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EVOLUTION EQUATIONS AND THEIR TRAJECTORY ATTRACTORS (*)

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A compact set $\mathfrak{A} \Subset E$ is said to be a global attractor of a semigroup $\{S(t),\ t \geq 0\}$ acting in a Banach or Hilbert space E if $\mathfrak{A}$ is strictly invariant with respect to $\{S(t)\} : S(t)\mathfrak{A} = \mathfrak{A}$ $\forall t \geq 0$ and $\mathfrak{A}$ attracts any bounded set $B \subset E : \mathbf{dist}(S(t)B, \mathfrak{A}) \to 0\ (t \to +\infty)$. A reach variety of works has been devoted to the study of global attractors of semigroups $\{S(t)\}$ corresponding to autonomous evolution equations including evolution equations arising in mathematical physics (*see*, for example, books [13], [24], [1], and the literature cited their). In the last few years, uniform attractors $\mathcal{A}$ of processes $\{U(t, \tau)\}$ corresponding to non-autonomous partial differential equations have been treated as well (*see* [20], [10], [14], [2], [3], [4]). Notice that a uniform attractor of a process $\{U(t, \tau)\}$ acting in E is a minimal compact set $\mathcal{A} \Subset E$ that attracts any bounded set $B \subset E$ uniformly w.r.t. $\tau \in \mathbb{R}$: $\sup_{\tau \in \mathbb{R}} \mathbf{dist}(U(t + \tau, \tau)B, \mathcal{A}) \to 0\ (t \to +\infty)$ (*see* [14], [2]).

The present paper deals with a trajectory attractor of a given non-autonomous evolution equation. The existence and the structure of trajectory attractors are treated. In particular we study evolution equations and systems arising in mathematical physics (for example, 3D Navier-Stokes system with time-dependent external force, nonlinear dissipative hyperbolic equation having an arbitrary polynomial growth of the nonlinear function $f(u, s)$ w.r.t. u, and other equations). It should be pointed out that we do not suppose the uniqueness solvability of the corresponding Cauchy problems.

Equations we study can be written in the following abstract operator form:

$$\partial_t u(t) = A_{\sigma(t)}(u), \quad t \geq 0. \tag{1}$$

Here $\sigma(s), s \geq 0$, is a functional parameter called *the time symbol* of equation (1). (We have replaced t by s). In applications to mathematical physics equations, a function $\sigma(s)$ consists of all time-dependent coefficients, terms, and right-hand sides of an equation under consideration. For example, for the dissipative hyperbolic equation:

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - f(u, t) + g(x, t),\ u|_{\partial\Omega} = 0,\ t \geq 0, \tag{2}$$

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the time symbol is $\sigma(s) = (f(v,s), g(x,s))$ ($v \in \mathbb{R}$, $x \in \Omega \Subset \mathbb{R}^n$, $s \geq 0$). (To reduce equation (2) to the form (1) one has to add a new variable $p = \partial_t u$). We assume that the functions $f(v,s)$ and $g(x,s)$ satisfy some general conditions providing the solvability of the Cauchy problem for (2) (*see* [19]). But, generally speaking, under these conditions, the corresponding solution $u(t) = u(x,t)$ need not be unique.

A trajectory attractor $\mathcal{A}$ is constructed for the family of equations (1). We start from the fact that the attractor $\mathcal{A}$ may not change when the initial symbol $\sigma_0(s)$ is replaced by any shifted symbol $\sigma_0(s+h)$, $h \geq 0$. This is why, together with the initial equation (1) having the symbol $\sigma_0(s)$, we consider the family of equations (1) with shifted symbols $\sigma_0(s+h)$, $h \geq 0$. This family contains also any symbol $\sigma(s)$ that is a limit of some sequence $\{\sigma_0(s+h_m) \mid h_m \geq 0\}_{m\in\mathbb{N}} : \sigma(s) = \lim_{m\to\infty} \sigma(s+h_m)$, where the limit is taken in an appropriate topological space $\Xi_+ = \{\xi(s), s \geq 0\}$. The family of such symbols $\{\sigma(s)\}$ is said to be *a hull* $\mathcal{H}_+(\sigma_0)$ of function $\sigma_0(s)$ in Ξ_+, *i.e.*

$$\mathcal{H}_+(\sigma_0) = [\{\sigma_0(s+h) \mid h \geq 0\}]_{\Xi_+}.$$

(Here $[\cdot]_{\Xi_+}$ means the closure in Ξ_+). We assume that the hull $\mathcal{H}_+(\sigma_0)$ is compact in Ξ_+. The topological space Ξ_+ is selected in such a way to provide the solvability of equation (1) with any symbol $\sigma(t) \in \mathcal{H}_+(\sigma_0)$. Usually, Ξ_+ is a space with some local convergence topology on any segment $[t_1, t_2] \subset \mathbb{R}_+$ (*see* section 6). In applications, for example to hyperbolic equation (2), any symbol $\sigma(s) = (f(v,s), g(x,s)) \in \mathcal{H}_+(\sigma_0) = \mathcal{H}_+(f_0(v,s), g_0(x,s))$ satisfies the same conditions as the initial symbol $\sigma_0 = (f_0(v,s), g_0(x,s))$ does (*see* section 7).

Next, for any equation (1) with a symbol $\sigma(s) \in \mathcal{H}_+(\sigma_0)$, we define some collection of its solutions $\mathcal{K}_\sigma^+ = \{u(s), s \geq 0\}$ belonging to the corresponding functional space $\mathcal{F}_+^a$. Here, we have replaced t by s. The set $\mathcal{K}_\sigma^+$ is called a trajectory space of the equation with a symbol σ. In application to a particular equation, a Banach space $\mathcal{F}_+^a$ is defined and a trajectory space $\mathcal{K}_\sigma^+$ consists of all weak solutions $u(s) \in \mathcal{F}_+^a$ of this equation which, in addition, satisfy some natural energy inequality. Any weak solution $u(s)$ resulting from the Faedo-Galerkin approximation method satisfies this inequality and therefore it belongs to $\mathcal{K}_\sigma^+$.

In Sections 7 and 8, the detailed description of trajectory spaces $\mathcal{K}_\sigma^+, \sigma \in \mathcal{H}_+(\sigma_0)$, is given for the hyperbolic equation (2) and for 3D Navier-Stokes system.

Consider the united trajectory space $\mathcal{K}^+ = \cup_{\sigma\in\mathcal{H}_+(\sigma_0)}\mathcal{K}_\sigma^+$. The translation semigroup $\{T(t), t \geq 0\}$ acts on $\mathcal{K}^+$ as a set of translations along the time axis: $T(t)u(s) = u(t+s)$. Notice that $T(t)\mathcal{K}_\sigma^+ \subseteq \mathcal{K}_{T(t)\sigma}^+$,(but $T(t)\mathcal{K}_\sigma^+ \nsubseteq \mathcal{K}_\sigma^+$) and therefore:

$$T(t)\mathcal{K}^+ \subseteq \mathcal{K}^+ \quad \forall t \geq 0. \tag{3}$$

Together with the Banach space $\mathcal{F}_+^a$, we introduce some topological space $\Theta_+ = \{\theta(s), s \geq 0\}$, $\mathcal{F}_+^a \subseteq \Theta_+$. Usually, the topology of Θ_+ is weaker than the topology of $\mathcal{F}_+^a$. The translation semigroup $\{T(t)\}$ is continuous in the topological space Θ_+. Let the space $\mathcal{K}^+$ is closed in Θ_+. In applications the topology of Θ_+ is a local weak convergence topology on any segment $[t_1, t_2] \subseteq \mathbb{R}_+$ (*see* section 2).

The global attractor (in the topology Θ_+) of the translation semigroup $\{T(t),\, t \geq 0\}$ acting on $\mathcal{K}^+$ is said to be *the trajectory attractor* $\mathcal{A}$ of the family of equations (1) with symbols $\sigma(s) \in \mathcal{H}_+(\sigma_0)$. More, precisely, the set $\mathcal{A} \subseteq \mathcal{K}^+$ is compact in Θ_+, it is strictly invariant with respect to $\{T(t)\}$: $T(t)\mathcal{A} = \mathcal{A}\ \forall t \geq 0$, and a set $T(t)B$ is attracted to $\mathcal{A}$ in the topology Θ_+ as $t \to +\infty$ for any set $B \subset \mathcal{K}^+$ bounded in $\mathcal{F}_+^a$. The latter means that for any neighbourhood $\mathcal{O}(\mathcal{A})$ in Θ_+ of the attractor $\mathcal{A}$ and any set $B \subset \mathcal{K}^+$ bounded in $\mathcal{F}_+^a$ there exist a number $t_0 = t_0(B, \mathcal{O}) > 0$ such that

$$T(t)B \subset \mathcal{O}(\mathcal{A})\ \forall t \geq t_0. \tag{4}$$

In applications, the attracting property (4) can be reduced to the form

$$\mathbf{dist}_L(T(t)B, \mathcal{A}) \to 0\ (t \to +\infty), \tag{5}$$

where L is an appropriate Banach or metric space containing $\mathcal{K}^+$.

In section 7, spaces $\mathcal{F}_+^a$, Θ_+ are described for the equation (2). It is proved that the translation semigroup $\{T(t)\}$ acting on the corresponding united trajectory space $\mathcal{K}^+$ possesses a compact (in Θ_+) absorbing set that is bounded in $\mathcal{F}_+^a$. This fact implies the trajectory attractor existence theorem for equation (2). In this case the attracting property (5) of the trajectory attractor $\mathcal{A}$ looks as follows: for any bounded (in $\mathcal{F}_+^a$) set $B \subset \mathcal{K}^+$ and for any $M > 0$

$$\mathbf{dist}_{L_\rho(0,M;E_{1-\delta})}(T(t)B, \mathcal{A}) \to 0\ (t \to +\infty),$$

where $\|u(x,s)\|^2_{E_{1-\delta}} = \|u(\cdot,s)\|^2_{H^{1-\delta}} + \|\partial_t u(\cdot,s)\|^2_{H^{-\delta}}$, $0 < \delta \leq 1$, $\rho > 1$, ρ is any number. In section 7, the structure of the trajectory attractor $\mathcal{A}$ of equation (2) is described as well. In particular it is shown that any solution $u(t), t \geq 0$, of equation (2) lying in the attractor $\mathcal{A}$ admit a bounded (in $\mathcal{F}_+^a$) prolongation to the whole time-axis $\{t \in \mathbb{R}\}$ as a solution $\tilde{u}(t), t \in \mathbb{R}$, of equation (2) with an appropriate symbol $\tilde{\sigma}(s) = (\tilde{f}(v,s), \tilde{g}(x,s))$, $s \in \mathbb{R}$.

For the 3D Navier-Stokes system (section 8), the attracting property (5) of the trajectory attractor $\mathcal{A}$ implies that for any bounded set $B \subset \mathcal{K}^+$:

$$\mathbf{dist}_{L_2(0,M;H^{1-\delta})}(T(t)B, \mathcal{A}) \to 0\ (t \to +\infty)\ \forall M > 0,\ 0 < \delta \leq 1. \tag{6}$$

Some properties of the trajectory attractor $\mathcal{A}$ are given in section 8. Notice that the work [21] is devoted to the study of global attractors of 3D Navier-Stokes systems.

The work [9] deals with trajectory attractors of non-autonomous reaction-diffusion systems, for which a solution of the Cauchy problem need not be unique. (The Lipschitz condition for the nonlinear interaction function is not required).

Let us formulate some corollaries from trajectory attractor existence theorems. In section 10 we study 3D Navier-Stokes system with a perturbed external force $g(x,s) + a(x,s)$. If, for example, the perturbation term $a(x,s)$ has a form: $a(x,s) = G(x)\sin(s^2)$, where $G(x) \in H$, then the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g(x,s))}$ of the non-perturbed system coincides with the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g(x,s)+a(x,s))}$ of the perturbed one. Thus, roughly speaking, the perturbation $a(x, s)$ having a weak zero convergence as $t \to +\infty$ does not effect on the trajectory attractor. Analogous results are valid for trajectory attractors of perturbed non-autonomous hyperbolic equations (2) and for trajectory attractors of other evolution equations and systems. One more important corollary involves the trajectory attractor $\mathcal{A}^N$ of the Faedo-Galerkin approximation system of order N for 3D Navier-Stokes system. Let $\mathcal{A}$ be the trajectory attractor of the origin 3D Navier-Stokes system. It is proved that the attractor $\mathcal{A}^N$ tends to $\mathcal{A}$ as $N \to \infty$ in the norm (6):

$$\mathbf{dist}_{L_2(0,M;H^{1-\delta})}(\mathcal{A}^N, \mathcal{A}) \to 0 \ (N \to \infty) \ \forall M > 0.$$

Finally notice that the proved general trajectory attractor existence theorem (section 4) can be applied to evolution equations for which the uniqueness theorem of the Cauchy problem takes place. In this case, bounded trajectory sets tend to the trajectory attractor in a stronger topology. Trajectory attractors for 2D Navier-Stokes systems have been studied in [8]. In particular, it has been proved that the attraction to $\mathcal{A}$ takes place in the strong topology of the space $L_\infty^{loc}(\mathbb{R}_+; H_1) \cap L_2^{loc}(\mathbb{R}_+; H_2) \cap \{\partial_t u \in L_2^{loc}(\mathbb{R}_+; H)\}$.

The main results of this paper are briefly outlined in [5]-[7].

1. Symbols of non-autonomous evolution equations

We consider non-autonomous evolution equations of the type:

$$\partial_t u = A_{\sigma(t)}(u), \ t \geq 0. \tag{1.1}$$

For any $s \in \mathbb{R}_+$ we are given an operator $A_{\sigma(s)}(\,.\,) : E \to E_0$, where E, E_0 are Banach spaces. The functional parameter $\sigma(s), s \in \mathbb{R}_+$, in (1.1) reflects the dependence on time of the equation. The function $\sigma(s)$ is called the *time symbol* (or the *symbol*) of equation (1.1). Values of $\sigma(s)$ belong to some Banach space Ψ, *i.e.* $\sigma(s) \in \Psi$ for any (or almost any) $s \in \mathbb{R}_+$ (*see* [2], [3], [7]).

For the non-autonomous dissipative hyperbolic equation (2), the symbol is the pair $\sigma(s) = (f(v, s), g(x, s)), s \geq 0$. The component $g(\,,\,s)$ takes its values in $L_2(\Omega)$ and values of $f(\,.\,, s)$ belong to the specially selected functional space $\mathcal{M} = \{\psi(v), v \in \mathbb{R}\}$ (*see* section 8).

For the Navier-Stokes system (*see* section 8)

$$\partial_t u = -\nu L u - B(u) + g(x, t), \ (\nabla, u) = 0, \ \ u|_{\partial\Omega} = 0, \ t \geq 0,$$

where $x \in \Omega \Subset \mathbb{R}^n, u = (u^1, \ldots, u^n), g = (g^1, \ldots, g^n)$, $(n = 2, 3)$, the external force $g(x, s) = \sigma(s), s \in \mathbb{R}_+$, is taken to be the time symbol. The symbol $\sigma(s) = g(\,\cdot\,, s)$ takes its values in the known space $H = \Psi$.

We assume that the symbol $\sigma(s)$ of equation (1.1), as a function of s, belongs to a topological space $\Xi_+ = \{\xi(s), s \geq 0 \mid \xi(s) \in \Psi$ for almost any $s \geq 0\}$. Usually, in applications, the topology in the space Ξ_+ is a local convergence topology on any segment $[t_1, t_2] \subset \mathbb{R}_+$. Different spaces Ξ_+ will be described in section 6 in more details. We assume that Ξ_+ is a Hausdorff topological space. This condition is valid when Ξ_+ is a metric space. The *translation semigroup* $\{T(t),\, t \geq 0\}$ acts on Ξ_+:

$$T(t)\xi(s) = \xi(t+s),\ s \in \mathbb{R},\ t \geq 0. \tag{1.2}$$

We assume that the mapping $T(t)$ is continuous in the topological space Ξ_+ for any $t \geq 0$.

Now consider a family of equations (1.1) with various symbols $\sigma(s)$ belonging to a set $\Sigma \subseteq \Xi_+$. The set Σ is called the *symbol space* of the family of equation (1.1). It is assumed that the set Σ, together with any symbol $\sigma(s) \in \Sigma$, contains all positive translations of $\sigma(s) : \sigma(t+s) = T(t)\sigma(s) \in \Sigma$ for any $t \geq 0$. So, the symbol space Σ is invariant with respect to the translation semigroup $\{T(t)\}$ in the following sense:

$$T(t)\Sigma \subseteq \Sigma\ \forall t \geq 0. \tag{1.3}$$

We suppose that the symbol space Σ with the topology from Ξ_+ is a metrizable space and the corresponding metric space is complete.

In such a manner, we shall study the family of equations (1.1) with symbols $\sigma(s)$ from the complete metric space Σ, $\Sigma \subseteq \Xi_+$ and the continuous translation semigroup $\{T(t)\}$, satisfying (1.3), acts on Σ.

Let us describe the typical symbol space in particular problems. We are given some fixed symbol $\sigma_0(s), s \geq 0$, (in applications, consisting of all time-dependent terms of the equation under consideration: external forces, parameters of mediums, interaction functions, control functions, etc.). Then one chooses appropriate enveloping topological space $\Xi_+ = \{\xi(s), s \geq 0\}$, such that $\sigma_0(s) \in \Xi_+$. Consider the closure in Ξ_+ of the following set: $\{T(t)\sigma_0(s),\ t \geq 0\} = \{\sigma_0(t+s),\ t \geq 0\}$. This closure is said to be the hull of the function $\sigma_0(s)$ in Ξ_+ and it is denoted as:

$$\mathcal{H}_+(\sigma_0) = [\{T(t)\sigma_0 \mid t \geq 0\}]_{\Xi_+}. \tag{1.4}$$

Evidently, $T(t)\mathcal{H}_+(\sigma_0) \subseteq \mathcal{H}_+(\sigma_0)$ for any $t \geq 0$.

DEFINITION 1.1. – *The function* $\sigma_0(s) \in \Xi_+$ *is said to be* translation-compact (tr.-c.) in Ξ_+ *if the hull* $\mathcal{H}_+(\sigma_0)$ *is compact in* Ξ_+.

Mostly in applications, we consider symbol spaces $\Sigma = \mathcal{H}_+(\sigma_0)$, where $\sigma_0(s)$ is a tr.-c. function in an appropriate topological space Ξ_+. If Ξ_+ is a Hausdorff space with a countable base of open sets then, by Uryson Theorem, a hull $\mathcal{H}_+(\sigma_0)$ of a tr.-c. function $\sigma_0(s)$ in Ξ_+ is a metrizable complete space. In section 6, translation-compactness criterions for various spaces Ξ_+ will be given.

2. Trajectory spaces of evolution equations

The aim of this article is to study solutions $u(s)$ of equations (1.1) being a function of $s \in \mathbb{R}_+$ as a whole. A set of all solutions is said to be *a trajectory space* $\mathcal{K}_\sigma^+$ of equation (1.1) with a symbol σ. Let us describe a trajectory space $\mathcal{K}_\sigma^+$ in more details. In all applications below, we shall strictly clarify the meaning of the expression: "a function

$u(s)$ is a solution of (1.1)". In this section we shall emphasize the needed properties of $\mathcal{K}_\sigma^+$ to construct the general theory of trajectory spaces.

At first, we consider solutions $u(s)$ of (1.1) defined on any fixed segment $[t_1, t_2]$ from $\mathbb{R}$. We are looking for solutions of (1.1) in a separable Banach space $\mathcal{F}_{t_1,t_2}$. We make the following assumptions. $\mathcal{F}_{t_1,t_2}$ consists of functions $f(s), s \in [t_1, t_2]$, such that $f(s) \in E$ for almost all $s \in [t_1, t_2]$. If $f(s) \in \mathcal{F}_{t_1,t_2}$ then $A_{\sigma(s)}(f(s)) \in \mathcal{D}_{t_1,t_2}$, where $\mathcal{D}_{t_1,t_2}$ is a larger Banach space, $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$. The space $\mathcal{D}_{t_1,t_2}$ contains functions with values in E_0 for almost all $s \in [t_1, t_2]$ ($E \subseteq E_0$, E and E_0 are Banach spaces). The derivative $\partial_t f(s)$ is a distribution with values in $E_0, \partial_t f(s) \in D'(]t_1, t_2[; E_0)$: $\mathcal{D}_{t_1,t_2} \subseteq D'(]t_1, t_2[; E_0)$. Finally, a function $u(s) \in \mathcal{F}_{t_1,t_2}$ is said to be a solution of (1.1) from $\mathcal{F}_{t_1,t_2}$ (on the segment $[t_1, t_2]$), if $\partial_t u(s) = A_{\sigma(s)}(u(s))$ in the distribution sense. Denote by $\mathcal{K}_\sigma^{t_1,t_2}$ the set of some solutions of (1.1) from $\mathcal{F}_{t_1,t_2}$. (Notice, that $\mathcal{K}_\sigma^{t_1,t_2}$ is not necessarily the set of all solutions from $\mathcal{F}_{t_1,t_2}$.) We suppose, that $\Pi_{t_1,t_2}\mathcal{K}_\sigma^{t_1',t_2'} \subseteq \mathcal{K}_\sigma^{t_1,t_2}$ for any $[t_1', t_2'] \supseteq [t_1, t_2]$, where $\Pi_{t_1,t_2} f$ denotes the restriction of f to the segment $[t_1, t_2]$.

DEFINITION 2.1. – *A function* $u(s), s \in \mathbb{R}_+$, *is said to be* a trajectory *of (1.1) if* $\Pi_{t_1,t_2} u(s) \in \mathcal{K}_\sigma^{t_1,t_2}$ *for any* $[t_1, t_2]$ *from* $\mathbb{R}_+$. *Denote by* $\mathcal{K}_\sigma^+$ *a set of some trajectories* $u(s), s \in \mathbb{R}_+$, *of equation* (1.1).

Other required properties of the trajectory space $\mathcal{K}_\sigma^+$ are given in section 3. Consider some examples of spaces $\mathcal{F}_{t_1,t_2}$ we shall study in applications.

Exemple 2.1. – $\mathcal{F}_{t_1,t_2} = C([t_1, t_2]; E)$, where $C([t_1, t_2]; E)$ is the space of continuous functions on $[t_1, t_2]$ with values in a Banach space E. The norm in $\mathcal{F}_{t_1,t_2}$ is:

$$\|f\|_{C([t_1,t_2];E)} = \max_{s\in[t_1,t_2]} \|f(s)\|_E. \tag{2.1}$$

Exemple 2.2. – **a)** $\mathcal{F}_{t_1,t_2} = L_p(t_1, t_2; E)$, $p \geq 1$. Here $L_p(t_1, t_2; E)$ is the space of functions $f(s)$, $s \in [t_1, t_2]$, p-power integrable in Bochner sense. The norm is:

$$\|f\|_{L_p(t_1,t_2;E)}^p = \int_{t_1}^{t_2} \|f(s)\|_E^p ds. \tag{2.2}$$

b) $\mathcal{F}_{t_1,t_2} = L_\infty(t_1, t_2; E)$ is the space of essentially bounded functions on $[t_1, t_2]$ with values in E,

$$\|f\|_{L_\infty(t_1,t_2;E)} = \mathbf{essup}_{s\in[t_1,t_2]} \|f(s)\|_E. \tag{2.3}$$

Other spaces $\mathcal{F}_{t_1,t_2}$, corresponding to specific equations and systems are given in sections 7 and 8. Notice, that in the above examples, spaces $\mathcal{F}_{t_1,t_2}$ are similar to $\mathcal{F}_{0,1}$. The corresponding similitude J is:

$$Jf(s) = f((t_2 - t_1)s + t_1). \tag{2.4}$$

Evidently, in Examples 2.1 and 2.2 one has:

$$\|Jf\|_{\mathcal{F}_{0,1}} = c\|f\|_{\mathcal{F}_{t_1,t_2}}, \tag{2.5}$$

where $c = c(| t_1 - t_2 |)$ does not depend on f. Considering the general scheme, we shall assume that relation (2.5) holds for the spaces $\mathcal{F}_{t_1,t_2}$.

Returning to equation (1.1), let we are given a trajectory space $\mathcal{K}_\sigma^+$ of this equation. For $\mathcal{K}_\sigma^+$, we consider enveloping spaces $\mathcal{F}_+^{loc}$ and $\mathcal{F}_+^a$.

DEFINITION 2.2. – **(i)** $\mathcal{F}_+^{loc} = \{f(s), s \in \mathbb{R}_+ \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2} \ \forall\ [t_1,t_2] \subseteq \mathbb{R}_+\}$; **(ii)** $\mathcal{F}_+^a = \{f(s) \in \mathcal{F}_+^{loc} \mid \|f\|_{\mathcal{F}_+^a} < +\infty\}$, *where*

$$\|f\|_{\mathcal{F}_+^a} = \sup_{t \geq 0} \|\Pi_{0,1} f(t+s)\|_{\mathcal{F}_{0,1}}. \tag{2.6}$$

Evidently, $\mathcal{F}_+^a$ *with norm (2.6) is a Banach space.*

REMARK 2.1. – *To define the equivalent norm in* $\mathcal{F}_+^a$ *one can use* $\mathcal{F}_{0,M}$ *in (2.6) instead of* $\mathcal{F}_{0,1}$ *since equality (2.5) takes place.*

Now we define a topology in $\mathcal{F}_+^{loc}$. The space $\mathcal{F}_+^{loc}$ with this topology, we shall denote as Θ_+^{loc}. Let we be given some topology in $\mathcal{F}_{t_1,t_2}$. Denote by Θ_{t_1,t_2} the topological space $\mathcal{F}_{t_1,t_2}$, endowed with this topology. Suppose, Θ_{t_1,t_2} is Hausdorff and Fréchet-Uryson topological space with a countable base. Let Θ_{t_1,t_2} be homeomorphic to $\Theta_{0,1}$ with respect to the similitude J (*see* (2.4)). For example, Θ_{t_1,t_2} can be $\mathcal{F}_{t_1,t_2}$ itself with the strong or weak (or even $*$-weak) convergence topology in a Banach space. The space Θ_{t_1,t_2} defines a local convergence topology in $\mathcal{F}_+^{loc}$.

DEFINITION 2.3. – Θ_+^{loc} *denotes the space* $\mathcal{F}_+^{loc}$ *with the local convergence topology on* Θ_{t_1,t_2} *for any* $[t_1,t_2] \subseteq \mathbb{R}_+$ *i.e., by the definition, a sequence* $\{f_n(s)\} \subset \mathcal{F}_+^{loc}$ *converges to* $f(s) \in \mathcal{F}_+^{loc}$ *as* $n \to \infty$ *in* Θ_+^{loc} *if* $\Pi_{t_1,t_2} f_n(s) \to \Pi_{t_1,t_2} f(s)$ $(n \to \infty)$ *in* Θ_{t_1,t_2} *for any* $[t_1,t_2] \subseteq \mathbb{R}_+$.

It is not hard to prove, that Θ_+^{loc} is Hausdorff and Fréchet-Uryson topological space with a countable base.

Consider some examples. In Example 2.1, $\mathcal{F}_+^{loc} = C(\mathbb{R}_+;E), \mathcal{F}_+^a = C_b(\mathbb{R}_+;E)$ with the norm $\|f\|_{\mathcal{F}_+^a} = \sup_{s\geq 0} \|f(s)\|_E$ (compare with (2.1)). Let Θ_{t_1,t_2} be $C([t_1,t_2];E)$ with the uniform convergence topology generated by the norm (2.1). Then, by definition, $f_n(s) \to f(s)$ as $n \to \infty$ in Θ_+^{loc} if $\Pi_{t_1,t_2} f_n(s) \to \Pi_{t_1,t_2} f(s)$ $(n \to \infty)$ in $C([t_1,t_2];E)$ for any $[t_1,t_2] \subseteq \mathbb{R}_+$.

In Example 2.2 **a)** $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+;E)$, $\mathcal{F}_+^a = L_p^a(\mathbb{R}_+;E)$, where $L_p^a(\mathbb{R}_+;E)$ is the space of function $f(s) \in L_p^{loc}(\mathbb{R}_+;E)$ such that

$$\|f\|_{L_p^a(\mathbb{R}_+;E)}^p = \sup_{t \geq 0} \int_t^{t+1} \|f(s)\|_E^p ds < +\infty.$$

Let Θ_{t_1,t_2} be the convergence topology with respect to the norm (2.2). Then $f_n(s) \to f(s)$ $(n \to \infty)$ in Θ_+^{loc} if $\Pi_{t_1,t_2} f_n(s) \to \Pi_{t_1,t_2} f(s)$ $(n \to \infty)$ in $L_p(t_1,t_2;E)$ for any $[t_1,t_2] \subseteq \mathbb{R}_+$.

Consider another topology in $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+;E)$. Assume E is a reflexive separable Banach space and $p > 1$ then $L_p(t_1,t_2;E)^* = L_q(t_1,t_2;E^*)$, where $1/p + 1/q = 1$ (*see* [12]). Now let $\Theta_{t_1,t_2} = L_{p,w}(t_1,t_2;E)$ be the space $L_p(t_1,t_2;E)$ with the weak convergence topology. Then $f_n(s) \to f(s)$ $(n \to \infty)$ in Θ_+^{loc} whenever $f_n(s) \to f(s)$ $(n \to \infty)$ weakly in $L_p(t_1, t_2; E)$ for any $[t_1, t_2] \subseteq \mathbb{R}_+$.

In Example 2.2 **b)** $\mathcal{F}_+^{loc} = L_\infty^{loc}(\mathbb{R}_+; E)$, $\mathcal{F}_+^a = L_\infty(\mathbb{R}_+; E)$. Let E be a reflexive separable Banach space then $L_\infty(t_1, t_2; E) = L_1(t_1, t_2; E^*)^*$ (*see* [12]). Let Θ_{t_1,t_2} be the space $L_\infty(t_1, t_2; E)$ with the $*$-weak convergence topology. Then $f_n(s) \to f(s)$ $(n \to \infty)$ in Θ_+^{loc} if $f_n(s) \to f(s)$ $(n \to \infty)$ $*$-weakly in $L_\infty(t_1, t_2; E)$ for any $[t_1, t_2] \subseteq \mathbb{R}_+$.

Let us give a simple compactness criterion in the topological space Θ_+^{loc}.

PROPOSITION 2.1. – *A set $B \subset \mathcal{F}_+^{loc}$ is compact in the topological space Θ_+^{loc} if and only if the set $\Pi_{t_1,t_2} B$ is compact in Θ_{t_1,t_2} for any $[t_1, t_2] \subset \mathbb{R}_+$.*

Proof. – The spaces Θ_{t_1,t_2}, and Θ_+^{loc} have countable topology base. Hence, one has to check the countable compactness. The necessity is evident. The sufficiency can be proved by the diagonalization method. □

The translation semigroup $\{T(t), t \geq 0\}$ acts on $\mathcal{F}_+^{loc}$ (and on $\mathcal{F}_+^a$) by the formula:

$$T(t)f(s) = f(t+s), \ t \geq 0. \tag{2.7}$$

PROPOSITION 2.2. – *The semigroup $\{T(t)\}$ is continuous in the topological space Θ_+^{loc}.*

Proof. – If $f_n(s) \to f(s)$ $(n \to \infty)$ in Θ_+^{loc} then $\Pi_{t_1,t_2} f_n(s) \to \Pi_{t_1,t_2} f(s)$ $(n \to \infty)$ in Θ_{t_1,t_2} for any $[t_1, t_2] \subseteq \mathbb{R}_+$. In particular, $\Pi_{t+t_1,t+t_2} f_n(s) \to \Pi_{t+t_1,t+t_2} f(s)$ $(n \to \infty)$ in $\Theta_{t+t_1,t+t_2}$ for any $t \geq 0$, *i.e.* $\Pi_{t_1,t_2} T(t) f_n(s) \to \Pi_{t_1,t_2} T(t) f(s)$ $(n \to \infty)$ in Θ_{t_1,t_2} for any $t \geq 0$, *i.e.* $T(t) f_n(s) \to T(t) f(s)$ $(n \to \infty)$ in Θ_+^{loc}. Hence, $T(t)$ is continuous in Θ_+^{loc}, since Θ_+^{loc} is a Fréchet-Uryson space. □

3. Trajectory attractors of non-autonomous evolution equations

It is considered a family of equations (1.1) with symbols $\sigma(s), s \in \mathbb{R}_+$, belonging to a symbol space Σ. Σ is a complete metric space. The invariant translation semigroup $\{T(t)\}$ acts on Σ (*see* (1.2) and (1.3)). Let we are given spaces $\mathcal{F}_{t_1,t_2}$ and Θ_{t_1,t_2} that satisfy the above condition from section 2. Using the described scheme, we construct the spaces $\mathcal{F}_+^{loc}$, $\mathcal{F}_+^a$, and Θ_+^{loc}. To each symbol $\sigma \in \Sigma$, there corresponds a trajectory space $\mathcal{K}_\sigma^+$. Suppose, $\mathcal{K}_\sigma^+ \neq \emptyset$ and $\mathcal{K}_\sigma^+ \subseteq \mathcal{F}_+^a$ for any $\sigma \in \Sigma$ *i.e.* any solution $u(s) \in \mathcal{K}_\sigma^+$ of equation (1.1) has finite norm (2.6). We shall study the family of trajectory spaces $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ corresponding to equations (1.1) with symbols $\sigma \in \Sigma$.

DEFINITION 3.1. – *The family of trajectory spaces $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ is said to be* translation-coordinated (tr.-coord.) *if for any $\sigma \in \Sigma$ and any $u \in \mathcal{K}_\sigma^+$*

$$T(t)u \in \mathcal{K}_{T(t)\sigma}^+ \quad \forall t \geq 0. \tag{3.1}$$

In applications to evolution partial differential equations, one proves property (3.1) as follows. If $u(s), s \geq 0$, is a solution of (1.1) then the function $T(h)u(s) = u(h+s), s \geq 0$, $(h \geq 0)$ is a solution of (1.1) with the shifted symbol $T(h)\sigma(s) = \sigma(h+s), s \geq 0$.

DEFINITION 3.2. – *The set* $\mathcal{K}_\Sigma^+ = \cup_{\sigma\in\Sigma}\mathcal{K}_\sigma^+$ *is called the united trajectory space of the family* $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$.

PROPOSITION 3.1. – *If the family* $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ *is tr.-coord. then the translation semigroup* $\{T(t)\}$ *takes* $\mathcal{K}_\Sigma^+$ *to itself:*

$$T(t)\mathcal{K}_\Sigma^+ \subseteq \mathcal{K}_\Sigma^+ \ \forall t \geq 0.$$

The proof follows from (3.1) since $T(t)\mathcal{K}_\sigma^+ \subseteq \mathcal{K}_{T(t)\sigma}^+$.

Notice that $\mathcal{K}_\Sigma^+ \subseteq \mathcal{F}_+^a \subseteq \Theta_+^{loc}$. Let us define a trajectory attractor of the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}_\Sigma^+$.

DEFINITION 3.3. – *A set* $P \subseteq \Theta_+^{loc}$ *is said to be* a uniformly (w.r.t. $\sigma \in \Sigma$) attracting set *for the family* $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ *in the topology* Θ_+^{loc} *if for any bounded in* $\mathcal{F}_+^a$ *set* B *and* $B \subseteq \mathcal{K}_\Sigma^+$, *the set* P *attracts* $T(t)B$ *as* $t \to +\infty$ *in the topology* Θ_+^{loc}, *i.e. for any neighbourhood* $\mathcal{O}(P)$ *in* Θ_+^{loc} *there exists* $t_1 \geq 0$ *such that* $T(t)B \subseteq \mathcal{O}(P)$ *for any* $t \geq t_1$.

DEFINITION 3.4. – *A set* $\mathcal{A}_\Sigma \subseteq \Theta_+^{loc}$ *is said to be* a uniform (w.r.t. $\sigma \in \Sigma$) trajectory attractor *of the translation semigroup* $\{T(t)\}$ *on* $\mathcal{K}_\Sigma^+$ *in the topology* Θ_+^{loc}, *if* $\mathcal{A}_\Sigma$ *is compact in* Θ_+^{loc}, $\mathcal{A}_\Sigma$ *is strictly invariant:* $T(t)\mathcal{A}_\Sigma = \mathcal{A}_\Sigma \ \forall t \geq 0$, *and* $\mathcal{A}_\Sigma$ *is a minimal uniformly attracting set for* $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$, *i.e.* $\mathcal{A}_\Sigma$ *belongs to any compact uniformly attracting set* P *of* $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\} : \mathcal{A}_\Sigma \subseteq P$.

To construct the trajectory attractor of the semigroup $\{T(t)\}$ on $\mathcal{K}_\Sigma^+$, the set $\mathcal{K}_\Sigma^+$ is to be closed in Θ_+^{loc}.

DEFINITION 3.5. – *The family* $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ *is called* (Θ_+^{loc}, Σ)*-closed, if the graph set* $\cup_{\sigma\in\Sigma}\mathcal{K}_\sigma^+ \times \{\sigma\}$ *is closed in the topological space* $\Theta_+^{loc} \times \Sigma$ *with a usual product topology.*

PROPOSITION 3.2. – *Let* Σ *be compact and* $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ *be* (Θ_+^{loc}, Σ)*-closed; then the united trajectory space* $\mathcal{K}_\Sigma^+$ *is closed in* Θ_+^{loc}.

Proof. – Let $u_n(s) \in \mathcal{K}_\Sigma^+$, *i.e.* $u_n(s) \in \mathcal{K}_{\sigma_n}^+$ for some σ_n and let $u_n \to u$ $(t \to \infty)$ in Θ_+^{loc}. We claim that $u \in \mathcal{K}_\Sigma^+$. The set Σ is compact, therefore, we may assume by refining that $\sigma_n \to \sigma$ $(n \to \infty)$ in Σ, $\sigma \in \Sigma$. But $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ is (Θ_+^{loc}, Σ)-closed, hence, $u \in \mathcal{K}_\sigma^+$, that is, $u \in \mathcal{K}_\Sigma^+$. □

PROPOSITION 3.3. – *If a continuous semigroup* $\{T(t)\}$ *acts on a compact metric space* Σ, $T(t)\Sigma \subseteq \Sigma$, $\forall t \geq 0$, *then the semigroup* $\{T(t)\}$ *possesses a global attractor in* Σ *which coincides with* ω*-limit set of the whole* Σ*:*

$$\omega(\Sigma) = \bigcap_{t\geq 0}\left[\bigcup_{h\geq t} T(h)\Sigma\right]_\Sigma, \ \omega(\Sigma) \subseteq \Sigma, \tag{3.2}$$

where $[\ .\ ]_\Sigma$ *means the closure in* Σ. *Moreower, we have:*

$$T(t)\omega(\Sigma) = \omega(\Sigma) \ \forall t \geq 0. \tag{3.3}$$

This statement is a well-known fact from the theory of attractors of semigroups acting in a metric spaces (*see*, for example, [1], [24], [13], [3]).

Besides the family of trajectory spaces $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$, we shall consider more slender family of trajectory spaces $\{\mathcal{K}_\sigma^+, \sigma \in \omega(\Sigma)\}$, which corresponds to the strictly invariant symbol space $\omega(\Sigma) \subseteq \Sigma$ since (3.3) is valid. Now we have the following result about the trajectory attractor of the family of equation (1.1).

THEOREM 3.1. – *Let Σ be a compact metric space and let a continuous translation semigroup $\{T(t), t \geq 0\}$ acts on $\Sigma : T(t)\Sigma \subseteq \Sigma$. Assume, the family $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$, $\mathcal{K}_\sigma^+ \subseteq \mathcal{F}_+^a$, corresponding to the equation (1.1) with symbols $\sigma \in \Sigma$, is tr.-coord. and (Θ_+^{loc}, Σ)-closed. Let there is a uniformly (w.r.t. $\sigma \in \Sigma$) attracting set P for $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ in Θ_+^{loc}, such that P is compact in Θ_+^{loc} and P is bounded in $\mathcal{F}_+^a$. Then the translation semigroup $\{T(t), t \geq 0\}$ acting on $\mathcal{K}_\Sigma^+$ possesses the uniform (w.r.t. $\sigma \in \Sigma$) trajectory attractor $\mathcal{A}_\Sigma \subseteq \mathcal{K}_\Sigma^+ \cap P$ which is strictly invariant:*

$$T(t)\mathcal{A}_\Sigma = \mathcal{A}_\Sigma \, \forall t \geq 0. \tag{3.4}$$

Moreover, we have:

$$\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)}, \tag{3.5}$$

where $\mathcal{A}_{\omega(\Sigma)}$ is the uniform (w.r.t. $\sigma \in \omega(\Sigma)$) *trajectory attractor of the family $\{\mathcal{K}_\sigma^+, \sigma \in \omega(\Sigma)\}$, $\mathcal{A}_{\omega(\Sigma)} \subseteq \mathcal{K}_{\omega(\Sigma)}^+$. The set $\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)}$ is compact in Θ_+^{loc} and bounded in $\mathcal{F}_+^a$.*

The proof of Theorem 3.1 will be given in section 11.

Theorem 3.1 shows that to construct the trajectory attractor one needs a uniformly attracting set P, compact in Θ_+^{loc} and bounded in $\mathcal{F}_+^a$. Usually, in application, a large ball $B_R = \{\|f\|_{\mathcal{F}_+^a} \leq R\}$ in $\mathcal{F}_+^a$ $(R \gg 1)$ serves as such attracting set. The attraction to B_R follows from the inequality:

$$\|T(t)u\|_{\mathcal{F}_+^a} \leq C(\|u\|_{\mathcal{F}_+^a})e^{-\gamma t} + R_0, \quad \gamma > 0, \tag{3.6}$$

for any $u \in \mathcal{K}_\Sigma^+$ and any $t \geq 0$, where $C(R)$ depends on R and R_0 does not depend on u. Usually, inequality (3.6) follows from *a priori* estimates for solutions of equation (1.1). If, in addition, a ball B_R in $\mathcal{F}_+^a$ is compact in Θ_+^{loc} then B_{2R_0} is the required compact uniformly attracting set.

COROLLARY 3.1. – *If $u(s) \in \mathcal{A}_\Sigma$ then $u(s)$ is tr.-c. in Θ_+^{loc}.*

Indeed, using (3.4), we get $T(t)u(s) \in \mathcal{A}_\Sigma$ for any $t \geq 0$, that is, the set $\{T(t)u(s) \mid t \geq 0\}$ is precompact in Θ_+^{loc}, *i.e.* $u(s)$ is tr.-c. in Θ_+^{loc} (*see* Definition 1.1). □

COROLLARY 3.2. – *For any $u_0 \in \mathcal{A}_\Sigma$ there exists a function $\gamma(l), l \in \mathbb{R}, \gamma(l) = (u_l, \sigma_l)$ where $u_l \in \mathcal{A}_\Sigma$, $\sigma_l \in \omega(\Sigma)$ such that $u_l \in \mathcal{K}_{\sigma_l}^+$ and $T(t)\gamma(l) = \gamma(t+l)$, $l \in \mathbb{R}$, $t \geq 0$.*

The proof is given in section 11.

COROLLARY 3.3. – *Assume the family $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ satisfy the following property: for some $R > 0$ the set $B_R \cap \mathcal{K}_\sigma^+ \neq \emptyset$ for all $\sigma \in \Sigma$. Here B_R is a ball in $\mathcal{F}_+^a$ having radius R. Then for any $\sigma \in \omega(\Sigma)$ there exists $u \in \mathcal{A}_\Sigma$ such that $u \in \mathcal{K}_\sigma^+$.*

The proof is given in section 11.

Corollaries 3.2 and 3.3 will be of particular assistance in the next section.

4. Structure of trajectory attractors

Let Σ be a compact symbol space, $\Sigma \Subset \Xi_+ = \{\xi(s), s \geq 0\}$, the semigroup $\{T(t)\}$ is continuous on Σ. Consider any symbol $\sigma \in \omega(\Sigma)$. The invariance property (3.3) implies that there is a symbol $\sigma_1(s), \sigma_1 \in \omega(\Sigma)$ such that $T(1)\sigma_1 = \sigma$. Consider the function $\hat{\sigma}(s), s \geq -1$, $\hat{\sigma}(s) = \sigma_1(s+1)$. Obviously, $\hat{\sigma}(s) \equiv \sigma(s)$ for $s \geq 0$, hence, $\hat{\sigma}(s)$ is a prolongation of $\sigma(s)$ on the semiaxis $[-1, +\infty[$. In such doing, there is $\sigma_2 \in \omega(\Sigma)$ such that $T(1)\sigma_2 = \sigma_1, T(2)\sigma_2 = \sigma$. Put $\hat{\sigma}(s) = \sigma_2(s+2)$ for $s \geq -2$. Evidently, the function $\hat{\sigma}(s)$ is well defined, since $\sigma_2(s+2) = \sigma_1(s+1)$ for $s \geq -1$. Continuing this process, we define $\hat{\sigma}(s) = \sigma_n(s+n)$ for $s \in [-n, +\infty[$, where $\sigma_n \in \omega(\Sigma)$ and $n \in \mathbb{N}$. We have defined a function $\hat{\sigma}(s), s \in \mathbb{R}$, which is a prolongation of the initial symbol $\sigma(s), s \in \mathbb{R}_+$. Moreover, the function $\hat{\sigma}(s)$ satisfies the following property: $\Pi_+\hat{\sigma}_t(s) \in \omega(\Sigma)$ for any $t \in \mathbb{R}$, where $\hat{\sigma}_t(s) = \hat{\sigma}(t+s)$. Here $\Pi_+ = \Pi_{0,\infty}$ is the restriction operator to the semiaxis $\mathbb{R}_+$.

DEFINITION 4.1. – *A function $\zeta(s), s \in \mathbb{R}$, is said to be a complete symbol in $\omega(\Sigma)$ if for any fixed $t \in \mathbb{R}$ the function $\zeta_t(s) = \zeta(t+s)$ has the following property: $\Pi_+\zeta_t(s) \in \omega(\Sigma)$, $(s \in \mathbb{R}_+)$.*

As it was showed above, for any symbol $\sigma \in \omega(\Sigma)$ there exists at least one complete symbol $\zeta(s) = \hat{\sigma}(s), s \in \mathbb{R}$, which is a prolongation of σ for negative s. Notice at once, that, in general, this prolongation need not be unique.

Now consider some complete symbol $\zeta(s), s \in \mathbb{R}$, in $\omega(\Sigma)$. It is easily seen that to $\zeta(s)$ there corresponds the family of operators $A_{\zeta(t)}(\cdot) : E \to E_0$, $A_{\zeta(t)}(\cdot) \equiv A_{\Pi_+\zeta_t}(\cdot)$, $t \in \mathbb{R}$. Consider the corresponding evolution equation on the whole axis:

$$\partial_t u = A_{\zeta(t)}(u), \; t \in \mathbb{R}. \tag{4.1}$$

In section 2 we have defined the set $\mathcal{K}_\zeta^{t_1,t_2}$ of solutions of equation (4.1) on the segment $[t_1, t_2] \in \mathbb{R}_+$ in the class $\mathcal{F}_{t_1,t_2}$. Now we extend this definition on any segment $[t_1, t_2] \subset \mathbb{R}$.

DEFINITION 4.2. – *A function $u(s), s \in \mathbb{R}$, is said to be a complete trajectory of equation (4.1) with the complete symbol $\zeta(s), s \in \mathbb{R}$, if $\Pi_{t_1,t_2}u(s) \in \mathcal{K}_\zeta^{t_1,t_2}$ for any $[t_1, t_2] \subset \mathbb{R}$.*

In section 2 we have introduced the spaces $\mathcal{F}_+^{loc}, \mathcal{F}_+^a$, and Θ_+^{loc}. In the same way, one determines spaces $\mathcal{F}^{loc}, \mathcal{F}^a$, and Θ^{loc}.

DEFINITION 4.3. – **i)** $\mathcal{F}^{loc} = \{f(s), s \in \mathbb{R} \mid \Pi_{t_1,t_2}f(s) \in \mathcal{F}_{t_1,t_2} \; \forall \, [t_1, t_2] \subseteq \mathbb{R}\}$;

ii) $\mathcal{F}^a = \{f(s) \in \mathcal{F}^{loc} \mid \|f\|_{\mathcal{F}^a} < +\infty\}$, where

$$\|f\|_{\mathcal{F}^a} = \sup_{t \in \mathbb{R}} \|\Pi_{0,1}f(t+s)\|_{\mathcal{F}_{0,1}}. \tag{4.2}$$

iii) T*opological space Θ^{loc} coincides (as a set) with $\mathcal{F}^{loc}$ and, by the definition, $f_n(s) \to f(s)$ $(n \to \infty)$ in Θ^{loc} if $\Pi_{t_1,t_2}f_n(s) \to \Pi_{t_1,t_2}f(s)$ $(n \to \infty)$ in Θ_{t_1,t_2} for any $[t_1, t_2] \subseteq \mathbb{R}$.*

DEFINITION 4.4. – *The kernel $\mathcal{K}_\zeta$ in the space $\mathcal{F}^a$ of the equation* (4.1) *with the complete symbol $\zeta(s), s \in \mathbb{R}$, is the union of all complete trajectories $u(s), s \in \mathbb{R}$, of the equation* (4.1) *bounded in $\mathcal{F}^a$ with respect to the norm* (4.2):

$$\|\Pi_{0,1}u(t+s)\|_{\mathcal{F}_{0,1}} \leq C_u, \;\; \forall t \in \mathbb{R}. \tag{4.3}$$

By $Z = Z(\Sigma)$ denote the set of all complete symbols in $\omega(\Sigma)$, $Z = \{\zeta(s), s \in \mathbb{R} \mid \Pi_+\zeta_t(s) \in \omega(\Sigma)\ \forall t \in \mathbb{R}\}$. Evidently, $\Pi_+ Z(\Sigma) = \omega(\Sigma)$. Let $\mathcal{K}_{Z(\Sigma)}$ denote the union of all kernels $\mathcal{K}_\zeta$ corresponding to all complete symbols $\zeta \in Z(\Sigma) : \mathcal{K}_{Z(\Sigma)} = \cup_{\zeta \in Z(\Sigma)} \mathcal{K}_\zeta$.

THEOREM 4.1. – *Let the conditions of Theorem 3.1 be valid. Then*

$$\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)} = \Pi_+ \cup_{\zeta \in Z(\Sigma)} \mathcal{K}_\zeta = \Pi_+ \mathcal{K}_{Z(\Sigma)}, \tag{4.4}$$

the set $\mathcal{K}_{Z(\Sigma)}$ is compact in Θ^{loc} and bounded in $\mathcal{F}^a$. If the family $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ satisfy the condition: for some ball B_R in $\mathcal{F}_+^a$ the set $B_R \cap \mathcal{K}_\sigma^+ \neq \emptyset$ for all $\sigma \in \Sigma$; then $\mathcal{K}_\zeta \neq \emptyset$ for any $\zeta \in Z(\Sigma)$.

Proof. – Let $\zeta \in Z(\Sigma)$ and $u(s) \in \mathcal{K}_\zeta$. Then $\Pi_+ u(t+s) \in \mathcal{K}^+_{\Pi_+\zeta_t}$ and $\Pi_+\zeta_t \in \omega(\Sigma)$. Consider a set $B = \{\Pi_+ u(h+s) \mid h \in \mathbb{R}\} \subseteq \mathcal{F}_+^a$. It is clear that B is bounded in $\mathcal{F}_+^a$ since u has finite norm (4.3). At the same time, the set B belongs to $\mathcal{K}_\Sigma^+$ and it is strictly invariant with respect to the translation semigroup $\{T(t)\}$. On the other hand, $\mathcal{A}_\Sigma$ attracts $T(t)B = B$ as $t \to +\infty$ *i.e.* B belongs to any neighbourhood of $\mathcal{A}_\Sigma$. But $\mathcal{A}_\Sigma$ is a compact set of the Hausdorff space Θ_+^{loc}. Therefore, $B \subseteq \mathcal{A}_\Sigma$, that is, $\mathcal{A}_\Sigma \supseteq \Pi_+\mathcal{K}_{Z(\Sigma)}$. Let us check the inverse inclusion. Let $u_0 \in \mathcal{A}_\Sigma$. Using Corollary 3.2 we construct the function $\gamma(l) = (u_l, \sigma_l), l \in \mathbb{R}$ such that $u_l \in \mathcal{A}_\Sigma$, $\sigma_l \in \omega(\Sigma)$, $u_l \in \mathcal{K}^+_{\sigma_l}$, and for any $t \geq 0$ $(T(t)u_l, T(t)\sigma_l) = T(t)\gamma(l) = \gamma(l+t) = (u_{l+t}, \sigma_{l+t})$ for any $l \in \mathbb{R}$. Put $\zeta(s) = \sigma_s(0)$, $u(s) = u_s(0)$. It follows easily that $\zeta(s) \in Z(\Sigma)$ and $u(s) \in \mathcal{K}_\zeta$. Hence, $u_0 = \Pi_+ u(s) \in \Pi_+\mathcal{K}_{Z(\Sigma)}$, so that $\mathcal{A}_\Sigma \subseteq \Pi_+\mathcal{K}_{Z(\Sigma)}$. Equality (4.4) is proved. Evidently, the set $\mathcal{K}_{Z(\Sigma)}$ is compact in Θ^{loc} because the set $\Pi_+\mathcal{K}_{Z(\Sigma)}$ is compact in Θ_+^{loc}.

We shall prove the second part of the Theorem. Let $\zeta \in Z(\Sigma)$ be any complete symbol, *i.e.* $\Pi_+\zeta(t+s) \in \omega(\Sigma)$ for any $t \in \mathbb{R}$. Corollary 3.3 implies that there is $v_n \in \mathcal{K}^+_{\Pi_+\zeta(-n+s)}$, $v_n \in \mathcal{A}_\Sigma$, for any $n \in \mathbb{N}$. Put $u_n(s) = v_n(n+s)$ for $s \geq -n$. Evidently, $u_n(s)$ is a solution of the equation (4.1) with the symbol $\zeta(s)$ for $s \geq -n$. More precisely, $u_n(s) \in \mathcal{K}_\zeta^{-n,\infty}$. It is not hard to prove that functions $\{v_n(s)\}_{n \geq M}$ form a precompact set in $\Theta^{loc}_{-M,\infty}$ for any $M \geq 0$. Using method of diagonalization, one can choose a subsequence $\{u_{n_i}(s)\}$ of $\{u_n(s)\}$ such that $u_{n_i}(s) \to u(s)$ $(n_i \to \infty)$ in Θ_{t_1,t_2} for some $u(s) \in \mathcal{F}^{loc}$ and for any $[t_1, t_2] \subseteq \mathbb{R}$. It can be proved that $\Pi_+ u(t+s) \in \mathcal{K}^+_{\Pi_+\zeta_t(s)} = \mathcal{K}^+_{\Pi_+\zeta(t+s)}$ for any $t \in \mathbb{R}$, since the family $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ is (Θ_+^{loc}, Σ)-closed. This yields that $u \in \mathcal{K}_\zeta$ and consequently $\mathcal{K}_\zeta \neq \emptyset$. □

Notice that the proof of Theorem 4.1 is rather long because, in general, the translation semigroup $\{T(t)\}$ does not satisfy the backward uniqueness property on Σ, *i.e.* the function $\sigma(s) \in \omega(\Sigma)$ can have different prolongations for negative s.

DEFINITION 4.5. – *The semigroup $\{T(t)\}$ satisfies* the backward uniqueness property on Σ *if the equality $T(t)\sigma_1 = T(t)\sigma_2$ for some $t \geq 0$ implies $\sigma_1 = \sigma_2$.*

If the semigroup $\{T(t)\}$ satisfies the backward uniqueness property on Σ then $\{T(t)\}$ is invertible on $\omega(\Sigma)$. In this case any symbol $\sigma(s) \in \omega(\Sigma)$ $(s \geq 0)$ has a unique prolongation $\hat{\sigma}(s)$ for $s < 0$ to the complete symbol. Therefore, one can identify $\sigma(s)$ with $\hat{\sigma}(s)$ and consider equation (4.1) in the whole axis $\mathbb{R}$ at once. In the next section we shall study equations having the symbol in the whole axis. Analogs of theorems from sections 1-4 will be given.

5. Non-autonomous equations with symbols on the whole time axis.

Consider equation (1.1) with a symbol $\zeta(s), s \in \mathbb{R}$, defined on the whole axis:

$$\partial_t u = A_{\zeta(t)}(u),\ t \in \mathbb{R}. \tag{5.1}$$

As in section 1, we assume that the time symbol $\zeta(s), s \in \mathbb{R}$, as a whole, is an element of the topological space $\Xi = \{\xi(s), s \in \mathbb{R} \mid \xi(s) \in \Psi \text{ for almost all } s \in \mathbb{R}\}$. The space Ξ is similar to Ξ_+. As usually, Ξ is a Hausdorff topological space. Let the translation group $\{T(t),\ t \in \mathbb{R}\}$ acts on Ξ :

$$T(t)\xi(s) = \xi(t+s),\ s \in \mathbb{R},\ t \in \mathbb{R}. \tag{5.2}$$

Let we are given some strictly invariant symbol space $Z \subseteq \Xi$:

$$T(t)Z = Z\ \forall t \in \mathbb{R}. \tag{5.3}$$

Suppose Z is a metrizable complete space. We study the family of equations (5.1) with symbols $\zeta(s)$ from Z.

In applications, symbol spaces appear as follows. We are given a fixed symbol $\zeta_1(s), s \in \mathbb{R},\ \zeta_1 \in \Xi$. Consider its complete hull:

$$\mathcal{H}(\zeta_1) = [\{T(t)\zeta_1(s) \mid t \in \mathbb{R}\}]_\Xi. \tag{5.4}$$

It is clear that $\mathcal{H}(\zeta_1)$ is strictly invariant, *i.e.* $T(t)\mathcal{H}(\zeta_1) = \mathcal{H}(\zeta_1),\ \forall t \in \mathbb{R}$.

DEFINITION 5.1. – *A function $\zeta \in \Xi$ is said to be* translation-compact (tr.-c.) in Ξ *if the complete hull $\mathcal{H}(\zeta)$ is compact in Ξ.*

Let $\zeta_1(s)$ be tr.-c. in Ξ. Consider the symbol space $Z = \mathcal{H}(\zeta_1)$. If Ξ is a Hausdorff space and it possesses a countable topology base then $\mathcal{H}(\zeta_1)$ is metrizable due to Uryson theorem.

Let be given spaces $\mathcal{F}_{t_1,t_2}$ and Θ_{t_1,t_2} for any $[t_1, t_2] \subset \mathbb{R}$, (*see* section 2). Using the usual scheme, we construct the spaces $\mathcal{F}_+^{loc}$, $\mathcal{F}_+^a$, Θ_+^{loc} and the spaces $\mathcal{F}^{loc}$, $\mathcal{F}^a$, Θ^{loc} (*see* section 4). To each symbol $\zeta \in Z$, there corresponds a trajectory space $\mathcal{K}_\zeta^+ \subseteq \mathcal{F}_+^a$. Solutions $u(s), s \geq 0$, from $\mathcal{K}_\zeta^+$ has finite norm (2.6). Like in section 4, consider also the kernel $\mathcal{K}_\zeta$ of the equation (5.1) consisting of the complete trajectories $u(s), s \in \mathbb{R}$, bounded in $\mathcal{F}^a$.

We shall study uniform (w.r.t. $\zeta \in Z$) trajectory attractor of the translation semigroup $\{T(t)\}$ for the family of trajectory spaces $\{\mathcal{K}_\zeta^+, \zeta \in Z\}$ corresponding to equations (5.1). As before, $\mathcal{K}_Z^+ = \cup_{\zeta \in Z}\mathcal{K}_\zeta^+$. Evidently, Propositions 3.1 and 3.2 take place. Notice that $\omega(Z) = Z$ for the translation semigroup $\{T(t)\}$ acting on Z.

Let us formulate the combined analog of Theorems 3.1 and 4.1.

THEOREM. – *Let Z be a compact metric space and let a continuous translation group $\{T(t), t \in \mathbb{R}\}$ acting on $\Sigma : T(t)Z = Z\ \forall t \in \mathbb{R}$. Assume, the family $\{\mathcal{K}_\zeta^+, \zeta \in Z\}$, $\mathcal{K}_\zeta^+ \subseteq \mathcal{F}_+^a$, corresponding to the equation (5.1) with symbols $\zeta \in Z$, is tr.-coord. and (Θ_+^{loc}, Z)-closed. Let there exist a uniformly (w.r.t. $\zeta \in Z$) attracting set P for $\{\mathcal{K}_\zeta^+, \zeta \in Z\}$ in Θ_+^{loc}, such that P is compact in Θ_+^{loc} and P is bounded in $\mathcal{F}_+^a$. Then the translation semigroup*

$\{T(t), t \geq 0\}$ acting on $\mathcal{K}_Z^+$ possesses the uniform (w.r.t. $\zeta \in Z$) trajectory attractor $\mathcal{A}_Z \subseteq \mathcal{K}_Z^+ \cap P$,

$$T(t)\mathcal{A}_Z = \mathcal{A}_Z \; \forall t \geq 0. \tag{5.5}$$

Moreover, we have:

$$\mathcal{A}_Z = \Pi_+ \mathcal{K}_Z = \Pi_+ \cup_{\zeta \in Z} \mathcal{K}_\zeta, \tag{5.6}$$

the set $\mathcal{K}_Z$ is compact in Θ^{loc} and bounded in $\mathcal{F}^a$. In particular, each complete trajectory $u(s), s \in \mathbb{R}$, from $\mathcal{F}^a$ is tr.-c. in Θ^{loc}.

b) *If the family $\{\mathcal{K}_\zeta^+, \zeta \in Z\}$ satisfy the condition: for some ball B_R in $\mathcal{F}_+^a$ the set $B_R \cap \mathcal{K}_\zeta^+ \neq \emptyset$ for all $\zeta \in Z$ then $\mathcal{K}_\zeta \neq \emptyset$ for any $\zeta \in Z$.*

Theorem 5.1 follows from Theorems 3.1 and 4.1. The property (5.3) essentially simplifies the proof of (5.6).

Let us briefly clarify the nature of attraction of bounded set B from $\mathcal{K}_Z^+$ to the trajectory attractor $\mathcal{A}_Z$.

COROLLARY 5.1. – *Under the assumptions of Theorem 5.1, let B be a bounded in $\mathcal{F}_+^a$ set from $\mathcal{K}_Z^+$; then for any $M > 0$ the set $\Pi_{0,M}T(t)B$ tends to $\Pi_{0,M}\mathcal{K}_Z = \Pi_{0,M} \cup_{\zeta \in Z} \mathcal{K}_\zeta$ in the topological space $\Theta_{0,M}$ as $t \to \infty$. For example, if $\Theta_{0,M}$ is a metrizable space then:*

$$\mathbf{dist}_{\Theta_{0,M}}(\Pi_{0,M}T(t)B, \Pi_{0,M}\mathcal{K}_Z) = \mathbf{dist}_{\Theta_{0,M}}(\Pi_{0,M}T(t)B, \Pi_{0,M}\mathcal{A}_Z) \to 0 \; (t \to \infty).$$

Here, as usually, the distance from a set X to a set Y in a metric space $\mathcal{M}$ defines as follows:

$$\mathbf{dist}_{\mathcal{M}}(X, Y) = \sup_{x \in X} \mathbf{dist}_{\mathcal{M}}(x, Y) = \sup_{x \in X} \inf_{y \in Y} \rho_{\mathcal{M}}(x, y),$$

where $\rho_{\mathcal{M}}(x, y)$ denotes the metric in $\mathcal{M}$.

6. Translation-compact functions

In this section we study various classes of tr.-c. functions. We shall present translation-compactness criterions and we shall consider some examples. In sections 1 and 5 we have defined the tr.-c. function on semiaxis $\mathbb{R}_+$ and on the whole axis $\mathbb{R}$. Properties of these functions are close, so we describe in details tr.-c. functions on semiaxis $\mathbb{R}_+$. All statements can be easily extended to the case of tr.-c. functions on $\mathbb{R}$.

1. Tr.-c. functions in $C(\mathbb{R}_+, \mathcal{M})$ and in $C(\mathbb{R}, \mathcal{M})$.

Let $\mathcal{M}$ be a complete metric space with metric $\rho_{\mathcal{M}}(\cdot, \cdot)$. Consider the space $\Xi_+ = C(\mathbb{R}_+, \mathcal{M})$ of continuous functions $f(s), s \in \mathbb{R}_+$, with values in $\mathcal{M}$. The space $C(\mathbb{R}_+, \mathcal{M})$ is equipped with a local uniform convergence topology on any segment of the time semi axis. By the definition, a sequence of functions $\{\sigma_n(t)\}_{n\in\mathbb{N}} \subset C(\mathbb{R}_+,\mathcal{M})$ converges to a function $\sigma(t)$ as $t\to\infty$ in $C(\mathbb{R}_+,\mathcal{M})$ if for any $[t_1,t_2]\subset\mathbb{R}_+$

$$\max_{s\in[t_1,t_2]} \rho_{\mathcal{M}}(\sigma_n(s),\,\sigma(s)) \to 0(n\to+\infty). \tag{6.1}$$

Similarly, one defines the space $\Xi = C(\mathbb{R},\mathcal{M})$. It follows easily that the topological space $C(\mathbb{R}_+,\mathcal{M})$ is metrizable by means of the Fréchet metric

$$\mu_1(\sigma_1,\sigma_2) = \sum_{n=1}^{\infty} a_n \frac{\mu_1^{(n)}(\sigma_1,\sigma_2)}{1+\mu_1^{(n)}(\sigma_1,\sigma_2)}, \tag{6.2}$$

where

$$\mu_1^{(n)}(\sigma_1,\sigma_2) = \max_{s\in[O,\,R_n]} \rho_{\mathcal{M}}(\sigma_n(s),\,\sigma(s)).$$

Here $\{R_n\}$ is any fixed non-decreasing sequence, $R_n\geq 0, R_n \to +\infty$ $(n\to\infty)$ and $\{a_n\}$ is any positive sequence such that $\sum_{n=1}^{\infty} a_n < \infty$. Notice, that the corresponding topology does not depends on sequences $\{R_n\}$, $\{a_n\}$. The metric space $C(\mathbb{R}_+,\mathcal{M})$ with metric (6.2) is complete.

Let $\sigma(s)\in C(\mathbb{R}_+,\mathcal{M})$, $\mathcal{H}_+(\sigma) = [\{\sigma(t+s)\mid t\geq 0\}]_{C(\mathbb{R}_+,\mathcal{M})}$. Bellow we study tr.-c. functions σ in $C(\mathbb{R}_+,\mathcal{M})$, *i.e.* $\mathcal{H}_+(\sigma)$ is compact in $C(\mathbb{R}_+,\mathcal{M})$.

LEMMA 6.1. – *Any tr.-c. function $\sigma(s)$ in $C(\mathbb{R}_+,\mathcal{M})$ is bounded, that is, $\rho_{\mathcal{M}}(\sigma(s),a)\leq R\ \forall s\geq 0$ for some $a\in\mathcal{M}$ and $R\in\mathbb{R}_+$.*

Proof. – Consider the sequence of functions $\sigma_n(s) = \sigma(s+n), n\in\mathbb{Z}_+$, on the segment $[0,1]$. The function $\sigma(s)$ is tr.-c. in $C(\mathbb{R}_+,\mathcal{M})$, therefore the sequence $\{\sigma_n(s)\}$ is precompact in $C([0,1],\mathcal{M})$. Arzelá-Ascoli compactness criterion implies that the sequence $\{\sigma_n(s)\}$ is bounded in $C([0,1],\mathcal{M})$, *i.e.* $\rho_{\mathcal{M}}(\sigma_n(s),a)\leq R\ \forall s\in[0,1]$ for some $a\in\mathcal{M}$ and $R\in\mathbb{R}_+$ or $\rho_{\mathcal{M}}(\sigma(s),a)\leq R\ \forall s\geq 0$. □

By $C_b(\mathbb{R}_+,\mathcal{M})$ (and $C_b(\mathbb{R},\mathcal{M})$) denote the space of bounded continuous functions with uniform convergence topology generated by the following metric:

$$\mu_b(\sigma_1,\sigma_2) = \sup_{s\in\mathbb{R}_+} \rho_{\mathcal{M}}(\sigma_1(s),\sigma_2(s)). \tag{6.3}$$

(To define metric in $C_b(\mathbb{R},\mathcal{M})$ one has to replace $\mathbb{R}_+$ in (6.3) by $\mathbb{R}$.)

PROPOSITION 6.1. – *A function $\sigma(s)$ is tr.-c. in $C(\mathbb{R}_+,\mathcal{M})$ if and only if* **(i)** *the set $\{\sigma(t)\mid t\in\mathbb{R}_+\}$ is precompact in $\mathcal{M}$;* **(ii)** *$\sigma(s)$ is uniformly continuous on $\mathbb{R}_+$, i.e. there exists a positive function $\alpha(s)\to 0$ $(s\to 0+)$ such that*

$$\rho_{\mathcal{M}}(\sigma(t_1),\sigma(t_2)) \leq \alpha(|t_1-t_2|)\ \ \forall t_1,t_2\in\mathbb{R}_+. \tag{6.4}$$

To prove Proposition 6.1 one is to consider the family of functions $\{\sigma(t+s),\ s\in[0,1]\mid t\geq 0\}$ in the space $C([0,1];\mathcal{M})$ and to apply the Arzelá-Ascoli compactness criterion. Let us formulate the main properties of tr.-c. functions in $C(\mathbb{R}_+,\mathcal{M})$.

PROPOSITION 6.2. – *Let $\sigma(s)$ be tr.-c. in $C(\mathbb{R}_+, \mathcal{M})$. Then:*

(i) *any function $\sigma_1 \in \mathcal{H}_+(\sigma)$ is also tr.-c. in $C(\mathbb{R}_+, \mathcal{M})$, moreover, $\mathcal{H}_+(\sigma_1) \subseteq \mathcal{H}_+(\sigma)$ (the inclusion can be strict);*

(ii) *the set $\mathcal{H}_+(\sigma)$ is bounded in $C_b(\mathbb{R}_+, \mathcal{M})$, that is, $\rho_{\mathcal{M}}(\sigma_1(s), a) \leq R \; \forall s \geq 0$ for any $\sigma_1 \in \mathcal{H}_+(\sigma)$, where a and R do not depend on σ_1;*

(iii) *the set $\mathcal{H}_+(\sigma)$ is equicontinuous on $\mathbb{R}_+$, i.e. any function $\sigma_1 \in \mathcal{H}_+(\sigma)$ satisfies (6.4) with one and the same function $\alpha(s)$;*

(iv) *Translation semigroup $\{T(t)\}$ is continuous on $\mathcal{H}_+(\sigma)$ in the topology of $C(\mathbb{R}_+, \mathcal{M})$;*

(v) $T(t)\mathcal{H}_+(\sigma) \subseteq \mathcal{H}_+(\sigma) \; \forall t \geq 0$.

In the same way, one formulates propositions about tr.-c. functions in $C(\mathbb{R}, \mathcal{M})$ by replacing $\mathbb{R}_+$ with $\mathbb{R}$ and $\mathcal{H}_+(\sigma)$ with $\mathcal{H}(\zeta)$. In point **(v)** the translation group $\{T(t), t \in \mathbb{R}\}$ is strictly invariant on $\mathcal{H}(\zeta) : T(t)\mathcal{H}(\zeta) = \mathcal{H}(\zeta) \; \forall t \in \mathbb{R}$.

Let us give an example of tr.-c. function that is not an almost periodic function.

EXAMPLE 6.1. – Let $\zeta(s) \in C_b(\mathbb{R}, \mathcal{M})$, and $\zeta(s) \to \zeta_+ \; (s \to +\infty)$, $\zeta(s) \to \zeta_- \; (s \to -\infty)$ in $\mathcal{M}$, $\zeta_+, \zeta_- \in \mathcal{M}$ $(\zeta_+ \neq \zeta_-)$. Then $\zeta(s)$ is tr.-c. in $C(\mathbb{R}, \mathcal{M})$, besides, $\mathcal{H}(\zeta) = \{\zeta(s+t) \mid t \in \mathbb{R}\} \cup \{\zeta_1(s) \equiv \zeta_+, \zeta_2(s) \equiv \zeta_-\}$. Evidently, $\zeta(s)$ is not an almost periodic function.

In the sequel, we shall need a class of tr.c. functions in $C(\mathbb{R}_+, \mathcal{M}_0)$ with values in a special space $\mathcal{M}_0$. Let $\mathcal{M}_0 = C(\mathbb{R}^N, \mathbb{R}^M)$ be the space of continuous vector-functions $f(v)$ with the domain $\mathbb{R}^N$ and with the range $\mathbb{R}^M$. The space $C(\mathbb{R}^N, \mathbb{R}^M)$ is equipped with a uniform convergence topology on any ball $B_R = \{v \in \mathbb{R}^N \mid |v|_{\mathbb{R}^N} \leq R\}$. So that, by the definition, $f_n(v) \to f(v) \; (n \to \infty)$ in $C(\mathbb{R}^N, \mathbb{R}^M)$ if:

$$(6.5) \qquad \|f_n - f\|_R = \max_{|v| \leq R} |f_n(v) - f_n(v)|_{\mathbb{R}^M} \to 0 \; (n \to \infty),$$

for any $R > 0$. It is easily seen that the above topology is metrizable by the use of the corresponding Fréchet metric.

PROPOSITION 6.3. – *A function $f(v, s) \in C(\mathbb{R}_+; \mathcal{M}_0)$ is tr.-.c in $C(\mathbb{R}_+; \mathcal{M}_0)$ if and only if for any $R > 0$ the function $f(v, s)$ is bounded and uniformly continuous on any semicylinder $Q_+(R) = \{(v, s) \mid v \in B_R, \; s \geq 0\}$, i.e. $|f(v, s)| \leq C(R) \; \forall (v, s) \in Q_+(R)$ and there is a function $\alpha_0(s, R)$, $\alpha_0(s, R) \to 0+ \;\; (t \to 0+)$ such that:*

$$(6.6) \; |f(v_1, s_1) - f(v_2, s_2)|_{\mathbb{R}^M} \leq \alpha_0(|v_1 - v_2| + |s_1 - s_2|, R) \; \forall (v_1, s_1), (v_2, s_2) \in Q_+(R).$$

To prove Proposition 6.3 one uses the following compactness criterion in $\mathcal{M}_0$: A set $\Sigma \Subset \mathcal{M}_0$ iff the set $\Sigma|_{B_R}$ is bounded and equicontinuous on B_R for any $R > 0$, where $|_{B_R}$ denotes the restriction operator on B_R.

2. Tr.-c. functions in $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$ and in $L_p^{loc}(\mathbb{R}; \mathcal{E})$.

Let $\mathcal{E}$ be a Banach space and $p \geq 1$. Consider the space $\Xi_+ = L_p^{loc}(\mathbb{R}_+; \mathcal{E})$ of functions $\sigma(s), s \in \mathbb{R}_+$, with values in $\mathcal{E}$ and $\sigma(s)$ is locally p-power integrable in Bochner sense.

In particular, for any segment $[t_1, t_2] \subset \mathbb{R}_+$

$$\int_{t_1}^{t_2} \|\sigma(s)\|_{\mathcal{E}}^p ds < +\infty.$$

The space $\Xi_+ = L_p^{loc}(\mathbb{R}_+; \mathcal{E})$ is supplemented by the local p-power mean convergence topology, *i.e.*, by the definition, $\sigma_n \to \sigma$ $(n \to \infty)$ in $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$ whenever $\int_{t_1}^{t_2} \|\sigma_n(s) - \sigma(s)\|_{\mathcal{E}}^p ds \to 0$ $(n \to \infty)$ for any $[t_1, t_2] \subset \mathbb{R}_+$. It is easily shown that $\Xi_+ = L_p^{loc}(\mathbb{R}_+; \mathcal{E})$ is a linear countably normable topological space. In particular, $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$ is metrizable and the corresponding metric space is complete. In the same way, one defines the topology in the space $\Xi = L_p^{loc}(\mathbb{R}; \mathcal{E})$.

Let us study tr.-c. functions in $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$. We shall use the compactness criterion in $L_p(0, 1; \mathcal{E})$ which is the generalization of the compactness criterion in $L_p(0, 1; \mathbb{R}^N)$ (*see*, for example, [22], [16]).

PROPOSITION 6.4. – *Let $p \geq 1$. A set Σ is precompact in $L_p(0, 1; \mathcal{E})$ if and only if:*

(i) *for any $[t_1, t_2] \subseteq [0, 1]$ the set $\left\{\int_{t_1}^{t_2} \psi(s)ds \mid \psi \in \Sigma\right\}$ is precompact in $\mathcal{E}$;*

(ii) *there exists a function $\alpha(s)$, $\alpha(s) \to 0+$ $(s \to 0+)$ such that:*

$$\int_{t_1}^{t_2} \|\psi(s) - \psi(s + l)\|_{\mathcal{E}}^p ds \leq \alpha(|l|) \; \forall \psi \in \Sigma.$$

The proof is standard.

Let $\sigma(s) \in L_p^{loc}(\mathbb{R}_+; \mathcal{E})$. Consider the value:

$$\eta_\sigma(h) = \sup_{t \in \mathbb{R}_+} \int_t^{t+h} \|\sigma(s)\|_{\mathcal{E}}^p ds. \tag{6.7}$$

LEMMA 6.2. – *Let $\sigma(s)$ be a tr.-c. function in $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$. Then $\eta_\sigma(h) < +\infty$ for any $h \geq 0$.*

The proof is analogous to one of Lemma 6.1.

For $h = 1$, the formula (6.7) defines the norm in the space $L_p^a(\mathbb{R}_+; \mathcal{E})$:

$$\|\sigma\|_{L_p^a(\mathbb{R}_+;\mathcal{E})}^p = \sup_{t \in \mathbb{R}_+} \int_t^{t+1} \|\sigma(s)\|_{\mathcal{E}}^p ds. \tag{6.8}$$

Lemma 6.2 implies that any tr.-c. function in $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$ belongs to $L_p^a(\mathbb{R}_+; \mathcal{E})$. Similarly to (6.8), one introduces the norm in $L_p^a(\mathbb{R}; \mathcal{E})$.

Proposition 6.4 implies the analogous tr.-c. criterion in $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$.

PROPOSITION 6.5. – *A function $\sigma(s)$ is tr.-c. in $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$ if and only if:*

(i) *for any $h > 0$ the set $\left\{\int_t^{t+h} \sigma(s)ds \mid t \in \mathbb{R}_+\right\}$ is precompact in $\mathcal{E}$;*

(ii) *there exists a function* $\alpha(s)$, $\alpha(s) \to 0+$ $(s \to 0+)$ *such that*

$$\int_t^{t+1} \|\sigma(s) - \sigma(s+l)\|_{\mathcal{E}}^p ds \le \alpha(|l|)\ \forall t \ge 0. \tag{6.9}$$

Let, as usually, $\mathcal{H}_+(\sigma)$ be a hull of σ in $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$.

PROPOSITION 6.6. – *Let* $\sigma(s)$ *be tr.-c. in* $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$. *Then:*

(i) *any function* $\sigma_1 \in \mathcal{H}_+(\sigma)$ *is also tr.-c. in* $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$, *moreover,* $\mathcal{H}_+(\sigma_1) \subseteq \mathcal{H}_+(\sigma)$ *(the inclusion can be strict);*

(ii) *the set* $\mathcal{H}_+(\sigma)$ *is bounded in* $L_p^a(\mathbb{R}_+; \mathcal{E})$, *and* $\eta_{\sigma_1}(h) \le \eta_\sigma(h)$ *for any* $\sigma_1 \in \mathcal{H}_+(\sigma)$;

(iii) *any function* $\sigma_1 \in \mathcal{H}_+(\sigma)$ *satisfies (6.9) with one and the same function* $\alpha(s)$;

(iv) *Translation semigroup* $\{T(t)\}$ *is continuous on* $\mathcal{H}_+(\sigma)$ *in the topology of* $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$;

(v) $T(t)\mathcal{H}_+(\sigma) \subseteq \mathcal{H}_+(\sigma)$ $\forall t \ge 0$.

Let us formulate some convenient sufficient conditions of functions to be tr.-c. in $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$ for particular spaces $\mathcal{E}$. Let $\mathcal{E} = L_2(\Omega)$, where $\Omega \Subset \mathbb{R}^n$ and $Q_{0,1} = \Omega \times [0,1]$. By $H^\delta(Q_{0,1})$ denote the Sobolev space of order $\delta > 0$. Let $\sigma(x,s) \in L_2^{loc}(\Omega \times \mathbb{R}_+) = L_2^{loc}(\mathbb{R}_+; L_2(\Omega))$. Let

$$\|\sigma(x, s+t)\|_{H^\delta(Q_{0,1})} \le M < +\infty\ \forall t \ge 0,$$

where M does not depend on t. Then $\sigma(x,s)$ is tr.–c. in $L_2^{loc}(\mathbb{R}_+; L_2(\Omega))$. This statement follows directly from the Sobolev embedding theorem. To formulate another sufficient condition, we need the following theorem from [11], [19].

THEOREM 6.1. – *Let* $\mathcal{E}_1 \Subset \mathcal{E} \subset \mathcal{E}_0$, *where* $\mathcal{E}_1$ *and* $\mathcal{E}_0$ *are reflexive Banach spaces. Consider the space* $W_{0,1} = \{\psi(s),\ s \in [0,1] \mid \psi(s) \in L_p(0,1;\mathcal{E}_1),\ \psi'(s) \in L_{p_0}(0,1;\mathcal{E}_0)\}$ *with the norm*

$$\|\psi\|_{W_{0,1}} = \left(\int_0^1 \|\psi(s)\|_{\mathcal{E}_1}^p ds\right)^{1/p} + \left(\int_0^1 \|\psi'(s)\|_{\mathcal{E}_0}^{p_0} ds\right)^{1/p_0},$$

where $p, p_0 > 1$. *Then* $W_{0,1} \Subset L_p(0,1;\mathcal{E})$.

Theorem 6.1 implies the following:

PROPOSITION 6.7. – *Let* $\sigma(s) \in L_p^{loc}(\mathbb{R}_+; \mathcal{E}_1)$, $\sigma'(s) \in L_{p_0}^{loc}(\mathbb{R}_+; \mathcal{E}_0)$, $(p, p_0 > 1)$ *and*

$$\int_t^{t+1} \left(\|\sigma(s)\|_{\mathcal{E}_1}^p + \|\sigma'(s)\|_{\mathcal{E}_0}^{p_0}\right) ds \le C\ \forall t \ge 0;$$

then $\sigma(s)$ *is tr.-c. in* $L_p^{loc}(\mathbb{R}_+; \mathcal{E})$.

Usually, in applications, $\mathcal{E} = L_2(\Omega)$, $\mathcal{E}_1 = H^{s_1}(\Omega)$, $\mathcal{E}_0 = H^{s_0}(\Omega)$, where $s_1 > 0$, $s_0 < 0$ and $\Omega \Subset \mathbb{R}^n$.

REMARK 6.1. – *Analogs of Propositions 6.5, 6.6, and 6.7 are valid for the space* $L_p^{loc}(\mathbb{R}; \mathcal{E})$ *if to replace* $\mathbb{R}_+$ *with* $\mathbb{R}$ *and* $\mathcal{H}_+(\sigma)$ *with* $\mathcal{H}(\zeta)$.

3. Tr.-c. functions in $L_{p,w}^{loc}(\mathbb{R}_+;\mathcal{E})$ and in $L_{p,w}^{loc}(\mathbb{R};\mathcal{E})$.

Let $\mathcal{E}$ be a reflexive separable Banach space and $p > 1$. By $L_{p,w}^{loc}(\mathbb{R}_+;\mathcal{E})$ denote the space $L_p^{loc}(\mathbb{R}_+;\mathcal{E})$ endowed with the local weak convergence topology. It is well known that a ball in a reflexive separable Banach is a weakly compact set. This fact implies the following tr.-c. criterion in $\Xi_+ = L_{p,w}^{loc}(\mathbb{R}_+;\mathcal{E})$.

PROPOSITION 6.8. – *Let $\mathcal{E}$ be a reflexive separable Banach space and $p > 1$. A function $\sigma(s) \in L_p^{loc}(\mathbb{R}_+;\mathcal{E})$ is tr.-c. in $L_{p,w}^{loc}(\mathbb{R}_+;\mathcal{E})$ if and only if $\sigma(s)$ is* translation-bounded *in $L_p^{loc}(\mathbb{R}_+;\mathcal{E})$, i.e.*

$$\|\sigma(s)\|_{L_p^a(\mathbb{R}_+;\mathcal{E})}^p = \sup_{t\geq 0}\int_t^{t+1} \|\sigma(s)\|_{\mathcal{E}}^p ds \leq C \ \forall t \geq 0, \tag{6.10}$$

where C does not depend on $t \in \mathbb{R}_+$.

Let $\sigma(s)$ be tr.-c. in $L_{p,w}^{loc}(\mathbb{R}_+;\mathcal{E})$. By $\mathcal{H}_+(\sigma)$ denote the hull of $\sigma(s)$ in $L_{p,w}^{loc}(\mathbb{R}_+;\mathcal{E})$.

LEMMA 6.3. – *The set $\mathcal{H}_+(\sigma)$ being a topological subset of $L_{p,w}^{loc}(\mathbb{R}_+;\mathcal{E})$ is metrizable and the corresponding metric space is complete.*

Lemma 6.3 follows from the fact that a ball in a separable Banach space with the weak topology is metrizable space.

Finally, let us formulate some properties of the translation semigroup $\{T(t)\}$ on $\mathcal{H}_+(\sigma)$.

PROPOSITION 6.9. – *Let $\sigma(s)$ be tr.-c. in $L_{p,w}^{loc}(\mathbb{R}_+;\mathcal{E})$. Then:*

(i) *any function $\sigma_1 \in \mathcal{H}_+(\sigma)$ is also* tr.-c. *in $L_{p,w}^{loc}(\mathbb{R}_+;\mathcal{E})$, moreover, $\mathcal{H}_+(\sigma_1) \subseteq \mathcal{H}_+(\sigma)$;*

(ii) *the set $\mathcal{H}_+(\sigma)$ is bounded in $L_p^a(\mathbb{R}_+;\mathcal{E})$, and $\eta_{\sigma_1}(h) \leq \eta_\sigma(h)$ for any $\sigma_1 \in \mathcal{H}_+(\sigma)$;*

(iii) *Translation semigroup $\{T(t)\}$ is continuous on $\mathcal{H}_+(\sigma)$ in the topology of $L_{p,w}^{loc}(\mathbb{R}_+;\mathcal{E})$;*

(iv) $T(t)\mathcal{H}_+(\sigma) \subseteq \mathcal{H}_+(\sigma)\ \forall t \geq 0.$

The proof is straightforward.

REMARK 6.2. – *Similarly, one constructs the theory of tr.-c. functions in $L_{p,w}^{loc}(\mathbb{R};\mathcal{E})$.*

4. Other tr.-c. functions.

In application we shall use also another spaces Ξ_+ and Ξ except $C(\mathbb{R}_+;\mathcal{M})$, $L_p^{loc}(\mathbb{R}_+;\mathcal{E})$, or $L_{p,w}^{loc}(\mathbb{R}_+;\mathcal{E})$. Sometimes, a symbol $\sigma(s)$ of an equation can be represented in the form: $\sigma(s) = (\sigma^{(1)}(s), \sigma^{(2)}(s))$ (or even with more components), where $\sigma^{(i)}(s)$ are tr.-c. functions in different spaces. For example, $\sigma^{(1)}(s)$ is tr.-c. in $C(\mathbb{R}_+;\mathcal{M})$ and $\sigma^{(2)}(s)$ is tr.-c. in $L_{p,w}^{loc}(\mathbb{R}_+;\mathcal{E})$. It is clear that $\sigma(s)$ is tr.-c. in $\Xi_+ = C(\mathbb{R}_+;\mathcal{M}) \times L_{p,w}^{loc}(\mathbb{R}_+;\mathcal{E})$ and the hull $\mathcal{H}_+(\sigma)$ of σ in Ξ_+ satisfies all the above properties described in Propositions 6.2 and 6.9.

7. Trajectory attractor for hyperbolic equation

In this section we shall apply the above theory to non-autonomous dissipative hyperbolic equations in a domain $\Omega \Subset \mathbb{R}^n$. We study an equation:

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - f(u,t) + g(x,t), \ u|_{\partial\Omega} = 0, \ t \geq 0. \tag{7.1}$$

Here $x \in \Omega \Subset \mathbb{R}^n$ and $\gamma > 0$. The time symbol of this equation is the pair $(f(v,s), g(.,s)) = \sigma(s)$. We assume, $g(x,s) \in L_2^{loc}(\mathbb{R}_+, L_2(\Omega))$ and the function $g(x,s)$ is translation-bounded in $L_2^{loc}(\mathbb{R}_+; L_2(\Omega))$:

$$|g|_a^2 = \|g\|^2_{L_2^b(\mathbb{R}_+;L_2(\Omega))} = \sup_{t \geq 0} \int_t^{t+1} |g(s)|^2 ds < C_0.$$

The non-linear term $f(v,s)$ satisfies the conditions: $f(v,s)$, $f_t'(v,s) \in C(\mathbb{R} \times \mathbb{R}_+)$ and

$$|f(v,s)| \leq \gamma_0(|v|^{p-1} + 1), \ p > 1, \ \gamma_0 > 0; \tag{7.2}$$

$$F(v,s) = \int_o^v f(w,s)dw, \ F(v,s) \geq \gamma_1 |v|^p - C_1; \forall v \in \mathbb{R}, s \in \mathbb{R}_+. \tag{7.3}$$

Besides, we assume that for some segment $I_0 = [\alpha_1, \alpha_2] \subset]0, \gamma[$

$$f(v,s)v \geq \gamma_2 F(v,s) + \frac{1}{\alpha} F_t'(v,s) - C_2, \quad \forall v \in \mathbb{R}, \ s \in \mathbb{R}_+, \tag{7.4}$$

$$0 < \alpha(\gamma - \alpha) < \lambda_1 \ \forall \alpha \in I_0, \tag{7.5}$$

where $\gamma_i > 0$, $C_i > 0$, $i = 0,1,2$. Here λ_1 is the first eigenvalue of the operator $-\Delta u$, $u|_{\partial\Omega} = 0$. Let $C_3 > C_1$ (*i.e.* $F(v,s) + C_3 > 0$) and the function $\Phi(v,s) = (F(v,s) + C_3)^{1/p}$ satisfies ($\beta(s) \geq 0$, $\beta(s) \to 0$ as $s \to 0$)

$$|\Phi(v_1,s_1) - \Phi(v_2,s_2)| \leq C_4(|v_1 - v_2| + \beta|s_1 - s_2|) \ \forall v_1, v_2 \in \mathbb{R}, \ s_i \geq 0 \tag{7.6}$$

Constants γ, γ_i, p, C_i and the interval $I_0 = [\alpha_1, \alpha_2]$ are assumed to be fixed.

Let $u(x,s) \in L_\infty^{loc}(\mathbb{R}_+; L_p(\Omega))$. It follows from (7.2) that $f(u(x,s),s) \in L_\infty^{loc}(\mathbb{R}_+; L_q(\Omega))$, where $1/p + 1/q = 1$. Moreover,

$$\|f(u(x,.),.)\|^q_{L_\infty(t_1,t_2;L_q(\Omega))} \leq \gamma_0' \Big(\|u(x,.)\|^p_{L_\infty(t_1,t_2;L_p(\Omega))} + 1 \Big) \ \forall [t_1,t_2] \subset \mathbb{R}_+. \tag{7.7}$$

On the other hand, if $u(x,s) \in L_\infty^{loc}(\mathbb{R}_+; H_0^1(\Omega))$ than $\Delta u(x,s) \in L_\infty^{loc}(\mathbb{R}_+; H^{-1}(\Omega))$ and $\Delta u(x,s) + g(x,s) \in L_2^{loc}(\mathbb{R}_+; H^{-1}(\Omega))$. So, if $p \leq 2$ then the right-hand side of equation (7.1) belongs to $L_2^{loc}(\mathbb{R}_+; H^{-1}(\Omega))$. Consider the case $p > 2$. The Sobolev embedding theorem implies, passing to the conjugate spaces, that $L_q(\Omega) \subset H^{-r}(\Omega)$, where $r \geq n(1/q - 1/2)$. If, in addition $r \geq 1$, then the right-hand side of (7.1) belongs to $L_2^{loc}(\mathbb{R}_+; H^{-r}(\Omega))$. Put $r \equiv \max\{1, n(1/q - 1/2)\}$ for any $p > 1$. We conclude, if $u(x,s) \in L_\infty^{loc}(\mathbb{R}_+; L_p(\Omega)) \cap L_\infty^{loc}(\mathbb{R}_+; H_0^1(\Omega))$ then equation (7.1) can be considered in the distribution sense of the space $D'(\mathbb{R}_+; H^{-r}(\Omega))$.

Notice at once that the number p can be arbitrary large.

DEFINITION 7.1. – *A function $u(x,s)$, $x \in \Omega$, $s \geq 0$, is said to be a weak solution of equation (7.1) if $u(x,s) \in L_\infty^{loc}(\mathbb{R}_+; L_p(\Omega)) \cap L_\infty^{loc}(\mathbb{R}_+; H_0^1(\Omega))$, $\partial_t u(x,s) \in L_\infty^{loc}(\mathbb{R}_+; L_2(\Omega))$ and $u(x,s)$ satisfies equation (7.1) in the distribution sense of the space $D'(\mathbb{R}_+; H^{-r}(\Omega))$, where $r \equiv \max\{1, n(1/q - 1/2)\}$ (see [19]).*

If $u(x,s)$ is a weak solution of (7.1) then, evidently, $u \in C(\mathbb{R}_+; L_2(\Omega))$ and $\partial_t u(x,s) \in C(\mathbb{R}_+; H^{-r}(\Omega))$.

LEMMA 7.1. – **(i)** (Lions-Magenes [18]) *Let X and Y be Banach spaces, such that $Y \subset X$ with a continuous injection. If $f(s) \in C([t_1 t_2]; X)$ and $f(s) \in L_\infty([t_1, t_2]; Y)$ then $f(s)$ is weakly continuous on $[t_1, t_2]$ with values in Y, i.e. for any $\psi \in Y^*$ the function $\langle \psi, f(t) \rangle \in C([t_1, t_2])$.*

(ii) *The function $\|f(s)\|_Y$ is lower semi-continuous on $[t_1, t_2]$, i.e. $\|f(t)\|_Y \leq \liminf_{s \to t} \|f(s)\|_Y$ for any $t \in [t_1, t_2]$.*

Indeed, if $f(s_n) \rightharpoonup f(t)$ $(s_n \to t)$ weakly in Y then $\|f(t)\|_Y \leq \liminf_{s_n \to t} \|f(s_n)\|_Y$.

COROLLARY 7.1. – *Let $u(s)$ be a weak solution of (7.1) then*

$$u \in C_w(\mathbb{R}_+; H_0^1(\Omega)),\ u \in C_w(\mathbb{R}_+; L_p(\Omega)),\ \partial_t u(x,s) \in C_w(\mathbb{R}_+; L_2(\Omega)), \tag{7.8}$$

moreover, for any $\alpha \in \mathbb{R}$ the function

$$\|u(s)\|^2 + |\partial_t u(s) + \alpha u(s)|^2 + \|u(s)\|_{L_p(\Omega)}^p \tag{7.9}$$

is lower semi-continuous on $\mathbb{R}_+$. Here and below $\|\,.\,\|$, $|\,.\,|$ denote the usual norms in $H_0^1(\Omega)$ and $L_2(\Omega)$ respectively.

Proof. – Property (7.8) follows directly from the part **(i)** of Lemma 7.1. The expression $(\|v\|^2 + |p + \alpha v|^2)^{1/2}$ defines an equivalent norm in $H_0^1(\Omega) \times L_2(\Omega)$. So, by the part **(ii)** of Lemma 7.1, $\|u(s)\|^2 + |\partial_t u(s) + \alpha u(s)|^2$ is lower semi-continuous on $\mathbb{R}_+$. For the same reason, $\|u(s)\|_{L_p(\Omega)}$ and $\|u(s)\|_{L_p(\Omega)}^p$ are lower semi-continuous on $\mathbb{R}_+$. The sum of semi-continuous functions is a semi-continuous function as well, so that, property (7.9) is valid. □

Let $v(x) \in H_0^1(\Omega) \cap L_p(\Omega)$ and $p(x) \in L_2(\Omega)$. Consider the nonlinear functional:

$$J_\alpha(v,p,s) = \int_\Omega \big(|\nabla v(x)|^2 + |p(x) + \alpha v(x)|^2 + 2F(v(x),s) \big) dx. \tag{7.10}$$

Due to (7.2) and (7.3), we get:

$$\begin{aligned} \|v\|^2 + |p + \alpha v|^2 + 2\gamma_1 \|v\|_{L_p(\Omega)}^p - C_1' &\leq J_\alpha(v,p,s) \\ &\leq \|v\|^2 + |p + \alpha v|^2 + 2\gamma_0'' \|v\|_{L_p(\Omega)}^p + C_0'. \end{aligned} \tag{7.11}$$

COROLLARY 7.2. – *Let $u(s)$ be a weak solution of (7.1) then the function*

$$z(s) = z_\alpha(u(\,.\,), s) \stackrel{\text{def}}{=} J_\alpha(u(s), \partial_t u(s), s) \tag{7.12}$$

is lower semi-continuous on $\mathbb{R}_+$.

Proof. – It is sufficient to prove that the function $\int_\Omega F(u(x,s),s)dx$ is lower semi-continuous. Consider the function $\phi(x,s) = (F(u(x,s),s) + C_3)^{1/p}$. Using (7.2), we get $\phi(x,s) \in L_\infty^{loc}(\mathbb{R}_+; L_p(\Omega))$. Taking into account (7.6), we have $\phi(x,s) \in C(\mathbb{R}_+; L_2(\Omega))$. Indeed, according to (7.6)

$$\int_\Omega |\phi(x,s_1) - \phi(x,s_2)|^2 dx \le C_4^2 \left(\int_\Omega |u(x,s_1) - u(x,s_2)|^2 dx + \beta^2 |s_1 - s_2| \right). \tag{7.13}$$

But $u(s) \in C(\mathbb{R}_+; L_2(\Omega))$ and therefore the right-hand side of (7.13) tends to zero as $s_2 \to s_1$. This mean that $\phi(x,s) \in C(\mathbb{R}_+; L_2(\Omega))$. If $p \le 2$ then, evidently, $\phi(x,s) \in C(\mathbb{R}_+; L_p(\Omega))$. If $p > 2$ then, by the part **(i)** of Lemma 7.1 $\phi(x,s) \in C_w(\mathbb{R}_+; L_p(\Omega))$ since $\phi(x,s) \in L_\infty^{loc}(\mathbb{R}_+; L_p(\Omega))$. Finally it follows from the part **(ii)** of Lemma 7.1 that the function $\|\phi(s)\|^p_{L_p(\Omega)}$ is lower semi-continuous. To conclude the proof notice that $\int_\Omega F(u(x,s),s)dx = \|\phi(s)\|^p_{L_p(\Omega)} - C_3\mu(\Omega)$. □

REMARK 7.1. – *For $\alpha = 0$ the functional $J_0(v,p,s)$ coincides with the energy-type integral of equation (7.1).*

Now we shall derive formally some differential inequality for the function $z(s)$. Later on we shall use this inequality to prove *a priori* estimate for the Faedo-Galerkin approximations of a solution of equation (7.1). Multiplying both sides of (7.1) by $\partial_t u(s) + \alpha u(s)$ and integrating over Ω, we obtain after standard formal calculations:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|\partial_t u(t) + \alpha u(t)|^2 + \|u(t)\|^2) \\ &\quad + (\gamma - \alpha)|\partial_t u(t) + \alpha u(t)|^2 - (\gamma - \alpha)\alpha(\partial_t u(t) + \alpha u(t), u) \\ &\quad + \alpha\|u(t)\|^2 + (f(u,t), \partial_t u(t) + \alpha u(t)) = (g, \partial_t u(t) + \alpha u(t)). \end{aligned} \tag{7.14}$$

Owing to (7.3) and (7.4), we get for $\alpha \in I_0$

$$\begin{aligned} &(f(u,t), \partial_t u(t) + \alpha u(t)) \\ &= \frac{d}{dt}\int_\Omega F(u(x,t),t)dx + \alpha(f(u,t),u(t)) - \int_\Omega F'_t(u(x,t),t)dx \\ &= \frac{d}{dt}\int_\Omega F(u,t)dx \\ &+ \alpha\gamma_2 \int_\Omega F(u,t)dx + \alpha \int_\Omega \left(f(u,t)u - \gamma_2 F(u,t) - \frac{1}{\alpha}F'_t(u(x,t),t) \right) dx \\ &\ge \frac{d}{dt}\int_\Omega F(u(t),t)dx + \alpha\gamma_2 \int_\Omega F(u(t),t)dx - \alpha C_2 \mu(\Omega). \end{aligned} \tag{7.15}$$

Combining (7.14) and (7.15) we obtain

$$\frac{1}{2}\frac{d}{dt}\left(|v(t)|^2 + \|u(t)\|^2 + 2\int_\Omega F(u(t),t)dx\right) + (\gamma-\alpha)|v(t)|^2 + \alpha\|u(t)\|^2$$
$$+ \alpha\gamma_2 \int_\Omega F(u(t),t)dx - (\gamma-\alpha)\alpha(v(t),u(t)) \le \alpha C_5 + (g(t),v(t)),$$

where $v = \partial_t u(t) + \alpha u(t)$ and $C_5 = C_2\mu(\Omega)$. We have: $(\gamma-\alpha)\alpha(v,u) \le (\gamma-\alpha)|v|^2/4 + (\gamma-\alpha)\alpha^2|u|^2$, $(g,v) \le (\gamma-\alpha)|v|^2/4 + |g|^2/(\gamma-\alpha)$. Consequently,

$$(7.16)\quad \frac{d}{dt}\left(|v(t)|^2 + \|u(t)\|^2 + 2\int_\Omega F(u(t),t)dx\right) + (\gamma-\alpha)|v(t)|^2 + 2\alpha\|u(t)\|^2$$
$$+ \alpha\gamma_2 \int_\Omega F(u(t),t)dx - 2(\gamma-\alpha)\alpha^2|u|^2 \le \frac{1}{\gamma-\alpha}|g|^2 + \alpha C_5.$$

Since $|u|^2 \le \|u\|^2/\lambda_1$, inequality (7.16) implies:

$$(7.17)\qquad \frac{d}{dt}z(t) + \delta_\alpha z(t) \le \frac{2}{\gamma-\alpha}|g(t)|^2 + \rho_\alpha, \quad \forall \alpha \in I_0,$$

where, owing to (7.5),

$$(7.18)\qquad z(t) = J_\alpha(u(s),\partial_t u(s),s) = |v(t)|^2 + \|u(t)\|^2 + 2\int_\Omega F(u(x,t),t)dx,$$

$$(7.19)\qquad \delta_\alpha = \min\{\gamma-\alpha,\ 2\alpha(1-(\gamma-\alpha)\alpha/\lambda_1),\ \alpha\gamma_2/2\},\ \delta_\alpha > 0,$$

$$(7.20)\qquad \rho_\alpha = \alpha C_5 + (\alpha\gamma_2/2 - \delta_\alpha)C_1\mu(\Omega).$$

PROPOSITION 7.1. – *If a function $z(t)$ satisfies (7.17) then*

$$(7.21)\qquad z(t) \le R_\alpha + z(0)e^{-\delta t},\ R_\alpha = \rho_\alpha\delta^{-1} + 2(\gamma-\alpha)^{-1}(1+\delta^{-1})|g|_a^2,\ \delta = \delta_\alpha.$$

Proof. – It follows from inequality (7.17) that

$$z(t)e^{\delta t} - z(0) \le \int_0^t \left(\rho_\alpha + \frac{2}{\gamma-\alpha}|g(s)|^2\right)e^{\delta s}ds = \rho_\alpha\frac{e^{\delta t}-1}{\delta} + \frac{2}{\gamma-\alpha}\int_0^t |g(s)|^2 e^{\delta s}ds.$$

For the last integral we have:

$$\int_0^t |g(s)|^2e^{\delta s}ds = \int_{t-1}^t |g(s)|^2e^{\delta s}ds + \int_{t-2}^{t-1} |g(s)|^2e^{\delta s}ds + \ldots$$
$$\le e^{\delta t}\int_{t-1}^t |g(s)|^2ds + e^{\delta(t-1)}\int_{t-2}^{t-1} |g(s)|^2ds + \ldots$$
$$\le e^{\delta t}(1 + e^{-\delta} + e^{-2\delta} + \ldots)|g|_a^2 = (1-e^{-\delta})^{-1}|g|_a^2e^{\delta t} \le (1+\delta^{-1})|g|_a^2e^{\delta t}.$$

Hence, $z(t)e^{\delta t} - z(0) \le (\rho_\alpha\delta^{-1} + 2(\gamma-\alpha)^{-1}(1+\delta^{-1})|g|_a^2)e^{\delta t}$, and finally $z(t) \le R_\alpha + z(0)e^{-\delta t}$. □

Let be given a fixed symbol $\sigma_0(s) = (f_0(v,s), g_0(x,s))$. The function $g_0(x,s)$ is translation-bounded in $L_2^{loc}(\mathbb{R}_+; L_2(\Omega))$. $\mathcal{H}_+(g_0)$ is the hull of the function g_0 in the space $L_{2,w}^{loc}(\mathbb{R}_+; L_2(\Omega))$. The function $g_0(x,s)$ is tr.-c. in $L_{2,w}^{loc}(\mathbb{R}_+; L_2(\Omega))$ and $\mathcal{H}_+(g_0) \Subset L_{2,w}^{loc}(\mathbb{R}_+; L_2(\Omega))$ (*see* section 6 subsection 3).

Let the function $f_0(v,s)$ satisfy conditions (7.2)-(7.6). Consider the space $\mathcal{M}_0 = \{(\psi(v), \psi_1(v)),\ v \in \mathbb{R} \mid (\psi, \psi_1) \in C(\mathbb{R}; \mathbb{R}^2)\}$ endowed with the following local uniconvergence topology. By the definition $(\psi^{(m)}, \psi_1^{(m)}) \to (\psi, \psi_1)$ $(m \to +\infty)$ in $\mathcal{M}_0$ if

$$\max_{|v|\le R} \left(|\psi^{(m)}(v) - \psi(v)| + |\psi_1^{(m)}(v) - \psi_1(v)| \right) \to 0\,(n \to +\infty)$$

for any $R > 0$. Evidently, the space $\mathcal{M}_0$ is metrizable by the Fréchet metric and the corresponding metric space is complete. Consider the space $C(\mathbb{R}_+, \mathcal{M}_0)$ of continuous functions with values in $\mathcal{M}_0$. Let $(f_0(v,s), f'_{0t}(v,s))$ be a tr.-c. function in $C(\mathbb{R}_+, \mathcal{M}_0)$.

By the tr.-c. criterion (*see* Proposition 6.3), the function $(f_0(v,s), f'_{0t}(v,s))$ is tr.c. in $C(\mathbb{R}_+, \mathcal{M}_0)$ if and only if for any $R > 0$ the function $(f_0(v,s), f'_{0t}(v,s))$ is bounded and uniformly continuous in the semi-cylinder $Q_+(R) = \{(v,s) \mid |v| \le R,\ s \ge 0\}$, *i.e.*

$$\begin{aligned}(7.22)\qquad & |f_0(v,s)| + |f'_0(v,s)| \le C_7(R)\ \forall (v,s) \in Q_+(R);\\ & |f_0(v_1,s_1) - f_0(v_2,s_2)| + |f'_{0t}(v_1,s_1) - f'_{0t}(v_2,s_2)|\\ & \quad \le \beta(|v_1 - v_2| + |s_1 - s_2|, R)\\ & \forall (v_1,s_1),(v_2,s_2) \in Q_+(R);\ \beta(s,R) \to 0+\ \ (s \to 0+).\end{aligned}$$

Let $\mathcal{H}_+(f_0)$ be the hull of the function $(f_0(s), f'_{0t}(s))$ in the space $C(\mathbb{R}_+, \mathcal{M}_0)$. For brevity sake, we shall write f_0 and f instead of (f_0, f'_{0t}) and (f, f'_t).

PROPOSITION 7.2. – *Any function $f \in \mathcal{H}_+(f_0)$ satisfies the conditions (7.2)- (7.6) with one and the same constants.*

Proof follows directly from (7.22).

Evidently, the symbol $\sigma_0(s) = (f_0(v,s), g_0(x,s))$ is a tr.-c. function in $\Xi_+ = C(\mathbb{R}_+, \mathcal{M}_0) \times L_{2,w}^{loc}(\mathbb{R}_+; L_2(\Omega))$.

Now consider the symbol space Σ of equation (7.1): $\Sigma = \mathcal{H}_+(\sigma_0)$, where $\mathcal{H}_+(\sigma_0)$ is the hull of the function $\sigma_0(s)$ in Ξ_+.

PROPOSITION 7.3. – *For any symbol $\sigma(s) = (f(v,s), g(x,s)) \in \mathcal{H}_+(\sigma_0)$*

i) $|g|_a^2 = \sup_{t\ge 0} \int_t^{t+1} |g(s)|^2 ds \le |g_0|_a^2$; **ii)** $f(v,s)$ *satisfies conditions* (7.2)- (7.6) *with one and the same constants.*

The proof follows from Proposition 6.9 and Proposition 7.2, since $\mathcal{H}_+(\sigma_0) \subseteq \mathcal{H}_+(f_0) \times \mathcal{H}_+(g_0)$

To any symbol $\sigma(s) = (f(v,s), g(x,s)) \in \mathcal{H}_+(\sigma_0)$, there corresponds the equation (7.1). We fix a number $M > 0$. Let us define a trajectory space $\mathcal{K}_\sigma^+(M)$ of the equation (7.1).

DEFINITION 7.2. – *The space $\mathcal{K}_\sigma^+(M)$ is the union of all weak solutions $u(s)$ of equation (7.1) (see Definition 7.1)) that satisfy the following property: for any positive $\alpha \in I_0$*

$$(7.23)\qquad z(t) \le R_\alpha(\sigma_0) + M \exp(-\delta_\alpha t)\ \forall t \ge 0,$$

where $z(t) = J_\alpha(u(s), \partial_t u(s), s)$ *(see (7.18)),* $R_a(\sigma_0) = \rho_\alpha \delta_\alpha^{-1} + 2(\gamma - \alpha)^{-1}(1 + \delta_\alpha^{-1})C_0$, $\delta_\alpha = \delta_\alpha(\sigma_0)$ *is determined in (7.19).*

PROPOSITION 7.4. – *Let* $u_0(x) \in H_0^1(\Omega) \cap L_p(\Omega)$, $p_0(x) \in L_2(\Omega)$, *and* $z_0 = J_\alpha(u_0, p_0, 0) \leq M$ *for any* $\alpha \in I_0$; *then for any* $\sigma \in \mathcal{H}_+(\sigma_0)$, $\sigma(s) = (f(v,s), g(x,s))$, *there exist at least one trajectory* $u(s) \in \mathcal{K}_\sigma^+(M)$ *such that*

$$u|_{t=0} = u_0(x),\ \partial_t u|_{t=0} = p_0(x). \tag{7.24}$$

Proof. – We construct $u(s) \in \mathcal{K}_\sigma^+(M)$ using the Faedo-Galerkin method [19]. Let $u_m(t) = \sum_{i=1}^m a_{j,m}(s) w_j$ be the Galerkin approximation, satisfying the following ordinary differential system

$$\partial_t^2 u_m + \gamma \partial_t u_m = P_m \Delta u_m - P_m f(u_n, t) + P_m g(x,t), \tag{7.25}$$

with the initial conditions

$$u_m|_{t=0} = u_{0m}(x),\ \partial_t u_m|_{t=0} = p_{0m}(x),$$

where P_m is the orthogonal projector from $L_2(\Omega)$ onto the linear span of functions $\{w_1(x), w_2(x), \ldots, w_m(x)\}$. Here $\{w_j(x)\}_{j\in\mathbb{N}}$ is a complete system of functions in $H_0^1(\Omega) \cap L_p(\Omega)$. We assume that $u_{0m}(x) \to u_0(x)$ $(m \to \infty)$ strongly in $H_0^1(\Omega) \cap L_p(\Omega)$ and $p_{0m}(x) \to p_0(x)$ $(m \to \infty)$ strongly in $L_2(\Omega)$. It is easy to prove that:

$$z_m(0) = J_\alpha(u_{0m}, p_{0m}, 0) \to J_\alpha(u_0, p_0, 0) = z_0\ (m \to \infty), \tag{7.26}$$

(*see* (7.2), (7.4), and (7.10)). The formulas (7.14)-(7.18) are correct for the functions $u_m(s)$. This is why, according to Proposition 7.1, the function $u_m(s)$ satisfies the inequality

$$z_m(t) \leq R_\alpha(\sigma_0) + z_m(0)e^{-\delta_\alpha t}\ \forall t \geq 0\ \forall \alpha \in I_0. \tag{7.27}$$

Using estimates (7.7), (7.11), and (7.27), we conclude that the sequence $\{u_m(s)\}$ is bounded in $L_\infty^a(\mathbb{R}_+; L_p(\Omega)) \cap L_\infty^a(\mathbb{R}_+; H_0^1(\Omega))$, $\{\partial_t u_m(x,s)\}$ is bounded in $L_\infty^a(\mathbb{R}_+; L_2(\Omega))$, $\{f(u_m(s), s)\}$ is bounded in $L_\infty^a(\mathbb{R}_+; L_q(\Omega))$, $\{(F(u_m(s), s) + C_3)^{1/p}\}$ is bounded in $L_\infty^a(\mathbb{R}_+; L_p(\Omega))$, and $\{\partial_t^2 u_m(x,s)\}$ is bounded in $L_2^a(\mathbb{R}_+; H^{-r}(\Omega))$.

Passing to a subsequence (which we label the same), we get: there exist a function $u(s) \in L_\infty^a(\mathbb{R}_+; L_p(\Omega)) \cap L_\infty^a(\mathbb{R}_+; H_0^1(\Omega))$, $\partial_t u(x,s) \in L_\infty^a(\mathbb{R}_+; L_2(\Omega))$, $\partial_t^2 u(x,s) \in L_2^a(\mathbb{R}_+; H^{-r}(\Omega))$ such that for any $[t_1, t_2] \in \mathbb{R}_+$, one has: $u_m(s) \rightharpoonup u(s)$ $(m \to \infty)$ $*$-weakly in $L_\infty(t_1, t_2; H_0^1(\Omega))$, and in $L_\infty(t_1, t_2; L_p(\Omega))$, $\partial_t u_m(s) \rightharpoonup \partial_t u(s)$ $(m \to \infty)$ $*$-weakly in $L_\infty(t_1, t_2; L_2(\Omega))$, $f(u_m(s), s) \rightharpoonup f(u(s), s)$ $(m \to \infty)$ $*$-weakly in $L_\infty(t_1, t_2; L_q(\Omega))$, $(F(u_m(s), s) + C_3)^{1/p} \rightharpoonup (F(u(s), s) + C_3)^{1/p}$ $(m \to \infty)$ $*$-weakly in $L_\infty(t_1, t_2; L_q(\Omega))$, and $\partial_t^2 u_m(s) \rightharpoonup \partial_t^2 u(s)$ $(m \to \infty)$ weakly in $L_2(t_1, t_2; H^{-r}(\Omega))$.

Passing to the limit in equation (7.25) we obtain that $u(s)$ is a weak solution of equation (7.1). To prove that: $u(s) \in \mathcal{K}_\sigma^+(M)$ we have to check (7.1) for $u(s)$. First of all we claim that:

$$\operatorname*{ess\,sup}_{s\in[t,t_2]} z(s) \leq \liminf_{m\to\infty} \operatorname*{ess\,sup}_{s\in[t,t_2]} z_m(s).$$

Indeed, using the above limit relations, we get:

$$\operatorname*{ess\,sup}_{s\in[t,t_2]}\int_\Omega (F(u(s),s)+C_3)dx \le \liminf_{m\to\infty}\operatorname*{ess\,sup}_{s\in[t,t_2]}\int_\Omega (F(u_m(s),s)+C_3)dx.$$

Similarly, we obtain:

$$\begin{aligned}&\operatorname*{ess\,sup}_{s\in[t,t_2]}\left(\|u(s)\|^2+|\partial_t u(s)+\alpha u(s)|^2\right)\\ \le\ &\liminf_{m\to\infty}\operatorname*{ess\,sup}_{s\in[t,t_2]}\left(\|u_m(s)\|^2+|\partial_t u_m(s)+\alpha u_m(s)|^2\right).\end{aligned}$$

So that, by virtue of (7.26) and (7.27),

$$\operatorname*{ess\,sup}_{s\in[t,t_2]} z(s)\le \liminf_{m\to\infty}\operatorname*{ess\,sup}_{s\in[t,t_2]} z_m(s)\le R_\alpha(\sigma_0)+Me^{-\delta_\alpha t}.$$

Since $z(s)$ is lower semi-continuous, $z(t)\le \operatorname*{ess\,sup}_{s\in[t,t_2]} z(s)\le R_\alpha(\sigma_0)+Me^{-\delta_\alpha t}$ and, finally, $u(s)\in\mathcal{K}_\sigma^+(M)$. □

PROPOSITION 7.5. – *The family* $\{\mathcal{K}_\sigma^+(M),\sigma\in\mathcal{H}_+(\sigma_0)\}$ *is tr.-coord. i.e.* $T(h)u\in\mathcal{K}_{T(h)\sigma}^+(M)$, $h\ge 0$, *for any* $u\in\mathcal{K}_\sigma^+(M)$.

Proof. – Let $u(s)\in\mathcal{K}_\sigma^+(M)$, $\sigma(s)=(f(v,s),g(x,s))$. Then, evidently, $u(s+h)\in\mathcal{K}_{\sigma_h}^+(M)$ is a solution of equation (7.1) with the symbol $\sigma_h(s)=\sigma(s+h)=(f(v,s+h),g(x,s+h))$. Since for $u(s)$

$$z(t)\le R_\alpha(\sigma_0)+Me^{-\delta_\alpha t}\ \forall t\ge 0\ \forall\alpha\in I_0,$$

then for $T(h)u(s)=u(s+h)$

$$T(h)z(t)=z(t+h)\le R_\alpha(\sigma_0)+Me^{-\delta_\alpha(t+h)}\le R_\alpha(\sigma_0)+Me^{-\delta_\alpha t}\ \forall t\ge 0,\ h\ge 0.\ \square$$

Let us describe the spaces $\mathcal{F}_+^{loc}$, $\mathcal{F}_+^a$, and Θ_+^{loc} for equation (7.1). By the definition, $\mathcal{F}_{t_1,t_2}=\{v(s),s\in\mathbb{R}_+\mid v\in L_\infty(t_1,t_2;L_p(\Omega)\cap H_0^1(\Omega)),\ \partial_t v\in L_\infty(t_1,t_2;L_2(\Omega)),\ \partial_t^2 v\in L_2(t_1,t_2;H^{-r}(\Omega))\ \}$. Denote by Θ_{t_1,t_2} the space $\mathcal{F}_{t_1,t_2}$ with the following convergence topology. By the definition, a sequence $\{v_m\}\subset\mathcal{F}_{t_1,t_2}$ converges to $v\in\mathcal{F}_{t_1,t_2}$ as $t\to\infty$ in Θ_{t_1,t_2} if $v_m(s)\rightharpoonup v(s)$ $(m\to\infty)$ $*$-weakly in $L_\infty(t_1,t_2;H_0^1(\Omega))$, $*$-weakly in $L_\infty(t_1,t_2;L_p(\Omega))$, $\partial_t v_m(s)\rightharpoonup\partial_t v(s)$ $(m\to\infty)$ $*$-weakly in $L_\infty(t_1,t_2;L_2(\Omega))$, and $\partial_t^2 v_m(s)\rightharpoonup\partial_t^2 v(s)$ $(m\to\infty)$ weakly in $L_2(t_1,t_2;H^{-r}(\Omega))$. It is easily seen that $\mathcal{F}_{t_1,t_2}$ is a Hausdorff and Fréchet-Uryson space with a countable topology base. Spaces $\mathcal{F}_{t_1,t_2}$ and Θ_{t_1,t_2} generate $\mathcal{F}_+^{loc}$, $\mathcal{F}_+^a$, and Θ_+^{loc}. Evidently, $\mathcal{F}_+^{loc}=L_\infty^{loc}(\mathbb{R}_+;L_p(\Omega)\cap H_0^1(\Omega))\cap\{\ \partial_t v\in L_\infty^{loc}(\mathbb{R}_+;L_2(\Omega))\}\cap\{\ \partial_t^2 v\in L_2^{loc}(\mathbb{R}_+;H^{-r}(\Omega))\}$, $\mathcal{F}_+^a=L_\infty(\mathbb{R}_+;L_p(\Omega)\cap H_0^1(\Omega))\cap\{\ \partial_t v\in L_\infty(\mathbb{R}_+;L_2(\Omega))\}\cap\{\ \partial_t^2 v\in L_2^a(\mathbb{R}_+;H^{-r}(\Omega))\}$, and $v_m(s)\rightharpoonup v(s)$ $(m\to\infty)$ in Θ_+^{loc} if $\Pi_{t_1,t_2}v_m(s)\rightharpoonup\Pi_{t_1,t_2}v(s)$ $(m\to\infty)$ in Θ_{t_1,t_2} for any $[t_1,t_2]\in\mathbb{R}_+$.

PROPOSITION 7.6. – *For any* $M\ge 0$ *the trajectory space* $\mathcal{K}_\sigma^+(M)$ *is bounded in* $\mathcal{F}_+^a$ *for any symbol* $\sigma\in\mathcal{H}_+(\sigma_0)$.

The proof follows directly from (7.23) and (7.11). Indeed, if $u(s) \in \mathcal{K}_\sigma^+(M)$ then:

$$\|u(t)\|^2 + |\partial_t u(t) + \alpha u(t)|^2 + 2\gamma_1 \|u(t)\|_{L_p(\Omega)}^p - C_1' \le z(t) \le R_\alpha(\sigma_0) + M \tag{7.28}$$

$$\forall t \ge 0.$$

It follows from equation (7.1) and estimate (7.7) that $\partial_t^2 u(s)$ is bounded in $L_2^a(\mathbb{R}_+; H^{-r}(\Omega))$.

PROPOSITION 7.7. – *The family of trajectory spaces* $\{\mathcal{K}_\sigma^+(M),\ \sigma \in \mathcal{H}_+(\sigma_0)\}$ *is* $(\Theta_+^{loc}, \mathcal{H}_+(\sigma_0))$*-closed, so that,* $\mathcal{K}_{\mathcal{H}_+(\sigma_0)}^+(M) = \cup_{\sigma \in \mathcal{H}_+(\sigma_0)} \mathcal{K}_\sigma^+(M)$ *is closed in* Θ_+^{loc}.

Proof. – Let we be given $u_m(s) \in \mathcal{K}_{\sigma_m}^+(M)$, $\sigma_m(s) = (f_m(v,s), g_m(x,s))$ such that

$$u_m(s) \to u(s)\ (m \to \infty) \text{ in } \Theta_+^{loc}, \tag{7.29}$$

$$f_m(v,s) \to f(v,s)\ (m \to \infty) \text{ in } C(\mathbb{R}_+; M_0) \text{ and} \tag{7.30}$$

$$g_m(s) \to g(s)\ (m \to \infty) \text{ in } L_{2,w}^{loc}(\mathbb{R}_+; L_2(\Omega)). \tag{7.31}$$

It follows from (7.29) and from embedding $H^1(\Omega \times [t_1, t_2]) \Subset L_2(\Omega \times [t_1, t_2])$ that, passing to the subsequence (which we label the same), $u_m(x,s) \to u(x,s)$ $(m \to \infty)$ for almost any $(x,s) \in \Omega \times \mathbb{R}_+$. Using (7.30), we get $f_m(u_m(x,s),s) \to f(u(x,s),s)$ $(m \to \infty)$ for almost any $(x,s) \in \Omega \times \mathbb{R}_+$. On the other hand, the sequence $\{f_m(u_m(x,s),s)\}$ is bounded in $L_q^a(\mathbb{R}_+; L_q(\Omega))$. From Lions lemma (*see* ([19], Chapter 1, Lemma 1.3), we conclude that $f_m(u_m(x,s),s) \rightharpoonup f(u(x,s),s)$ $(m \to \infty)$ weakly in $L_q(t_1, t_2; L_q(\Omega))$ for any $[t_1, t_2] \subset \mathbb{R}_+$. Therefore, passing to the limit in the equation (7.1) with the symbol $\sigma_m(s) = (f_m(v,s), g_m(x,s))$ for the weak solution $u_m(s)$, we get that the function $u(s)$ is a weak solution of the equation with the symbol $\sigma(s) = (f(v,s), g(x,s))$. It is easy to prove similar to Proposition 7.4 that in the inequality,

$$z_m(t) \le R_\alpha(\sigma_0) + Me^{-\delta_\alpha t}\ \forall t \ge 0,$$

we may pass to the limit and get

$$z(t) \le R_\alpha(\sigma_0) + Me^{-\delta_\alpha t}\ \forall t \ge 0,$$

where $z(t)$ corresponds to the solution $u(s)$. Hence, $u(s) \in \mathcal{K}_\sigma^+(M)$.

By Proposition 3.2, the set $\mathcal{K}_{\mathcal{H}_+(\sigma_0)}^+(M)$ is closed in Θ_+^{loc}. □

Let us fix some appropriate $\alpha = \alpha_0 \in I_0$. Consider the set

$$P = \{u(s) \in \mathcal{F}_+^a \mid z(t) \le 2R_{\alpha_0}(\sigma_0)\},$$

where $z(s)$ corresponds to $u(s)$ by formula (7.12). Owing to (7.11) and (7.7), the set P is bounded in $\mathcal{F}_+^a$ and it is compact in Θ_+^{loc}. Inequality (7.23) implies that the set

P is a uniformly (w.r.t. $\sigma \in \mathcal{H}_+(\sigma_0)$) attracting (and even absorbing) set of the family $\{\mathcal{K}_\sigma^+(M),\ \sigma \in \mathcal{H}_+(\sigma_0)\}$ for any $M > 0$.

Therefore, Theorems 3.1, 4.1, and 5.1 are applicable to the family $\{\mathcal{K}_\sigma^+(M),\ \sigma \in \mathcal{H}_+(\sigma_0)\}$. Let $\omega(\mathcal{H}_+(\sigma_0))$ denote the global attractor of the semigroup $\{T(t)\}$ on $\mathcal{H}_+(\sigma_0)$. Let $Z(\sigma_0) \stackrel{def}{=} Z(\mathcal{H}_+(\sigma_0))$ be the set of all complete symbols in $\mathcal{H}_+(\sigma_0)$, *i.e.* the set of functions $\zeta(s), s \in \mathbb{R}, \zeta(s) \in \Xi = C(\mathbb{R}, \mathcal{M}_0) \times L_{2,w}^{loc}(\mathbb{R}; L_2(\Omega))$ such that $\zeta_t \in \omega(\mathcal{H}_+(\sigma_0))$ for any $t \in \mathbb{R}$, where $\zeta_t(s) = \Pi_+\zeta(s+t), s \geq 0$. To any complete symbol $\zeta(s) = (f(v,s), g(x,s)) \in Z(\sigma_0)$ there corresponds, by Definition 4.2, the kernel $\mathcal{K}_\zeta$ of equation (7.1). $\mathcal{K}_\zeta$ consists of all weak solutions $u(s), s \in \mathbb{R}$, of the equation

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - f(u,t) + g(x,t),\ t \in \mathbb{R}, \tag{7.32}$$

that are bounded in the space $\mathcal{F}^a = L_\infty(\mathbb{R}; L_p(\Omega) \cap H_0^1(\Omega)) \cap \{\,\partial_t v \in L_\infty(\mathbb{R}; L_2(\Omega))\} \cap \{\,\partial_t^2 v \in L_2^a(\mathbb{R}; H^{-r}(\Omega))\}$.

THEOREM 7.1. – *Let $\sigma_0(s) = (f_0(v,s), g_0(x,s)), s \in \mathbb{R}_+$, where the function $g_0(x,s)$ is translation-bounded in $L_2^{loc}(\mathbb{R}_+; L_2(\Omega))$ and $f_0(v,s)$ satisfies conditions (7.2)-(7.6), (7.22), i.e. $\sigma_0(s)$ is tr.-c. in $\Xi_+ = C(\mathbb{R}_+, \mathcal{M}_0) \times L_{2,w}^{loc}(\mathbb{R}_+; L_2(\Omega))$. Let $\Sigma = \mathcal{H}_+(\sigma_0)$ be the symbol space of equation (7.1). Then for any $M > 0$ the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}_{\mathcal{H}_+(\sigma_0)}^+(M) = \cup_{\sigma \in \mathcal{H}_+(\sigma_0)} \mathcal{K}_\sigma^+(M)$ possesses a uniform (w.r.t. $\sigma \in \mathcal{H}_+(\sigma_0)$) trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\sigma_0)} \subseteq P$. The set $\mathcal{A}_{\mathcal{H}_+(g_0)}$ is bounded in $\mathcal{F}_+^a$ and compact in Θ_+^{loc}. $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ does not depend on M and*

$$\mathcal{A}_{\mathcal{H}_+(\sigma_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(\sigma_0))} = \Pi_+ \bigcup_{\zeta \in Z(\sigma_0)} \mathcal{K}_\zeta = \Pi_+ \mathcal{K}_{Z(\sigma_0)}. \tag{7.33}$$

The kernel $\mathcal{K}_\zeta$ is not empty for any $\zeta \in Z(\sigma_0)$. The set $\mathcal{K}_{Z(g_0)}$ is bounded in $\mathcal{F}^a$ and compact in Θ^{loc}. Moreower, for any $u \in \mathcal{K}_\zeta$

$$z(t) = J_\alpha(u(.,t), \partial_t u(.,t), t) \leq R_\alpha(\sigma_0),\ \forall t \in \mathbb{R},\ \forall \alpha \in I_0.$$

Notice, the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ does not depend on M, since $T(t)\mathcal{K}_\Sigma^+(M_1) \subseteq \mathcal{K}_\Sigma^+(M)$ for any $M_1 > M$, when $t >> 1$. Indeed, it follows from (7.23) that for any $u \in \mathcal{K}_\sigma^+(M_1)$ the function $T(h)u \in \mathcal{K}_{T(h)\sigma}^+(M_1 e^{-\delta_0 h})$, where $\delta_0 = \delta_0(I_0) > 0$. So that, $T(t)u \in \mathcal{K}_\Sigma^+(M)$ if $t >> 1$.

Analogous result is valid when the symbol $\sigma_0(s) = (f_0(v,s), g_0(x,s))$ is defined on the whole time axis $s \in \mathbb{R}$, *i.e.* let σ_0 be tr.-c. in $\Xi = C(\mathbb{R}, \mathcal{M}_0) \times L_{2,w}^{loc}(\mathbb{R}; L_2(\Omega))$. Consider the symbol space $Z = \mathcal{H}(\sigma_0)$, where $\mathcal{H}(\sigma_0)$ is a hull of σ_0 in Ξ. The translation group $\{T(t), t \in \mathbb{R}\}$ acts on $\mathcal{H}(\sigma_0) : T(t)\mathcal{H}(\sigma_0) = \mathcal{H}(\sigma_0)\ \forall t \in \mathbb{R}$. To each symbol $\sigma \in \mathcal{H}(\sigma_0)$ there corresponds the trajectory space $\mathcal{K}_\sigma^+$ of equation (7.1). For the family $\{\mathcal{K}_\sigma^+,\ \sigma \in \mathcal{H}(\sigma_0)\}$ Theorem 5.1 is applicable. For any $\sigma \in \mathcal{H}(\sigma_0)$, by $\mathcal{K}_\sigma$ denote the kernel of equation (7.1). $\mathcal{K}_\sigma$ consists of all weak solutions $u(s), s \in \mathbb{R}$, of equation (7.32) that are bounded in $\mathcal{F}^a$.

THEOREM 7.2. – *Let $\sigma_0(s)$ be tr.-c. in $\Xi = C(\mathbb{R}, \mathcal{M}_0) \times L_{2,w}^{loc}(\mathbb{R}; L_2(\Omega))$. Let $Z = \mathcal{H}(\sigma_0)$ be a symbol space of equation (7.1). Then for any $M > 0$ the translation semigroup*

$\{T(t)\}$ acting on $\mathcal{K}^+_{\mathcal{H}(\sigma_0)}(M) = \cup_{\sigma\in\mathcal{H}(\sigma_0)}\mathcal{K}^+_\sigma(M)$ possesses a uniform (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) trajectory attractor $\mathcal{A}_{\mathcal{H}(\sigma_0)} \subseteq P$. The set $\mathcal{A}_{\mathcal{H}(g_0)}$ is bounded in $\mathcal{F}^a_+$ and compact in Θ^{loc}_+. $\mathcal{A}_{\mathcal{H}(g_0)}$ does not depend on M and

$$\mathcal{A}_{\mathcal{H}(\sigma_0)} = \Pi_+ \bigcup_{\sigma\in\mathcal{H}(\sigma_0)} \mathcal{K}_\sigma = \Pi_+\mathcal{K}_{\mathcal{H}(\sigma_0)}.$$

The kernel $\mathcal{K}_\sigma$ is not empty for any $\sigma \in \mathcal{H}(\sigma_0)$. The set $\mathcal{K}_{\mathcal{H}(\sigma_0)}$ is bounded in $\mathcal{F}^a$ and compact in Θ^{loc}.

8. Trajectory attractors for non-autonomous 3D Navier-Stokes system

Excluding the pressure, the Navier-Stokes system in the semicylinder $Q_+ = \Omega \times \mathbb{R}_+$ can be written in the form:

$$\partial_t u + \nu L u + B(u) = g(x,t),\ (\nabla, u) = 0,\ u|_{\partial\Omega} = 0,\ x \in \Omega,\ t \geq 0, \tag{8.1}$$

where, as usually, $x = (x_1, \dots, x_n)$, $u = u(x,t) = (u^1, \dots, u^n)$, $g = g(x,t) = (g^1, \dots, g^n)$, $n = 2, 3$. L is the Stokes operator: $Lu = -P\Delta u$; $B(u) = B(u,u)$, $B(u,v) = P(u,\nabla)v = P\sum_{i=1}^n u_i\partial_{x_i}v$, $\nu > 0$ (*see* [17], [19], [23], [25]). By H and V denote the closure in $(L_2(\Omega))^n$ and $(H^1_0(\Omega))^n$ of the set $\mathcal{V}_0 = \{v \mid v \in (C^\infty_0(\Omega))^n,\ (\nabla, v) = 0\}$; P denotes the orthogonal projector in $(L_2(\Omega))^n$ onto the Hilbert space H. The scalar product in H is $(u,v) = \int_\Omega (u(x), v(x))dx$ and the norm $|u| = (u,u)^{1/2}$. Let $V' = (V)^*$ be the dual space of V. For any $v \in V'$ the expression $\langle v, u\rangle$ means the value of the functional v on a vector $u \in V$. The operator L is an isomorphism from V into V'. The scalar product in V is $((u,v)) = \langle Lu, v\rangle$ and the norm is $\|u\| = \langle Lu, u\rangle^{1/2}$.

The external force $g(x,t)$ is the time symbol of equation (8.1): $\sigma(s) = g(\,.\,,t)$. Suppose

$$g(x,t) \in L^{loc}_2(\mathbb{R}_+, V'). \tag{8.2}$$

To describe a trajectory space $\mathcal{K}^+_g$ of equation (8.1), we shall study weak solutions of this equation on any segment $[t_1, t_2] \subset \mathbb{R}_+$ to begin with.

The operator $B(u)$ takes V to V' and the following inequality is valid:

$$|\langle B(u), v\rangle| \leq c_0\|u\|^2\|v\|\ \forall u, v \in V \tag{8.3}$$

and therefore $|B(u)|_{V'} \leq c_0\|u\|^2$. Thereby if a function $u(s) \in L_2(t_1, t_2; V)$ then $B(u(s)) \in L_1(t_1, t_2; V')$. Besides, $\nu Lu \in L_2(t_1, t_2; V')$. Hence, all the terms of equation (8.1) (excluding $\partial_t u$) belongs to $L_1(t_1, t_2; V')$. Consider these functions as distributions with values in V' from the space $D'(]t_1, t_2[; V')$. A function $u(s) \in L_2(t_1, t_2; V)$ is said to be a weak solution of equation (8.1) on the segment $[t_1, t_2]$ if the derivative $\partial_t u$ satisfies (8.1) in the sense of the distribution space $D'(]t_1, t_2[; V')$ (*see* [19], [23]). If $u(s) \in L_2(t_1, t_2; V)$ and $u(s)$ is a weak solution of (8.1) then $\partial_t u(s) \in L_1(t_1, t_2; V')$ and therefore $u(s) \in C([t_1, t_2]; V')$. So for any $t \in [t_1, t_2]$ the value of the function u

at moment t is meaningful. In particular, the initial value problem $u|_{t=t_1} = u_0$ makes sence. If, in addition, $u(s) \in L_\infty(t_1, t_2; H)$ then, by Lemma 8.1, $u(s) \in C_w([t_1, t_2]; H)$ and we may assume that $u_0 \in H$.

THEOREM 8.1. – **(i)** *Let $g \in L_2(t_1, t_2; V')$ and $u_0 \in H$. Then there exists a weak solution $u(s)$ of equation (8.1) belonging to the space $L_2(t_1, t_2; V) \cap L_\infty(t_1, t_2; H)$ such that $u(t_1) = u_0$ and $u(s)$ satisfies the inequality:*

$$(8.4) \qquad \frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 \leq \langle g(t), u(t)\rangle,\ t \in]t_1, t_2[.$$

The inequality (8.4) should be read as follows: for any function $\psi(s) \in C_0^\infty(]t_1, t_2[), \psi \geq 0$,

$$(8.5) \qquad -\frac{1}{2}\int_{t_1}^{t_2} |u(s)|^2\psi'(s)ds + \nu\int_{t_1}^{t_2} \|u(s)\|^2\psi(s)ds \leq \int_{t_1}^{t_2} \langle g(s), u(s)\rangle\psi(s)ds.$$

(ii) *For $n = 2$ the weak solution $u(s)$ of (8.1) from the space $L_2(t_1, t_2; V) \cap L_\infty(t_1, t_2; H)$ with the initial data $u(t_1) = u_0$ is unique and*

$$(8.6) \qquad \frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 = \langle g(t), u(t)\rangle,\ t \in]t_1, t_2[,$$

where the function $|u(t)|^2$ is absolutely continuous and (8.6) is valid almost everywhere in $[t_1, t_2]$.

For $n = 2$ the existence and uniqueness theorem is well known (*see* [17], [19]). For $n = 3$ the existence theorem was proved in [15] and for the spaces we use in [19] (*see* also [23]). Below we outline the proof of (8.4) and (8.5) for a weak solution $u(s)$ resulting from the Faedo-Galerkin method.

Proof. – We are looking for an approximative solution $u_m(x, s)$ of equation (8.1), $u_m(x, s) = \sum_{i=1}^m a_{j,m}(s)w_j$, where $a_{j,m}(s)$ are absolutely continuous scalar functions on $[t_1, t_2]$. Here $\{w_j\}_{j\in\mathbb{N}}$ is a basis in V. The function $u_m(x, s)$ satisfies the ordinary differential system:

$$(8.7) \qquad \frac{du_m}{dt} + \nu P_m L u_m + P_m B(u_m) = P_m g(x, t),\ u_m(t_1) = u_{0,m},$$

where $u_{0,m} \rightharpoonup u_0$ $(m \to \infty)$ weakly in H, so that, $\{u_{0,m}\}$ is bounded in H. As usually, it is straightforward (*see* [19], [23]) that

$$(8.8) \qquad |u_m(t)|^2 + \nu\int_{t_1}^{t} \|u_m(s)\|^2 ds \leq |u_{m,0}|^2 + \frac{1}{\nu}\int_{t_1}^{t} \|g(s)\|_{V'}^2 ds.$$

It follows easily that (8.8) is valid for any $t \in [t_1, t_2]$, and the sequence $\{u_m(s)\}$ remains in a bounded set of $L_2(t_1, t_2; V) \cap L_\infty(t_1, t_2; H)$, since $|u_{m,0}|^2$ is bounded. So by refining, we may assume that there exists a function $u(s) \in L_2(t_1, t_2; V) \cap L_\infty(t_1, t_2; H)$ such that $u_m(s) \rightharpoonup u(s)$ $(m \to \infty)$ weakly in $L_2(t_1, t_2; V)$ and $*$-weakly in $L_\infty(t_1, t_2; H)$. From (8.3) and (8.7) it follows that $\{\partial_t u_m(s)\}$ is bounded in $L_1(t_1, t_2; V')$. Due to a compactness theorem (*see* [19], [23]), we extract a subsequence $\{u_m(s)\}$ (which we label the same) strongly convergent to $u(s)$ in $L_2(t_1, t_2; H)$. The passage to the limit allows us to conclude that $u(s)$ is a weak solution of (8.1). In addition, notice that $u_m(s) \to u(s)$ $(m \to \infty)$ in $C_w([t_1, t_2]; H)$, so that, $u(t_1) = u_0$. (For more details, *see* [19], [23]).

Let us prove inequality (8.5); from

$$\int_{t_1}^{t} (|u_m(s)| - |u(s)|)^2 ds \leq \int_{t_1}^{t} |u_m(s) - u(s)|^2 ds,$$

it follows that $|u_m(s)| \to |u(s)|$ $(m \to \infty)$ strongly in $L_2(t_1, t_2)$. In particular, by refining, $|u_m(s)|^2 \to |u(s)|^2$ $(m \to \infty)$ almost everywhere in $[t_1, t_2]$. Now let $\psi(s) \in C_0^\infty(]t_1, t_2[)$ and $\psi \geq 0$. It follows from (8.8) that functions $|u_m(s)|^2 \psi'(s)$ have a majorant on $[t_1, t_2]$. The Lebesgue theorem implies that

$$(8.9) \qquad \int_{t_1}^{t} |u_m(s)|^2 \psi'(s) ds \to \int_{t_1}^{t} |u(s)|^2 \psi'(s) ds \ (m \to \infty).$$

Notice that $u_m(s)(\psi(s))^{1/2} \rightharpoonup u(s)(\psi(s))^{1/2}$ $(m \to \infty)$ weakly in $L_2(t_1, t_2; V)$. Thereby,

$$(8.10) \qquad \int_{t_1}^{t} \|u(s)\|^2 \psi(s) ds \leq \liminf_{m \to \infty} \int_{t_1}^{t} \|u_m(s)\|^2 \psi(s) ds.$$

Finally, using (8.7), we get

$$(8.11) \quad -\frac{1}{2} \int_{t_1}^{t_2} |u_m(s)|^2 \psi'(s) ds + \nu \int_{t_1}^{t_2} \|u_m(s)\|^2 \psi(s) ds = \int_{t_1}^{t_2} \langle g(s), u_m(s) \rangle \psi(s) ds.$$

Combining (8.9) and (8.5), we pass to the limit in (8.11) and obtain (8.10). This complete the proof. □

Now we describe trajectory spaces $\mathcal{K}_g^{t_1, t_2}$ for equation (8.1).

DEFINITION 8.1. – *The space $\mathcal{K}_g^{t_1, t_2}$ is the union of all weak solutions $u(s)$ of (8.1) from $L_2(t_1, t_2; V) \cap L_\infty(t_1, t_2; H)$ for which inequality (8.5) is valid.*

Notice that in 2D case, any weak solution satisfies (8.6) and hence (8.5).

Using Definition 2.1, one defines the trajectory space $\mathcal{K}_g^+$. Evidently, $\mathcal{K}_g^+$ is the union of all weak solutions $u(s) \in L_2^{loc}(\mathbb{R}_+; V) \cap L_\infty^{loc}(\mathbb{R}_+; H)$ that satisfy inequality (8.5) for any $\psi(s) \in C_0^\infty(\mathbb{R}_+)$, $\psi \geq 0$.

COROLLARY 8.1. – *Let $g \in L_2(t_1, t_2; V')$ and $u \in \mathcal{K}_g^{t_1, t_2}$. Then:*

$$(8.12) \qquad -\int_{t_1}^{t_2} |u(s)|^2 \psi'(s) ds + \nu \int_{t_1}^{t_2} \|u(s)\|^2 \psi(s) ds \leq \frac{1}{\nu} \int_{t_1}^{t_2} \|g(s)\|_{V'}^2 \psi(s) ds$$

for any $\psi(s) \in C_0^\infty(]t_1, t_2[)$, $\psi \geq 0$.

This inequality follows from (8.5).

COROLLARY 8.2. – **(i)** *Let $g \in L_2^{loc}(\mathbb{R}_+; V')$ and $u_0 \in H$. Then there exists a trajectory $u(s) \in \mathcal{K}_g^+$ such that $u(0) = u_0$;* **(ii)** *for $n = 2$ this trajectory is unique.*

Indeed, the solution $u_m(s)$ of equation (8.7) is defined for $s \in \mathbb{R}_+$. Using diagonalization method, we may extract from $\{u_m(s)\}$ a subsequence that converges to a weak solution $u(s), s \geq 0$, for any segment $[t_1, t_2] \subset \mathbb{R}_+$. Evidently, $u(s) \in \mathcal{K}_g^+$.

PROPOSITION 8.1. – *Let $u_m(s) \in \mathcal{K}_{g_m}^{t_1,t_2}$ is a solution of (8.1) with the external force $g_m(x, s) \in L_2(t_1, t_2; V')$. Let $u_m \rightharpoonup u$ $(m \to \infty)$ weakly in $L_2(t_1, t_2; V)$ and $*$-weakly in $L_\infty(t_1, t_2; H)$. Suppose that:* **(i)** *for $n = 3$, $g_m \rightharpoonup g$ $(m \to \infty)$ weakly in $L_2(t_1, t_2; H)$ or $g_m \to g$ $(m \to \infty)$ strongly in $L_2(t_1, t_2; V')$;* **(ii)** *for $n = 2$, $g_m \rightharpoonup g$ $(m \to \infty)$ weakly in $L_2(t_1, t_2; V')$. Then $u \in \mathcal{K}_g^{t_1,t_2}$.*

Proof. – Similarly to the proof of Theorem 8.1, we can extract a subsequence from $\{u_m(s)\}$, strongly convergent to $u(s)$ in $L_2(t_1, t_2; H)$. Thus in the equation $\partial_t u_m + \nu L u_m + B(u_m) = g_m(x, t)$, we can pass to the limit and get: $\partial_t u + \nu L u + B(u) = g(x, t)$, so that, $u(s)$ is a weak solution of (8.1). The point **(ii)** is proved. In order to conclude that $u(s) \in \mathcal{K}_g^{t_1,t_2}$ for $n = 3$, we have to show (8.5) under the condition that any pair $u_m(s), g_m(s)$ satisfy this inequality, *i.e.*:

$$(8.13)\quad -\frac{1}{2}\int_{t_1}^{t_2} |u_m(s)|^2 \psi'(s) ds + \nu \int_{t_1}^{t_2} \|u_m(s)\|^2 \psi(s) ds \leq \int_{t_1}^{t_2} \langle g_m(s), u_m(s)\rangle \psi(s) ds,$$

for any function $\psi(s) \in C_0^\infty(]t_1, t_2[), \psi \geq 0$. The limit relations (8.9) and (8.10) are valid for the terms in the left-hand side of (8.13). So, we can make the passage to the limit in the left-hand side of (8.13). We claim that the right-hand side of (8.5) tends to the right-hand side of (8.5). Let $g_m \rightharpoonup g$ $(m \to \infty)$ weakly in $L_2(t_1, t_2; H)$. We have:

$$\begin{aligned} &\left| \int_{t_1}^{t_2} \langle g_m(s), u_m(s)\rangle \psi(s) ds - \int_{t_1}^{t_2} \langle g(t), u(s)\rangle \psi(s) ds \right| \\ &\leq c\left(\int_{t_1}^{t_2} |u_m(s) - u(s)|^2 ds \right)^{1/2} \left(\int_{t_1}^{t_2} |g_m(s)|^2 ds \right)^{1/2} \\ &+ \left| \int_{t_1}^{t_2} \langle g_m(s) - g(s), u(s)\rangle \psi(s) ds \right|. \end{aligned}$$

The first term in the last sum tends to zero, since $u_m(s) \to u(s)$ $(m \to \infty)$ strongly in $L_2(t_1, t_2; H)$ and $\{g_m(s)\}$ is bounded in $L_2(t_1, t_2; H)$. The last term tends to zero, because $g_m \rightharpoonup g$ $(m \to \infty)$ weakly in $L_2(t_1, t_2; H)$.

Analogously we argue when $g_m \to g$ $(m \to \infty)$ strongly in $L_2(t_1, t_2; V')$. □

Let us describe the spaces $\mathcal{F}_+^{loc}$, $\mathcal{F}_+^a$, and Θ_+^{loc} for equation (8.1). Consider the Hilbert space $\mathcal{X}^\gamma(\mathbb{R}; H) = \{v(s) \mid v(s) \in L_2(\mathbb{R}, H), |\tau|^\gamma \hat{v}(\tau) \in L_2(\mathbb{R}_\tau, H)\}$, where $1/2 > \gamma > 0$ and $\hat{v}(\tau)$ is a Fourier transform of $v(s) : \hat{v}(\tau) = \int_{-\infty}^{+\infty} v(s) e^{-2\pi i s \tau} ds$. The norm in $\mathcal{X}^\gamma(\mathbb{R}; H)$ is

$$\|v\|^2_{\mathcal{X}^\gamma(\mathbb{R};H)} = \|\hat{v}\|^2_{L_2(\mathbb{R},H)} + \||\tau|^\gamma \hat{v}\|^2_{L_2(\mathbb{R},H)},$$

(*see* [19]). By $\mathcal{X}^\gamma(t_1, t_2; H)$ denote a subspace of function from $\mathcal{X}^\gamma(\mathbb{R}; H)$ with the support contained in $[t_1, t_2] : \mathcal{X}^\gamma(t_1, t_2; H) = \{v \in \mathcal{X}^\gamma(\mathbb{R}; H) \mid \mathbf{supp}(v) \subseteq [t_1, t_2]\}$. One says, by the definition, that the space $\mathcal{X}^\gamma(t_1, t_2; H)$ consists of functions $v \in L_2(t_1, t_2; H)$ that possesses a fractional derivative of order γ

belonging to $L_2(t_1,t_2;H)$. We shall use the space $\mathcal{X}^{\gamma,loc}(\mathbb{R}_+;H) = \{v(s) \mid v(s) \in L_2^{loc}(\mathbb{R}_+,H),\ \chi_{t_1,t_2}(s)v(s) \in \mathcal{X}^\gamma(t_1,t_2;H)\ \forall[t_1,t_2] \subset \mathbb{R}_+\}$, where χ_{t_1,t_2} is a characteristic function of the interval $[t_1,t_2]$. Let, by the definition, $\mathcal{X}^{\gamma,a}(\mathbb{R}_+;H) = \{v(s) \mid v \in \mathcal{X}^{\gamma,loc}(\mathbb{R}_+;H),\ \|v\|^2_{\mathcal{X}^{\gamma,a}(\mathbb{R}_+;H)} < +\infty\}$, where

$$\|v\|_{\mathcal{X}^{\gamma,a}(\mathbb{R}_+;H)} = \sup_{t\geq 0} \|\chi_{t,t+1}v\|_{\mathcal{X}^\gamma(\mathbb{R};H)}. \tag{8.14}$$

Let $u \in L_2(t_1,t_2;V)$ be a weak solution of (8.1). It can be checked (*see* [19]) that the function u possesses a fractional derivative of order $\gamma = 1/4 - \varepsilon$ in $L_2(t_1,t_2;H)$ for any ε, $0 < \varepsilon < 1/4$.

On the other hand, due to the well-known inequality

$$|B(u)|_{V'} \leq C_1\|u\|^{3/2}|u|^{1/2}, \tag{8.15}$$

(*see*, for example, [23]) we have that $B(u) \in L_{4/3}(t_1,t_2;V')$, if, in addition, $u \in L_\infty(t_1,t_2;H)$. Therefore, from equation (8.1) we obtain $\partial_t u \in L_{4/3}(t_1,t_2;V')$, where $\partial_t u$ is a distribution from $D'(]t_1,t_2[;V')$. For $n = 2$

$$|B(u)|_{V'} \leq C_2\|u\||u|, \tag{8.16}$$

(*see* [17], [23]). Thereby, $\partial_t u \in L_2(t_1,t_2;V')$.

DEFINITION 8.2. – *The space $\mathcal{F}_{t_1,t_2} = \{v \mid v \in L_2(t_1,t_2;V) \cap L_\infty(t_1,t_2;H) \cap \mathcal{X}^\gamma(t_1,t_2;H), \partial_t v \in L_p(t_1,t_2;V')\}$, where $p = 4/n$, $n = 2,3$, and γ is fixed, $0 < \gamma < 1/4$. The norm in the space $\mathcal{F}_{t_1,t_2}$ is*

$$\|v\|_{\mathcal{F}_{t_1,t_2}} = \|v\|_{L_2(t_1,t_2;V)} + \|v\|_{L_\infty(t_1,t_2;H)} + \|v\|_{\mathcal{X}^\gamma(t_1,t_2;H)} + \|\partial_t v\|_{L_p(t_1,t_2;V')} \tag{8.17}$$

Evidently, $\mathcal{F}_{t_1,t_2}$ is a Banach space.

REMARK 8.1. – *The space $\mathcal{F}_{t_1,t_2}$ can be chosen by different ways. This reflects the specific character of nonlinear equation (8.1). For example, from (8.2) it follows that a weak solution $u(s)$ possesses a fractional derivative of order $\gamma' = 1/2 - \varepsilon'$ in $L_2(t_1,t_2;V')$ for any ε', $0 < \varepsilon' < 1/2$. So, one can add in the definition 8.2 of $\mathcal{F}_{t_1,t_2}$ the condition: $u(s) \in \mathcal{X}^{\gamma'}(t_1,t_2;V')$. For the sake of definiteness, we shall consider the space $\mathcal{F}_{t_1,t_2}$ with the norm (8.17).*

The spaces $\mathcal{F}_{t_1,t_2}$ generate, by Definition 2.2, the spaces $\mathcal{F}_+^{loc}$ and $\mathcal{F}_+^a$. It is easy to *see* that $\mathcal{F}_+^{loc} = L_2^{loc}(\mathbb{R}_+;V) \cap L_\infty^{loc}(\mathbb{R}_+;H) \cap \mathcal{X}^{\gamma,loc}(\mathbb{R}_+;H) \cap \{v \mid \partial_t v \in L_p^{loc}(\mathbb{R}_+;V')\}$ and $\mathcal{F}_+^a = L_2^a(\mathbb{R}_+;V) \cap L_\infty(\mathbb{R}_+;H) \cap \mathcal{X}^{\gamma,a}(\mathbb{R}_+;H) \cap \{v \mid \partial_t v \in L_p^a(\mathbb{R}_+;V')\}$, $p = 4/n$, $n = 2,3$. By Θ_{t_1,t_2} denote the space $\mathcal{F}_{t_1,t_2}$ with the following convergence topology.

DEFINITION 8.3. – *A sequence $\{v_n\} \subset \mathcal{F}_{t_1,t_2}$ converges to $v \in \mathcal{F}_{t_1,t_2}$ as $n \to \infty$ in Θ_{t_1,t_2} if $v_n(s) \rightharpoonup v(s)$ $(n \to \infty)$ weakly in $L_2(t_1,t_2;V)$, $*$-weakly in $L_\infty(t_1,t_2;H)$, weakly in $\mathcal{X}^\gamma(t_1,t_2;H)$, and $\partial_t v_n(s) \rightharpoonup \partial_t v(s)$ $(n \to \infty)$ weakly in $L_p(t_1,t_2;V')\}$.*

It is easy to prove that Θ_{t_1,t_2} is a Hausdorff and Fréchet-Uryson space with a countable topology base and Θ_{t_1,t_2} is homeomorphic to $\Theta_{0,1}$ with respect to the similitude J (*see* (2.4)). The spaces Θ_{t_1,t_2} define the topological space Θ_+^{loc} (*see* Definition 2.3).

Let us describe the symbol space Σ of equation (8.1). Consider some fixed external force $g_0(s) \in L_2^{loc}(\mathbb{R}_+, V')$. Suppose $g_0(s)$ is tr.-c. function in $L_{2,w}^{loc}(\mathbb{R}_+, V')$. By compactness criterion (Proposition 6.8), this is equivalent to the condition:

$$(8.18) \qquad \|g_0\|^2_{L_2^a(\mathbb{R}_+;V')} = \sup_{t\in\mathbb{R}_+} \int_t^{t+1} \|g(s)\|^2_{V'} ds < +\infty,$$

i.e. $g_0(s)$ is translation-bounded in $L_2^{loc}(\mathbb{R}_+, V')$, $g_0(s) \in L_2^a(\mathbb{R}_+, V')$. Let Σ be the hull of the function $g_0(s)$ in the space $\Xi_+ = L_{2,w}^{loc}(\mathbb{R}_+, V')$, that is, $\Sigma = \mathcal{H}_+(g_0) \equiv [\{g_0(s+t) \mid t \geq 0\}]_{L_{2,w}^{loc}(\mathbb{R}_+,V')}$. It follows from Lemma 6.3 that $\Sigma = \mathcal{H}_+(g_0)$ is a complete metric space. By Proposition 6.9 , the translation semigroup $\{T(t)\}$ is continuous on $\mathcal{H}_+(g_0)$ and $T(t)\mathcal{H}_+(g_0) \subseteq \mathcal{H}_+(g_0)$ $\forall t \geq 0$, moreower, for any $g \in \mathcal{H}_+(g_0)$ we have

$$(8.19) \qquad \|g\|^2_{L_2^a(\mathbb{R}_+;V')} \leq \|g_0\|^2_{L_2^a(\mathbb{R}_+;V')}.$$

Sometimes in the sequel, we shall assume that the function $g_0(s)$ is tr.-c. in a space with stronger topology. For instance, $g_0(s)$ is tr.-c. in $L_2^{loc}(\mathbb{R}_+, V')$ or in $L_{2,w}^{loc}(\mathbb{R}_+, H)$. It is not hard to show that, in these spaces, a hull $\Sigma = \mathcal{H}_+(g_0)$ of $g_0(s)$ coincides with the hull of $g_0(s)$ in a weaker space $L_{2,w}^{loc}(\mathbb{R}_+, V')$. Moreover, the set $\mathcal{H}_+(g_0)$, as a subset in $L_{2,w}^{loc}(\mathbb{R}_+, V')$, is homeomorphic to the set in the corresponding stronger space $L_2^{loc}(\mathbb{R}_+, V')$ or $L_{2,w}^{loc}(\mathbb{R}_+, H)$.

To each symbol $g \in \mathcal{H}_+(g_0)$ there corresponds, by Definitions 8.1 and 2.1, the trajectory space $\mathcal{K}_g^+$ of equation (8.1).

PROPOSITION 8.2. – *If $g_0(s)$ is tr.-c. in $L_{2,w}^{loc}(\mathbb{R}_+, V')$ then:*

(i) *$\mathcal{K}_g^+ \in \mathcal{F}_+^a$ for any $g \in \mathcal{H}_+(g_0)$;*

(ii) *for any $u(s) \in \mathcal{K}_g^+$,*

$$(8.20) \qquad \|T(t)u(\,.\,)\|_{\mathcal{F}_+^a} \leq C_0\|u(\,.\,)\|^2_{L_\infty(0,1;H)} \exp(-\alpha t) + R_0 \ \forall t \geq 1,$$

where $\alpha = \nu\lambda_1$, λ_1 is the first eigenvalue of the operator L, C_0 depends on λ_1, ν, and R_0 depends on λ_1, ν, $\|g_0\|^2_{L_2^a(\mathbb{R}_+;V')}$.

The proof of Proposition 8.2 will be given in the next section.

Put $\mathcal{K}_\Sigma^+ = \cup_{g\in\mathcal{H}_+(g_0)}\mathcal{K}_g^+$, $\Sigma = \mathcal{H}_+(g_0)$. The translation semigroup $\{T(t)\}$ acts on $\mathcal{K}_\Sigma^+$. By Proposition 2.1 a ball B_R in $\mathcal{F}_+^a$ is compact in Θ_+^{loc}.

PROPOSITION 8.3. – *Let $g_0(s)$ be* tr.-c. *in $L_{2,w}^{loc}(\mathbb{R}_+, V')$ for $n = 2$ and $g_0(s)$ be* tr.-c. *in $L_2^{loc}(\mathbb{R}_+, V')$ or in $L_{2,w}^{loc}(\mathbb{R}_+, H)$* for *$n = 3$; then the family $\{\mathcal{K}_g^+,\ g \in \Sigma\}$ is $(\Theta_+^{loc}, \mathcal{H}_+(g_0))$-closed and $\mathcal{K}_\Sigma^+$ is closed in Θ_+^{loc}.*

Proposition 8.3 follows directly from Propositions 8.1 and 3.2.

REMARK 8.2. – *We are unaware whether or not the set $\mathcal{K}_\Sigma^+$ is closed in Θ_+^{loc} when $g_0(s)$ is tr.-c. in $L_{2,w}^{loc}(\mathbb{R}_+, V')$ for $n = 3$. To provide this condition we shall consider later on wider spaces of solutions $\mathcal{K}_\sigma^{t_1,t_2}(i) \supseteq \mathcal{K}_\sigma^{t_1,t_2}$ (see Definition 8.4 below).*

In such a way, under the conditions of Proposition 8.3, theorems 3.1, 4.1, and 5.1 are applicable. Let $\omega(\mathcal{H}_+(g_0))$ denote the global attractor of the semigroup $\{T(t)\}$ on

$\mathcal{H}_+(g_0)$. Let $Z(g_0) =^{def} Z(\mathcal{H}_+(g_0))$ be the set of all complete symbols in $\mathcal{H}_+(g_0)$, *i.e.* the set of functions $\zeta(s), s \in \mathbb{R}, \zeta(s) \in L_2^{loc}(\mathbb{R}, V')$ such that $\zeta_t \in \omega(\mathcal{H}_+(g_0))$ for any $t \in \mathbb{R}$, where $\zeta_t(s) = \Pi_+\zeta(s+t), s \geq 0$. To any complete symbol $\zeta(s) \in Z(g_0)$ there corresponds, by Definition 4.4, the kernel $\mathcal{K}_\zeta$ of equation (8.1). $\mathcal{K}_\zeta$ consists of all weak solutions $u(s), s \in \mathbb{R}$, of the equation

$$\partial_t u + \nu L u + B(u) = \zeta(x,t),\ t \in \mathbb{R}, \tag{8.21}$$

that satisfy inequality (8.5) for any $[t_1, t_2] \subset \mathbb{R}$ and that are bounded in the space $\mathcal{F}^a = L_2^a(\mathbb{R}; V) \cap L_\infty(\mathbb{R}; H) \cap \mathcal{X}^{\gamma,a}(\mathbb{R}; H) \cap \{v \mid \partial_t v \in L_p^a(\mathbb{R}; V')\}$.

THEOREM 8.2. – *Let $g_0(s)$ be tr.-c. in $L_{2,w}^{loc}(\mathbb{R}_+, V')$ for $n = 2$ and $g_0(s)$ be tr.-c. in $L_2^{loc}(\mathbb{R}_+, V')$ or in $L_{2,w}^{loc}(\mathbb{R}_+, H)$ for $n = 3$; then the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}_\Sigma^+$ possesses a uniform (w.r.t. $g \in \mathcal{H}_+(g_0)$) trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$. The set $\mathcal{A}_{\mathcal{H}_+(g_0)}$ is bounded in $\mathcal{F}_+^a$ and compact in Θ_+^{loc}. Moreower,*

$$\mathcal{A}_{\mathcal{H}_+(g_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(g_0))} = \Pi_+ \bigcup_{\zeta \in Z(g_0)} \mathcal{K}_\zeta = \Pi_+ \mathcal{K}_{Z(g_0)}. \tag{8.22}$$

The kernel $\mathcal{K}_\zeta$ is not empty for any $\zeta \in Z(g_0)$; the set $\mathcal{K}_{Z(g_0)}$ is bounded in $\mathcal{F}^a$ and compact in Θ^{loc}.

Proof. – It is clear that the family of trajectory spaces $\{\mathcal{K}_g^+,\ g \in \mathcal{H}_+(g_0)\}$ is tr.-coord. in the sense of Definition 3.1: $T(t)\mathcal{K}_g^+ \subseteq \mathcal{K}_{T(t)g}^+,\ t \geq 0$, for any $g \in \mathcal{H}_+(g_0)$. By Proposition 8.3 the family $\{\mathcal{K}_g^+,\ g \in \mathcal{H}_+(g_0)\}$ is $(\Theta_+^{loc}, \mathcal{H}_+(g_0))$-closed. Thanks to (8.20), the set $\{v \in \mathcal{F}_+^a \mid \|v(\,.\,)\|_{\mathcal{F}_+^a} \leq 2R_0\}$ is a uniformly (w.r.t. $g \in \mathcal{H}_+(g_0)$) absorbing set of the family $\{\mathcal{K}_g^+,\ g \in \mathcal{H}_+(g_0)\}$. The ball B_{2R_0} is compact in Θ_+^{loc} and bounded in $\mathcal{F}_+^a$. Thus the conditions of Theorems 3.1 and 4.1 are valid and Theorem 8.2 is proved. □

Analogous result is true when the external force $g_0(s)$ is defined on the whole time axis. Let $g_0 \in L_2^{loc}(\mathbb{R}, V')$ and let $g_0(s)$ be translation-bounded in $L_2^a(\mathbb{R}, V')$:

$$\|g\|_a^2 = \|g\|_{L_2^a(\mathbb{R};V')}^2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|g(s)\|_{V'}^2 ds.$$

Consider the symbol space $Z = \mathcal{H}(g_0)$, where $\mathcal{H}(g_0)$ is a hull of g_o in $L_{2,w}^{loc}(\mathbb{R}, V')$. The translation group $\{T(t), t \in \mathbb{R}\}$ acts on $\mathcal{H}(g_0) : T(t)\mathcal{H}(g_0) = \mathcal{H}(g_0)\ \forall t \in \mathbb{R}$. To each symbol $g \in \mathcal{H}(g_0)$ there corresponds the trajectory space $\mathcal{K}_g^+$ of equation (8.1). In fact, $\mathcal{K}_g^+$ depends on $\Pi_+ g(s), s \geq 0$, only. For the family $\{\mathcal{K}_g^+,\ g \in \mathcal{H}(g_0)\}$, Propositions 8.2 and 8.3 are valid if to replace $\mathcal{H}_+(g_0)$ with $\mathcal{H}(g_0)$. Hence Theorem 5.1 is applicable. For any $g \in \mathcal{H}(g_0)$, by $\mathcal{K}_g$ denote the kernel of equation (8.21). $\mathcal{K}_g$ consists of all weak solutions $u(s), s \in \mathbb{R}$, of equation (8.21) that are bounded in $\mathcal{F}^a$ and satisfy inequality (8.5) for any $\psi \in C_0^\infty(\mathbb{R}), \psi \geq 0$.

THEOREM 8.3. – *Let $g_0(s)$ be tr.-c. in $L_{2,w}^{loc}(\mathbb{R}, V')$ for $n = 2$ and $g_0(s)$ be tr.-c. in $L_2^{loc}(\mathbb{R}, V')$ or in $L_{2,w}^{loc}(\mathbb{R}, H)$ for $n = 3$; then the translation semigroup $\{T(t)\}$ acting on*

$\mathcal{K}^+_{\mathcal{H}(g_0)} = \cup_{g\in\mathcal{H}(g_0)}\mathcal{K}^+_g$ *possesses a uniform (w.r.t. $g \in \mathcal{H}(g_0)$) trajectory attractor $\mathcal{A}_{\mathcal{H}(g_0)}$. The set $\mathcal{A}_{\mathcal{H}(g_0)}$ is bounded in $\mathcal{F}^a_+$ and compact in Θ^{loc}_+. Moreower, we have:*

$$\mathcal{A}_{\mathcal{H}(g_0)} = \Pi_+ \bigcup_{g\in\mathcal{H}(g_0)} \mathcal{K}_g = \Pi_+\mathcal{K}_{\mathcal{H}(g_0)}. \tag{8.23}$$

The kernel $\mathcal{K}_g$ is not empty for any $g \in \mathcal{H}(g_0)$. The set $\mathcal{K}_{\mathcal{H}(g_0)}$ is bounded in $\mathcal{F}^a$, compact in Θ^{loc}, and any $u \in \mathcal{K}_{\mathcal{H}(g_0)}$ is a tr.-c. function in Θ^{loc}.

Remark 8.3. – *If a function $g_0 \in L_2^{loc}(\mathbb{R}, V')$ satisfies the conditions of Theorem 8.3 then, evidently, the function $\Pi_+g_0 \in L_2^{loc}(\mathbb{R}_+, V')$ satisfies the conditions of Theorem 8.2 and $\mathcal{H}_+(\Pi_+g_0) \subseteq \Pi_+\mathcal{H}(g_0)$ (the inclusion can be strict). Consequently,*

$$\mathcal{A}_{\mathcal{H}_+(\Pi_+g_0)} \subseteq \mathcal{A}_{\mathcal{H}(g_0)}.$$

Now, we consider wider spaces $\mathcal{K}_g^{t_1,t_2}(i), g \in \mathcal{H}_+(g_0)$, to generalize Theorems 8.1 and 8.2 for $n = 3$ when $g_0(s)$ is translation bounded function in $L_2^{loc}(\mathbb{R}_+, V')$ only. We shall use inequality (9.13) from Corollary 9.4.

Definition 8.4. – *Let $g_0 \in L_2^{loc}(\mathbb{R}_+, V')$ be translation bounded in $L_2^{loc}(\mathbb{R}_+, V')$ and $\mathcal{H}_+(g_0)$ is a hull of g_0 in $L^{loc}_{2,w}(\mathbb{R}_+, V')$. The space $\mathcal{K}_g^{t_1,t_2}(i)$, $g \in \mathcal{H}_+(g_0)$, is the union of weak solutions $u(s)$ of (8.1) from $L_2(t_1, t_2; V) \cap L_\infty(t_1, t_2; H)$ that satisfy the inequality:*

$$|u(t)|^2e^{\alpha t} - |u(\tau)|^2e^{\alpha\tau} + \nu\int_\tau^t \left(\|u(s)\|^2 - \lambda_1|u(s)|^2\right)e^{\alpha s}ds \le \frac{\beta_\alpha(g_0)}{\alpha\nu}\left(e^{\alpha t} - e^{\alpha\tau}\right), \tag{8.24}$$

for any $\tau \in \mathbb{R}_+\backslash Q_{u,g}$ and any $t \ge \tau + 1$, $t, \tau \in [t_1, t_2]$, where $\mu(Q_{u,g}) = 0$, where

$$\beta_\alpha(g_0) = \sup_{h\in[1,2]}\sup_{t\ge 0}\left(\frac{\alpha\int_0^h \|g_0(s+t)\|^2_{V'}e^{\alpha s}ds}{e^{\alpha h} - 1}\right).$$

The right-hand side of (8.24) contains the value $\beta_\alpha(g_0)$ unlike $\beta_\alpha(g)$ in (9.10). It is proved in Section 9 that $\beta_\alpha(g) \le \beta_\alpha(g_0)$ for any $g \in \mathcal{H}_+(g_0)$, (*see* Remark 9.1). Thereby we get:

Proposition 8.4. – *$\mathcal{K}_g^{t_1,t_2}(i) \supseteq \mathcal{K}_g^{t_1,t_2}$ and $\mathcal{K}_g^+(i) \supseteq \mathcal{K}_g^+$ for any $g \in \mathcal{H}_+(g_0)$.*

Here, evidently, $\mathcal{K}_g^+(i)$ consists of all weak solutions $u(s)$ of (8.1) from $L_2^{loc}(\mathbb{R}_+, V) \cap L_\infty^{loc}(\mathbb{R}_+, H)$ that satisfy (8.24) for almost every $\tau \in \mathbb{R}_+$ and any $t \ge \tau + 1$.

Proposition 8.2 is valid for the family $\{\mathcal{K}_g^+(i), g \in \mathcal{H}_+(g_0)\}$ with the same constants if in **i)** and **ii)** to replace $\mathcal{K}_g^+$ with $\mathcal{K}_g^+(i)$. The proof is absolutely the same (*see* Remark 9.2).

Proposition 8.5. – *The family $\{\mathcal{K}_g^+(i), g \in \mathcal{H}_+(g_0)\}$ is $(\Theta^{loc}_+, \mathcal{H}_+(g_0))$-closed and $\mathcal{K}^+_{\mathcal{H}_+(g_0)}(i) = \cup_{g\in\mathcal{H}_+(g_0)}\mathcal{K}_g^+(i)$ is closed in Θ^{loc}_+.*

Proof. – Let $u_m \in \mathcal{K}_{g_m}^{t_1,t_2}(i)$, $g_m \in \mathcal{H}_+(g_0)$, and $u_m \to u$ $(m \to \infty)$ weakly in $L_2(t_1, t_2; V)$, $*$-weakly in $L_\infty(t_1, t_2; H), \partial_t u_m \to \partial_t u$ $(m \to \infty)$ weakly in $L_{4/3}(t_1, t_2; V')$, and $g_m \to g$ $(m \to \infty)$ weakly in $L_2(t_1, t_2; V')$. Then, evidently u is a weak solution of (8.1) with the symbol g. Finally, if u_m satisfy (8.24) then, passing to the limit in the left-hand sides of (8.24) (the right-hand sides do not depend on m), we obtain that $u(s)$ satisfies (8.24) as well and hence $u \in \mathcal{K}_g^{t_1,t_2}(i)$. □

Thus, Theorems 3.1, 4.1, and 5.1 are applicable. Theorems 8.1 and 8.2 are valid with $\{\mathcal{K}_g^+,\ g \in \mathcal{H}_+(g_0)\}$ being replaced by the family $\{\mathcal{K}_g^+(i),\ g \in \mathcal{H}_+(g_0)\}$. We formulate the analog of Theorem 8.1.

THEOREM 8.4. – *Let $g_0(s)$ be tr.-c. in $L_{2,w}^{loc}(\mathbb{R}_+, V')$; then the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+_{\mathcal{H}_+(g_0)}(i)$ possesses a uniform (w.r.t. $g \in \mathcal{H}_+(g_0)$) trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}(i)$. The set $\mathcal{A}_{\mathcal{H}_+(g_0)}(i)$ is bounded in $\mathcal{F}_+^a$ and compact in Θ_+^{loc}. Moreower, we have:*

$$\mathcal{A}_{\mathcal{H}_+(g_0)}(i) = \mathcal{A}_{\omega(\mathcal{H}_+(g_0))}(i) = \Pi_+ \bigcup_{\zeta \in Z(g_0)} \mathcal{K}_\zeta(i) = \Pi_+ \mathcal{K}_{Z(g_0)}(i).$$

The kernel $\mathcal{K}_\zeta(i)$ is not empty for any $\zeta \in Z(g_0)$; The set $\mathcal{K}_{Z(g_0)}(i)$ is bounded in $\mathcal{F}^a$ and compact in Θ^{loc}.

We conclude the section with few remarks about the character of attraction to a trajectory attractor. Notice that the topology of space Θ_{t_1,t_2}, considered above, is stronger than the strong topology of the space $L_2(t_1, t_2; H^\delta)$, $H^\delta = \left(H^\delta(\Omega)\right)^n$, $0 \le \delta < 1$. So, Theorems 8.1, 8.2, and 8.3 imply

COROLLARY 8.3. – *For any set $B \subset \mathcal{K}^+_{\mathcal{H}_+(g_0)}$ (or $B \subset \mathcal{K}^+_{\mathcal{H}_+(g_0)}(i)$), bounded in $\mathcal{F}_+^a$, one has*

$$\mathbf{dist}_{L_2(0,M;H^\delta)}\left(\Pi_{0,M}T(t)B,\ \Pi_{0,M}\mathcal{K}_{Z(g_0)}\right) \to 0\ (t \to +\infty),$$

for any $M > 0$, $0 \le \delta < 1$.

Let us formulate one more result. By $\vec{v} = (v_1, \ldots, v_N) \in (H)^N$ denote an arbitrary vector-function. Let $J_{\vec{v}}$ be the map of $\mathcal{F}_+^{loc}$ to $C(\mathbb{R}_+; \mathbb{R}^N)$ such that $J_{\vec{v}}(u)(s) = (\langle u(s), v_1\rangle, \langle u(s), v_2\rangle, \ldots, \langle u(s), v_N\rangle)$.

COROLLARY 8.4. – *For any set $B \subset \mathcal{K}^+_{\mathcal{H}_+(g_0)}$ (or $B \subset \mathcal{K}^+_{\mathcal{H}_+(g_0)}(i)$), bounded in $\mathcal{F}_+^a$, one has*

$$\mathbf{dist}_{C([0,M];\mathbb{R}^N)}\left(\Pi_{0,M}J_{\vec{v}}(T(t)B),\ \Pi_{0,M}J_{\vec{v}}\left(\mathcal{K}_{Z(g_0)}\right)\right) \to 0\ (t \to +\infty),$$

for any $M > 0$.

9. Proof of Proposition 8.2

1) First of all we establish few auxiliary lemmas and propositions.

LEMMA 9.1. – *Let $f(s) \in L_1(a, b)$. The following conditions are equivalent:*

(i) *for any $\varphi \in C_0^\infty(]a, b[)$, $\varphi(s) \ge 0$, one has*

$$\int_a^b f(s)\varphi'(s)ds \ge 0. \tag{9.1}$$

(ii) *the function $f(s)$ almost everywhere is equal to a monotone non-increasing function on $[a, b]$, that is, $f(t) \le f(\tau)$ for any $t, \tau \in [a, b]\backslash Q$, $\tau \le t$, where a set Q has zero measure, $\mu(Q) = 0$.*

Proof. – Suppose **(i)** is satisfied. Let $\omega(s) \in C_0^\infty(]-1,+1[)$, $\omega(s) \geq 0$, $\omega(-s) = \omega(s)$, and $\int_{-1}^{+1} \omega(s)ds = 1$. Consider the function $\omega_\rho(s) = \frac{1}{\rho}\omega(\frac{s}{\rho})$, $\omega_\rho(s) \in C_0^\infty(]-\rho,+\rho[)$. Put $f(s) \equiv 0$ for $s \notin [a,b]$. Consider the averaging function $f_\rho(t) \equiv \int_a^b \omega_\rho(t-s)f(s)ds = \int_{t-\rho}^{t+\rho} \omega_\rho(t-s)f(s)ds$. Evidently, $f_\rho(s) \in C_0^\infty(]a-\rho, b+\rho[)$. It is well known that

$$\|f_\rho - f\|_{L_1(a,b)} \to 0 \ (\rho \to 0+). \tag{9.2}$$

(*See* [16].) Consider the function $f_\rho(t)$ on the segment $[a+\rho, b-\rho]$. By (9.1), we have:

$$f_\rho(t) = \int_a^b \frac{d}{dt}\omega_\rho(t-s)f(s)ds = -\int_a^b \frac{d}{ds}\omega_\rho(t-s)f(s)ds \leq 0,$$

because, for any fixed $t \in [a+\rho, b-\rho]$, the expression $\omega_\rho(t-s)$, as a function of s, belongs to $C_0^\infty(]a,b[)$ and $\omega_\rho(s) \geq 0$. Thus, $f_\rho(t)$ is a monotone non-increasing function on the segment $[a+\rho, b-\rho]$. Due to (9.2), there is a sequence $\rho_m \to 0+$ $(m \to \infty)$ such that $f_{\rho_m}(t) \to f(t)$ $(m \to \infty)$ almost everywhere in $t \in [a,b]$. Put $Q = \{t \mid f_{\rho_m}(t) \nrightarrow f(t)\ (m \to \infty)\}$. Evidently, $\mu(Q) = 0$. Let $t, \tau \in]a,b[\backslash Q$, $t \geq \tau$. Chose ρ such that $t, \tau \in [a+\rho, b-\rho]$. Then the function $f_{\rho_m}(t)$ non-increases on the segment $[a+\rho, b-\rho]$ when $\rho_m \leq \rho$, *i.e.* $f_{\rho_m}(t) \leq f_{\rho_m}(\tau)$. Passing to the limit, we obtain that $f(t) \leq f(\tau)$ for $t \geq \tau$; $t, \tau \in]a,b[\backslash Q$ and **(ii)** is established.

Let us show that **(ii)** implies **(i)**. Let $f(t)$ is a monotone non-increasing function on $[a,b]$ and $\varphi \in C_0^\infty(]a,b[)$, $\varphi \geq 0$. Then $\mathbf{supp}(\varphi) \subseteq [a+h_0, b-h_0]$ for some $h_0 > 0$. Notice that $f(s-h) \geq f(s)$ for any $s \in [a+h_0, b]$, and every $h : 0 < h \leq h_0$. Therefore,

$$\int_a^b (f(s-h) - f(s))\varphi(s)ds \geq 0.$$

Hence

$$\int_a^b f(s-h)\varphi(s)ds \geq \int_a^b f(s)\varphi(s)ds \Longrightarrow \int_a^b f(s)\varphi(s+h)ds \geq \int_a^b f(s)\varphi(s)ds,$$

$$0 \leq h \leq h_0.$$

Consequently, the function $J(h) = \int_a^b f(s)\varphi(s+h)ds \geq J(0)$ when $h \leq h_0$. But $J(h) \in C^\infty([0+h_0])$ and therefore, $J'(0) \geq 0$. Since $J'(h) = \int_a^b f(s)\varphi'(s+h)ds$, we get $\int_a^b f(s)\varphi'(s)ds = J'(0) \geq 0$. This completes the proof of Lemma. □

COROLLARY 9.1. – *Let $g \in L_2(t_1, t_2; V')$ and $u(s) \in L_2(t_1, t_2; V) \cap L_\infty(t_1, t_2; H)$. Then the condition (8.5) is equivalent to the following one: there is a set $Q \subset [t_1, t_2]$ of zero measure such that:*

$$\frac{1}{2}\left(|u(t)|^2 - |u(\tau)|^2\right) + \nu\int_\tau^t \|u(s)\|^2 ds \leq \int_\tau^t \langle g(s), u(s)\rangle ds \tag{9.3}$$

for any $t, \tau \in [t_1, t_2]\backslash Q$, $t \geq \tau$.

Proof. – Consider the function $f(t) = \frac{1}{2}|u(t)|^2 + \nu \int_{t_1}^t \|u(s)\|^2 ds - \int_{t_1}^t \langle g(s), u(s)\rangle ds$. Evidently, $f \in L_1(t_1, t_2)$. Integrating in (8.5) by part, we obtain that (8.5) is equivalent to

$$\int_{t_1}^{t_2} f(s)\varphi'(s)ds \geq 0, \tag{9.4}$$

for any $\varphi \in C_0^\infty(]t_1, t_2[)$, $\varphi(s) \geq 0$. By Lemma 9.1, (9.4) is equivalent to (9.3). □

By the same argument, Corollary 8.1 implies the following statement.

COROLLARY 9.2. – *Let $g \in L_2(t_1, t_2; V')$ and $u \in \mathcal{K}_g^{t_1,t_2}$. Then there is a set $Q \subset [t_1, t_2]$ of zero measure such that*

$$|u(t)|^2 - |u(\tau)|^2 + \nu \int_\tau^t \|u(s)\|^2 ds \leq \frac{1}{\nu}\int_\tau^t \|g(s)\|_{V'}^2 ds \tag{9.5}$$

for any $\tau \in [t_1, t_2]\backslash Q,\ t \geq \tau$.

Inequality (9.5) is valid for any $t \geq \tau$ since the function $f(t) = |u(t)|^2 + \nu \int_{t_1}^t \|u(s)\|^2 ds - \int_{t_1}^t \langle g(s), u(s)\rangle ds$ is lower semicontinuous when $u(s)$ is a weak solution of (8.1) (*see* Lemma 7.1 **(ii)**). Indeed, $u(t) \in C_w([t_1, t_2], H)$ implies, by Lemma 7.1 **(ii)**, that $|u(t)|^2$ is lower semicontinuous. Other summands of $f(t)$ is continuous w.r.t. $t \in [t_1, t_2]$. Therefore, $f(t)$ itself is lower semicontinuous.

LEMMA 9.2. – *Let $y(s), a(s) \in L_1^{loc}(0, +\infty)$ and*

$$-\int_0^{+\infty} y(s)\psi'(s)ds + \alpha\int_0^{+\infty} y(s)\psi(s)ds \leq \int_0^{+\infty} a(s)\psi(s)ds, \tag{9.6}$$

for any $\psi \in C_0^\infty(\mathbb{R}_+)$, $\psi(s) \geq 0$, where $\alpha \in \mathbb{R}$; then

$$y(t)e^{\alpha t} - y(\tau)e^{\alpha\tau} \leq \int_\tau^t a(s)e^{\alpha s}ds, \tag{9.7}$$

for any $t, \tau \in \mathbb{R}_+\backslash Q$, $t \geq \tau$, where $\mu(Q) = 0$.

Proof. – Substituting $\psi(s) = \varphi(s)e^{\alpha s}$ in (9.6) we get

$$-\int_0^{+\infty} e^{\alpha s}y(s)\varphi'(s)ds \leq \int_0^{+\infty} a(s)e^{\alpha s}\varphi(s)ds \tag{9.8}$$

for any $\varphi \in C_0^\infty(\mathbb{R}_+)$, $\varphi(s) \geq 0$. Put $f(t) = y(t) - \int_0^t a(s)e^{\alpha s}ds$. Integrating by part in the right-hand side of (9.8) , we obtain (9.4) since

$$\begin{aligned}\int_0^{+\infty} a(s)e^{\alpha s}\varphi(s)ds &= \int_0^{+\infty} \frac{d}{ds}\left(\int_0^s a(\theta)e^{\alpha\theta}d\theta\right)\varphi(s)ds \\ &= -\int_0^{+\infty}\left(\int_0^s a(\theta)e^{\alpha\theta}d\theta\right)\varphi'(s)ds.\end{aligned}$$

Applying Lemma 9.1, we get (9.7). Lemma is proved. □

Let λ_1 be the first eigenvalue of the operator L; it is obvious that:

(9.9) $$\lambda_1|v|^2 \leq \|v\|^2 \ \forall v \in V.$$

Now let $u \in \mathcal{K}_g^+$, where $g \in L_2^{loc}(\mathbb{R}_+; V')$. It follows from (8.12) that:

$$-\int_0^{+\infty} |u(s)|^2\psi'(s)ds + \nu\lambda_1 \int_0^{+\infty} |u(s)|^2\psi(s)ds$$
$$\leq \int_0^{+\infty} \left(\frac{1}{\nu}\|g(s)\|_{V'}^2 - \nu\big[\|u(s)\|^2 - \lambda_1|u(s)|^2\big]\right)\psi(s)ds.$$

Applying Lemma 9.2 for $y(s) = |u(s)|^2$, $a(s) = \frac{1}{\nu}\|g(s)\|_{V'}^2 - \nu[\|u(s)\|^2 - \lambda_1|u(s)|^2]$, and $\alpha = \lambda_1\nu$, we obtain

COROLLARY 9.3. – *If $u \in \mathcal{K}_g^+$, where $g \in L_2^{loc}(\mathbb{R}_+; V')$, then we obtain:*

(9.10)
$$|u(t)|^2e^{\alpha t} - |u(\tau)|^2e^{\alpha\tau} + \nu\int_\tau^t \left(\|u(s)\|^2 - \lambda_1|u(s)|^2\right)e^{\alpha s}ds \leq \frac{1}{\nu}\int_\tau^t \|g(s)\|_{V'}^2e^{\alpha s}ds,$$

for any $t, \tau \in \mathbb{R}_+\backslash Q_u$, $t \geq \tau$, where $\mu(Q_u) = 0$.

Notice that inequality (9.10) is valid for any $\tau \in \mathbb{R}_+\backslash Q_u$, $t \geq \tau$, since $|u(t)|^2$ is lower semicontinuous on $\mathbb{R}_+$.

From this point on we assume that $g(s)$ is translation-bounded in $L_2^{loc}(\mathbb{R}_+, V')$ (*see* (8.19)). Consider the value

(9.11) $$\beta_\alpha(g) = \sup_{h\in[1,2]}\sup_{t\geq 0}\left(\frac{\alpha\int_0^h \|g(s+t)\|_{V'}^2e^{\alpha s}ds}{e^{\alpha h}-1}\right), \ \alpha > 0.$$

It is easy to check that:

(9.12) $$\frac{\alpha}{e^\alpha - 1}\|g\|_{L_2^a(\mathbb{R}_+;V')}^2 \leq \beta_\alpha(g) \leq \frac{2\alpha e^\alpha}{e^\alpha - 1}\|g\|_{L_2^a(\mathbb{R}_+;V')}^2 \leq 2(1+\alpha)\|g\|_{L_2^a(\mathbb{R}_+;V')}^2.$$

On the other hand, if $\|g(s)\|_{V'}^2 \equiv \|g_0\|_{V'}^2$ does not depend on time then $\beta_\alpha(g) \equiv \|g_0\|_{V'}^2$.

COROLLARY 9.4. – *Let $u(s) \in \mathcal{K}_g^+$ and let $g(s)$ be translation-bounded in $L_2^{loc}(\mathbb{R}_+; V')$. Then*

(9.13)
$$|u(t)|^2e^{\alpha t} - |u(\tau)|^2e^{\alpha\tau} + \nu\int_\tau^t \left(\|u(s)\|^2 - \lambda_1|u(s)|^2\right)e^{\alpha s}ds \leq \frac{\beta_\alpha(g)}{\alpha\nu}\left(e^{\alpha t} - e^{\alpha\tau}\right),$$

for any $\tau \in \mathbb{R}_+\backslash Q_u$, $t \geq \tau + 1$, where $\mu(Q_u) = 0$.

Proof. – To prove (9.13) we have to estimate the right-hand side of (9.10). Let $\tau \in \mathbb{R}_+\backslash Q_u$, $t \geq \tau + 1$. There is an integer $m \in \mathbb{N}$ such that $1 \leq (t-\tau)/m < 2$. Denote $h = (t-\tau)/m$ and $t_i = \tau + ih$, $1 = 0, \ldots, m-1$. We get:

$$\int_\tau^t \|g(s)\|_{V'}^2e^{\alpha s}ds = \sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} \|g(s)\|_{V'}^2e^{\alpha s}ds = \sum_{i=0}^{m-1} e^{\alpha t_i}\int_0^h \|g(s+t_i)\|_{V'}^2e^{\alpha s}ds$$
$$\leq \frac{\beta_\alpha(g)}{\alpha}\sum_{i=0}^{m-1} e^{\alpha t_i}\left(e^{\alpha h} - 1\right) = \frac{\beta_\alpha(g)}{\alpha}\sum_{i=0}^{m-1}\left(e^{\alpha t_{i+1}} - e^{\alpha t_i}\right) = \frac{\beta_\alpha(g)}{\alpha}\left(e^{\alpha t} - e^{\alpha\tau}\right). \square$$

REMARK 9.1. – *If $g_0(s)$ is translation-bounded in $L_2^{loc}(\mathbb{R}_+, V')$ then, evidently, $\beta_\alpha(g) \le \beta_\alpha(g_0)$ for any $g \in \mathcal{H}_+(g_0)$.*

2) Now let us prove inequality (8.20), the main part of Proposition 8.2. To get (8.20) we shall derive analogous inequalities for $\|T(t)u\|_{L_\infty(\mathbb{R}_+;H)}$, for $\|T(t)u\|_{L_2^a(\mathbb{R}_+;V)}$, for $\|T(t)u\|_{\mathcal{N}^{\gamma,a}(\mathbb{R}_+;H)}$, and for $\|T(t)\partial_t u\|_{L_{4/3}^a(\mathbb{R}_+;V')}$, since $\|T(t)u\|_{\mathcal{F}_+^a}$ is a sum of these terms.

COROLLARY 9.5. – *Let $g_0(s)$ be translation-bounded in $L_2^{loc}(\mathbb{R}_+;V')$, then*

(i)

$$\|T(t)u\|^2_{L_\infty(\mathbb{R}_+,H)} \le \|u\|^2_{L_\infty(0,1;H)} \exp(-\alpha t) + R_1 \ \forall t \ge 1 \tag{9.14}$$

for any $g \in \mathcal{H}_+(g_0)$ and $u(s) \in \mathcal{K}_g^+$, where $R_1 = (\alpha\nu)^{-1}\beta_\alpha(g_0)$, $\alpha = \lambda_1\nu$.

(ii)

$$\|T(t)u\|^2_{L_2^a(\mathbb{R}_+,V)} \le (1+\alpha)\nu^{-1}\|u\|^2_{L_\infty(0,1;H)} \exp(-\alpha t) + R_2 \ \forall t \ge 1, \tag{9.15}$$

where $R_2 = (2e^\alpha - 1)\nu^{-1}R_1$.

Proof. – **(i)** (9.14) follows directly from (9.13) since, by (9.9), the integral in the left-hand side of (9.13) is positive.

(ii) Integrating (9.14) over $[t, t+1]$, we obtain:

$$\alpha \int_t^{t+1} |u(s)|^2 e^{\alpha s} ds \le \alpha \|u\|^2_{L_\infty} + R_1\alpha \int_t^{t+1} e^{\alpha s} ds = \alpha\|u\|^2_{L_\infty} + R_1(e^\alpha - 1)e^{\alpha t}, \tag{9.16}$$

where $\|u\|^2_{L_\infty} = \|u\|^2_{L_\infty(0,1;H)}$. Combining (9.13) and (9.14), we get

$$\begin{aligned}\nu \int_t^{t+1} \big(\|u(s)\|^2 - \lambda_1|u(s)|^2\big)e^{\alpha s} ds &\le R_1(e^{\alpha(t+1)} - e^{\alpha t}) + |u(t)|^2 e^{\alpha t} \\ &\le R_1(e^\alpha - 1)e^{\alpha t} + \|u\|^2_{L_\infty} + R_1 e^{\alpha t} = \|u\|^2_{L_\infty} + R_1 e^\alpha e^{\alpha t}.\end{aligned}$$

Therefore

$$\nu \int_t^{t+1} \|u(s)\|^2 e^{\alpha s} ds \le \alpha \int_t^{t+1} |u(s)|^2 e^{\alpha s} ds + \|u\|^2_{L_\infty} + R_1 e^\alpha e^{\alpha t}.$$

Taking into account (9.16) we conclude that

$$\nu \int_t^{t+1} \|u(s)\|^2 e^{\alpha s} ds \le (1+\alpha)\|u\|^2_{L_\infty} + R_1(2e^\alpha - 1)e^{\alpha t},$$

so that,

$$\nu \int_t^{t+1} \|u(s)\|^2 ds \le e^{-\alpha t}\nu \int_t^{t+1} \|u(s)\|^2 e^{\alpha s} ds \le (1+\alpha)\|u\|^2_{L_\infty} e^{-\alpha t} + R_1(2e^\alpha - 1),$$

and (9.15) is proved. □

REMARK 9.2. – *To estimate* $\|T(t)u\|^2_{L_2^a(\mathbb{R}_+,V)}$ *we have used (9.13) only. At the same time, it is easily seen from (9.5) that*

$$\nu\int_t^{t+1}\|u(s)\|^2ds \le |u(t)|^2 + \nu^{-1}\int_t^{t+1}\|g(s)\|^2_{V'}ds.$$

Then by (9.14),

$$\|T(t)u\|^2_{L_2^a(\mathbb{R}_+,V)} \le \nu^{-1}\|u\|^2_{L_\infty(0,1;H)}e^{-\alpha t} + \nu^{-1}\Big(R_1 + \nu^{-1}\|g_0\|^2_{L_2^a(\mathbb{R}_+;V')}\Big).$$

Using (8.15), it follows from (9.14) and (9.15) that

$$(9.17)\quad \left(\int_t^{t+1}\|B(u(s))\|^{4/3}_{V'}ds\right)^{3/4} \le C_1\left(\int_t^{t+1}|u(s)|^{2/3}\|u(s)\|^2ds\right)^{3/4}$$
$$\le C_1 \operatorname*{\mathbf{essup}}_{s\in[t,t+1]}|u(s)|^{1/2}\left(\int_t^{t+1}\|u(s)\|^2ds\right)^{3/4}$$
$$\le C_1\big(\|u\|^2_{L_\infty}e^{-\alpha t} + R_1\big)^{1/4}\big((1+\alpha)\nu^{-1}\|u\|^2_{L_\infty}e^{-\alpha t} + R_2\big)^{3/4}$$
$$\le C_1\big(\big(1+(1+\alpha)\nu^{-1}\big)\|u\|^2_{L_\infty}e^{-\alpha t} + R_1 + R_2\big) = C_4\|u\|^2_{L_\infty}e^{-\alpha t} + R_3.$$

Equation (8.1) implies that

$$\left(\int_t^{t+1}\|\partial_t u(s)\|^{4/3}_{V'}ds\right)^{3/4} \le \nu\left(\int_t^{t+1}\|Lu(s)\|^{4/3}_{V'}ds\right)^{3/4} + \left(\int_t^{t+1}\|B(u(s))\|^{4/3}_{V'}ds\right)^{3/4}$$
$$+\left(\int_t^{t+1}\|g(s)\|^{4/3}_{V'}ds\right)^{3/4} \le \nu\left(\int_t^{t+1}\|u(s)\|^2ds\right)^{1/2} + \left(\int_t^{t+1}\|B(u(s))\|^{4/3}_{V'}ds\right)^{3/4}$$
$$+\left(\int_t^{t+1}\|g(s)\|^2_{V'}ds\right)^{1/2}$$
$$\le \nu\big((1+\alpha)\nu^{-1}\|u\|^2_{L_\infty}e^{-\alpha t} + R_2\big)^{1/2}$$
$$+ C_4\|u\|^2_{L_\infty}e^{-\alpha t} + R_3 + \|g_0\|_{L_2^a(\mathbb{R}_+;V')}.$$

In the last inequality we have used (9.15) and (9.17). Finally, we get:

COROLLARY 9.6. – *For any* $g\in\mathcal{H}_+(g_0)$ *and* $u(s)\in\mathcal{K}_g^+$

$$\|T(t)\partial_t u\|_{L^a_{4/3}(\mathbb{R}_+,V')} \le C_5\|u\|^2_{L_\infty(0,1;H)}\exp(-\alpha t) + R_4\ \forall t\ge 1.$$

Let us estimate $\|T(t)u\|_{\mathcal{X}^{\gamma,a}(\mathbb{R}_+;H)}$ (*see* 8.14)). Put $w(s) = \chi_{t,t+1}(s)u(s)$, $f(s) = \chi_{t,t+1}(s)(-\nu Lu - B(u) + g)$. Let $\hat{w}(\tau)$ and $\hat{f}(\tau)$ be the Fourier transforms of $w(s)$

and $f(s)$. We have:

$$(9.18)\qquad \sup_{\tau\in\mathbb{R}}\|\hat{f}(\tau)\|_{V'} \le \int_t^{t+1}\|f(s)\|_{V'}ds \le \nu\int_t^{t+1}\|u(s)\|ds + \int_t^{t+1}\|B(u(s))\|_{V'}ds + \int_t^{t+1}\|g(s)\|_{V'}ds$$
$$\le \nu\left(\int_t^{t+1}\|u(s)\|^2ds\right)^{1/2} + \left(\int_t^{t+1}\|B(u(s))\|_{V'}^{4/3}ds\right)^{3/4} + \left(\int_t^{t+1}\|g(s)\|_{V'}^2ds\right)^{1/2} \le C_5\|u\|^2_{L_\infty(0,1;H)}\exp(-\alpha t) + R_5.$$

By virtue of equation (8.1)

$$(9.19)\qquad \frac{d}{dt}w(s) = f(s) + u(t)\delta_t(s) - u(t+1)\delta_{t+1}(s),$$

where δ_t, δ_{t+1} are the Dirac distributions at t and $t+1$ (*see* [19]). Similarly to ([19]), by the Fourier transform, (9.19) gives

$$(9.20)\qquad 2\pi i\tau\hat{w}(\tau) = \hat{f}(\tau) + u(t)e^{-2\pi i t\tau} - u(t+1)e^{-2\pi i(t+1)\tau},\ \tau\in\mathbb{R}.$$

We multiply (9.20) by $\hat{w}(\tau)$ in H :

$$(9.21)\qquad 2\pi i\tau|\hat{w}(\tau)|^2 = \left\langle \hat{f}(\tau), \hat{w}(\tau)\right\rangle + \langle u(t), \hat{w}(\tau)\rangle e^{-2\pi i t\tau} - \langle u(t+1), \hat{w}(\tau)\rangle e^{-2\pi i(t+1)\tau}.$$

Hence, using (9.21), we get (as in [19]) that:

$$\||\tau|^\gamma\hat{w}(\tau)|^2_{L_2(\mathbb{R}_\tau;H)} \le 2C'(t)C_7(\gamma)\left(\int_t^{t+1}\|u(s)\|^2ds\right)^{1/2} + 2\int_t^{t+1}|u(s)|^2ds,$$

where $C'(t) = C_6\|u\|^2_{L_\infty}e^{-\alpha t} + R_6$. Using (9.15), we obtain

COROLLARY 9.7. – *For any $g\in\mathcal{H}_+(g_0)$ and $u(s)\in\mathcal{K}_g^+$*

$$\|T(t)u\|_{\mathcal{X}^{\gamma,a}(\mathbb{R}_+;H)} \le C_7\|u\|^2_{L_\infty(0,1;H)}\exp(-\alpha t) + R_7\ \forall t\ge 1.$$

Finally, inequality (8.20) follows from Corollaries 9.5, 9.6, and 9.7. Proposition 8.2 is proved. □

10. Some applications

In this section we study some perturbation and approximation problems for 3D Navier-Stokes system and for the dissipative hyperbolic equation considered in the previous sections. We prove that the trajectory attractors of these equations are stable with respect to small perturbation of their symbols. In some cases, when the time shift of the perturbing symbol on the value $h > 0$ tends to zero as $h \to +\infty$ in a weak sense, it is shown that perturbation does not effect to the trajectory attractors: the trajectory attractor of the perturbed equation coincides with the trajectory attractor of the non-perturbed one. If the symbol of the equation under consideration contains a small parameter ε, then we establish that the trajectory attractor $\mathcal{A}_\varepsilon$ tends from below (in the corresponding topology) as $\varepsilon \to 0$ to the trajectory attractor $\mathcal{A}_0$ of the limit equation. This limit behaviour is valid even through the equations without the uniqueness theorem of the Cauchy problem. Besides, we investigate trajectory attractors $\mathcal{A}^{(N)}$ of Galerkin approximation systems of the above equations. We prove that $\mathcal{A}^{(N)}$ converge from bellow as $N \to \infty$ to the trajectory attractor of the origin equation. In what follows we only sketch the ideas of the proofs. The detailed description will be given in the other publication.

1. Trajectory attractors of perturbed equations

a) Consider the Navier-Stokes system (8.1) with a perturbed external force (symbol) $g_0(x,s) = g_{01}(x,s) + g_{02}(x,s)$. We assume that both functions $g_{01}(x,s)$ and $g_{02}(x,s)$ are tr.-c. in $L_{2,w}^{loc}(\mathbb{R}_+; H)$, or, equivalently, they are bounded in $L_2^a(\mathbb{R}_+; H)$. Denote $\mathcal{H}_+(g_{0i}), i = 1, 2$, the hulls of these functions. As usually, the translation semigroup $\{T(h) \mid h \geq 0\}$ acts on $\mathcal{H}_+(g_{0i}) : T(t)g_i(x,s) = g_i(x, h+s)$. Let $\omega(\mathcal{H}_+(g_{0i}))$ be the ω-limit sets of these hulls (w.r.t. $\{T(h)\}$). Assume that:

$$T(h)g_{02}(x,s) = g_{02}(x, h+s) \to 0 \ (h \to +\infty) \text{ in } L_{2,w}^{loc}(0,1; H),$$

i.e.

$$T(h)g_{02}(x,s) \to 0 \ (h \to +\infty) \text{ in } L_{2,w}^{loc}(\mathbb{R}_+; H). \tag{10.1}$$

Consequently, $\omega(\mathcal{H}_+(g_{02})) = \{0\}$ and therefore

$$\omega(\mathcal{H}_+(g_0)) = \omega(\mathcal{H}_+(g_{01})). \tag{10.2}$$

THEOREM 10.1. – *Under the above conditions, the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ of the perturbed 3D Navier-Stokes system coincides with the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_{01})}$ of the non-perturbed system:*

$$\mathcal{A}_{\mathcal{H}_+(g_0)} = \mathcal{A}_{\mathcal{H}_+(g_{01})}. \tag{10.3}$$

The proof follows from formulas (3.5), (8.22), and (10.2) because

$$\mathcal{A}_{\mathcal{H}_+(g_0)} = \mathcal{A}_{\mathcal{H}_+(g_{01}+g_{02})} = \mathcal{A}_{\omega(\mathcal{H}_+(g_{01}+g_{02}))} = \mathcal{A}_{\omega(\mathcal{H}_+(g_{01}))} = \mathcal{A}_{\mathcal{H}_+(g_{01})}. \tag{10.4}$$

As an example, consider the perturbing external force:

$$g_{02}(x,s) = G(x)\sin s^2, \tag{10.5}$$

where $G(x) \in H$. Evidently, $G(x)\sin(t+h)^2 \to 0$ $(h \to +\infty)$ weakly in $L_{2,w}^{loc}(t_1, t_2; H)$ for any $[t_1, t_2] \subseteq \mathbb{R}_+$ and (10.1) takes place. Roughly speaking, more and more rapidly oscillating term $g_{02}(x,s)$ does not effect to the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_{01})}$.

b) Consider the family of hyperbolic equations (7.1) with symbols $\sigma(s) = \sigma_1(s) + \sigma_2(s) = (f_1(v,s) + f_2(v,s), g_1(x,s) + g_2(x,s))$. Here $\sigma(s) \in \mathcal{H}_+(\sigma_{01} + \sigma_{02})$ $(i = 1, 2)$. The functions $f_{0i}(v,s)$ $(i = 1, 2)$ satisfy (7.2)-(7.6) and they are tr.-c. in $C(\mathbb{R}_+; \mathcal{M}_0)$ (*see* section 7). Assume that the perturbing function $f_{02}(v,s)$ satisfies the condition of type (10.1):

$$\max_{|v|\le R} (|f_{02}(v,s)| + |f'_{02t}(v,s)|) \le \beta(R,s),\ \beta(R,h) \to 0\ (h \to +\infty)\ \forall R > 0,$$

i.e.

$$T(h) f_{02}(v,s) \to 0\ (h \to +\infty) \text{ in } C(\mathbb{R}_+; \mathcal{M}_0). \tag{10.6}$$

The functions $g_{0i}(x,s)$ are tr.-c. in $L^{loc}_{2,w}(\mathbb{R}_+; H)$, $(H = L_2(\Omega))$. Besides, the function $g_{02}(x,s)$ satisfies (10.1). Let $\mathcal{H}_+(\sigma_{0i})$ $(i = 1, 2)$ be the hulls of symbols σ_{0i} in $C(\mathbb{R}_+; \mathcal{M}_0) \times L^{loc}_{2,w}(\mathbb{R}_+; H)$.

Similarly to (10.2), one gets

$$\omega(\mathcal{H}_+(\sigma_0)) = \omega(\mathcal{H}_+(\sigma_{01} + \sigma_{02})) = \omega(\mathcal{H}_+(\sigma_{01})). \tag{10.7}$$

THEOREM 10.2. – *Under the above conditions, the trajectory attractor of the equation (7.1) with the symbol* $\sigma_0(s) = \sigma_{01}(s) + \sigma_{02}(s) = (f_{01}(v,s) + f_{02}(v,s), g_{01}(x,s) + g_{02}(x,s))$ *is*

$$\mathcal{A}_{\mathcal{H}_+(\sigma_0)} = \mathcal{A}_{\mathcal{H}_+(\sigma_{01})}. \tag{10.8}$$

The proof is similar to one of Theorem 10.1 and it uses formulas (3.5), (7.33), and (10.7).

Notice that the perturbing function $g_{02}(x,s)$ can be of the type (10.5) and

$$f_{02}(v,s) = \alpha(s) f(v),\ \alpha(h) \to 0\ (h \to +\infty),$$

where $f(v)$ satisfies (7.2)-(7.6) and it can be of any power $p > 1$ with respect to v.

Consider one more example. Let $f_{01}(v,s) \equiv f_{01}(v)$ and $g_{01}(x,s) \equiv g_{01}(x)$ do not depend on time s and $f_{02}(v,s)$, $g_{02}(x,s)$ satisfy the same conditions as above. Let also $g_{01}(x) \in L_2(\Omega)$, $f_{01}(v)$ satisfies (7.2)-(7.6), and the following inequality be valid: $|f_{01v}(v)| \le C_0(1 + |v|^{2/(n-2)})$ (for $n \ge 3$). In this case, for the autonomous equation (7.1) with $f = f_{01}$ and $g = g_{01}$, the uniqueness theorem of the Cauchy problem takes place (*see* [19]). What is more, we assume that this equation possesses a finite number of equilibrium points $\{z_1(x), \ldots, z_N(x)\} : \Delta z_i(x) - f_1(z_i) + g_1(x) = 0,\ z_i|_{\partial\Omega} = 0\ (i = 1, \ldots, N)$, and all of them are hyperbolic. Then the trajectory attractor $\mathcal{A}_{(f_{01}, g_{01})}$ consists of all complete trajectories $\{u(s),\ s \in \mathbb{R}\}$ of this equation that lies on the union of unstable manifolds $M^u(z_i)$ passing through $z_i(x)$, $\lim_{s\to-\infty} u(s) = z_i$ $(i = 1, \ldots, N)$:

$$\mathcal{A}_{(f_1, g_1)} = \Pi_+ \bigcup_{i=1}^{N} \{u(s), s \in \mathbb{R} \mid u(s) \in C_b(\mathbb{R}, E),\ u(s) \in M^u(z_i)\}.$$

This fact follows from the results of section 7 and [1]. Then (10.8) is valid.

2. Dependence of trajectory attractors on a small parameter.

a) Consider equation (7.1) with a symbol $\sigma^0(s,\varepsilon) = (f_0(v,s) + \varepsilon f_1(v,s), g_0(x,s) + \varepsilon g_1(x,s))$, where functions $f_i(v,s)$ $(i = 1,2)$ satisfy (7.2)-(7.6) and they are tr.-c. in $C(\mathbb{R}_+; \mathcal{M}_0)$. Let also $g_i(x,s)$ $(i = 1,2)$ be tr.-c. in $L_{2,w}^{loc}(\mathbb{R}_+; H)$. To construct the trajectory attractor for the equation (7.1) with a symbol $\sigma^0(s,\varepsilon)$, we study the family equations (7.1) with symbols $\sigma(s,\varepsilon) = \sigma(\varepsilon) \in \Sigma(\varepsilon) = \mathcal{H}_+(\sigma_0 + \varepsilon\sigma_1)$, where $\sigma_0(s) = (f_0(v,s), g_0(x,s))$, $\sigma_1(s) = (f_1(v,s), g_1(x,s))$, and $\varepsilon \in [0, \varepsilon_0]$. The hull $\mathcal{H}_+(\sigma)$ is taken in the space $\Xi_+ = C(\mathbb{R}_+; \mathcal{M}_0) \times L_{2,w}^{loc}(\mathbb{R}_+; H)$. For any $\sigma(\varepsilon) \in \Sigma(\varepsilon), \varepsilon \in [0,\varepsilon_0]$, denote $\mathcal{K}_{\sigma(\varepsilon)}^+(M) \subset \mathcal{F}_+^a$ the trajectory space of the equation (7.1) (*see* Definition 7.2 with C_i being replaced by $C_i(1+\varepsilon_0)$). According to Theorem 7.1, the translation semigroup $\{T(t)\}$, acting on the united trajectory space $\mathcal{K}_\varepsilon^+(M) = \mathcal{K}_{\Sigma(\varepsilon)}^+(M) = \cup_{\sigma(\varepsilon)\in\Sigma(\varepsilon)}\mathcal{K}_{\sigma(\varepsilon)}^+(M)$, possesses the trajectory attractor $\mathcal{A}_{\Sigma(\varepsilon)}$ in the topology Θ_+ which was described in section 7. The set $\mathcal{A}_{\Sigma(\varepsilon)}$ does not depends on M. Consider the semigroup $\{S(t), t \geq 0\}$ acting on the extended phase space $\mathcal{F}_+^a \times [0,\varepsilon_0]$ and on $\bigcup_{0\leq\varepsilon\leq\varepsilon_0} \mathcal{K}_\varepsilon^+(M) \times \{\varepsilon\}$ by the formula:

$$S(t)(u_{\sigma(\varepsilon)}(s), \varepsilon) = (u_{\sigma(\varepsilon)}(s+t), \varepsilon),$$

where $u_{\sigma(\varepsilon)}(s) \in \mathcal{K}_{\sigma(\varepsilon)}^+(M)$.

The following statement generalizes Theorem 7.1.

THEOREM 10.3. – *Let the symbol $\sigma^0(s,\varepsilon) = (f_0(v,s) + \varepsilon f_1(v,s), g_0(x,s) + \varepsilon g_1(x,s))$ satisfies the above conditions. Then the semigroup $\{S(t), t \geq 0\}$ acting on $\bigcup_{0\leq\varepsilon\leq\varepsilon_0} \mathcal{K}^+(\varepsilon) \times \{\varepsilon\}$ possesses the global attractor $\mathcal{A}$ with the following properties:*

i) *the set $\mathcal{A}$ is compact in $\Theta_+ \times [0,\varepsilon_0]$;*

ii) *the set $\mathcal{A}$ is a union of trajectory attractors $\mathcal{A}_{\Sigma(\varepsilon)} \times \{\varepsilon\}$, $\varepsilon \in [0,\varepsilon_0]$:*

$$\mathcal{A} = \bigcup_{\varepsilon\in[0,\varepsilon_0]} \mathcal{A}_{\Sigma(\varepsilon)} \times \{\varepsilon\},$$

where $\mathcal{A}_{\Sigma(\varepsilon)}$ is the trajectory attractor of the family of equations (10.1) *with symbols $\sigma \in \Sigma(\varepsilon)$;*

iii) *the trajectory attractors $\mathcal{A}_{\Sigma(\varepsilon)}$ converge to the trajectory attractor $\mathcal{A}_{\Sigma(0)} = \mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ as $\varepsilon \to 0$ in the topology Θ_+. In particular, we have:*

$$\mathbf{dist}_{L_r(0,R;E_{1-\delta})}\left(\Pi_{0,R}\mathcal{A}_{\Sigma(\varepsilon)}, \Pi_{0,R}\mathcal{A}_{\mathcal{H}_+(\sigma_0)}\right) \to 0 \ (\varepsilon \to 0) \ \forall R > 0,\ 0 < \delta \leq 1, r > 1.$$

The proof is similar to one given in [1] but a little bit longer.

3. Convergence of trajectory attractors of Galerkin approximation systems.

Consider ones more 3D Navier-Stokes system (8.1) with an external force $g_0(x,s)$ that is translation-bounded function in $L_2^{loc}(\mathbb{R}_+; H)$. Let $\mathcal{H}_+(g_0)$ be a hull of it. Let we be given some complete system $\{w_j(x)\}$ of functions in V. Let P_m be the orthogonal projector from H onto the space H_m spanned by $\{w_j(x)\}_{j=1}^m$. Consider the Faedo-Galerkin approximation system (8.7) of order m. The symbol of this system is $P_m g_0(x,s)$

and this function is tr.-c. in $L^{loc}_{2,w}(\mathbb{R}_+; H)$ as well. It follows easily that this system of m ordinary differential equations possesses a trajectory space $\mathcal{K}^+_g \subset \mathcal{F}^a_+$ for any $g \in \mathcal{H}_+(P_m g_0)$ and the family $\{\mathcal{K}^+_g,\ g \in \mathcal{H}_+(P_m g_0)\}$ satisfies the same properties as the family corresponding to the origin symbol $g_0(x, s)$ described is section 8. Therefore, the results of section 4 are applicable and the analog of Theorem 8.2 takes place: the Galerkin approximation system possess the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(P_m g_0)\}} = \mathcal{A}^{(m)}$ in the space $L^{loc}_{2,w}(\mathbb{R}_+; V) \cap L^{loc}_{\infty,*w}(\mathbb{R}_+; H) \cap \mathcal{X}^{\gamma,loc}_w(\mathbb{R}_+; H) \cap \{v \mid \partial_t v \in L^{loc}_{4/3,w}(\mathbb{R}_+; V')\} = \Theta^{loc}_+$. The set $\mathcal{A}^{(m)}$ is compact in Θ^{loc}_+ and uniformly bounded in $\mathcal{F}^a_+$:

$$\|\mathcal{A}^{(m)}\|_{\mathcal{F}^a_+} \le C. \tag{10.9}$$

THEOREM 10.4. – *The trajectory attractors $\mathcal{A}^{(m)}$ of the Faedo-Galerkin approximation system (8.7) converge as $m \to \infty$ (in Θ^{loc}_+) to the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ of the origin system (8.1) in the following sence: fore any neighbourhood $\mathcal{O}(\mathcal{A}_{\mathcal{H}_+(g_0)})$ (in Θ^{loc}_+) there exists a number $N = N(\mathcal{O})$ such that:*

$$\mathcal{A}^{(m)} \subset \mathcal{O}(\mathcal{A}_{\mathcal{H}_+(g_0)})\ \forall m \ge N.$$

In particular, for any $R > 0$,

$$\mathbf{dist}_{L_2(0,R;H_{1-\delta})}\left(\Pi_{0,R}\mathcal{A}^{(m)}, \Pi_{0,R}\mathcal{A}_{\mathcal{H}_+(g_0)}\right) \to 0\ (m \to \infty),\ 0 < \delta \le 1.$$

The proof makes use the standard reasoning.

The similar result is valid for the trajectory attractor $\mathcal{A}^{(m)}$ of the Faedo-Galerkin approximation system corresponding to dissipative hyperbolic equation (7.1) having a nonlinear function $f(v, s)$ with an arbitrary polynomial growth with respect to v.

11. Proofs of Theorem 3.1, Corollary 3.2 and Corollary 3.3

PROOFS OF THEOREM 3.1. – We are given the Banach space $\mathcal{F}^a_+$ with the norm (2.6). Notice that we don't use the topology generated by this norm (it is very strong); we use the norm (2.6) to define bounded sets in $\mathcal{F}^{loc}_+$ only. The space $\mathcal{F}^a_+$, as a set, belongs to the topological space Θ^{loc}_+. The translation semigroup $\{T(t)\}$ acts on Θ^{loc}_+. Each mapping $T(t)$ is continuous in the topology of Θ^{loc}_+. In the space $\mathcal{F}^a_+$ we consider the united trajectory space $\mathcal{K}^+_\Sigma$. By Proposition 3.1, the set $\mathcal{K}^+_\Sigma$ is closed in Θ^{loc}_+ and, by Proposition 3.1 the set $\mathcal{K}^+_\Sigma$ is invariant w.r.t. $\{T(t)\} : T(t)\mathcal{K}^+_\Sigma \subseteq \mathcal{K}^+_\Sigma\ \forall t \ge 0$. Let P be a set from Θ^{loc}_+ that attracts bounded (in $\mathcal{F}^a_+$) sets $B \subset \mathcal{K}^+_\Sigma$ as $t \to \infty$ in the topology Θ^{loc}_+. The set P is assume to be bounded in $\mathcal{F}^a_+$ and compact in Θ^{loc}_+. Notice, the set P need not belong to $\mathcal{K}^+_\Sigma$.

We can not apply directly Proposition 3.3 (or another theorem about attractors of semigroups in Banach space) in the described situation, because, generally speaking, the topology Θ^{loc}_+ does not coincides with the strong topology of $\mathcal{F}^a_+$. The theorem similar to Theorem 3.1 was proved in [1], where the theory of (F, D)-attractors was studied. However the assumptions of those Theorem differs from the assumptions of Theorem 3.1.

This is why we present here the complete proof of Theorem 3.1; the reasoning follows the usual general scheme (*see* [1], [24], [13], etc.).

We recall that the topological space Θ_+^{loc} is a Hausdorff space. We separate the proof into few steps.

Le B be a bounded set in $\mathcal{F}_+^a$, $B \subseteq \mathcal{K}_\Sigma^+$. Consider ω-limit set of B in Θ_+^{loc} :

$$\omega(B) = \bigcap_{t\geq 0} \left[\bigcup_{h\geq t} T(h)B \right]_{\Theta_+^{loc}} . \tag{11.1}$$

Here $[\,.\,]_{\Theta_+^{loc}}$ means the closure in Θ_+^{loc}.

1) Let us show that $y \in \omega(B)$ if and only if for any neighbourhood $V(y) = V$ (in Θ_+^{loc}) of the point y there exist two sequences $\{x_n\} \subseteq B$ and $\{t_n\} \subseteq \mathbb{R}_+$, $t_n \to +\infty$ $(n \to \infty)$, such that $T(t_n)x_n \in V$.

Indeed, let $y \in \omega(B)$, then, for any $t \geq 0$, the point y is a point of tangency of the set $\bigcup_{h\geq t} T(h)B$. So, any neighbourhood $V(y)$ contains a point from $\bigcup_{h\geq t} T(h)B$ for any $t \geq 0$. Therefore, there are sequences $\{x_n\} \subseteq B$, $t_n \to +\infty$ $(n \to \infty)$, such that $T(t_n)x_n \in V$. Let us prove the converse statement. Let for any $V(y)$ there are $\{x_n\} \subseteq B$, $t_n \to +\infty$ $(n \to \infty)$, such that $T(t_n)x_n \in V$. But $T(t_n)x_n \subseteq \bigcup_{h\geq t} T(h)B$ when $t_n \geq t$. Hence, y is a point of tangency of $\bigcup_{h\geq t} T(h)B$ for any $t \geq 0$, *i.e.* $y \in \left[\bigcup_{h\geq t} T(h)B\right]_{\Theta_+^{loc}}$ for any $t \geq 0$ and thereby $y \in \omega(B)$. (Notice, if Θ_+^{loc} is a space with the first axiom of countability, for example, a metric space, then $y \in \omega(B)$ $\Longleftrightarrow \exists \{x_n\} \subseteq B, \{t_n\} \subseteq \mathbb{R}_+$, $T(t_n)x_n \to y$ $(t_n \to \infty)$. Usually, one utilizes this property to prove Proposition 3.3 and equality (3.3)).

It follows from (11.1) that $\omega(B)$ is closed in Θ_+^{loc}.

2) Let us prove that $\omega(B) \neq \emptyset$ and $\omega(B)$ attracts $T(t)B$ as $t \to \infty$ in Θ_+^{loc}. Let $\{x_n\} \subseteq B$ and $t_n \to +\infty$ $(n \to \infty)$. Consider the set $M = \{y_n\} \cup P$, where $y_n = T(t_n)x_n$. Let us prove, that M is countably compact. Let $\{V_n\}$ be any countable covering of M. Evidently, it covers P. But, P is compact. Consider a finite subcovering $\{V_{n_i} \mid i = 1, \ldots, N\}$ of M. Denote $V = \bigcup_{i=1}^N V_{n_i}$. Since P attracts $T(t)B$, there exists N_1 such that $y_n = T(t_n)x_n \in T(t_n)B \subset V$ for $n \geq N_1$. Therefore, $\{V_{n_i} \mid i = 1, \ldots, N\}$ covers $M \backslash \{y_1, \ldots, y_{N_1}\}$. Adding the finite number of open sets that cover the finite set $\{y_1, \ldots, y_{N_1}\}$, we obtain the finite subcovering of M. Hence, the set M is countably compact. This means, by the definition, that the set $\{y_n\}$ has a limit point, if $\{y_n\}$ is infinite. By refining, we may assume that all points $\{y_n\}$ are different. Let the set $\{y_n\}$ is infinite and y is a limit point of $\{y_n\}$. We claim that $y \in \omega(B)$. Let V be any neighbourhood of y. Then there is $y_{n_1} \in V$, $y_{n_1} \neq y$. The space Θ_+^{loc} is a Hausdorff space. So, there is a neighbourhood $W \subset V$ of y that $y_{n_1} \notin W$. Similarly, there exist $y_{n_2} \in W$, $y_{n_2} \neq y$. Using this procedure, we get a subsequence $\{y_{n_i}\} \subseteq V$, *i.e.* $T(t_{n_i})x_{n_i} \in V$, $x_{n_i} \in B$, $t_n \to +\infty$ $(n \to \infty)$. So, by virtue of **1)**, $y \in \omega(B)$. If $\{y_n\}$ is finite, then, evidently, for some y, $y_n = y$ infinitely many times, *i.e.* $T(t_{n_i})x_{n_i} = y_{n_i} = y$ for some subsequence $\{n_i\}$ and, hence, $y \in \omega(B)$. Finally, $\omega(B) \neq \emptyset$.

Let us show that $\omega(B)$ attracts $T(t)B$. Assume the converse. There are a neighbourhood $\mathcal{O}(\omega(B))$ and sequences $\{x_n\} \subseteq B$, $t_n \to +\infty$ $(n \to \infty)$ such that $y_n = T(t_n)x_n \notin$

$\mathcal{O}(\omega(B))$. Similarly to the point **1)**, one proves that the set $\{y_n\}$ has a limit point y and $y \in \omega(B)$. But $\mathcal{O}(\omega(B))$ is also a neighbourhood of y. So, $y_N \in \mathcal{O}(\omega(B))$ for some N. Contradiction.

3) Now let us prove that $\omega(B) \subseteq P$, $\omega(B)$ is compact, and $\omega(B)$ is a minimal compact set that attracts $T(t)B$ as $t \to \infty$, *i.e.* any compact attracting set contains $\omega(B)$. Let $y \in \omega(B)$ and $y \notin P$. Since Θ_+^{loc} is a Hausdorff space, for any $x \in P$, there exist a neighbourhood V_x of x and a neighbourhood W_x of y such that $V_x \cap W_x = \emptyset$. The family of open sets $\{V_x \mid x \in P\}$ covers P. Consider a finite subcovering $\{V_{x_i} \mid i = 1, \dots, N\}$. Put $V = \bigcup_{i=1}^N V_{x_i}$, $W = \bigcap_{i=1}^N W_{x_i}$. Then $P \subseteq V$, $y \in W$, $V \cap W = \emptyset$. Since $y \in \omega(B)$, $T(t_n)x_n \in W$ for some sequences $\{x_n\} \subseteq B$, $t_n \to +\infty$ $(n \to \infty)$ (*see* point **1)**). It follows from the attracting property of P that $T(t_n)x_n \in V$ if $t_n \geq t' = t'(V)$, *i.e.* $V \cap W \neq \emptyset$. Contradiction. Therefore, $y \in P$ and $\omega(B) \subseteq P$. By (11.1), the set $\omega(B)$ is closed and, hence, $\omega(B)$ is compact in Θ_+^{loc}. By the similar way one proves that $\omega(B)$ belongs to any compact attracting set P', *i.e.* $\omega(B)$ is the minimal compact attracting set for B.

4) It follows from the continuity of the semigroup $\{T(t)\}$ that the set $\omega(B)$ is strictly invariant:

$$T(t)\omega(B) = \omega(B) \ \forall t \geq 0. \tag{11.2}$$

Indeed, let $y \in \omega(B)$. Fix any $t_0 \geq 0$. Consider any neighbourhood $V(z)$ of the point $z = T(t_0)y$. The mapping $T(t_0)$ is continuous, therefore, there exists a neighbourhood $W(y)$ of y such that $T(t_0)W(y) \subseteq V(z)$. For the neighbourhood $W(y)$ there are $\{x_n\} \subseteq B$, $t_n \to +\infty$ $(n \to \infty)$ such that $T(t_n)x_n \in W(y)$ (*see* point **1)**). Then $T(t_0 + t_n)x_n = T(t_0)T(t_n)x_n \in V(z)$ and, once more, by **1)**, $z = T(t_0)y \in \omega(B)$, *i.e.* $T(t_0)\omega(B) \subseteq \omega(B)$. Let us check the inverse inclusion. We have proved that $T(t_0)\omega(B)$ is compact and it attracts B. The first is evident, since the continuous image $T(t_0)\omega(B)$ of a compact set $\omega(B)$ is a compact set. Let V be any open set that contains $T(t_0)\omega(B)$. Put $W = T(t_0)^{-1}V$. Evidently, W is an open set, $\omega(B) \subseteq W$, and $T(t_0)W \subseteq V$. The set $\omega(B)$ attracts B, *i.e.* for W there is t' such that $T(t)B \subseteq W$ when $t \geq t'$. Then $T(t_0 + t)B = T(t_0)T(t)B \subseteq V \ \forall t \geq t'$. Therefore, $T(t_0)\omega(B)$ attracts B. Using point **3)**, we get: $\omega(B) \subseteq T(t_0)\omega(B)$ and (11.2) is proved.

5) We now proceed to a trajectory attractor construction; put:

$$\mathcal{A}_\Sigma = \left[\bigcup_{B \subset \mathcal{K}_\Sigma^+} \omega(B)\right]_{\Theta_+^{loc}} = [\mathcal{A}_0]_{\Theta_+^{loc}}, \tag{11.3}$$

where the union is taken in all sets $B \subset \mathcal{K}_\Sigma^+$, bounded in $\mathcal{F}_+^a$. In virtue of **2)**, the set $\mathcal{A}_\Sigma$ attracts any bounded set $B \subset \mathcal{K}_\Sigma^+$. At the same time, by **3)**, $\mathcal{A}_\Sigma \subseteq P$, $\mathcal{A}_\Sigma$ is compact in Θ_+^{loc}, and it is the minimal compact attracting set. Therefore, $\mathcal{A}_\Sigma$ is the trajectory attractor.

6) Let us show that $\mathcal{A}_\Sigma$ is strictly invariant with respect to $\{T(t)\}$, *i.e.* (3.4) takes place. Consider the set $\mathcal{A}_0 = \bigcup_{B \subset \mathcal{K}_\Sigma^+} \omega(B)$. By (11.2), we get: $T(t)\mathcal{A}_0 = \mathcal{A}_0 \ \forall t \geq 0$. At the same time, $\mathcal{A}_0 \subseteq P$ and $\mathcal{A}_0$ is bounded in $\mathcal{F}_+^a$. Therefore P attracts $\mathcal{A}_0$. Consider

$\omega(\mathcal{A}_0)$. By **4)**, $\omega(\mathcal{A}_0)$ is invariant: $T(t)\omega(\mathcal{A}_0) = \omega(\mathcal{A}_0)$. By the definition of ω-limit set, according to (11.3), we have:

$$\omega(\mathcal{A}_0) = \bigcap_{t\geq 0}\left[\bigcup_{h\geq t} T(h)\mathcal{A}_0\right]_{\Theta_+^{loc}} = \bigcap_{t\geq 0}[\mathcal{A}_0]_{\Theta_+^{loc}} = [\mathcal{A}_0]_{\Theta_+^{loc}} = \mathcal{A}_\Sigma.$$

Hence, $\mathcal{A}_\Sigma$ is strictly invariant as well.

7) Finally, let us prove that $\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)}$. Notice that the proved above part of Theorem 3.1 is also applicable to the family $\{\mathcal{K}_\sigma^+, \sigma \in \omega(\Sigma)\}$. In particular, on $\mathcal{K}_{\omega(\Sigma)}^+ = \bigcup_{\sigma\in\omega(\Sigma)} \mathcal{K}_\sigma^+$, there exists a trajectory attractor $\mathcal{A}_{\omega(\Sigma)}$ of the translation semigroup $\{T(t)\}$ in Θ_+^{loc}. The set $\mathcal{A}_{\omega(\Sigma)}$ is strictly invariant with respect to $\{T(t)\}$. By (11.3), $\mathcal{A}_{\omega(\Sigma)} \subseteq \mathcal{A}_\Sigma$. To prove the inverse inclusion we have to check that $\mathcal{A}_\Sigma \subseteq \mathcal{K}_{\omega(\Sigma)}^+$, since $\mathcal{A}_\Sigma$ is strictly invariant and it is bounded in $\mathcal{F}_+^a$. Let $y \in \mathcal{A}_\Sigma$. Then, by (3.4), for any $n \in \mathbb{N}$, there exist $u_n \in \mathcal{A}_\Sigma$ such that $T(n)u_n = y$. Moreower, $u_n \in \mathcal{K}_{\sigma_n}^+$ for some $\sigma_n \in \Sigma$. It is easy to show that the sequence $\{T(n)\sigma_n\}$ has a limit point $\sigma \in \omega(\Sigma)$, *i.e.* $\sigma'_{n_i} = T(n_i)\sigma_{n_i} \to \sigma$ $(n \to \infty)$ in Σ for some subsequence $\{n_i\}$. Since the family $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ is translation-coordinated, $y = T(n_i)u_{n_i} \in \mathcal{K}_{T(n_i)\sigma_{n_i}}^+ = \mathcal{K}_{\sigma'_i}^+$. The family $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ is (Θ_+^{loc}, Σ)-closed, therefore, $y \in \mathcal{K}_\sigma^+$, *i.e.* $y \in \mathcal{K}_{\omega(\Sigma)}^+$. Consequently, $\mathcal{A}_\Sigma \subseteq \mathcal{K}_{\omega(\Sigma)}^+$. Theorem 3.1 is proved. □

PROOF OF COROLLARY 3.2. – First of all, remark that $u \in \mathcal{K}_\sigma^+$, $u' \in \mathcal{K}_{\sigma'}^+$, $T(1)u' = u$ do not imply $T(1)\sigma' = \sigma$. Therefore the statement is meaningful. So, let $u_0 \in \mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)}$. By (3.2) and (3.4), it follows that for any $n \in \mathbb{N}$ there exists $u^n \in \mathcal{A}_\Sigma$, $u^n \in \mathcal{K}_{\sigma^n}^+$, such that $T(n)u^n = u_0$. Consider the sequence $\{T(n)\sigma^n\}_{n\in\mathbb{N}}$ from the compact set $\omega(\Sigma)$. It has a limit point $\sigma_0 \in \omega(\Sigma)$ and, therefore, for some subsequence $T(n_{i,0})\sigma^{n_{i,0}} \to \sigma_0$ $(n_{i,0} \to \infty)$. On the other hand, $T(n_{i,0})u^{n_{i,0}} = u_0$ and the family $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ is (Θ_+^{loc}, Σ)-closed. This implies that $u_0 \in \mathcal{K}_{\sigma_0}^+$. Consider the sequences $\{T(n_{i,0}-1)\sigma^{n_{i,0}}\}$, $\{T(n_{i,0}-1)u^{n_{i,0}}\}$. Both of them possess limit points σ_{-1} and u_{-1} in $\omega(\Sigma)$ and $\mathcal{A}_\Sigma$ respectively. By refining, we may assume that $T(n_{i,1}-1)\sigma^{n_{i,1}} \to \sigma_{-1}$, $T(n_{i,1}-1)u^{n_{i,1}} \to u_{-1}$ $(n_{i,1} \to \infty)$, where $\{n_{i,1}\}$ is a subsequence of $\{n_{i,0}\}$. (The sets $\omega(\Sigma)$ and $\mathcal{A}_\Sigma$ belong to Fréchet-Uryson spaces Σ and Θ_+^{loc} respectively.) Extending this procedure, we obtain, for any $k \in \mathbb{N}$, points $\sigma_{-k} \in \omega(\Sigma)$, $u_{-k} \in \mathcal{A}_\Sigma$ such that $T(n_{i,k}-k)\sigma^{n_{i,k}} \to \sigma_{-k}$, $T(n_{i,k}-k)u^{n_{i,k}} \to u_{-k}$ $(n_{i,k} \to \infty)$, where $\{n_{i,k}\}$ is a subsequence of $\{n_{i,k-1}\}$, and $u_{-k} \in \mathcal{K}_{\sigma_{-k}}^+$. Using the diagonalization method , we put $m_i = n_{i,i}$. Thus, for any $k \in \mathbb{N}$,. we have $T(m_i - k)\sigma^{m_i} \to \sigma_{-k}$, $T(m_i - k)u^{m_i} \to u_{-k}$ $(m_i \to \infty)$. Let us prove that $T(1)\sigma_{-k} = \sigma_{-(k-1)}$ and $T(1)u_{-k} = u_{-(k-1)}$. Indeed, since the mapping $T(1)$ is continuous we get $T(1)T(m_i-k)\sigma^{m_i} = T(m_i-(k-1))\sigma^{m_i} \to T(1)\sigma_{-k}$ $(m_i \to \infty)$. But $T(m_i-(k-1))\sigma^{m_i} \to \sigma_{-(k-1)}$ $(m_i \to \infty)$. Consequently, $T(1)\sigma_{-k} = \sigma_{-(k-1)}$. For the same reason, $T(1)u_{-k} = u_{-(k-1)}$. Let us produce the function $\gamma(l)$, $l \in \mathbb{R}$. Put $\gamma(l) = (T(l)u_0, T(l)\sigma_0)$ for $l \geq 0$ and $\gamma(l) = (u_l, \sigma_l)$ for $l = -1, -2, \ldots$, where σ_{-l} and u_{-l} are constructed above. Finally, put $\gamma(l) = (T(k+l)u_{-k}, T(l+k)\sigma_{-k})$ for $l \in]-k, -(k-1)[$. Points $u_l \in \mathcal{K}_{\sigma_l}^+$ for any $l \in \mathbb{N}$, because the family $\{\mathcal{K}_\sigma^+, \sigma \in \Sigma\}$ is tr.-coord. It follows easily that $T(t)\gamma(l) = \gamma(l+t)$ for any $l \in \mathbb{R}$, $t \geq 0$. This concludes the proof. □

PROOF OF COROLLARY 3.3. – Let $\sigma \in \omega(\Sigma)$. By (3.3), it follows that for any $n \in \mathbb{N}$ there exist $\sigma_n \in \omega(\Sigma)$ such that $T(n)\sigma_n = \sigma$. Consider any element $u_n \in \mathcal{K}^+_{\sigma_n}$ such that $u_n \in B_R$. Similar to the proof of Theorem 3.1, one can show that the sequence $\{T(n)u_n\}$ has a limit point $u \in \omega(B_R)$. Since Θ^{loc}_+ is a Fréchet-Uryson space, u is the limit of some subsequence $v_k = T(n_k)u_{n_k} : T(n_k)u_{n_k} \to u$ $(k \to \infty)$ in Θ^{loc}_+. By tr.-coord. property, $v_k \in \mathcal{K}^+_{T(n_k)\sigma_{n_k}} = \mathcal{K}^+_\sigma$. It is readily seen that $u \in \mathcal{K}^+_\sigma$, because $\{\mathcal{K}^+_\sigma, \sigma \in \Sigma\}$ is (Θ^{loc}_+, Σ)-closed. Corollary 3.3 is proved. □

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