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An Optimal Threshold Control for a BMAP/SM/1 System with Map Disaster Flow

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Abstract—A BMAP/SM/1 queueing system with two operation modes, a Markov disaster flow, and a modified threshold control strategy is studied. The stationary state probability distribution of the imbedded Markov chain is determined. An algorithm for finding the optimal modified threshold control strategy for the system is designed.

1. INTRODUCTION

Modern information networks, owing to their increasing development rate, require a more adequate description of the random processes taking place in them. Different priorities of information flows in these networks and the need for high-quality information processing necessitate the optimization of dynamic reallocation of network resources. This, in turn, stimulates enhanced interest in queue control systems. A particular case of such systems is a system with two operation modes, in which the first operation mode is cheaper than the second mode, but service is slower. The operation mode is chosen according to some strategy (for example, single-, many-threshold or hysteresis strategy) for minimizing the economic quality criterion applied to evaluate the performance of the system. Control systems are reviewed in [1–5].

Real queueing systems, even communication networks, are not reliable and diasters may occur in them, thereby violate the operation of the system and, in particular, lead to loss of several or all customers. Mathematical models of such systems describe disasters leading to instantaneous departure of all customers, including the customer under service, from the system. Such disasters are particular cases of the so-called negative customer introduced by Gelenbe in 1991 [6]. Works on systems with negative customers and disasters are listed in [6–10].

Thus, the study of control systems with disasters has become pivotal. The economic quality criterion for these systems includes a penalty for the loss of customers per unit time and a dynamic control (in particular, switching to a faster operation mode) to reduce the queue length, and thereby reduce cost due to the loss of customers.

At present, great attention is paid to systems with BMAP flows [11–21], because the BMAP model aids in more exactly describing flows in modern communication networks since information flows are neither stationary, nor ordinary, nor nonhereditary.

In this paper, we study a system with BMAP flow of customers, a Markov flow of diasters, and two operation modes. A modified threshold strategy is applied to control the system.

2. MATHEMATICAL MODEL

Let us consider a single-server queueing system with waiting and two operation modes. A flow of diasters arrive at the system, causing instantaneous departure of all customers, including the customer under service, from the system.

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The operation of the system in the rth mode, $r = \overline{1,2}$, is described as follows. The input is a BMAP flow, which is defined by a control process ν_t—a continuous-time Markov chain with finite state space $\{0, 1, \dots, W\}$ and matrix generating function $D^{(r)}(z) = \sum\limits_{k=0}^{\infty} D_k^{(r)} z^k$, $|z| \leq 1$. Such a description for the BMAP flow is given in [11]. We assume that the matrix function $D^{(r)}(z)$ satisfies the Lucantoni conditions [11].

Let $\mathbf{b}^{(r)}$ denote the stationary distribution vector of the Markov chain ν_t. The row vector $\mathbf{b}^{(r)}$ is the unique solution of the system of equations

$$\mathbf{b}^{(r)} D^{(r)}(1) = \mathbf{0}, \quad \mathbf{b}^{(r)}\mathbf{1} = 1.$$

Here $\mathbf{0}$ is a row vector of appropriate dimension and consists of zeros, and $\mathbf{1}$ is a column vector consisting of units. The dimension of vectors in what follows depends on the context.

The mean intensity $\lambda^{(r)}$ of the BMAP flow in the rth mode is given by the formula

$$\lambda^{(r)} = \mathbf{b}^{(r)} \frac{dD^{(r)}(z)}{dz}\bigg|_{z=1} \mathbf{1}.$$

Customer service times are determined as sequential times of sojourn of a semi-Markov process m_t, $t \geq 0$ in its states. The state space of the process m_t is $\{1, \dots, M\}$ and its semi-Markov kernel is $B^{(r)}(x) = \|B_{m,m'}^{(r)}(x)\|_{m,m'=\overline{1,M}}$. The function $B_{m,m'}^{(r)}(x)$ defines the probability that the process exists in the current state for a time not greater than x and passes to the state m', provided its current state is m, $m, m' = \overline{1,M}$. We assume that the kernel $B^{(r)}(x)$ satisfies the Neuts condition [22] and Lucantoni–Neuts condition [12]. Let $P^{(r)} = B^{(r)}(\infty)$ denote the transition probability matrix of the imbedded chain for the semi-Markov process m_t. We assume that the process m_t changes its state according to the matrix $P^{(r)}$ at service of completion instants, irrespective of whether the service has been completed successfully or interrupted by the arrival of a disaster.

Diasters arrive at the system in a MAP flow controlled by a continuous-time Markov chain η_t with state space $\{0, 1, \dots, N\}$ and matrix generating function $F^{(r)}(z) = F_0^{(r)} + F_1^{(r)} z, |z| \leq 1$, $r = \overline{1,2}$. Like Jain and Sigman [10], we assume that all customers instantaneously quit the system, including the customer under service, when a disaster arrives at the system. We also assume that upon arrival of a disaster the server resumes his work after a random time having a distribution function $G(t)$. We consider two cases: (a) customers arriving in the renewal period accumulate in the system and (b) cusomers arriving in the renewal period are lost.

We assume that the disasters arriving when the system is empty or in the renewal period are ignored and have no influence on the operation of the system.

The operation of the system is evaluated by the economic quality criterion

$$C = a\Lambda L + c_1\Phi_1 + c_2\Phi_2 + hR, \tag{1}$$

where a is the penalty for the sojourn of a customer in the system, L is the mean number of customers in the system at the instant of departure of a customer, Λ^{-1} is the mean time between instants of departure of customers, Φ_r is the mean coefficient of utilization per unit time for the rth mode, c_r is the cost per unit time of utilization of the rth mode, $r = \overline{1,2}$, R is the mean number of lost customers per unit time, and h is the penalty for a lost customer. We assume that $a > 0$, $c_1 \leq c_2$, and $h \geq 0$.

According to [23], the threshold control strategy is optimal for the operation of an M/G/1 system without disasters. Although the optimality of threshold strategies in the class of Markov strategies for systems with disasters is yet to be demonstrated, it is reasonable to apply a threshold control strategy for our system. The threshold strategy is defined as follows. The system changes its operation mode only at the instants of departure of customers from the system. An integer $j \geq 0$, called the threshold, is defined. If the number i of customers in the system at a departure instant satisfies the inequality $i \leq j$, then the next customer is served in the first operation mode; if $i > j$, then the next customer is served in the second operation mode.

This strategy presupposes that the busy period of a system always begins with operation in the first mode. But this period for our system may begin with the arrival of a batch of customers (if the system is not busy with renewing the server) or completion of the server renewal period (if customers arrive in this period). Thus, the queue at the instant of commencement of a busy period may be very long and the use of the first (slow) operation mode in this case is not desirable, because there is a risk of losing a large number of customers due to the arrival of a diaster. Therefore, we consider a modified threshold strategy, which is described as follows. A system changes its operation mode either at the instant of successful completion of service (if there are customers in the system) or at the instant of commencement of a busy period of the system; the mode is chosen along the same lines as in the classical threshold strategy. We also assume that the system operates in the first mode when it is idle or busy with renewing the server.

Our problem now is to design an algorithm for finding the optimal threshold j minimizing the quality criterion (1). For this purpose, we apply the so-called direct method. We take a threshold j and find the stationary probability distribution for the imbedded Markov chain describing the behavior of the system. Then we compute the value of the quality criterion for a given value of the threshold and then find the optimal threshold value minimizing the quality criterion.

3. STATIONARY PROBABILITY DISTRIBUTION OF THE IMBEDDED MARKOV CHAIN

Taking a fixed threshold $j \geq 0$, let us study the behavior of the system at successful service completion instants and renewal completion instants. Let t_n be such an nth instant. Consider the random process

$$\{i_n, \nu_n, \eta_n, m_n\}, \quad n \geq 1,$$

where i_n is the number of customers in the system at the instant $t_n + 0$, $i_n \geq 0$, ν_n is the state of the control process ν_t of the BMAP flow at the instant t_n, $\nu_n = \overline{0,W}$, η_n is the state of the control process of arrivals of disasters η_t at the instant t_n, $\eta_n = \overline{0,N}$, and m_n is the state of the control process of service m_t at the instant $t_n + 0$, $m_n = \overline{1,M}$.

The process $\{i_n, \nu_n, \eta_n, m_n\}$, $n \geq 1$, is a Markov chain. To find its transition probabilities, let us order the states of the three-dimensional process $\{\nu_n, \eta_n, m_n\}$, $n \geq 1$, lexicographically, using the notation

$$\begin{gathered} P\{(i,\nu,\eta,m) \to (l,\nu',\eta',m')\} \\ = P\{i_{n+1} = l, \nu_{n+1} = \nu', \eta_{n+1} = \eta', m_{n+1} = m' | i_n = i, \nu_n = \nu, \eta_n = \eta, m_n = m\}, \\ i, l \geq 0,\ \nu, \nu' = \overline{0,W}, \quad \eta, \eta' = \overline{0,N}, \quad m, m' = \overline{1,M}. \end{gathered}$$

Let us also introduce the matrices

$$P_{i,l} = \left\| P\{(i,\nu,\eta,m) \to (l,\nu',\eta',m')\} \right\|_{\nu,\nu'=\overline{0,W},\eta,\eta'=\overline{0,N},m,m'=\overline{1,M}}.$$

Lemma. *The one-step transition probability matrices $P_{i,l}$, $i, l \geq 0$, are of the form*

$$P_{0,0} = \Psi_1 \Omega_0^{(1)} + \sum_{k=1}^{j} \Psi_k S^{(1)} H_0 + \sum_{k=j+1}^{\infty} \Psi_k S^{(2)} H_0,$$

$$P_{0,l} = \sum_{k=1}^{l+1} \Psi_k \Omega_{l-k+1}^{(1)}, \quad 0 < l \leq j-1,$$

$$P_{0,j+l} = \sum_{k=1}^{j} \Psi_k \Omega_{j+l-k+1}^{(1)} + \sum_{k=1}^{l+1} \Psi_{j+k} \Omega_{l-k+1}^{(2)}, \quad l > 0,$$

$$P_{i,l} = \begin{cases} S^{(r)} H_l, & 0 \leq l < i-1 \\ \Omega_{l-i+1}^{(r)} + S^{(r)} H_l, & l \geq i-1, \end{cases} \qquad r = \begin{cases} 1, & 0 < i \leq j \\ 2, & i > j. \end{cases}$$

• The elements of the matrix $\Omega_l^{(r)}$ are the probabilities that during the service of a customer in the rth mode, l customers and no disaster arrive, and the random process $\{\nu_n, \eta_n, m_n\}$ passes from the state $\{\nu, \eta, m\}$ to the state $\{\nu', \eta', m'\}$, $l \geq 0$, $r = \overline{1,2}$. These matrices are determined from the expansion

$$\sum_{l=0}^{\infty} \Omega_l^{(r)} z^l = \int_0^{\infty} e^{D^{(r)}(z)t} \otimes e^{F_0^{(r)}t} \otimes dB^{(r)}(t) = \widehat{\beta}_r(z), \quad r = \overline{1,2},$$

where $\otimes$ and $\oplus$ denote the Kronecker product and sum, respectively. Their definitions and properties are given, for example, in [24].

Computation of the matrices $\Omega_l^{(r)}$, $l \geq 0$, $r = \overline{1,2}$, is described by Locantoni in [11].

• The elements of the matrix $S^{(r)}$ are the probabilities that during the service of a customer in the rth mode, a disaster arrives and the process $\{\nu_n, \eta_n, m_n\}$ passes from the state $\{\nu, \eta, m\}$ to the state $\{\nu', \eta', m'\}$, $r = \overline{1,2}$. These matrices are defined by

$$S^{(r)} = \int_0^{\infty} e^{D^{(r)}(1)t} \otimes \left(e^{F_0^{(r)}t} F_1^{(r)} \right) \otimes (P^{(r)} - B^{(r)}(t)) dt, \quad r = \overline{1,2}.$$

• The elements of the matrices Ψ_k have the following probability meaning: the busy period of the system begins with the arrival of a batch of k, $k \geq 1$, customers and the process $\{\nu_n, \eta_n, m_n\}$ passes from the state $\{\nu, \eta, m\}$ to the state $\{\nu', \eta', m'\}$ when the system is idle. These matrices are defined by

$$\Psi_k = \int_0^{\infty} \left(e^{D_0^{(1)}t} D_k^{(1)} \right) \otimes e^{F^{(1)}(1)t} dt \otimes I_M = -\left[\left(D_0^{(1)} \oplus F^{(1)}(1) \right)^{-1} \left(D_k^{(1)} \otimes I_{N+1} \right) \right] \otimes I_M, \quad k \geq 1,$$

where I_i is a unit matrix of size i and I is a unit matrix of size $K = (W+1)(N+1)M$.

• The elements of the matrix H_k, $k \geq 0$, are the transition probabilities of the process $\{\nu_n, \eta_n, m_n\}$ corresponding to the arrival of k customers during the server renewal period if customers accumulate in the server renewal period. If customers are lost in the renewal period, H_0 denotes the matrix of transition probabilities of the process $\{\nu_n, \eta_n, m_n\}$ in the renewal period. The generating function $H(z)$ of the matrices H_k, $k \geq 0$, is of the form

$$H(z) = \sum_{k=0}^{\infty} H_k z^k = \int_0^{\infty} e^{D^{(1)}(z)t} \otimes e^{F^{(1)}(1)t} dG(t) \otimes I_M$$

if customers accumulate in the renewal period, and of the form

$$H(z) = \sum_{k=0}^{\infty} H_k z^k = \int_0^{\infty} e^{D^{(1)}(1)t} \otimes e^{F^{(1)}(1)t} dG(t) \otimes I_M$$

if customers are lost in the renewal period.

Let us consider the stationary probabilities

$$p(i,\nu,\eta,m) = \lim_{n\to\infty} P\{i_n = i, \nu_n = \nu, \eta_n = \eta, m_n = m\}, \qquad (2)$$
$$i \geq 0, \quad \nu = \overline{0,W}, \quad \eta = \overline{0,N}, \quad m = \overline{1,M}.$$

Since a flow of disasters, which empty the system, arrives at our system, probabilities (2) exist for any positive arrival and service parameters.

Let us introduce the vectors

$$\mathbf{p}(i,\nu,\eta) = (p(i,\nu,\eta,1),\ldots,p(i,\nu,\eta,M)),$$
$$\mathbf{p}(i,\nu) = (\mathbf{p}(i,\nu,0),\ldots,\mathbf{p}(i,\nu,N)),$$
$$\mathbf{p}_i = (\mathbf{p}(i,0),\ldots,\mathbf{p}(i,W))$$

and partial generating functions

$$\mathbf{P}_1(z) = \sum_{i=0}^{j} \mathbf{p}_i z^i, \quad \mathbf{P}_2(z) = \sum_{i=j+1}^{\infty} \mathbf{p}_i z^i, \quad |z| \leq 1.$$

Theorem 1. *The partial generating functions $\mathbf{P}_1(z)$ and $\mathbf{P}_2(z)$ satisfy the functional equation*

$$\mathbf{P}_2(z)\left(zI - \widehat{\beta}_2(z)\right)$$
$$= (\mathbf{p}_0 U + \mathbf{q})H(z)z + \mathbf{p}_0\left[(\Psi_1(z) - I)\widehat{\beta}_1(z) + \Psi_2(z)\widehat{\beta}_2(z)\right] + \mathbf{P}_1(z)\left(\widehat{\beta}_1(z) - zI\right), \qquad (3)$$

where $\Psi_1(z) = \sum_{k=1}^{j} \Psi_k z^k$, $\Psi_2(z) = \sum_{k=j+1}^{\infty} \Psi_k z^k$, *and*

$$U = (\Psi_1(1) - I)S^{(1)} + \Psi_2(1)S^{(2)}, \qquad (4)$$
$$\mathbf{q} = \mathbf{P}_1(1)S^{(1)} + \mathbf{P}_2(1)S^{(2)}. \qquad (5)$$

The proof of Theorem 1 is given in the Appendix.

Corollary 1. *The generating function $\mathbf{P}_1(z)$ is defined by the expression*

$$\mathbf{P}_1(z) = \mathbf{p}_0 Y(z) + (\mathbf{p}_0 U + \mathbf{q})Q(z), \qquad (6)$$

where $Y(z) = \sum_{i=0}^{j} Y_i z^i$, $Q(z) = \sum_{i=0}^{j} Q_i z^i$, $|z| \leq 1$, *and the matrices* Y_i *and* Q_i, $i = \overline{0,j}$, *are defined by the recurrent relations*

$$Y_0 = I, \quad Y_{i+1} = \left(Y_i - \sum_{k=1}^{i+1} \Psi_k \Omega_{i-k+1}^{(1)} - \sum_{k=1}^{i} Y_k \Omega_{i-k+1}^{(1)}\right)(\Omega_0^{(1)})^{-1}, \ i = \overline{0,j-1},$$
$$Q_0 = O, \quad Q_{i+1} = \left(Q_i - H_i - \sum_{k=1}^{i} Q_k \Omega_{i-k+1}^{(1)}\right)(\Omega_0^{(1)})^{-1}, \quad i = \overline{0,j-1}. \qquad (7)$$

The proof of Corollary 1 is given in the Appendix.

Thus, the function $\mathbf{P}_1(z)$ is known up to the vectors $\mathbf{p}_0$ and $\mathbf{q}$. Let us find the relationship between them.

Substituting $z = 1$ into equality (3) and using notation (5) for $\mathbf{q}$, after simple transformations we obtain

$$(\mathbf{P}_1(z) + \mathbf{P}_2(1))(I - A_2) = \mathbf{p}_0[(\Psi_1(1) - I)A_1 + \Psi_2(1)A_2] + \mathbf{P}_1(1)(A_1 - A_2), \tag{8}$$

where $A_r = \widehat{\beta}_r(1) + S^{(r)}H(1)$, $r = \overline{1,2}$.

Note that the matrices $H(1)$ and $\widehat{\beta}_r(1)+S^{(r)}$, $r = \overline{1,2}$, are stochastic since they are the transition probability matrices of control processes in the renewal period and service time of a customer in the rth mode, respectively. Therefore, the matrix $A_r = \widehat{\beta}_r(1) + S^{(r)}H(1)$ is stochastic and the matrix $I - A_r$, $r = \overline{1,2}$, is degenerate. In particular, the matrix A_2 has a stochastic eigenvector $\mathbf{a}_2$ corresponding to the eigenvalue 1, i.e., $\mathbf{a}_2 A_2 = \mathbf{a}_2$, $\mathbf{a}_2\mathbf{1} = 1$.

According to the theory of matrices, the matrix $I - A_2 + \mathbf{1a}_2$ is nondegenerate.

Since the vector $\mathbf{P}_1(1) + \mathbf{P}_2(1)$ is stochastic, we find that

$$(\mathbf{P}_1(1) + \mathbf{P}_2(1))\mathbf{1a}_2 = \mathbf{a}_2. \tag{9}$$

Adding (9) to both sides of relation (8) and then right multiplying the expression thus obtained by the matrix $Z = (I - A_2 + \mathbf{1a}_2)^{-1}$, after simple transformations we obtain

$$\mathbf{P}_2(1) = \mathbf{a}_2 + \mathbf{p}_0[(\Psi_1(1) - I)A_1 + \Psi_2(1)A_2]Z + \mathbf{P}_1(1)(A_1 - I - \mathbf{1a}_2)Z. \tag{10}$$

Now substituting the expressions for $\mathbf{P}_1(1)$ (6) and $\mathbf{P}_2(1)$ (10) into equality (5), we obtain

Corollary 2. *The vector* $\mathbf{q}$ *is expressed in terms of the vector* $\mathbf{p}_0$ *as*

$$\mathbf{q} = \mathbf{p}_0 T V + \mathbf{a}_2 S^{(2)} V, \tag{11}$$

where

$$T = (Y(1) + UQ(1))A^* + [(\Psi_1(1) - I)A_1 + \Psi_2(1)A_2]ZS^{(2)},$$
$$V = (I - Q(1)A^*)^{-1}, \quad A^* = S^{(1)} + (A_1 - I - \mathbf{1a}_2)ZS^{(2)}.$$

Substituting relations (6) and (11) into equality (3), we obtain an algorithm for computing the unknown vector $\mathbf{p}_0$. This algorithm is based on the analyticity of the generating function $\mathbf{P}_2(z)$ inside the unit circle $|z| < 1$. By Theorem 3 in [25], the equation

$$\det\left(zI - \widehat{\beta}_2(z)\right) = 0$$

has exactly K roots inside the unit circle $|z| < 1$ and no roots on the boundary of this circle, because the matrix $\widehat{\beta}_2(z)$ substochastic. Denoting these roots by z_k with respective multiplicity n_k, $k = \overline{1,J}$, $\sum_{k=1}^{J} n_k = K$, where J is the number of distinct roots, we find that the vector $\mathbf{p}_0$ satisfies the system of linear algebraic equations

$$\mathbf{p}_0 \frac{d^n}{dz^n}\left\{\left[(\Psi_1(z) - I)\widehat{\beta}_1(z) + \Psi_2(z)\widehat{\beta}_2(z) + Y(z)(\widehat{\beta}_1(z) - zI) + (U + TV)\widetilde{A}(z)\right]\mathrm{adj}(zI - \widehat{\beta}_2(z))\right\}\Bigg|_{z=z_k}\mathbf{e}_1$$
$$= -\mathbf{a}_2 S^{(2)} V \frac{d^n}{dz^n}\left\{\widetilde{A}(z)\mathrm{adj}(zI - \widehat{\beta}_2(z))\right\}\Bigg|_{z=z_k}\mathbf{e}_1, \quad n = \overline{0, n_k - 1}, \quad k = \overline{1,J}, \tag{12}$$

where $\widetilde{A}(z) = zH(z) + Q(z)(\widehat{\beta}_1(z) - zI)$, $\mathbf{e}_1 = (1, 0, \cdots, 0)^{\mathrm{T}}$, and adj denotes the matrix of algebraic adjuncts. Using the approach of [25, 26], we can show that system (12) has a unique solution.

4. PROBABILITY OF SUCCESSFUL SERVICE OF A CUSTOMER AND MEAN INTER-DEPARTURE TIME

Certain customers may quit the system unserved or partially served upon arrival of a disaster. Therefore, let us find the probability of successful service of an arbitrary customer.

Let t_n be the nth instant of departure of a customer from the system. Note that it is the instant of successful completion of service or the instant of arrival of a disaster. Consider the random process $\xi_n = \{i_n, \nu_n, \eta_n, m_n, c_n\}$, $n \geq 1$, where the components of the process $\{\nu_n, \eta_n, m_n\}$, $n \geq 1$, have the same meaning as in Section 3, whereas the process c_n, $n \geq 1$, is defined as follows:

(1) $c_n = 1$ if t_n is an instant of successful completion of service and i_n, $i_n \geq 1$, is the number of customers in the system at the instant $t_n + 0$ and

(2) $c_n = 0$ if t_n is an instant of arrival of a disaster and i_n, $i_n \geq 1$, is the number of customers that quit the system at the instant t_n due to the arrival of a disaster.

The random process ξ_n, $n \geq 1$ is a Markov chain. Let us denote its stationary probabilities by

$$r(i,\nu,\eta,m) = \lim_{n\to\infty} P\{i_n = i,\ \nu_n = \nu,\ \eta_n = \eta,\ m_n = m,\ c_n = 1\},\ i_n \geq 0,$$
$$k(i,\nu,\eta,m) = \lim_{n\to\infty} P\{i_n = i,\ \nu_n = \nu,\ \eta_n = \eta,\ m_n = m,\ c_n = 0\},\ i_n \geq 1,$$
$$\nu = \overline{0,W}, \quad \eta = \overline{0,N}, \quad m = \overline{1,M}.$$

Ordering the states of the process $\{\nu_n, \eta_n, m_n\}$, $n \geq 1$, lexicographically, consider the vectors

$$\mathbf{r}(i,\nu,\eta) = (r(i,\nu,\eta,1),\ldots,r(i,\nu,\eta,M)), \qquad \mathbf{k}(i,\nu,\eta) = (k(i,\nu,\eta,1),\ldots,k(i,\nu,\eta,M)),$$
$$\mathbf{r}(i,\nu) = (\mathbf{r}(i,\nu,0),\ldots,\mathbf{r}(i,\nu,N)), \qquad \mathbf{k}(i,\nu) = (\mathbf{k}(i,\nu,0),\ldots,\mathbf{k}(i,\nu,N)),$$
$$\mathbf{r}_i = (\mathbf{r}(i,0),\ldots,\mathbf{r}(i,W)), \qquad \mathbf{k}_i = (\mathbf{k}(i,0),\ldots,\mathbf{k}(i,W)),$$

and their generating functions $\mathbf{R}_1(z) = \sum\limits_{i=0}^{j} \mathbf{r}_i z^i$, $\mathbf{R}_2(z) = \sum\limits_{i=j+1}^{\infty} \mathbf{r}_i z^i$, $\mathbf{K}(z) = \sum\limits_{i=1}^{\infty} \mathbf{k}_i z^i$, $|z| \leq 1$.

Theorem 2. *The vector generating functions $\mathbf{R}_1(z)$, $\mathbf{R}_2(z)$ and $\mathbf{K}(z)$ satisfy the relations*

$$\mathbf{K}(1) = \mathbf{p}_0(U + TV) + \mathbf{a}_2 S^{(2)} V, \tag{13}$$

$$\mathbf{R}_1(z) = \mathbf{p}_0 Y(z) + \mathbf{K}(1)(Q(z) - H^{(1)}(z)), \tag{14}$$

$$\mathbf{R}_2(z)\left(zI - \widehat{\beta}_2(z)\right) = \left[\mathbf{p}_0(\Psi_1(z) - I) + \mathbf{K}(1)H^{(1)}(z)\right]\widehat{\beta}_1(z)$$
$$+ \left[\mathbf{p}_0\Psi_2(z) + \mathbf{K}(1)H^{(2)}(z)\right]\widehat{\beta}_2(z) + \mathbf{R}_1(z)\left(\widehat{\beta}_1(z) - zI\right), \tag{15}$$

$$\mathbf{K}(z) = \left[\mathbf{p}_0(\Psi_1(z) - I) + \mathbf{R}_1(z) + \mathbf{K}(1)H^{(1)}(z)\right] S^{(1)}(z)$$
$$+ \left[\mathbf{p}_0\Psi_2(z) + \mathbf{R}_2(z) + \mathbf{K}(1)H^{(2)}(z)\right] S^{(2)}(z), \tag{16}$$

where $H^{(1)}(z) = \sum\limits_{i=0}^{j} H_i z^i$, $H^{(2)}(z) = \sum\limits_{i=j+1}^{\infty} H_i z^i$,

$$S^{(r)}(z) = \sum_{l=0}^{\infty} S_l^{(r)} z^l = \int_0^{\infty} e^{D^{(r)}(z)t} \otimes \left(e^{F_0^{(r)}t} F_1^{(r)}\right) \otimes \left(P^{(r)} - B^{(r)}(t)\right) dt, \quad r = \overline{1,2}.$$

The proof of Theorem 2 is given in the Appendix.

Let P_+ denote the probability that a customer is successfully served. Using the ergodic theorem for functionals defined on Markov chains (see, for example, [27]), we obtain the following result.

Theorem 3. *The probability P_+ of successful service of a customer is $P_+ = \frac{(\mathbf{R}_1(1)+\mathbf{R}_2(1))\mathbf{1}}{\mathbf{T}\mathbf{1}}$, where*

$$\mathbf{T} = \mathbf{R}_1(1) + \mathbf{R}_2(1) + \mathbf{K}'(1), \tag{17}$$

if customers accumulate in the renewal period, and

$$\mathbf{T} = \mathbf{R}_1(1) + \mathbf{R}_2(1) + \mathbf{K}'(1) + \lambda^{(1)} g\mathbf{K}(1), \tag{18}$$

if customers are lost in the renewal period, $g = \int\limits_0^\infty t\,dG(t)$ is the mean renewal time, and the vectors $\mathbf{R}_1(1)$, $\mathbf{R}_2(1)$, $\mathbf{K}(1)$, and $\mathbf{K}'(1)$ are defined by the relations (13)–(16).

Let Λ^{-1} denote the mean inter-departure time. Using the theory of Markov renewal processes, we obtain the following result.

Theorem 4. *The mean length Λ^{-1} of the interval between the instants of departure of customers is defined by the equality $\Lambda^{-1} = \frac{1}{\lambda^{(1)}}\left[\mathbf{T} + \left(\lambda^{(1)} - \lambda^{(2)}\right)(\mathbf{P}_2(1) + \mathbf{p}_0\Psi_2(1))\right]\overline{b}_2^{(1)}\mathbf{1}$, where*

$$\overline{b}_2^{(1)} = \int\limits_0^\infty e^{D^{(2)}(1)t} \otimes e^{F_0^{(2)}t} \otimes t\,dB^{(2)}(t) + \int\limits_0^\infty e^{D^{(2)}(1)t} \otimes \left(te^{F_0^{(2)}t}F_1^{(2)}\right) \otimes (P^{(2)} - B^{(2)}(t))\,dt,$$

and the vector $\mathbf{T}$ is defined by equalities (17) *and* (18).

5. QUALITY CRITERION

The mean number L of customers in the system at a given customer departure instant is given by the formula

$$L = (\mathbf{R}_1'(1) + \mathbf{R}_2'(1))\mathbf{1}. \tag{19}$$

Using the ergodic theorem for functionals defined on Markov chains [27], we can show that

$$\Phi_2 = \Lambda(\mathbf{P}_2(1) + \mathbf{p}_0\Psi_2(1))\overline{b}_2^{(1)}\mathbf{1}, \quad \Phi_1 = 1 - \Phi_2. \tag{20}$$

The mean number R of lost customers per unit time is computed by the formula

$$R = \left(\lambda^{(1)}\Phi_1 + \lambda^{(2)}\Phi_2\right)(1 - P_+). \tag{21}$$

Substituting (19)–(21) into (1), we obtain the value of the quality criterion for a given value of the threshold j. Using the algorithm for computing the value of the quality criterion for any fixed threshold value, we can find the optimal threshold j^* minimizing the quality criterion. The optimal value of the quality criterion is determined by the trial and error method in the threshold value domain $[0, \widehat{J}]$, where $\widehat{J}$ is defined by the condition $\sum\limits_{i=0}^{\widehat{J}} \mathbf{p}_i\mathbf{1} = 1 - \varepsilon_c$, and ε_c is the accuracy of computer-aided approximation of numbers. The value of $C(j^*)$ thus obtained is optimal (at least for a given computation accuracy).

6. A NUMERICAL EXAMPLE

Let the BMAP customer flow be defined by the matrices

$$D_0^{(1)} = \begin{pmatrix} -4.4 & 2.4 \\ 2.8 & -7.8 \end{pmatrix}, \quad D_1^{(1)} = D_2^{(1)} = \begin{pmatrix} 1 & 0.0 \\ 0 & 2.5 \end{pmatrix},$$

$$D_0^{(2)} = \begin{pmatrix} -2.2 & 1.2 \\ 4.8 & -7.8 \end{pmatrix}, \quad D_1^{(2)} = D_2^{(2)} = \begin{pmatrix} 0.5 & 0.0 \\ 0.0 & 1.5 \end{pmatrix}.$$

The customer flow intensity is 5.1 and 2.1 in the first and second mode, respectively.

The disaster flow is defined by the matrices

$$F_0^{(1)} = \begin{pmatrix} -0.35 & 0.15 \\ 0.24 & -0.42 \end{pmatrix}, \quad F_1^{(1)} = \begin{pmatrix} 0.2 & 0.00 \\ 0.0 & 0.18 \end{pmatrix},$$

$$F_0^{(2)} = \begin{pmatrix} -0.26 & 0.16 \\ 0.27 & -0.35 \end{pmatrix}, \quad F_1^{(2)} = \begin{pmatrix} 0.1 & 0.00 \\ 0.0 & 0.08 \end{pmatrix}.$$

The disaster flow intensity is 0.2 and 0.1 in the first and second mode, respectively.

We also assume that the service control process has a semi-Markov kernel of the type $B^{(r)}(t) = \mathrm{diag}\left\{B_1^{(r)}(t), \ldots, B_M^{(r)}(t)\right\} P^{(r)}$, $r = \overline{1,2}$, where $\mathrm{diag}\{c_1, \ldots, c_M\}$ is a diagonal matrix with diagonal elements $c_1, \ldots, c_M$, $P^{(r)}$ is a stochastic matrix, and $B_i^{(r)}(t)$, $i = \overline{1,M}$ are distribution functions.

Let us take $M = 2$, $B_i^{(r)}(t) = 1 - e^{-\mu_i^{(r)} t}$, $i = \overline{1,M}$, $r = \overline{1,2}$, $\mu_1^{(1)} = 3$, $\mu_2^{(1)} = 3.5$, $\mu_1^{(2)} = 4$, $\mu_2^{(2)} = 4.5$, and

$$P^{(1)} = P^{(2)} = \begin{pmatrix} 0.60 & 0.40 \\ 0.35 & 0.65 \end{pmatrix}.$$

The mean service time is 0.31 and 0.24 in the first and second mode, respectively.

The server renewal time is exponentially distributed with parameter g^{-1}. Customers accumulate in the system in the renewal period. The cost coefficients are $a = 1.4$, $c_1 = 10$, $c_2 = 100$, and $h = 10$.

Figure shows the quality criterion (1) versus the threshold j for different mean renewal times g.

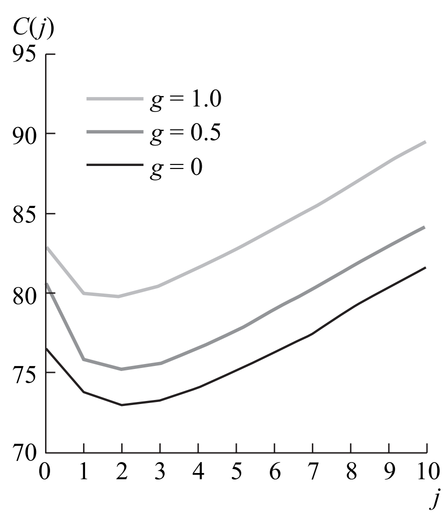

Figure.

Let C_r denote the value of the quality criterion only for the rth mode, $r = \overline{1,2}$. The optimal value of the quality criterion C^*, the values of C_1 and C_2 for the values of g given above are tabulated below. The optimal threshold value j^* is 2.

Table

g	C^*	C_1	C_2
0	72.97	80.33	105.83
0.5	75.23	83.86	105.96
1	79.85	89.55	106.32

Comparing C^* and $C_1 = \min\{C_1, C_2\}$, we find that the relative gain from the modified threshold control strategy is more than 9%.

7. CONCLUSIONS

A BMAP/SM/1 system with two operation modes, a MAP disaster flow, and a modified threshold control strategy is studied. The stationary state probability distribution of the system is determined and used in finding the relation between the quality criterion and threshold. The optimal threshold is determined numerically. The results are illustrated by a numerical example.

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APPENDIX

Proof of Theorem 1. Using the total probability formula and Lemma, we find that the stationary state probability vectors $\mathbf{p}_i$, $i \geq 0$, of the imbedded Markov chain are defined by the equations

$$\mathbf{p}_i = \mathbf{p}_0 \sum_{k=1}^{i+1} \Psi_k \Omega_{i-k+1}^{(1)} + \sum_{k=1}^{i+1} \mathbf{p}_k \Omega_{i-k+1}^{(1)} + \sum_{k=1}^{j} \mathbf{p}_k S^{(1)} H_i \sum_{k=j+1}^{\infty} \mathbf{p}_k S^{(2)} H_i + \mathbf{p}_0 \sum_{k=1}^{j} \Psi_k S^{(1)} H_i$$
$$+ \mathbf{p}_0 \sum_{k=j+1}^{\infty} \Psi_k S^{(2)} H_i, \quad i = \overline{0, j-1}, \tag{A.1}$$

$$\mathbf{p}_{j+l} = \mathbf{p}_0 \sum_{k=1}^{j} \Psi_k \Omega_{j+l-k+1}^{(1)} + \mathbf{p}_0 \sum_{k=1}^{l+1} \Psi_{j+k} \Omega_{l-k+1}^{(2)} + \sum_{k=1}^{j} \mathbf{p}_k \Omega_{j+l-k+1}^{(1)} + \sum_{k=1}^{l+1} \mathbf{p}_{j+k} \Omega_{l-k+1}^{(2)}$$
$$+ \sum_{k=1}^{j} \mathbf{p}_k S^{(1)} H_{j+l} + \sum_{k=j+1}^{\infty} \mathbf{p}_k S^{(2)} H_{j+l} + \mathbf{p}_0 \sum_{k=1}^{j} \Psi_k S^{(1)} H_{j+l} + \mathbf{p}_0 \sum_{k=j+1}^{\infty} \Psi_k S^{(2)} H_{j+l}, \quad l \geq 0.$$

Multiplying Eqs. (A.1) by suitable powers of z and summing the result, by virtue of notation (4) and (5), we obtain Eq (3). This completes the proof of Theorem 1.

Proof of Corollary 1. Expressing the function $\mathbf{P}_1(z)$ in (3), let us expand the expression thus obtained into a series. Denoting the expansions of the matrix functions $(\Psi_1(z) - I)\widehat{\beta}_1(z)(zI - \widehat{\beta}_1(z))^{-1}$ and $zH(z)(zI - \widehat{\beta}_1(z))^{-1}$ by Y_i and Q_i, $i = \overline{0, j}$, respectively, we obtain expression (6). It is a simple matter to show that the matrices Y_i and Q_i, $i = \overline{0, j}$, satisfy the recurrent relations (7). This completes the proof of Corollary 1.

Proof of Theorem 2. The vectors $\mathbf{p}_i$, $\mathbf{r}_i$, $i \geq 0$, and $\mathbf{K}(1)$ satisfy the system of equations

$$\mathbf{r}_i = \left(\mathbf{r}_0 + \sum_{m=1}^{\infty} \mathbf{k}_m H_0\right) \sum_{n=1}^{i+1} \Psi_n \Omega_{i-n+1}^{(1)} + \sum_{n=1}^{i+1} \mathbf{r}_n \Omega_{i-n+1}^{(1)} + \sum_{m=1}^{\infty} \mathbf{k}_m \sum_{n=1}^{i+1} H_n \Omega_{i-n+1}^{(1)}, \quad i = \overline{0, j-1},$$

$$\mathbf{r}_{j+l} = \left(\mathbf{r}_0 + \sum_{m=1}^{\infty} \mathbf{k}_m H_0\right) \left(\sum_{n=1}^{j} \Psi_n \Omega_{j+l-n+1}^{(1)} + \sum_{n=1}^{l+1} \Psi_{j+n} \Omega_{l-n+1}^{(2)}\right) + \sum_{n=1}^{j} \mathbf{r}_n \Omega_{j+l-n+1}^{(1)}$$
$$+ \sum_{n=1}^{l+1} \mathbf{r}_{j+n} \Omega_{l-n+1}^{(2)} + \sum_{m=1}^{\infty} \mathbf{k}_m \left(\sum_{n=1}^{j} H_n \Omega_{j+l-n+1}^{(2)} + \sum_{n=1}^{l+1} H_{j+n} \Omega_{l-n+1}^{(2)}\right), \; l \geq 0, \tag{A.2}$$

$$\mathbf{k}_i = \left(\mathbf{r}_0 + \sum_{m=1}^{\infty} \mathbf{k}_m H_0\right) \sum_{n=1}^{i} \Psi_n S_{i-n}^{(1)} + \sum_{n=1}^{i} \mathbf{r}_n S_{i-n}^{(1)} + \sum_{m=1}^{\infty} \mathbf{k}_m \sum_{n=1}^{i} H_n S_{i-n}^{(1)}, \; i = \overline{1, j},$$

$$\mathbf{k}_{j+l} = \left(\mathbf{r}_0 + \sum_{m=1}^{\infty} \mathbf{k}_m H_0\right) \left(\sum_{n=1}^{j} \Psi_n S_{j+l-n}^{(1)} + \sum_{n=1}^{l} \Psi_{j+n} S_{l-n}^{(2)}\right) + \sum_{n=1}^{j} \mathbf{r}_n S_{j+l-n}^{(1)}$$
$$+ \sum_{n=1}^{l} \mathbf{r}_{j+n} S_{l-n}^{(2)} + \sum_{m=1}^{\infty} \mathbf{k}_m \left(\sum_{n=1}^{j} H_n S_{j+l-n}^{(1)} + \sum_{n=1}^{l} H_{j+n} S_{l-n}^{(2)}\right), \quad l \geq 1,$$

where

$$S_l^{(r)} = \int_0^{\infty} P^{(r)}(l,t) \otimes \left(e^{F_0^{(r)} t} F_1^{(r)}\right) \otimes \left(P^{(r)} - B^{(r)}(t)\right) dt$$

is the transition probability matrix of the process $\{\nu_n, \eta_n, m_n\}$, $n \geq 1$, corresponding to the arrival of l customers in the period of service in the rth mode, when this service is terminated due to the arrival of a disaster, and $P^{(r)}(l,t)$ is the transition probability of the process ν_t in time t corresponding to the arrival of l customers in the rth mode, $l \geq 0$, $r = \overline{1,2}$.

Multiplying equalities (A.2) by suitable powers of z and summing them, by virtue of the relation $\mathbf{p}_0 = \mathbf{r}_0 + \mathbf{K}(1)H_0$, we obtain the functional Eqs. (15) and (16).

It is easily seen that the vectors $\mathbf{p}_i$, $\mathbf{r}_i$, $i \geq 0$, and $\mathbf{K}(1)$ are interrelated by the expression $\mathbf{p}_i = \mathbf{r}_i + \mathbf{K}(1)H_i$, $i \geq 0$. Hence it follows that

$$\mathbf{P}_r(z) = \mathbf{R}_r(z) + \mathbf{K}(1)H^{(r)}(z), \quad r = \overline{1,2}. \tag{A.3}$$

Substituting $z = 1$ into equality (16) and using relations (4), (5), and (A.3), we obtain formula (13). Then equality (6) for the function $\mathbf{P}_1(z)$ takes the form

$$\mathbf{P}_1(z) = \mathbf{p}_0 Y(z) + \mathbf{K}(1)Q(z). \tag{A.4}$$

From equalities (A.3) and (A.4) we obtain expression (14).

This completes the proof of Theorem 2.

REFERENCES

1. Rykov, V.V., *Upravlyaemye sistemy massovogo obsluzhivaniya* (Queue Control Systems), *Itogi Nauki Tekhn.*, 1975, vol. 10, pp. 43–153.
2. Fainberg, M.A. and Fainberg, E.A., Control in Queueing Systems, *Zarubezh. Elektronika*, 1975, no. 3, pp. 3–34.
3. Dudin, A.N., Optimal Control for a $M^x/G/1$ Queue with Two Operation Modes, *Probab. Eng. Inf. Sci.*, 1997, vol. 11, pp. 225–265.
4. Dudin, A.N. and Nishimura, S., Optimal Control for a BMAP/G/1 Queue with Two Service Modes, *Math. Probl. Eng.*, 1999, vol. 5, pp. 255–273.
5. Nobel, R., A Regenerative Approach to Analysis of an $M^X/G/1$ Queue with Two Service Modes, *Avtom. Vychisl. Tekh.*, 1998, no. 1, pp. 3–14.
6. Gelenbe, E., Product Form Networks with Negative and Positive Customers, *J. Appl. Prob.*, 1991, vol. 28, pp. 655–663.
7. Artalejo, J., G-networks: A Versatile Approach for Work Removal in Queueing Networks, *Eur. J. Oper. Res.*, 2000, vol. 126, pp. 233–249.
8. Dudin, A.N. and Nishimura, S., A BMAP/SM/1 Queueing System with Markovian Arrival of Disasters, *J. Appl. Prob.*, 1999, vol. 36, no. 3, pp. 868–881.
9. Dudin, A.N. and Karolik, A.V., A BMAP/SM/1 Queue with Markovian Input of Disasters and Non-Instantaneous Recovery, *Performance Evaluat.*, 2001, vol. 45, pp. 19–32.
10. Jain, G. and Sigman, K., A Pollaczeck–Khinchine Formula for M/G/1 Queues with Disasters, *J. Appl. Prob.*, 1996, vol. 33, pp. 1191–1200.
11. Lucantoni, D.M., New Results on the Single-Server Queue with a Batch Markovian Arrival Process, *Commun. Stat. Stochastic Models*, 1991, vol. 7, pp. 1–46.
12. Lucantoni, D.M. and Neuts, M.F., Some Steady-State Distributions for the BMAP/SM/1 Queue, *Commun. Stat. Stochastic Models*, 1994, vol. 10, pp. 575–598.
13. Bocharov, P.P., Pechinkin, A.V., and Phong, N.H., Stationary Probabilities of the States of a BMAP/G/1/r System with Repeated Customers and Priority Service for Primary Customers, *Avtom. Telemekh.*, 2000, no. 8, pp. 68–78.
14. Bocharov, P.P., Phong, N.H., and Hak, T., Analysis of a $MAP_2/G_2/1/r$ Queueing System with Relative Priority and Queue-Length Dependent Service, *Vestn. Ross. Univ. Druzhby Narodov*, 1999, no. 1, pp. 92–109.
15. Bocharov, P.P. and Phong, N.H., Analysis of a $MAP_2/G_2/1/r$ Queueing System with Absolute Priority, *Avtom. Telemekh.*, 1997, no. 11, pp. 102–117.
16. Kolyadenkova, L.G., Pechinkin, A.V., and Trishechkin, S.P., A $MAP_2/G_2/1/r$ System with Absolute Priority and Common Queue, *Vestn. Ross. Univ. Druzhby Narodov*, 2000, no. 1, pp. 72–90.
17. Bocharov, P.P., Analysis of a Finite Queue with Markov Input Flow Dependent on the State of the System and Arbitrary Service, *Avtom. Telemekh.*, 1995, no. 12, pp. 60–70.
18. Bocharov, P.P., Analysis of a MAP/G/1/r Queueing System of Finite Buffer, *Vestn. Ross. Univ. Druzhby Narodov*, 1995, no. 1, pp. 52–67.
19. Bocharov, P.P. and Phong, N.H., Analysis of a $MAP_2/G_2/1/r$ Queueing System with Relative Priority, *Vestn. Ross. Univ. Druzhby Narodov, Prikl. Mat. Inf.*, 1996, no. 2, pp. 67–84.
20. Pechinkin, A.V., A Queueing System with Markov Input Flow and Random Selection of a Customer from the Queue, *Avtom. Telemekh.*, 2000, no. 9, pp. 90–96.
21. Bocharov, P.P. and Pechinkin, A.V., *Teoriya massovogo obsluzhivaniya* (Queueing Theory), Moscow: Ross. Univ. Druzhby Narodov, 1995.

22. Neuts, M.F., *Structured Stochastic Matrices of* M/G/1 *Type and Their Applications*, New York: Marcel Dekker, 1989.

23. Tijms, H., On the Optimality of a Switch-over with Exponential Controlling the Queue Size in a M/G/1 Queue with Variable Service Rate, *Lect. Notes Comput. Sci.*, 1976.

24. Graham, A., *Kronecker Products and Matrix Calculus with Applications*, Chichester: Ellis Horwood, 1981.

25. Gail, H.R., Hantler, S.L., and Taylor, B.A., Spectral Analysis of M/G/1 and G/M/1 Type Markov Chains, *Adv. Appl. Probab.*, 1991, vol. 28, pp. 114–165.

26. Gail, H.R., Hantler, S.L., Sidi, M., and Taylor, B.A., Linear Independence of Root Equations for M/G/1 Type of Markov Chains, *Queueing Syst.*, 1995, vol. 20, pp. 321–339.

27. Skorokhod, A.V., *Teoriya veroyatnostei i sluchainykh protsessov* (Theory of Probability and Random Processes), Kiev: Vysshaya Shkola, 1980.

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