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STABLE ALGORITHM FOR STATIONARY DISTRIBUTION CALCULATION FOR A $BMAP/SM/1$ QUEUEING SYSTEM WITH MARKOVIAN ARRIVAL INPUT OF DISASTERS

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Abstract

Disaster arrival into a queueing system causes all customers to leave the system instantaneously. Numerically stable algorithm for calculating the stationary state distribution of embedded Markov chain for the $BMAP/SM/1$ queue with a MAP input of disasters is presented.

Keywords: $BMAP/SM/1$-type queue; disaster; censored Markov chain; stable algorithm

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1. Introduction

The theory of queues with negative arrivals was originated by paper of E. Gelenbe [7]. Detailed review of achieved results is presented in [1]. Disaster is a special kind of negative arrival which causes immediate departure of all customers (including one processed on the server) from the queue. In addition to [1], the paper [8] can be recommended for related references.

General model of the $BMAP/SM/1$ type with the MAP input of disasters was investigated by A.N. Dudin and S. Nishimura [5]. A key point for calculation of all performance characteristics of the model is calculation of the stationary state distribution of the embedded at customers departure epochs Markov chain. The procedure, which is presented in [5], exploits the analyticity of the vector generating

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function of the stationary distribution in a unit disc of a complex plane. It includes calculation of roots of some function in the unit disc. In case of high dimensionality of the probability vectors (it is defined by the product of dimensionalities of the state spaces of the underlying processes of the BMAP input, semi-Markovian (SM) service and MAP flow of disasters), root determination problems and problems relating to computation of derivatives can arise. After the probability vector corresponding to a zero state of the queue length is calculated, all other probability vectors are calculated recurrently from the equilibrium equations. This procedure is not very stable because it includes subtraction operation.

Shortcomings of such an approach are well-known and alternative approach for calculation of the stationary state probability vectors of similar Markov chains was elaborated by M. Neuts and V. Ramaswami, see, e. g. [12], [13]. However, this alternative procedure was elaborated only for so called skip-free to the left multi-dimensional Markov chains which have upper Hessenberg block transition matrix. In case of the disaster appearance, the embedded Markov chain does not belong to the class of skip-free to the left Markov chains. So, the approach elaborated in [13] can not be applied.

This approach exploits the idea of so called censored Markov chains, see, e. g. [9], [10]. Using the same idea, in this paper we present a stable procedure for calculation of the stationary distribution of embedded Markov chain for the $BMAP/SM/1$ system with disasters.

The rest of the paper consists of the following. Brief description of the model is presented in section 2. The embedded Markov chain under consideration is defined in section 3. In section 4, the algorithm for calculation the stationary distribution that stems from [5] is given and the alternative stable algorithm is presented. Section 5 contains three numerical examples illustrating the stationary distribution calculation by means of the present algorithm and algorithm from [5]. Some conclusions basing on more extensive computer comparison of the mentioned algorithms are made.

2. Model

Description of the model coincides with the one given in [5]. For convenience, we maintain denotations of that paper.

We consider a single-server system with an unlimited waiting space. The input to the system is a batch Markovian arrival process (BMAP). The set of the BMAP's is dense in the set of all Markov point processes, so the BMAP's suit for adequate modelling a wide range of flows in real life systems, telecommunication networks in particular. The BMAP is determined by the directing process $\nu_t, t \geq 0$ having a state space $\{0, 1, \ldots, L\}$ and by a matrix generating function $D(z) = \sum_{k=0}^{\infty} D_k z^k, |z| \leq 1$ where the $(L+1) \times (L+1)$ matrices D_k define the intensities of the process $\nu_t, t \geq 0$ transitions which are accompanied by generation of the batch consisting of k customers arriving into a system, $k \geq 0$. For more information about a BMAP, see [3] and [11].

A service process is of SM (semi-Markovian) type. It means that the successive service times are the sojourn times of a semi-Markovian process $m_t, t \geq 0$. This process has a state space $\{1, ..., M\}$ and a semi-Markovian kernel $B(t)$.

The arrival of disasters to the system is a MAP (partial case of the BMAP allowing only non-group arrivals) which is determined by the directing process $\eta_t, t \geq 0$ with a state space $\{0, 1, ..., N\}$ and a matrix generating function $F(z) = F_0 + F_1 z, |z| \leq 1$. Disaster arrival causes immediate removal of all customers from the system. If the system is empty at a disaster arrival epoch, this disaster is ignored by the system.

3. Embedded Markov chain

Let t_n be the n-th epoch of customer departures from the system. It's a service completion epoch or a disaster arrival epoch at a busy period.

Consider the following four-dimensional Markov chain:

$$\xi_n = \{i_n, \nu_n, \eta_n, m_n\},\ n \geq 1,$$

where i_n is a queue length at the epoch $t_n + 0, i_n \geq 0$; ν_n is the state of arrival directing process ν_t at the epoch t_n, $\nu_n = \overline{0, L}$; η_n is the state of disasters directing process η_n at the epoch $t_n + 0, \eta_n = \overline{0, N}$ and m_n is the state of service directing process m_t at the epoch $t_n + 0, m_n = \overline{1, M}$.

Denote by $P\{(i,\nu,\eta,m) \to (l,\nu',\eta',m')\}$ the one-step transition probabilities of Markov chain $\xi_n, n \geq 1$:

$$P\{(i,\nu,\eta,m) \to (l,\nu',\eta',m')\} =$$
$$= P\{i_{n+1} = l, \nu_{n+1} = \nu', \eta_{n+1} = \eta', m_{n+1} = m' | i_n = i, \nu_n = \nu, \eta_n = \eta, m_n = m\},$$
$$i,l \geq 0,\ \nu,\nu' = \overline{0,L},\ \eta,\eta' = \overline{0,N},\ m,m' = \overline{1,M}.$$

Enumerate the states of the Markov chain $\xi_n, n \geq 1$ in lexicographic order and form the matrices $P_{i,l}$, consisting of transition probabilities $P\{(i,\nu,\eta,m) \to (l,\nu',\eta',m')\}$. These matrices are defined by formulas:

$$P_{i,l} = \begin{cases} \Omega_{l-i+1}, & \text{if } i > 0,\ l > 0,\ l \geq i-1, \\ S, & \text{if } i > 0,\ l = 0, \\ \Omega_0 + S, & \text{if } i = 1,\ l = 0, \\ \sum_{j=1}^{l+1} \Psi_j \Omega_{l-j+1}, & \text{if } i = 0,\ l > 0, \\ \Psi_1 \Omega_0 + \sum_{j=1}^{\infty} \Psi_j S, & \text{if } i = 0,\ l = 0, \\ O, & \text{if } l < i-1,\ l \neq 0, \end{cases} \tag{1}$$

where the matrices $\Omega_l, l \geq 0$ are defined by the matrix expansion

$$\sum_{l=0}^{\infty} \Omega_l z^l = \hat{\beta}(z) = \int_0^{\infty} e^{D(z)t} \otimes e^{F_0 t} \otimes dB(t), \tag{2}$$

the matrices S and $\Psi_j, j \geq 0$ are defined as:

$$S = \int_0^{\infty} e^{D(1)t} \otimes \left(e^{F_0 t} F_1\right) \otimes (B(\infty) - B(t))dt, \tag{3}$$

$$\Psi_j \quad = \quad \int_0^{\infty} \left(e^{D_0 t} D_j\right) \otimes e^{F(1)t} dt \otimes E_M. \tag{4}$$

Here $\otimes$ is symbol of the Kronecker product, $E_\bullet$ denotes an identity matrix of corresponding size. Index can be omitted if its value is clear from context.

Formulas (1)–(4) along with the probabilistic sense of involved matrices are presented in [5].

4. Main result

Due to the disaster presence, under the evident assumptions about the irreducibility and non-periodicity of directing processes $\nu_t, \eta_t, m_t, t \geq 0$ and existence of positive finite means of their sojourn times in all states the stationary state probabilities $\pi(i,\nu,\eta,m) = \lim_{n\to\infty} P\{i_n = i, \nu_n = \nu, \eta_n = \eta, m_n = m\}$ exist.

Denote by $\boldsymbol{\pi}_i$ the vector of stationary probabilities $\pi(i,\nu,\eta,m)$ listed in the lexicographic order, $i \geq 0$.

In [5], the algorithm for calculation of unknown vector $\boldsymbol{\pi}_0$ is presented. As follows from [5], the rest of the vectors $\boldsymbol{\pi}_i, i \geq 1$ can be calculated as:

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_0 F_i + \boldsymbol{\pi}^* Q_i, \quad i \geq 1, \tag{5}$$

where $\boldsymbol{\pi}^*$ is a stochastic row eigenvector of the matrix $S + \hat{\beta}(1)$ (its explicit form is presented in [5]) and the matrices $F_i, Q_i, \geq 1$ are calculated recurrently:

$$F_1 = (E - \Psi_1\Omega_0 - (\Psi - E)ZS)\Omega_0^{-1}, \tag{6}$$

$$Q_1 = -S\Omega_0^{-1}, \tag{7}$$

$$F_{l+1} = \left(F_l - \sum_{n=1}^{l+1} \Psi_n \Omega_{l-n+1} - \sum_{n=1}^{l} F_n \Omega_{l-n+1}\right)\Omega_0^{-1}, \quad l \geq 1, \tag{8}$$

$$Q_{l+1} = \left(Q_l - \sum_{n=1}^{l} Q_n \Omega_{l-n+1}\right)\Omega_0^{-1}, \quad l \geq 1, \tag{9}$$

where

$$\Psi = -((D_0 \oplus F(1))^{-1}((D(1) - D_0) \otimes E_{N+1})) \otimes E_M,$$
$$Z = (E - S - \hat{\beta}(1) + \mathbf{1}\boldsymbol{\pi}^*)^{-1},$$

$\mathbf{1}$ is column vector consisting of 1.

Since recursions (8), (9) contain subtraction operation, it is expectable that they are not stable in computer realization. Our numerical experience confirms this instability.

Also, non-singularity of the matrix Ω_0 is required in (6)-(9). Although this assumption can be not very restrictive (see [4]), sometimes it does not hold.

Thus, below we present the alternative algorithm for a stable calculation of vectors $\boldsymbol{\pi}_i, i \geq 0$.

Let G be the matrix, which characterizes transitions of components $\{\nu_n, \eta_n, m_n\}$ of the Markov chain $\xi_n, n \geq 1$ in the time interval during which the state of the component

i_n changes from $k+1$ to k and no disaster appears, $k \geq 0$. It is easy to verify that the matrix G does not depend on the value of k, $k \geq 0$ and satisfies the matrix equation

$$G = \hat{\beta}(G) = \sum_{l=0}^{\infty} \Omega_l G^l. \tag{10}$$

This equation can be rewritten in the form

$$G = (E - \Omega_1)^{-1} \left(\Omega_0 + \sum_{l=2}^{\infty} \Omega_l G^l \right)$$

and its solution can be found as $G = \lim_{n\to\infty} G_n$, where the matrices $G_n, n \geq 1$ are calculated recurrently:

$$G_{n+1} = (E - \Omega_1)^{-1} \left(\Omega_0 + \sum_{l=2}^{\infty} \Omega_l G_n^l \right), \quad n \geq 0$$

with appropriate initial condition (in our case, e.g., $G_0 = 0$), see, e.g. [12].

Matrix $E - \Omega_1$ is not singular due to H'Adamard or O. Taussky theorem (see [6]) as the matrix Ω_1 is substochastic.

More involved procedures for solving equations of type (10) can be found, e.g. in [14].

Let H be the matrix that characterizes the transitions of components $\{\nu_n, \eta_n, m_n\}$ of the Markov chain $\xi_n, n \geq 1$ in the interval which starts from the state k of the component i_n and finishes by reaching the state 0 without visiting the state $k-1$, $k \geq 1$ (due to a disaster arrival). This matrix does not depend on k and is calculated as:

$$H = (G - E)(\hat{\beta}(1) - E)^{-1} S. \tag{11}$$

Non-singularity of the inverted matrix also follows from H'Adamard's theorem. Formula (11) for the matrix H is derived basing on the following equation which is clear from this matrix definition:

$$H = S + \big(P_{k,k} + \sum_{n=k+1}^{\infty} \sum_{i=k}^{n} G^{n-i}\big) H.$$

Theorem 1. *The probability vectors $\boldsymbol{\pi}_i, i \geq 0$ are calculated by the following way:*

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_0 \Phi_i, \ i \geq 1, \tag{12}$$

where the matrices Φ_i are calculated recurrently:

$$\Phi_0 = E, \qquad \Phi_k = \big(\overline{V}_k + \sum_{i=1}^{k-1} \Phi_i \overline{Y}_k^{(i)}\big)\big(E - \overline{Y}_k^{(k)}\big)^{-1}, \quad k \geq 1, \tag{13}$$

where the matrices $\overline{Y}_k^{(i)}, \overline{V}_k$ are calculated as

$$\overline{Y}_k^{(i)} = \sum_{l=0}^{\infty} \Omega_{l+k-i+1} G^l, \quad i = \overline{1,k}, \qquad \overline{V}_k = \sum_{l=0}^{\infty} P_{0,l+k} G^l, \quad k \geq 1, \tag{14}$$

vector $\boldsymbol{\pi}_0$ satisfies the system

$$\boldsymbol{\pi}_0 (E - \overline{V}_0) = \mathbf{0}, \qquad \boldsymbol{\pi}_0 \sum_{i=0}^{\infty} \Phi_i \mathbf{1} = 1, \tag{15}$$

where

$$\overline{V}_0 = \sum_{k=0}^{\infty} P_{0,k} G^k + \sum_{k=1}^{\infty} P_{0,k} \sum_{i=0}^{k-1} G^i H, \tag{16}$$

$\mathbf{0}$ *is row-vector consisting of 0.*

Present the sketch of the proof. The notion of a censored Markov chain [9],[10] is exploited.

Brief definition of a censored Markov chain for a scalar case is the following. Let $\zeta_n, n \geq 1$ be the Markov chain with the state space $\{0, 1, \dots\}$ and one-step transition probability matrix P. Fix some integer $k, k \geq 0$ and consider the process $\zeta_n^{(k)}, n \geq 1$ defined on the subset $\{0, 1, \dots, k\}$. Its trajectories are obtained from the trajectories of the original Markov chain $\zeta_n, n \geq 1$ by removing all epochs when $\zeta_n > k$. The process $\zeta_n^{(k)}, n \geq 1$ is called a censored process. It is known from [9],[10] that this process is the Markov chain as well and its stationary probability vector is the stationary probability vector of the original Markov chain $\zeta_n, n \geq 1$ restricted to the states $\{0, 1, \dots, k\}$ and normalized to sum 1.

Consider the Markov chain $\xi_n, n \geq 1$ describing the queue under consideration and fix some integer $k, k \geq 1$. Construct the censored Markov chain $\xi_n^{(k)}, n \geq 1$. Its trajectories are obtained from the trajectories of the original Markov chain $\xi_n = \{i_n, \nu_n, \eta_n, m_n\}, n \geq 1$ by means of deleting the moments when the value of component i_n is greater than k. Analyzing transitions of the censored chain, we see that the part of equilibrium equations for the components of the invariant probability vector of the censored Markov chain coincides to the equilibrium equations of original Markov chain $\xi_n, n \geq 1$ but the last equation (which takes into account the transition of components $\{\nu_n, \eta_n, m_n\}$ of the chain $\xi_n, n \geq 1$ during the removed periods of time) has the form

$$\boldsymbol{\pi}_k = \boldsymbol{\pi}_0 \bar{V}_k + \sum_{i=1}^{k} \boldsymbol{\pi}_i \bar{Y}_k^{(i)}, \tag{17}$$

where the matrices $\bar{V}_k, \bar{Y}_k^{(i)}$ are defined by formulas (14).

As mentioned above, the entries $\boldsymbol{\pi}_0, \ldots, \boldsymbol{\pi}_k$ of the stationary probability vector of Markov chain $\xi_n, n \geq 1$ coincide up to the normalizing constant with stationary probability vectors of the censored Markov chain $\xi_n^{(k)}, n \geq 1$. So, they satisfy the system (17) as well.

Since we have fixed $k, k \geq 1$ arbitrarily, the system (17) holds for any k. It implies relations (12), (13). Now, by putting $k = 0$ and taking into account normalization condition we get formulas (15), (16).

It concludes the outline of the proof.

As above, non-singularity of the matrix $E - \bar{Y}_k^{(k)}$ follows from H'Adamard's theorem. It follows from the theorem of Lederman [2] that the matrix $(E - \bar{Y}_k^{(k)})^{-1}$ has only non-negative entries. So, recursions (13) do not involve subtraction operations and are numerically stable.

Remark. It is possible to avoid infinite sums in formulas (14), (16) because we can calculate sums like $\sum_{l=0}^{\infty} \Omega_l G^l$, $\sum_{l=0}^{\infty} P_{0,l} G^l$ analytically. However, it is more convenient from computational point of view to work with such sums.

5. Numerical results

To illustrate the work of the presented algorithm for the stationary state distribution calculation and its comparison with algorithm elaborated in [5] we present the following numerical results.

Example 1. The BMAP-input is characterized by the matrices

$$D_0 = \begin{pmatrix} -1.45 & 0.45 \\ 0.6 & -2.6 \end{pmatrix}, \quad D_1 = D_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}.$$

This BMAP has the fundamental rate $\lambda = 2.14$ and the correlation coefficient $c_c = 0.035$.

We assume that the kernel $B(t)$ that describes the service process has the form

$$B(t) = diag\{B_1(t), \dots, B_M(t)\}P.$$

Here $diag\{a_1, \dots, a_M\}$ denotes a diagonal matrix with the diagonal entries $a_1, \dots, a_M$; P is a stochastic matrix and $B_i(t), i = \overline{1, M}$ are distribution functions.

In our example we set $M = 2$ and

$$B_m(t) = \sum_{i=1}^{k} q_i^{(m)} \int_0^t \frac{\gamma_i^{(m)} (\gamma_i^{(m)} \tau)^{h_i^{(m)}-1}}{(h_i^{(m)} - 1)!} e^{-\gamma_i^{(m)} \tau} d\tau, \quad m = \overline{1, M},$$
$$P = \begin{pmatrix} 0.65 & 0.35 \\ 0.45 & 0.55 \end{pmatrix},$$

$k = 2$, $q_1^{(1)} = 0.6$, $q_2^{(1)} = 0.4$, $q_1^{(2)} = q_2^{(2)} = 0.5$, $\gamma_1^{(1)} = 15$, $\gamma_2^{(1)} = \gamma_1^{(2)} = 10$, $\gamma_2^{(2)} = 8$, $h_1^{(1)} = 1$, $h_2^{(1)} = 2$, $h_1^{(2)} = 3$, $h_2^{(2)} = 4$.

The average service time is equal to 0.242.

For the disasters flow we assume that $N = 2$ and

$$F_0 = \begin{pmatrix} -0.25 & 0.15 \\ 0.24 & -0.33 \end{pmatrix}, \qquad F_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.09 \end{pmatrix}.$$

The flow of disasters has the fundamental rate $\varphi = 0.096$ and the correlation coefficient $\bar{c}_c = 9.84 \cdot 10^{-5}$.

The key point of the both algorithms for calculation of the stationary state distribution is the forming the system of linear algebraic equations for the entries of the vector $\boldsymbol{\pi}_0$.

Using the algorithm presented in [5] we get the roots $z_1 = 0.087$, $z_2 = 0.096$, $z_3 = 0.128$, $z_4 = 0.142$, $z_5 = 0.647$, $z_6 = 0.698$, $z_7 = 0.838$, $z_8 = 0.956$ and the system for the entries or the vector $\boldsymbol{\pi}_0$ in the form

$$\boldsymbol{\pi}_0 \begin{pmatrix} -0.00019 & 0.00021 & -0.0016 & 0.0053 & -0.33 & 0.5 & -1.7 & 2.2 \\ 0.00029 & -0.00033 & 0.0024 & -0.0074 & -0.17 & 0.29 & -1.4 & 2.1 \\ 0.0003 & 0.00022 & 0.0026 & 0.0055 & 0.52 & 0.51 & 2.8 & 2.3 \\ -0.00045 & -0.00034 & -0.0037 & -0.0076 & 0.27 & 0.3 & 2.2 & 2.1 \\ 0.00054 & -0.00061 & -0.00076 & 0.0024 & 0.50 & -0.61 & -1.1 & 1.4 \\ -0.00082 & 0.00091 & 0.0011 & -0.0033 & 0.26 & -0.36 & -0.88 & 1.3 \\ -0.00084 & -0.00062 & 0.0011 & 0.0025 & -0.79 & -0.63 & 1.7 & 1.4 \\ 0.0012 & 0.00094 & -0.0017 & -0.0034 & -0.41 & -0.36 & 1.3 & 1.3 \end{pmatrix} = (-5.5\cdot 10^{-8}, 7.8\cdot 10^{-7}, 4.7\cdot 10^{-7}, -1.8\cdot 10^{-5}, 1.5\cdot 10^{-4}, -2.5\cdot 10^{-3}, -5.2\cdot 10^{-3}, 0.61). \quad (18)$$

When we use the algorithm based on formulas (12)-(16) we get from the system (15) the following relation:

$$\boldsymbol{\pi}_0 \begin{pmatrix} 2.28 & -0.24 & -0.05 & -0.03 & -0.14 & -0.09 & -0.02 & -0.01 \\ 3.33 & 0.72 & -0.06 & -0.05 & -0.13 & -0.11 & -0.03 & -0.02 \\ 2.29 & -0.05 & 0.64 & -0.21 & -0.04 & -0.02 & -0.13 & -0.08 \\ 3.35 & -0.08 & -0.26 & 0.75 & -0.05 & -0.04 & -0.11 & -0.09 \\ 2.66 & -0.10 & -0.02 & -0.01 & 0.60 & -0.23 & -0.04 & -0.02 \\ 4.05 & -0.15 & -0.04 & -0.03 & -0.26 & 0.76 & -0.04 & -0.03 \\ 2.68 & -0.03 & -0.13 & -0.08 & -0.06 & -0.04 & 0.63 & -0.22 \\ 4.08 & -0.05 & -0.15 & -0.13 & -0.07 & -0.06 & -0.23 & 0.78 \end{pmatrix} = (1, 0, \dots, 0). \quad (19)$$

Both systems (18) and (19) have the same solution defined as

$$\boldsymbol{\pi}_0 = (0.0645, 0.048, 0.0402, 0.0298, 0.0551, 0.0395, 0.0342, 0.0244).$$

Example 2. Now we assume that BMAP-input is defined by the matrices

$$D_0 = \begin{pmatrix} -2.15 & 0.39 & 0.66 & 0.6 \\ 0.56 & -2.32 & 0.26 & 0.5 \\ 0.04 & 0.28 & -2 & 0.18 \\ 1.44 & 1.8 & 0.4 & -5.64 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.125 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.375 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} 0.375 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0 \\ 0 & 0 & 1.125 & 0 \\ 0 & 0 & 0 & 1.5 \end{pmatrix}.$$

This BMAP has the fundamental rate $\lambda = 2.14$ and the correlation coefficient $c_c = 0.016$.

The service process is the same as in previous example and the disasters flow is characterized by the matrices

$$F_0 = \begin{pmatrix} -0.5 & 0.15 & 0.2 & 0.05 \\ 0.12 & -0.45 & 0.12 & 0.12 \\ 0.07 & 0.35 & -0.64 & 0.14 \\ 0.18 & 0.12 & 0.24 & -0.43 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.09 & 0 & 0 \\ 0 & 0 & 0.08 & 0 \\ 0 & 0 & 0 & 0.07 \end{pmatrix}.$$

The flow of disasters has the fundamental rate $\varphi = 0.086$ and the correlation coefficient $\bar{c}_c = 2.25 \cdot 10^{-4}$.

In this case the dimensionality of the vectors $\boldsymbol{\pi}_i, i \geq 0$ is 32 and using the algorithm elaborated in [5] we did not get a result after two days of computation on PC Pentium II.

Using the algorithm defined by the formulas (12)-(16) we have got the values of the first 40 vectors of the stationary state distribution with the accuracy 10^{-6} during two hours. Present for example the value of the vector $\boldsymbol{\pi}_0$

$$\boldsymbol{\pi}_0 = (0.006, 0.0046, 0.0098, 0.0075, 0.0061, 0.0046, 0.0043, 0.0033, 0.0103, 0.0077,$$
$$0.017, 0.0126, 0.0105, 0.0078, 0.0074, 0.0055, 0.0183, 0.0134, 0.03, 0.0219, 0.0184,$$
$$0.0134, 0.013, 0.0094, 0.0036, 0.0026, 0.0059, 0.0043, 0.0036, 0.0026, 0.0026, 0.0018).$$

To compare the stability of the both algorithms for calculation of vectors $\boldsymbol{\pi}_i, i > 0$ we return to the data of example 1.

Present for example the form of matrices F_1 and F_3 which are calculated by relations (6), (8).

$$F_1 = \begin{pmatrix} 2.97 & -3.25 & -0.08 & 0.15 & -0.25 & 0.52 & -0.01 & -0.015 \\ -2.70 & 5.70 & 0.04 & -0.32 & 0.12 & -1.09 & 8.7\cdot 10^{-5} & 0.01 \\ -0.13 & 0.25 & 3.01 & -3.33 & -0.02 & -0.02 & -0.24 & 0.53 \\ 0.06 & -0.52 & -2.72 & 5.88 & 0.0001 & 0.02 & 0.12 & -1.09 \\ -0.27 & 0.70 & -0.01 & -0.02 & 3.29 & -4.6 & -0.07 & 0.19 \\ 0.16 & -1.37 & 0.0001 & 0.02 & -3.01 & 8.15 & 0.03 & -0.39 \\ -0.01 & -0.03 & -0.26 & 0.71 & -0.11 & 0.31 & 3.33 & -4.7 \\ 0.0002 & 0.04 & 0.16 & -1.39 & 0.06 & -0.62 & -3.03 & 8.36 \end{pmatrix},$$

$$F_3 = \begin{pmatrix} 142.6 & -251.6 & -12.8 & 32.6 & -50.1 & 141.5 & 3.1 & -14.5 \\ -201.8 & 363.1 & 19.4 & -49.6 & 77.1 & -218.7 & -5.3 & 23.5 \\ -20.6 & 52.2 & 149.5 & -269.1 & 5.0 & -23.2 & -51.8 & 149.2 \\ 31.1 & -79.4 & -212.3 & 389.7 & -8.5 & 37.6 & 79.9 & -231.2 \\ -63.8 & 181.5 & 4.4 & -19.6 & 256.7 & -580.7 & -21.4 & 69.5 \\ 98.7 & -280.3 & -7.4 & 31.8 & -380.2 & 870.5 & 33.7 & -109.3 \\ 7.1 & -31.5 & -66.1 & 192.1 & -34.2 & 111.2 & 268.2 & -617.9 \\ -11.9 & 51.1 & 102.7 & -297.3 & 53.9 & -174.9 & -398.2 & 929.1 \end{pmatrix}.$$

The entries of these matrices have different signs and grow in modulus when the index of the matrix F_i increases. E.g., the entries of the matrix F_6 are already of order 10^5. It leads to numerical instability and the vector $\boldsymbol{\pi}_{10}$ calculated by means of using formulas (5)-(9) is already negative.

Correspondingly, the matrices Φ_1 and Φ_3, which are used in (12), have the following values:

$$\Phi_1 = \begin{pmatrix} 0.263 & 0.155 & 0.036 & 0.023 & 0.112 & 0.069 & 0.021 & 0.013 \\ 0.226 & 0.222 & 0.048 & 0.044 & 0.118 & 0.107 & 0.029 & 0.025 \\ 0.058 & 0.037 & 0.241 & 0.142 & 0.034 & 0.022 & 0.100 & 0.061 \\ 0.076 & 0.071 & 0.199 & 0.197 & 0.047 & 0.041 & 0.101 & 0.093 \\ 0.107 & 0.070 & 0.020 & 0.014 & 0.306 & 0.182 & 0.032 & 0.021 \\ 0.149 & 0.135 & 0.037 & 0.032 & 0.236 & 0.220 & 0.042 & 0.037 \\ 0.033 & 0.023 & 0.095 & 0.061 & 0.052 & 0.033 & 0.287 & 0.170 \\ 0.059 & 0.051 & 0.128 & 0.117 & 0.068 & 0.059 & 0.212 & 0.199 \end{pmatrix},$$

$$\Phi_3 = \begin{pmatrix} 0.033 & 0.024 & 0.008 & 0.006 & 0.031 & 0.023 & 0.008 & 0.006 \\ 0.077 & 0.072 & 0.021 & 0.019 & 0.069 & 0.062 & 0.021 & 0.017 \\ 0.012 & 0.010 & 0.029 & 0.021 & 0.013 & 0.010 & 0.026 & 0.019 \\ 0.034 & 0.031 & 0.065 & 0.061 & 0.032 & 0.028 & 0.058 & 0.052 \\ 0.030 & 0.024 & 0.008 & 0.006 & 0.071 & 0.051 & 0.013 & 0.010 \\ 0.076 & 0.068 & 0.022 & 0.019 & 0.133 & 0.121 & 0.031 & 0.026 \\ 0.013 & 0.011 & 0.025 & 0.020 & 0.021 & 0.016 & 0.063 & 0.045 \\ 0.036 & 0.031 & 0.064 & 0.057 & 0.049 & 0.042 & 0.116 & 0.106 \end{pmatrix}.$$

All entries are non-negative. The entries of matrices $\Phi_i, i \geq 1$ decrease when i increases. E.g., the entries of the matrix Φ_6 are of order 10^{-3}. It allows us to calculate the first 40 vectors of the stationary state distribution with the accuracy 10^{-6} during 10 minutes.

For example the vectors $\boldsymbol{\pi}_i, i = \overline{1,8}$ calculated by formula (12) have the following values:

$$\boldsymbol{\pi}_1 = (0.0469, 0.0356, 0.0294, 0.0223, 0.0454, 0.0331, 0.0284, 0.0207),$$

$$\boldsymbol{\pi}_2 = (0.0216, 0.0181, 0.0135, 0.0113, 0.0257, 0.0207, 0.0161, 0.0130),$$

$$\boldsymbol{\pi}_3 = (0.0134, 0.0115, 0.0084, 0.0072, 0.0179, 0.0148, 0.0112, 0.0093),$$

$$\boldsymbol{\pi}_4 = (0.0078, 0.0068, 0.0049, 0.0043, 0.0115, 0.0097, 0.0072, 0.0061),$$

$$\boldsymbol{\pi}_5 = (0.0049, 0.0042, 0.0031, 0.0027, 0.0076, 0.0065, 0.0048, 0.0041),$$

$$\boldsymbol{\pi}_6 = (0.0030, 0.0026, 0.0019, 0.0017, 0.0049, 0.0042, 0.0031, 0.0027),$$

$$\boldsymbol{\pi}_7 = (0.0019, 0.0017, 0.0012, 0.0010, 0.0032, 0.0028, 0.0020, 0.0017),$$

$$\boldsymbol{\pi}_8 = (0.0012, 0.0011, 0.0008, 0.0007, 0.0021, 0.0018, 0.0013, 0.0011).$$

As the result of other extensive numerical experiments the following conclusions can be stated:

- in the region of parameters space where both algorithms work, calculated probabilities coincide with a high accuracy that confirms the correctness of the presented algorithm;
- the presented algorithm works stably for all values of the input, service and disaster parameters;
- the stability of calculations basing on algorithm from [5] depends essentially on : (a) dimension of vectors, (b) traffic intensity ρ (product of the BMAP fundamental rate and average service time), (c) whether the service process is actually semi-Markovian one of just recurrent (i. e. the kernel $B(t)$ is just a distribution function). In case of vectors dimension greater than 10, the procedure of calculating the vector $\boldsymbol{\pi}_0$ may not work. In case of recurrent service, the procedure for calculating the vectors $\boldsymbol{\pi}_i, i \geq 1$ is not so bad as it is reported sometimes in literature. In case ρ is about 0,4–0,6 and $(L+1)(N+1) = 4$ about 30 first probability vectors are calculated correctly. In case of heavy traffic ($\rho = 0,95$), about 40 vectors are correct. In case of SM-service, the situation is much worse. For $(L+1)(N+1)M = 8$, only about 10 vectors are calculated well;
- calculation of matrices involved into formulas (1)–(4) with accuracy 10^{-10} and neglecting the values of order $10^{-k}, k \leq 10$ in (14)–(16) provides accuracy about

10^{-k} in calculation of $\boldsymbol{\pi}_0$ and even higher accuracy in calculation the rest of vectors $\boldsymbol{\pi}_l, l \geq 1$.

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