$ and $y_0(x) = (u_0(x), p_0(x)) = y(x,0)$. By E, we denote the space of vector functions $y(x) = (u(x), p(x))$, where $u(x) \in H_0^1(\Omega)$ and $p \in L_2(\Omega)$. The norm in this space is

$$\|y\|_E = \left(\|\nabla u\|^2_{L_2(\Omega)} + \|p\|^2_{L_2(\Omega)}\right)^{1/2}.$$

Note that $y(x,t) \in E$ for all $t \ge 0$. Problem (6), (7), (15) is equivalent to the following system:

$$\begin{cases} \partial_t u = p, \\ \partial_t p = -\gamma p + \Delta u - f(u) + g, \end{cases} \qquad \begin{cases} u|_{t=0} = u_0, \\ p|_{t=0} = p_0, \end{cases} \tag{16}$$

which is of the form (5) with $A(y) = A(u,p) = (p, -\gamma p + \Delta u - f(u) + g(x))$.

If $y_0 \in E$, then problem (16) has a unique solution $y(t) \in C_b(\mathbb{R}_+; E)$. Thus, problem (16) generates a semigroup $\{S(t)\}$ acting in the space E.

Let us define the global attractor of a semigroup $\{S(t)\}$ acting in a Banach space E.

Definition 3. A compact set $\mathcal{A}$ from E is said to be the *global attractor* of the semigroup $\{S(t)\}$ if

(1) the set $\mathcal{A}$ is strictly invariant with respect to $\{S(t)\}$, that is,

$$S(t)\mathcal{A} = \mathcal{A} \quad \text{for all } t \ge 0,$$

(2) the set $\mathcal{A}$ attracts the set $S(t)B$ as $t \to +\infty$, where B is an arbitrary bounded (in the space E) set of initial conditions $\{y_0(x)\} = B$:

$$\mathrm{dist}_E(S(t)B, \mathcal{A}) \to 0, \quad t \to +\infty.$$

Here, $\mathrm{dist}_E(A_1, A_2) = \sup\limits_{a_1 \in A_1} \inf\limits_{a_2 \in A_2} \|a_1 - a_2\|_E$ is the Hausdorff distance between the sets A_1 and A_2 in the space E.

Condition (2) can be reformulated as follows: for any $\varepsilon > 0$, there is a number $T = T(\varepsilon, B)$ such that $S(t)B \subseteq \mathcal{O}_\varepsilon(\mathcal{A})$ for all $t \geq T$, where $\mathcal{O}_\varepsilon(A)$ denotes the ε-neighborhood of the set A in E. It follows from Definition 3 that the global attractor $\mathcal{A}$ attracts all the solutions $y(x,t) = S(t)y_0(x)$ as $t \to +\infty$ uniformly with respect to any bounded set $B = \{y_0(x)\}$ of initial data. It is easy to see that the global attractor is unique, if it exists.

Consider the semigroup $\{S(t)\}$ of equation (6), acting in the space $E = H_0^1(\Omega) \times L_2(\Omega)$. The theorem on the existence of the global attractor for a dissipative hyperbolic equation was originally proved in the works of Babin and Vishik (see [5,9]).

Proposition 2. *The semigroup $\{S(t)\}$ of the problem* (6), (7), (15) *has a global attractor $\mathcal{A} \Subset E$.*

For the proof, see [5,7,9].

Consider the dissipative sine-Gordon equation ($\beta > 0$)

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - \beta \sin(u) + g(x), \quad u|_{\partial\Omega} = 0. \tag{17}$$

The corresponding semigroup $\{S(t)\}$ has a global attractor $\mathcal{A}$. Let us formulate the conditions that provide the trivial global attractor consisting of the unique stationary solution of equation (17). We denote by λ_1 the first eigenvalue of the operator $-\Delta$ with zero boundary conditions.

Proposition 3. *Assume that the inequalities*

$$\beta < \lambda_1, \qquad \gamma^2 > \gamma_0^2 := 2\left(\lambda_1 - \sqrt{\lambda_1^2 - \beta^2}\right) \tag{18}$$

hold. Then equation (17) *has a unique stationary solution $\bar{u}(x) \in H_0^1(\Omega)$; that is,*

$$\Delta \bar{u} - \beta \sin(\bar{u}) + g(x) = 0, \quad \bar{u}|_{\partial\Omega} = 0. \tag{19}$$

Moreover, this solution is asymptotically stable; i.e., for any solution $y(x,t) = (u(x,t), \partial_t u(x,t)) = S(t)y_0 = S(t)(u_0(x), p_0(x))$ of the wave equation, we have the inequality

$$\|y(\cdot\,, t) - z(\cdot)\|_E \leq C\|y_0(\cdot) - z(\cdot)\|_E e^{-\delta t}, \tag{20}$$

where $z(x) = (\bar{u}(x), 0)$, $C > 0$, $\delta > 0$, and C and δ are independent of y_0.

Proposition 3 is proved in [12]. Thus, in this case, the global attractor coincides with the stationary point: $\mathcal{A} = \{z\}$. For $\beta > \lambda_1$, the stationary point becomes unstable. This leads to the appearance of new stable and unstable stationary points connected by different heteroclinic orbits. Then the global attractor is the union of all finite-dimensional unstable manifolds issuing from all the stationary points of the equation. Such attractors are called regular attractors. Detailed investigation of the general equation (6) is given in [5,6]. This theory uses the method of the Lyapunov function that can be explicitly constructed for equations (6) and (17).

Let us study the ε-entropy and fractal dimension of the global attractor of equation (6). We need the general theorem from the theory of global attractors of evolution equations. Let a semigroup

$\{S(t)\}$ be given acting in a Hilbert space E. Consider a compact set X in E, $X \Subset E$. Let the set X be strictly invariant with respect to $\{S(t)\}$, that is, $S(t)X = X$ for all $t \geq 0$. (For example, $X = \mathcal{A}$, where $\mathcal{A}$ is the global attractor of the semigroup.) We assume that the semigroup $\{S(t)\}$ is *uniformly quasidifferentiable on* X in the following sense: for any $t \geq 0$ and for every $y \in X$, there is a linear bounded operator $L(t,y)\colon E \to E$ (*quasidifferential*) such that

$$\|S(t)y_1 - S(t)y - L(t,y)(y_1 - y)\|_E \leq \gamma(\|y_1 - y\|_E, t)\|y_1 - y\|_E \tag{21}$$

for all $y, y_1 \in X$ and the function $\gamma = \gamma(\xi, t) \to 0+$ as $\xi \to 0+$ for every fixed $t \geq 0$. We assume that the linear operators $L(t,y)$ are generated by the variational equation

$$\partial_t z = A_y(y(t))z, \qquad z|_{t=0} = z_0 \in E, \tag{22}$$

where $y(t) = S(t)y_0$, $y_0 \in X$, $A_y(\cdot)$ is the formal derivative of the operator $A(\cdot)$ with respect to y, and the domain E_1 of the operator $A_y(y(t))$ is dense in E. It is necessary that the linear problem (22) is uniquely solvable for any $z_0 \in E$ for all $y_0 \in X$. By our assumption, in (21), the quasidifferentials $L(t, y_0)z_0 = z(t)$, where $z(t)$ is the solution of equation (22).

Let $m \in \mathbb{N}$ and let $L\colon E_1 \to E$ be a linear (unbounded) operator. The following number is called the *m-trace* of the operator L:

$$\mathrm{Tr}_m L = \sup_{\{\varphi_i\}_{i=1,\dots,m}} \sum_{i=1}^{m} (L\varphi_i, \varphi_i), \tag{23}$$

where the infimum is taken over all orthonormal (in E) families of vectors $\{\varphi_i\}_{i=1,\dots,m}$ belonging to E_1.

Definition 4. We set

$$\widetilde{q}_j = \overline{\lim_{t\to+\infty}} \sup_{y_0\in X} \frac{1}{t} \int_0^t \mathrm{Tr}_j(A_y(y(s))\, ds, \quad j = 1, 2, \dots, \tag{24}$$

where $y(t) = S(t)y_0$.

Theorem 1. *Assume that the semigroup $\{S(t)\}$ acting in E has a compact strictly invariant set X and is uniformly quasidifferentiable on X. Let the inequalities*

$$\widetilde{q}_j \leq q_j, \quad j = 1, 2, 3, \dots,$$

hold, where the numbers $\widetilde{q}_j$ are defined in (24). Assume that the function q_j is $\cap$-concave in j. Let m be the smallest integer such that $q_{m+1} < 0$ (i.e., $q_m \geq 0$). Set

$$d = m + \frac{q_m}{q_m - q_{m+1}}.$$

Then, for every $\delta > 0$, there exist $\alpha \in (0,1)$ and $\varepsilon_0 > 0$ such that the inequality

$$\mathbf{H}_\varepsilon(X) \leq (d + \delta)\log_2\left(\frac{\varepsilon_0}{\alpha\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(X), \quad \forall \varepsilon < \varepsilon_0, \tag{25}$$

holds for the ε-entropy $\mathbf{H}_\varepsilon(X)$ of the set X. Furthermore, the set X has finite fractal dimension and

$$\mathbf{d}_F(X) \leq d. \tag{26}$$

The proof of this theorem is given in [13]. It is bases on the study of volume contraction properties under the action of the quasidifferentials of the semigroup operators. Estimates for the Hausdorff dimension of invariant sets, which are similar to (25), were proved in [5, 7, 14–16].

Remark 2. In the recent work [17], estimate (25) was proved for $q_j = \widetilde{q}_j$ without the assumption that the function $\widetilde{q}_j$ is concave in j. The number $\mathbf{d}_L = m + \dfrac{\widetilde{q}_m}{\widetilde{q}_m - \widetilde{q}_{m+1}}$ is conventionally called the Lyapunov dimension of the set X. In [7, 14], it was proved that $\mathbf{d}_H(X) \le \mathbf{d}_L(X)$. In [17], it was shown that $\mathbf{d}_F(X) \le \mathbf{d}_L(X)$. A similar result was obtained in [18].

The books [5–7] contain many evolution equations of mathematical physics and mechanics, for which the global attractors were constructed, and upper estimates were proved for the Hausdorff and fractal dimension of these attractors.

In this paper, we apply Theorem 1 to study the ε-entropy of the global attractor of the dissipative wave equation (6). For brevity, we only consider the case $n = 3$. We assume that the function $f(v) \in C^2(\mathbb{R})$ and satisfies the conditions (8)–(10), where $\rho < 2$. Furthermore, we assume that

$$|f'_u(u_1) - f'_u(u_2)| \le C(|u_1|^{2-\varkappa} + |u_2|^{2-\varkappa} + 1)|u_1 - u_2|^{\varkappa}, \quad 0 < \varkappa \le 1. \tag{27}$$

The Hilbert space $E = H_0^1(\Omega) \times L_2(\Omega)$ is the phase space for this equation. We also denote by E_1 the space $E_1 = H^2(\Omega) \times H_0^1(\Omega)$ with norm $\|y\|_{E_1} = \left(\|u\|_2^2 + \|p\|_1^2\right)^{1/2}$.

We consider the semigroup $\{S(t)\}$ in E generated by problem (16). By Proposition 2, this semigroup has a global attractor $\mathcal{A} \Subset E$. In [5, 7], it was proved that the set $\mathcal{A}$ is bounded in E_1:

$$\|w\|_{E_1} \le M, \quad \forall w \in \mathcal{A},$$

where the constant M is independent of w. Then, by the Sobolev embedding theorem,

$$\|u(\cdot)\|_{C(\overline{\Omega})} \le M_1, \quad \forall w = (u(\cdot), p(\cdot)) = w(\cdot) \in \mathcal{A}. \tag{28}$$

Theorem 2. *For the ε-entropy of the global attractor $\mathcal{A}$ of problem* (16)*, we have the estimate*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \le \frac{C(M_1)}{\eta^3} \log_2\left(\frac{\varepsilon_0}{\alpha\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}), \quad \forall \varepsilon < \varepsilon_0, \tag{29}$$

where α and ε_0 are some positive numbers and $\eta = \min\{\gamma/4, \lambda_1/(2\gamma)\}$. For the fractal dimension of $\mathcal{A}$, we have the estimate

$$\mathbf{d}_F\mathcal{A} \le \frac{C(M_1)}{\eta^3}. \tag{30}$$

Proof. Following [7, 10], it is convenient to introduce the new variables

$$w = (u, v) = (u, p + \eta u), \qquad \eta = \min\{\gamma/4, \lambda_1/(2\gamma)\}.$$

Then (16) is equivalent to the system

$$\partial_t w = A(w) = Lw - G(w), \qquad w|_{t=0} = w_0, \tag{31}$$

where $w_0 \in E$, $G(w) = (0, f(u) - g(x))$, and

$$L = \begin{pmatrix} -\eta I & I \\ \Delta + \eta(\gamma - \eta) & -(\gamma - \eta)I \end{pmatrix}. \tag{32}$$

Using condition (27), one can prove that the semigroup $\{S(t)\}$ is uniformly quasidifferentiable on $\mathcal{A}$, and its quasidifferential $L(t; w_0)z_0 = z(t)$ satisfies the variational equation of problem (31)

$$\partial_t z = Lz - G_w(w)z = A_w(w(t))z, \quad z|_{t=0} = z_0, \quad z = (r, q), \tag{33}$$

where $G_w(w)z = (0, f'(u)r)$ (see, for example, [7]).

We estimate the trace of the operator $A_w(w(t))$. Consider the sum

$$\sum_{i=1}^{j}(A_w(w(t))\zeta_i, \zeta_i)_E. \tag{34}$$

Here, $\zeta_i = (r_i, q_i)$ is the orthonormal family in E. Let us estimate the summands in (34):

$$\begin{aligned}(A_w(w(t))\zeta_i, \zeta_i)_E &= (L\zeta_i, \zeta_i) - (f'(u)r_i, q_i)\\ &\le -(\eta/2)\|\zeta_i\|_E^2 + C_0(M_1)\|r_i\|_0\|q_i\|_0\\ &\le -\eta/4\left(\|r_i\|_1^2 + \|q_i\|_0^2\right) + (C_0^2(M_1)/\eta)\|r_i\|_0^2.\end{aligned} \tag{35}$$

The parameter η is chosen so that the operator L is negative:

$$(L\zeta_i, \zeta_i) \le -\eta/2\|\zeta_i\|_E^2.$$

We have also used the inequality

$$\sup\{\|f'(u(\cdot))\|_{C_b(\Omega)}:\ (u(\cdot), p(\cdot)) = w(\cdot) \in \mathcal{A}\} \le C_0(M_1) \tag{36}$$

(see (28)). The system ζ_i is orthonormal in E; from (35), we get

$$\begin{aligned}\sum_{i=1}^{j}(A_w(w(t),t)\zeta_i, \zeta_i)_E &\le -(\eta/4)j + (C_0^2(M_1)/\eta)\sum_{i=1}^{j}\|r_i\|_0^2\\ &\le -(\eta/4)j + (C_0^2(M_1)/\eta)\sum_{i=1}^{j}\lambda_i^{-1}\\ &\le -(\eta/4)j + (C_1(M_1)/\eta)j^{1/3},\end{aligned} \tag{37}$$

where $C_1(M_1) = c_1C_0^2(M_1)$, and λ_i, $i = 1, \dots, j$, are the first j eigenvalues of the operator $-\Delta u$, $u|_{\partial\Omega} = 0$, written in nondecreasing order with their multiplicities. It is known that $\lambda_i \ge c_0 i^{2/3}$; therefore, $\sum\limits_{i=1}^{j}\lambda_i^{-1} \le c_1 j^{1/3}$. In the second inequality of (37), we used the estimate

$$\sum_{i=1}^{j}\|r_i\|_0^2 \le \sum_{i=1}^{j}\lambda_i^{-1},$$

proved in [7]. Hence,

$$\mathrm{Tr}_j A_w(w(t)) \le \varphi(j) = -(\eta/4)j + (C_1(M_1)/\eta)j^{1/3},$$

where the function $q_j = \varphi(j)$ is concave in j. Chose the minimal m such that $q_{m+1} < 0$. We set $d = m + \dfrac{q_m}{q_m - q_{m+1}}$. Since the function q_j is concave, we have the inequality $d < d^*$, where d^* is the root of the equation

$$\varphi(d^*) = 0, \quad d^* = \frac{8C_1(M_1)^{3/2}}{\eta^3} = \frac{C(M_1)}{\eta^3}, \quad \text{where} \quad C(M_1) = 8C_1(M_1)^{3/2}.$$

Now it remains to take in (25) the number $\delta = d^* - d$ and use Theorem 1. △

Consider the sine-Gordon equation with $f(u) = \beta \sin(u)$. It is clear that the constant $C_0(M_1)$ equals β in inequality (36), and therefore $C_1(M_1) = c_1\beta^2$; that is, $C(M_1) = 8c_1^{3/2}\beta^3 = c\beta^3$. Thus, estimates (29) and (30) for the sine-Gordon equation have the form

$$\mathbf{H}_\varepsilon(\mathcal{A}) \le c\frac{\beta^3}{\eta^3}\log_2\left(\frac{\varepsilon_0}{\alpha\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}), \quad \forall \varepsilon < \varepsilon_0,$$
$$\mathbf{d}_F(\mathcal{A}) \le c\frac{\beta^3}{\eta^3},$$

where the constant c depends on Ω.

Remark 3. Using Theorem 1, estimates for the ε-entropy and fractal dimension of global attractors of various equations and systems of mathematical physics are obtained (see [5, 7, 13]).

3. GLOBAL ATTRACTORS OF NONAUTONOMOUS EQUATIONS

In this section, we study the global attractors of nonautonomous evolution equations of the form

$$\partial_t y = A(y, t), \quad y|_{t=\tau} = y_\tau \in E, \quad t \ge \tau. \tag{38}$$

The nonlinear operator $A(y, t)$ depends on the function y, its partial derivatives in x and also on time $t \in \mathbb{R}$. The initial condition y_τ belonging to the Banach space E is posed at $t = \tau$, where τ is an arbitrary fixed number. We assume that, for any $\tau \in \mathbb{R}$ and any $y_\tau \in E$, problem (38) has a unique solution $y(t)$ such that $u(t) \in E$ for all $t \ge \tau$. We consider the two-parameter family of nonlinear operators $\{U(t, \tau)\}$, $t \ge \tau$, $\tau \in \mathbb{R}$, in E, constructed by the formula

$$U(t, \tau)y_\tau = y(t), \quad t \ge \tau, \quad \tau \in \mathbb{R}, \quad y_\tau \in E, \tag{39}$$

where $y(t)$ is a solution of (38) with initial data $y_\tau \in E$. The family of operators $\{U(t, \tau)\}$ is called the *process* generated by problem (38). It has the following properties: (1) $U(\tau, \tau) = \mathrm{Id}$ for all $\tau \in \mathbb{R}$; (2) $U(t, s) \circ U(s, \tau) = U(t, \tau)$ for all $t \ge s \ge \tau$, $\tau \in \mathbb{R}$. If the operators $A(y, t)$ in (38) do not depend on time, then the process $\{U(t, \tau)\}$ is the semigroup $U(t, \tau) = S(t - \tau)$, whose properties were considered in Section 2.

As the main example, we consider the nonautonomous wave equation with dissipation

$$\partial_t^2 u + \gamma\partial_t u = \Delta u - f(u) + g_0(x, t), \qquad u|_{\partial\Omega} = 0, \tag{40}$$
$$u|_{t=\tau} = u_\tau(x) \in H_0^1(\Omega), \qquad \partial_t u|_{t=\tau} = p_\tau(x) \in L_2(\Omega), \tag{41}$$

which differs from equation (6) in the external force $g_0(x, t)$, which depends on time. We assume that $g_0(x, t) \in C_b(\mathbb{R}; L_2(\Omega))$, i.e.,

$$\sup_{t\in\mathbb{R}} \|g_0(\cdot\,, t)\|_{L_2(\Omega)} \le C. \tag{42}$$

All other terms in equations (40), (41) satisfy the conditions given for the autonomous equations (6), (7). In particular, the nonlinear function $f(u)$ satisfies inequalities (8)–(10). An analog of Proposition 1 takes place.

Proposition 4. *If $u_\tau \in H_0^1(\Omega)$ and $p_\tau \in L_2(\Omega)$, then, under assumptions* (8)–(10) *and* (42), *problem* (40), (41) *has a unique solution $u(x, t) \in C_b(\mathbb{R}_\tau; H_0^1(\Omega))$, $\partial_t u(x, t) \in C_b(\mathbb{R}_\tau; L_2(\Omega))$ in the space $D'(\mathbb{R}_\tau; H^{-1}(\Omega))$. Moreover, the second derivative $\partial_t^2 u(x, t) \in L_\infty(\mathbb{R}_\tau; H^{-1}(\Omega))$. Here we denote $\mathbb{R}_\tau = [\tau, +\infty)$.*

The proof is given in [13] (see also [5–8]).

Similarly to the autonomous case, problem (40) can be written in the form (38), where $y = (u,p) = (u,\partial_t u)$ and $A(y,t) = A(u,p,t) = (p, -\gamma p + \Delta u - f(u) + g_0(x,t))$. If $y_\tau = (u_\tau, p_\tau) \in E = H_0^1(\Omega)\times L_2(\Omega)$, then the solution $y(t) = (u(t),p(t)) = (u(t),\partial_t u(t)) \in E$ for all $t \geq \tau$. Therefore, problem (40) generates the process $\{U(t,\tau)\}$ acting in E by formula (39).

Let us define the *global attractor* $\mathcal{A}$ of the process $\{U(t,\tau)\}$. We denote by $\mathcal{B}(E)$ the family of all bounded sets in E. A set $B_0 \subset E$ is said to be *absorbing* for the process $\{U(t,\tau)\}$ if, for any set $B \in \mathcal{B}(E)$, there is a number $h = h(B)$ such that

$$U(t,\tau)B \subseteq B_0 \quad \text{for all } t,\tau; \quad t-\tau \geq h. \tag{43}$$

A set $P \subset E$ is said to be *attracting* for the process $\{U(t,\tau)\}$ if, for every $\varepsilon > 0$, the set $\mathcal{O}_\varepsilon(P)$ is absorbing for this process (here and below, $\mathcal{O}_\varepsilon(M)$ denotes the ε-neighborhood of a set M in the space E). The attracting property can be reformulated as follows: for any set $B \in \mathcal{B}(E)$,

$$\sup_{\tau\in\mathbb{R}} \operatorname{dist}_E\left(U\left(\tau+h,\tau\right)B, P\right) \to 0, \quad h \to +\infty. \tag{44}$$

The process $\{U(t,\tau)\}$ is called *asymptotically compact* if it has a compact attracting set.

Definition 5. A closed set $\mathcal{A} \subset E$ is said to be the *global attractor* of the process $\{U(t,\tau)\}$ if it is attracting for the process $\{U(t,\tau)\}$ and satisfies the property of minimality: the set $\mathcal{A}$ belongs to any closed attracting set of this process.

It is clear that a process has at most one global attractor. This notion was introduced in [19] (see also [13, 20–22]).

Proposition 5. *If a process $\{U(t,\tau)\}$ is asymptotically compact, then it has the global attractor, which is compact in E: $\mathcal{A} \Subset E$.*

This proposition is proved in [13], where it is also established that

$$\mathcal{A} = \omega(P) := \bigcap_{h\geq 0}\left[\bigcup_{t-\tau\geq h} U(t,\tau)P\right]_E, \tag{45}$$

where P is an arbitrary compact attracting set of the process. In formula (45), the square brackets $[\,\cdot\,]_E$ mean the closure in the space E.

Consider the process $\{U(t,\tau)\}$ corresponding to the nonautonomous wave equation (40). In [13], it is proved that, under condition (42), this process has a *bounded* in E *absorbing* set. The proof uses the main energy a priori estimates of this problem. However, condition (42) is not sufficient for constructing a *compact* in E *attracting* set. Therefore, supplementary conditions for the function $g_0(x,t)$ are needed to construct the global attractor of this equation, which will be given below.

To describe the general structure of the global attractor of a process we need some additional notions. A function $y(s)$, $s \in \mathbb{R}$, with values in E is called a *complete trajectory* of the process $\{U(t,\tau)\}$ if

$$U(t,\tau)y(\tau) = y(t) \quad \text{for all } t \geq \tau, \quad \tau \in \mathbb{R}. \tag{46}$$

A complete trajectory $y(s)$ is called *bounded* if the set $\{y(s),\ s \in \mathbb{R}\}$ is bounded in E.

Definition 6. The kernel $\mathcal{K}$ of a process $\{U(t,\tau)\}$ is the family of all bounded complete trajectories of this process:

$$\mathcal{K} = \left\{y(\cdot) \mid y \text{ satisfies (46) and } \|y(s)\|_E \leq C_y,\ \forall s \in \mathbb{R}\right\}.$$

The set

$$\mathcal{K}(t) = \{y(t) \mid y(\cdot) \in \mathcal{K}\} \subset E, \quad t \in \mathbb{R},$$

is called the kernel section at time moment t.

It is easy to verify the following property.

Proposition 6. *If a process $\{U(t,\tau)\}$ has the global attractor $\mathcal{A}$, then all kernel sections belong to $\mathcal{A}$:*

$$\bigcup_{t\in\mathbb{R}} \mathcal{K}(t) \subseteq \mathcal{A}. \tag{47}$$

Note that, in the general case, inclusion (47) is strict, i.e., there are points of the global attractor $\mathcal{A}$ that are not values of any bounded complete trajectory of the original equation (38). However, below we will show that such points are values of bounded complete trajectories of equations that are "contiguous" to the initial equation. To describe this "contiguous" equations, we introduce a notion of a *time symbol* of the considered equation. We assume that all terms of equation (38) that explicitly depend on time t can be written as a function $\sigma(t)$, $t \in \mathbb{R}$, with values in a Banach space Ψ. We rewrite equation (38) in the following form:

$$\partial_t u = A_{\sigma(t)}(u), \quad y|_{t=\tau} = y_\tau \in E, \quad t \geq \tau. \tag{48}$$

The function $\sigma(t)$ is called the time symbol of the equation. For example, in the nonautonomous wave equation (40), the symbol is the function $g_0(\cdot\,,t)$, $\sigma(t) = g_0(\cdot\,,t)$ with values in the space $L_2(\Omega) = \Psi$. For simplicity, we assume that $\sigma(t) \in C(\mathbb{R};\Psi)$.

The symbol of the initial equation (38) is denoted by $\sigma_0(t)$. Together with this equation, which has symbol $\sigma_0(t)$, we also consider equations (48) with symbols $\sigma(t) = \sigma_0(t+h)$ for any $h \in \mathbb{R}$. Moreover, we consider also equations whose symbols $\sigma(t)$ are limits of sequences of the form $\sigma_0(t+h_n)$ as $n \to \infty$. The limit is taken in the space $C(\mathbb{R};\Psi)$ in the topology $C^{\mathrm{loc}}(\mathbb{R};\Psi)$ defined as follows. By definition, a sequence of functions $\{\xi_n(t)\}$ from $C(\mathbb{R};\Psi)$ converges to the function $\xi(t)$ as $n \to \infty$ in the topology $C^{\mathrm{loc}}(\mathbb{R};\Psi)$ if, for any fixed $M > 0$,

$$\max_{t\in[-M,M]} \|\xi_n(t) - \xi(t)\|_\Psi \to 0, \quad n \to \infty.$$

This local uniform convergence topology in the space $C(\mathbb{R};\Psi)$ is metrizable, and the corresponding metric space is complete (see [13]).

Definition 7. The set

$$\mathcal{H}(\sigma_0) = [\{\sigma_0(t+h) \mid h \in \mathbb{R}\}]_{C^{\mathrm{loc}}(\mathbb{R};\Psi)} \tag{49}$$

is called the hull of the function $\sigma_0(t)$ in $C^{\mathrm{loc}}(\mathbb{R};\Psi)$. Here, as usual, $[\,\cdot\,]_{C^{\mathrm{loc}}(\mathbb{R};\Psi)}$ denotes the closure in the space $C^{\mathrm{loc}}(\mathbb{R};\Psi)$.

Consider the family of equations (48) with symbols $\sigma(t)$ belonging to the hull $\mathcal{H}(\sigma_0)$ of the symbol $\sigma_0(t)$ of the original equation (38). We assume that $\sigma_0(t)$ is a *translation-compact* function in $C^{\mathrm{loc}}(\mathbb{R};\Psi)$.

Definition 8. A function $\sigma_0(t) \in C^{\mathrm{loc}}(\mathbb{R};\Psi)$ is said to be translation-compact in the space $C^{\mathrm{loc}}(\mathbb{R};\Psi)$ if its hull $\mathcal{H}(\sigma_0)$ is compact in $C^{\mathrm{loc}}(\mathbb{R};\Psi)$.

Consider some examples of translation-compact functions.

Example 1. Let the function $\sigma_0(t)$ be almost periodic with values in the Banach space Ψ. By the definition, this means that its hull $\mathcal{H}(\sigma_0)$ is compact in the space $C_b(\mathbb{R};\Psi)$ with uniform convergence topology on the entire time axis $\mathbb{R}$ (see [23]). Clearly, the topology $C_b(\mathbb{R};\Psi)$ is stronger than the topology $C^{\mathrm{loc}}(\mathbb{R};\Psi)$. Thus, if a set $\mathcal{H}(\sigma_0)$ is compact in $C_b(\mathbb{R};\Psi)$, then it is also compact in $C^{\mathrm{loc}}(\mathbb{R};\Psi)$; i.e., the function $\sigma_0(t)$ is translation compact in $C^{\mathrm{loc}}(\mathbb{R};\Psi)$.

Example 2. Quasiperiodic functions are important particular cases of almost periodic functions with values in Ψ. A function $\sigma_0(t) \in C^{\mathrm{loc}}(\mathbb{R};\Psi)$ is called quasiperiodic if it has the form

$$\sigma_0(t) = \phi\left(\alpha_1 t, \alpha_2 t, \ldots, \alpha_k t\right) = \phi\left(\boldsymbol{\alpha} t\right), \quad \phi\left(\boldsymbol{\alpha} t\right) \in \Psi, \quad \forall t \in \mathbb{R}, \tag{50}$$

where the function $\phi\left(\boldsymbol{\omega}\right) = \phi\left(\omega_1, \omega_2, \ldots, \omega_k\right)$ is continuous and 2π-periodic with respect to each variable $\omega_i \in \mathbb{R}$:

$$\phi\left(\omega_1, \ldots, \omega_i + 2\pi, \ldots, \omega_k\right) = \phi\left(\omega_1, \ldots, \omega_i + 2\pi, \ldots, \omega_k\right), \quad i = 1, \ldots, k.$$

For $k = 1$, we get periodic functions. Let $\mathbb{T}^k = [\mathbb{R} \bmod 2\pi]^k$ denote the k-dimensional torus. Then $\phi(\boldsymbol{\omega}) \in C(\mathbb{T}^k;\Psi)$. We assume that the components of the vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ in (50) are rationally independent numbers (otherwise we can reduce the number of independent arguments ω_i in the representation (50)). It is easy to show that the hull of the function $\sigma_0(t)$ in $C_b(\mathbb{R};\Psi)$ consists of the functions

$$\left\{\phi(\boldsymbol{\alpha} t + \boldsymbol{\omega}_1) \mid \boldsymbol{\omega}_1 \in \mathbb{T}^k\right\} = \mathcal{H}(\sigma_0). \tag{51}$$

Consequently, the hull $\mathcal{H}(\sigma_0)$ is a continuous image of the k-dimensional torus $\mathbb{T}^k$. In particular, if the function $\phi(\boldsymbol{\omega})$ is smooth, then the fractal dimension of the set $\mathcal{H}(\sigma_0)$ does not exceeds k, i.e.,

$$\mathbf{d}_F\left(\mathcal{H}(\sigma_0)\right) \le \mathbf{d}_F\left(\mathbb{T}^k\right) = k, \tag{52}$$

and is equal to k in the generic case (inequality in (52) can be strict).

We now present a simple example of a translation-compact function in $C^{\mathrm{loc}}(\mathbb{R};\Psi)$, which is not almost periodic or quasiperiodic.

Example 3. Assume that the function $\sigma_0(t) \in C_b(\mathbb{R};\Psi)$ has the following property: $\sigma_0(t) \to \sigma_+$ $(t \to +\infty)$ and $\sigma_0(t) \to \sigma_-$ $(t \to -\infty)$, where $\sigma_+, \sigma_- \in \Psi$, $\sigma_+ \ne \sigma_-$. Then the function σ_0, evidently, is not almost periodic, while the function $\sigma_0(s)$ is translation-compact in $C^{\mathrm{loc}}(\mathbb{R};\Psi)$ and its hull is $\mathcal{H}(\sigma_0) = \{\sigma_0(s+h) \mid h \in \mathbb{R}\} \cup \{\sigma_-(t), \sigma_+(t)\}$, where $\sigma_\pm(t) \equiv \sigma_\pm$ for all $t \in \mathbb{R}$. Other examples of translation-compact functions are given in [13].

Now consider the family of equations (48) with symbols $\sigma(t) \in \mathcal{H}(\sigma_0)$, where $\sigma_0(t)$ is a translation compact function in $C^{\mathrm{loc}}(\mathbb{R};\Psi)$. We assume that, for any symbol $\sigma \in \mathcal{H}(\sigma_0)$, the Cauchy problem (48) is uniquely solvable for each $\tau \in \mathbb{R}$ and arbitrary initial data $y_\tau \in E$. Therefore, we have a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, acting in the space E. The family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, is called $(E \times \mathcal{H}(\sigma_0), E)$-continuous if, for any t and $\tau, t \ge \tau$, the mapping $(y,\sigma) \mapsto U_\sigma(t,\tau)y$ from $E \times \mathcal{H}(\sigma_0)$ into E is continuous.

Let us formulate the main theorem on the structure of the global attractor of equation (38) with translation-compact symbol $\sigma_0(t)$. The process corresponding to this symbol is denoted by $\{U_{\sigma_0}(t,\tau)\}$.

Theorem 3. *Assume that the function $\sigma_0(t)$ is translation compact in $C^{\mathrm{loc}}(\mathbb{R};\Psi)$. Let the process $\{U_{\sigma_0}(t,\tau)\}$ be asymptotically compact, and let the corresponding family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, be $(E \times \mathcal{H}(\sigma_0), E)$-continuous. Then the process $\{U_{\sigma_0}(t,\tau)\}$ has the global attractor $\mathcal{A} \Subset E$, and the identity*

$$\mathcal{A} = \bigcup_{\sigma \in \mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(0) = \bigcup_{\sigma \in \mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(t) \tag{53}$$

holds, where $\mathcal{K}_\sigma$ is the kernel of the process $\{U_\sigma(t,\tau)\}$ with symbol $\sigma \in \mathcal{H}(\sigma_0)$. Here, t is an arbitrary fixed number. The kernel $\mathcal{K}_\sigma$ is nonempty for every symbol $\sigma \in \mathcal{H}(\sigma_0)$.

A detailed proof of Theorem 3 can be found in [13, 22].

We apply Theorem 3 to the wave equation (40) and obtain the following result.

Proposition 7. *Let the external force $g_0(\cdot\,,t)$ be a translation-compact function in the space $C^{\mathrm{loc}}(\mathbb{R};L_2(\Omega))$. Then the process $\{U_{g_0}(t,\tau)\}$ of problem* (40), (41) *has the global attractor $\mathcal{A}\Subset E=H_0^1(\Omega)\times L_2(\Omega)$ and, moreover,*

$$\mathcal{A}=\bigcup_{g\in\mathcal{H}(g_0)}\mathcal{K}_g(0),\tag{54}$$

where $\mathcal{K}_g$ is the kernel of the wave equation with external force $g(\cdot\,,t)\in\mathcal{H}(g_0)$.

The proof is given in [13]. For the construction of this global attractor, the fact is used that the process $\{U_{g_0}(t,\tau)\}$ can be represented as a sum of two terms, namely, a compact term and exponentially vanishing one. This property is due to the presence of the dissipative term $\gamma\partial_t u$ in the equation and to the translation compactness of the function $g_0(\cdot\,,t)$.

Consider the nonautonomous sine-Gordon equation

$$\partial_t^2 u+\gamma\partial_t u=\Delta u-\beta\sin(u)+g_0(x,t),\quad u|_{\partial\Omega}=0,\tag{55}$$

which is a particular case of equation (40) and, therefore, has the global attractor $\mathcal{A}$ under the assumption that $g_0(\cdot\,,t)$ is translation-compact in $C^{\mathrm{loc}}(\mathbb{R};L_2(\Omega))$. Let us formulate a nonautonomous analog of Proposition 3.

Proposition 8. *Let $g_0(\cdot\,,t)$ be an almost periodic function with values in $L_2(\Omega)$. Furthermore, let $\beta<\lambda_1$ and $\gamma>\gamma_0$ (see* (18)*). Then equation* (55) *has a unique almost periodic solution $u_0(x,t)$, $t\in\mathbb{R}$:*

$$\|u_0(\cdot\,,t)\|_{H_0^1(\Omega)}\le C,\qquad \|\partial_t u_0(\cdot\,,t)\|_{L_2(\Omega)}\le C,\quad \forall t\in\mathbb{R},$$

which is asymptotically stable; i.e., for every solution $y(t)=(u(t),\partial_t u(t))=U_{g_0}(t,\tau)y_\tau$ of equation (55), *we have the inequality*

$$\|y(t)-z(t)\|_E\le C\|y_\tau-z(\tau)\|_E e^{-\delta(t-\tau)},\tag{56}$$

where $z(t)=(u_0(t),\partial_t u_0(t))$, $C>0$, $\delta>0$, and the constants C and δ are independent of y_τ.

If the function $g_0(x,t)=\phi(x,\alpha_1 t,\alpha_2 t,\dots,\alpha_k t)$ is quasiperiodic, then the solution $u_0(x,t)$ is also quasiperiodic with the same collection of rationally independent frequencies, that is, $u_0(x,t)=\Phi(x,\alpha_1 t,\alpha_2 t,\dots,\alpha_k t)$, where $\Phi(x,\boldsymbol{\omega})\in C(\mathbb{T}^k;E)$ is a periodic function.

For the proof, see [12].

Corollary 1. *Under the assumptions of Proposition* 8, *the global attractor*

$$\mathcal{A}=[\{z(t)\mid t\in\mathbb{R}\}]_E\,,\quad \textit{where}\quad z(t)=(u_0(\cdot\,,t),\partial_t u_0(\cdot\,,t)).\tag{57}$$

Proof. It follows from (56) that the set defined on the right-hand side of (57) is attracting for the process $\{U_{g_0}(t,\tau)\}$. Furthermore, it is easy to see that this set belongs to any closed attracting set, i.e., is minimal; therefore, we get (57). △

Corollary 2. *If $g_0(x,t)=\phi\,(x,\alpha_1 t,\alpha_2 t,\dots,\alpha_k t)$ is a quasiperiodic Lipschitz-continuous function, then we have the following estimate for the fractal dimension $\mathbf{d}_F(\mathcal{A})$ of the global attractor $\mathcal{A}$:*

$$\mathbf{d}_F(\mathcal{A})=\mathbf{d}_F(\mathcal{H}(g_0))\le k.\tag{58}$$

For its ε-entropy, we have the inequality

$$\mathbf{H}_\varepsilon(\mathcal{A})\lesssim k\log\left(\frac{1}{\varepsilon}\right).\tag{59}$$

If $g_0(x,t)$ is an almost periodic function with infinite number of rationally independent frequencies, then $\mathbf{d}_F(\mathcal{A})=+\infty$.

In the next section we study the ε-entropy of the global attractor $\mathcal{A}$ of nonautonomous equation (38) with a generic translation compact symbol $\sigma_0(t)$. These results will be applied to equations (40) and (55).

4. ε-ENTROPY OF GLOBAL ATTRACTORS OF NONAUTONOMOUS EQUATIONS

We study the family of equations (48) with $\sigma(t) \in \mathcal{H}(\sigma_0)$. We assume that the original symbol $\sigma_0(t)$ is a translation-compact function in the space $C^{\mathrm{loc}}(\mathbb{R};\Psi)$. Consider the corresponding family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, acting in E. We assume that the conditions of Theorem 3 hold. Then the process $\{U_{\sigma_0}(t,\tau)\}$ has a global attractor $\mathcal{A}$, which is of the form (53).

The problem is to study the ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A}) = \mathbf{H}_\varepsilon(\mathcal{A},E)$ of the global attractor $\mathcal{A}$ in the space E. We assume that the behavior of the ε-entropy of the set $\Pi_{0,l}\mathcal{H}(\sigma_0)$ in the space $C([0,l];\Psi)$ is known. Here, $\Pi_{0,l}$ denotes the restriction operator onto the segment $[0,l]$.

Let us formulate some additional necessary conditions for the process $\{U_{\sigma_0}(t,\tau)\}$. First of all, we have to generalize the property of quasidifferentiability introduced for semigroups in Section 2. Let $\{U(t,\tau)\}$ be a process in E. The space E is assumed to be Hilbert. Consider the kernel $\mathcal{K}$ of this process. The definition of a kernel of a process implies the invariance property of the kernel sections:

$$U(t,\tau)\mathcal{K}(t) = \mathcal{K}(\tau), \quad \forall t \geq \tau, \quad \tau \in \mathbb{R}. \tag{60}$$

Definition 9. A process $\{U(t,\tau)\}$ in E is called *uniformly quasidifferentiable on* $\mathcal{K}$ if there exists a family of linear bounded operators $\{L(t,\tau,y)\}$, where $y \in \mathcal{K}(\tau)$, $t \geq \tau$, $\tau \in \mathbb{R}$, such that

$$\|U(t,\tau)y_1 - U(t,\tau)y - L(t,\tau,y)(y_1 - y)\|_E \leq \gamma(\|y_1 - y\|_E, t-\tau)\|y_1 - y\|_E \tag{61}$$

for all $y, y_1 \in \mathcal{K}$, where the function $\gamma = \gamma(\xi,s) \to 0+$ as $\xi \to 0+$ for each fixed $s \geq 0$.

We assume that the process $\{U_{\sigma_0}(t,\tau)\}$ is uniformly quasidifferentiable on the kernel $\mathcal{K}_{\sigma_0}$ and its quasidifferentials are generated by the variational equation

$$\partial_t z = A_{\sigma_0 y}(y(t))z, \quad z|_{t=\tau} = z_\tau \in E, \tag{62}$$

where $y(t) = U_{\sigma_0}(t,\tau)y_\tau$, $y_\tau \in \mathcal{K}_{\sigma_0}(\tau)$, that is, $L(t,\tau,y_\tau)z_\tau = z(t)$, where $z(t)$ is a solution of problem (62). It is assumed that this Cauchy problem is uniquely solvable for all $y_\tau \in \mathcal{K}_{\sigma_0}(\tau)$ and for every $z_\tau \in E$. Similarly to (24), we introduce the numbers

$$\widetilde{q}_j = \overline{\lim_{t\to+\infty}} \sup_{\tau\in\mathbb{R}} \sup_{y_0\in\mathcal{K}(\tau)} \frac{1}{t}\int_0^t \mathrm{Tr}_j(A_{\sigma_0 y}(y(s)))\,ds, \tag{63}$$

where $y(t) = U_{\sigma_0}(t,\tau)y_\tau$, and the trace $\mathrm{Tr}_j(L)$ of a linear operator L is defined in (23).

We also assume the validity of the following Lipschitz condition for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$:

$$\|U_{\sigma_1}(h,0)y - U_{\sigma_2}(h,0)y\|_E \leq C(h)\|\sigma_1 - \sigma_2\|_{C([0,h];\Psi)}, \quad \forall \sigma_1,\sigma_2 \in \mathcal{H}(\sigma_0), \quad y \in \mathcal{A}, \quad h \geq 0. \tag{64}$$

Let us formulate the main result.

Theorem 4. *Let the assumptions of Theorem* 3 *be valid. Assume that the process* $\{U_{\sigma_0}(t,\tau)\}$ *is uniformly quasidifferentiable on* $\mathcal{K}_{\sigma_0}$, *its quasidifferentials are generated by the variational equation* (62), *and the numbers* $\widetilde{q}_j$ (*see* (63)) *satisfy the inequalities*

$$\widetilde{q}_j \leq q_j, \quad j = 1,2,3,\ldots. \tag{65}$$

Assume also that the Lipschitz condition (64) *holds for the family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, *and the function* q_j *is concave in* j. *Let* m *be the smallest number such that* $q_{m+1} < 0$ (*i.e.,* $q_m \geq 0$). *Denote*

$$d = m + q_m/(q_m - q_{m+1}).$$

Then, for every $\delta > 0$, there exist numbers $\alpha \in (0,1)$, $\varepsilon_0 > 0$, $h \geq 0$, such that

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leq (d+\delta)\log_2\left(\frac{\varepsilon_0}{\alpha\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon\alpha}{4C(h)}}\left(\Pi_{0,h\log_{1/\alpha}(\frac{\varepsilon_0}{\alpha\varepsilon})}\mathcal{H}(\sigma_0)\right), \quad \forall\varepsilon < \varepsilon_0. \tag{66}$$

The function $C(h)$ is taken from the Lipschitz condition (64)*.*

The proof is given in [13]. We now formulate some important corollaries.

Corollary 3. *Assume that the function $\sigma_0(t)$ is almost periodic, that is, the hull $\mathcal{H}(\sigma_0)$ is compact in $C_b(\mathbb{R};\Psi)$. Then inequality* (66) *admits a simpler form:*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leq (d+\delta)\log_2\left(\frac{\varepsilon_0}{\alpha\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon\alpha}{4C(h)}}\left(\mathcal{H}(\sigma_0)\right), \quad \forall\varepsilon < \varepsilon_0, \tag{67}$$

where $\mathbf{H}_\epsilon(\mathcal{H}(\sigma_0))$ is the ϵ-entropy of the hull $\mathcal{H}(\sigma_0)$ in the space $C_b(\mathbb{R};\Psi)$.

Indeed, the ϵ-entropy of $\Pi_{0,l}\mathcal{H}(\sigma_0)$ in $C([0,l];\Psi)$ does not exceed the ϵ-entropy of the set $\mathcal{H}(\sigma_0)$ in the space $C_b(\mathbb{R};\Psi)$. Estimate (67) shows that, for a generic almost periodic function $\sigma_0(t)$ having infinitely many rationally independent frequencies, the main contribution to the estimate for the ε-entropy of the global attractor $\mathcal{A}$ is made by the ε/L-entropy of the hull $\mathcal{H}(\sigma_0)$, where $L = \dfrac{4C(h)}{\alpha}$. However, if the function $\sigma_0(t)$ has a finite number of frequencies, i.e., is quasiperiodic, then the contribution of this quantity is comparable with the contribution of the term $d\log_2\left(\dfrac{\varepsilon_0}{\alpha\varepsilon}\right)$. This leads to the finite dimensionality of the global attractor.

Corollary 4. *Let, in the assumptions of Theorem 4, the function $\sigma_0(t)$ be quasiperiodic of the form $\sigma_0(t) = \phi(\alpha_1 t, \alpha_2 t, \ldots, \alpha_k t) = \phi(\boldsymbol{\alpha} t)$, where $\phi(\omega_1,\omega_2,\ldots,\omega_k) = \phi(\boldsymbol{\omega}) \in C^{\mathrm{Lip}}(\mathbb{T}^k;\Psi)$. Then estimate* (67) *becomes*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leq (d+\delta)\log_2\left(\frac{\varepsilon_0}{\alpha\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + k\log_2\left(\frac{8C(h)}{K\alpha\varepsilon}\right), \quad \forall\varepsilon < \varepsilon_0, \tag{68}$$

where K is the Lipschitz constant from the inequality

$$\|\phi(\boldsymbol{\omega}_1) - \phi(\boldsymbol{\omega}_2)\|_\Psi \leq K\|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|_{\mathbb{R}^k}, \quad \forall \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{T}^k.$$

Moreover,

$$\mathbf{d}_F(\mathcal{A}) \leq d + k. \tag{69}$$

Proof. If $\sigma_1, \sigma_2 \in \mathcal{H}(\sigma_0)$, then $\sigma_i(t) = \phi(\boldsymbol{\alpha} t + \boldsymbol{\omega}_i)$ for some $\boldsymbol{\omega}_i \in \mathbb{T}^k$, $i = 1,2$ (see (51)). Therefore,

$$\begin{aligned}\|\sigma_1 - \sigma_2\|_{C_b(\mathbb{R};\Psi)} &\equiv \sup_{t\in\mathbb{R}}\|\sigma_1(t) - \sigma_2(t)\|_\Psi \\ &= \sup_{t\in\mathbb{R}}\|\phi(\boldsymbol{\alpha} t + \boldsymbol{\omega}_1) - \phi(\boldsymbol{\alpha} t + \boldsymbol{\omega}_2)\|_\Psi \leq L|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2|_{\mathbb{T}^k}.\end{aligned}$$

Hence,

$$N_\epsilon(\mathcal{H}(\sigma_0)) \leq N_{L\epsilon}(\mathbb{T}^k).$$

It is known that the torus $\mathbb{T}^k$ with Euclidean metrics can be covered by at most $\left(\dfrac{2}{\rho}\right)^k$ balls of radius $\rho < 1$ (see [24]). Thus, for $\rho = L\epsilon$, we obtain

$$N_\epsilon(\mathcal{H}(\sigma_0)) \leq \left(\frac{2}{L\epsilon}\right)^k,$$

$$\mathbf{H}_\epsilon(\mathcal{H}(\sigma_0)) \le k\log_2\left(\frac{2}{L\epsilon}\right), \quad \forall\epsilon < L^{-1}.$$

Substituting $\epsilon = \dfrac{\varepsilon\alpha}{4C(h)}$ into (67), we get (68). It remains to note that inequality (68) implies the following estimate:

$$\mathbf{d}_F(\mathcal{A}) \equiv \varlimsup_{\varepsilon\to 0+} \frac{\mathbf{H}_\varepsilon(\mathcal{A})}{\log_2(1/\varepsilon)} \le d + k + \delta, \quad \forall\delta > 0.$$

Consequently, inequality (69) also holds. △

Recall that, in the autonomous case with $k = 0$, estimate (26) is an analog of estimate (69) with $X = \mathcal{A}$: $\mathbf{d}_F(\mathcal{A}) \le d$. In the nonautonomous case, when $k \ne 0$, estimate (69) with the number k of rationally independent frequencies of the function $\sigma_0(t)$ added to d is valid.

Consider two important characteristics of a compact set X in the space E introduced in [1]. The number

$$\mathbf{df}(X,E) = \mathbf{df}(X) = \varlimsup_{\varepsilon\to 0+} \frac{\log_2(\mathbf{H}_\varepsilon(X))}{\log_2\log_2(1/\varepsilon)} \tag{70}$$

is called the *functional dimension* of the set X in E, and the number

$$\mathbf{q}(X,E) = \mathbf{q}(X) = \varlimsup_{\varepsilon\to 0+} \frac{\log_2(\mathbf{H}_\varepsilon(X))}{\log_2(1/\varepsilon)} \tag{71}$$

is called its *metric order* in E. It is easy to see that $\mathbf{df}(X) = 1$ and $\mathbf{q}(X) = 0$ if $\mathbf{d}_F(X) < +\infty$. Thus, the values $\mathbf{df}(X)$ and $\mathbf{q}(X)$ characterize infinite dimensional sets. Some examples of calculation of these values are given in [1] (see also [25]).

Corollary 5. *Let $\sigma_0(t)$ be an almost periodic function. Then*

$$\mathbf{df}(\mathcal{A}) \le \mathbf{df}\left(\mathcal{H}(\sigma_0), C_b(\mathbb{R};\Psi)\right), \tag{72}$$

$$\mathbf{q}(\mathcal{A}) \le \mathbf{q}\left(\mathcal{H}(\sigma_0), C_b(\mathbb{R};\Psi)\right). \tag{73}$$

Let us now briefly explain the application of Theorem 4 and Corollaries 3–5 to the dissipative wave equation (40). In Section 3, it was proved that this equation has the global attractor $\mathcal{A}$ in $E = H_0^1(\Omega) \times L_2(\Omega)$ of the form (54).

Changing the variables $w = (u,v) = (u, p + \eta u)$, we rewrite equation (40) as

$$\partial_t w = A(w) = Lw - G_{g_0(t)}(w), \qquad w|_{t=\tau} = w_\tau, \tag{74}$$

where the operator L is defined by formula (32) and $G_{g_0(t)}(w) = (0, f(u) - g_0(x,t))$. Consider the case $n = 3$. The function $f(u)$ satisfies (27). The variational equation for (74) has the form

$$\partial_t z = Lz - G_{g_0 w}(w(t))z = A_{g_0 w}(w(t))z, \quad z|_{t=\tau} = z_\tau, \quad z = (r,q), \tag{75}$$

where $G_{g_0 w}(w(t))z = (0, f'_u(u)r)$. Similarly to the autonomous case (see [5]), we prove that the process $\{U_{g_0}(t,\tau)\}$ of problem (74) is uniformly quasidifferentiable, and its quasidifferentials are generated by system (75). Using the reasoning from the proof of Theorem 2, we get the following estimate for the numbers $\widetilde{q}_j$:

$$\widetilde{q}_j \le q_j = -(\eta/4)j + (C_2(M_1)/\eta)j^{1/3} \tag{76}$$

(see (34)–(37)), where M_1 is due to the inequality

$$\sup\left\{\|u(\cdot\,,t)\|_{C(\mathbf{\Omega})},\ t\in\mathbb{R},\ (u(\cdot),\partial_t u(\cdot)) \in \mathcal{K}_{g_0}\right\} \le M_1.$$

In [13], it was shown that the family of processes $\{U_g(t,\tau)\}$ $g \in \mathcal{H}(g_0)$, satisfies the Lipschitz condition (64). Applying Theorem 4, we obtain the following result.

Theorem 5. *If the function $g_0(x,t)$ is translation-compact in the space $C^{\rm loc}(\mathbb{R};L_2(\Omega))$, then the ε-entropy of the global attractor $\mathcal{A}$ of problem* (74) *satisfies the inequality*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \le \frac{C(M_1)}{\eta^3}\log_2\left(\frac{\varepsilon_0}{\alpha\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon\alpha}{4C(h)}}\left(\Pi_{0,h\log_{1/\alpha}\left(\frac{\varepsilon_0}{\alpha\varepsilon}\right)}\mathcal{H}(\sigma_0)\right), \quad \forall\varepsilon<\varepsilon_0, \tag{77}$$

where α, ε_0, and h are some positive numbers.

For equation (40), Corollaries 3–5 also hold in the cases where the external force is almost periodic or quasiperiodic. It is also easy to prove estimates for the ε-entropy and fractal dimension of the nonautonomous sine-Gordon equation.

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