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A Queueing System with Two Operation Modes and a Disaster Flow: Its Stationary State Probability Distribution

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Abstract—This paper is concerned with a controllable queueing in which the operation mode changes, depending on the current queue length, and an additional flow of diasters arrive. A disaster may completely empty the system, i.e., all customers, including the customer under service, may quit the system. Its stationary state probability distribution is determined, which is helpful in optimizing the threshold control of the operation of the system.

1. INTRODUCTION

Queueing models quite adequately describe the service situations encountered in normal life, industry, communication networks, etc. Since the classical queueing models, which have been studied in detail, disregard parametric changes in time, their application is rather restricted.

The recent practical problems encountered in economic and administrative control of large cities, complex technological processes, and information systems have stimulated the development of research in systems with changing operation parameters, viz., queueing systems with controllable operation modes. Dynamic changes in operation modes are typical of these systems. These changes are realized via control for improving certain quality criteria of operation of the system.

As a result of emergence of new control and communication systems, there is an acute need for a more adequate description of the random processes in these systems, and this resulted in the appearance of new mathematical models for queueing systems with changing operation modes. Such systems are interesting both from theoretical and practical viewpoints. Systems with controllable operation modes are briefly reviewed in [1–5].

Disasters may occur in real queueing systems and, as a result, customers may be lost. Mathematical models of such queueing systems describe disasters that result in instantaneous departure of all customers, including the customer under service. Such disasters are a particular case of the so-called negative customers. A system with such customers is investigated in [6]. Queueing systems with disasters are reviewed in [7, 8].

2. A MATHEMATICAL MODEL

Let us consider a single-server queueing system with waiting. The system may operate in two modes and an additional flow of disasters may arrive, thereby resulting in instantaneous exit of all customers, including the customer under service, from the system.

The intervals between arrivals of customers at the system in the rth operation mode are exponentially distributed with parameter λ_r, intervals between occurrences of disasters are exponentially distributed with parameter φ_r, and the service time has the distribution function $B_r(t)$ with the Laplace–Stieltjes transform $B_r^*(s)$ and finite initial instants $r = \overline{1,2}$.

The operation mode may change at the instants when customers quit the system. The operation of the system is described by the quality criterion

$$I = aL + c_1P_1 + c_2P_2 + gR, \tag{1}$$

where L is the mean number of customers at the instant of departure of a customer, a is the penalty for a customer to reside in the system per unit time, P_r, $r = \overline{1,2}$, is the mean utilization factor of the rth operation mode, c_r, $r = \overline{1,2}$, is the cost of utilization per unit time of the rth mode, R is the mean number of customers lost per unit time, and g is the penalty per lost customer.

We assume that the cost indexes satisfy the conditions $a > 0$, $c_1 \le c_2$, $g \ge 0$.

The optimal operation control strategy for $M/G/1$ systems without disasters, as shown in [9], is the threshold strategy. Although the optimality of threshold strategies for Markov strategies for systems with disasters is yet to be demonstrated, it is worthwhile to find the optimal strategy for this system in the class of threshold strategies. These strategies are defined as follows. An integer j, called the threshold, is taken. If the number i of customers in the system at an instant of departure of a customer satisfies inequality $i \le j$, then the succeeding customer is served in operation mode I. If $i > j$, then the succeeding customer is served in operation mode II.

Our problem now is to design an algorithm for determining the stationary state probability distribution for the system and the value of the quality criterion for a fixed threshold.

3. STATIONARY DISTRIBUTION OF AN EMBEDDED MARKOV CHAIN

Let j be an arbitrary fixed threshold. We use the following notation: $N(\tau_n)$, $n \ge 1$, is the number of customers in the system at the nth departure instant,

$$\pi_i = \lim_{n\to\infty} P\{N(\tau_n) = i\}$$

is the stationary probability for the system to contain i, $i \ge 0$, customers at the departure instant,

$$f_i^{(l)} = \int_0^\infty \frac{(\lambda_l t)^i}{i!} e^{-\lambda_l t} e^{-\varphi_l t}\, dB_l(t)$$

is the probability that i, $i \ge 0$, customers arrive at the system in the course of service of a customer and no disaster takes place in the course of operation in mode l, $l = \overline{1,2}$, and

$$H_r = \int_0^{+\infty} (1 - e^{-\varphi_r t}) dB_r(t)$$

is the probability that a disaster occurs in the system in the course of service of a customer in the rth operation mode, $r = \overline{1,2}$.

Let us introduce the partial generating functions

$$\Pi_1(z) = \sum_{m=0}^{j} \pi_m z^m, \quad |z| < 1,$$

$$\Pi_2(z) = \sum_{m=j+1}^{\infty} \pi_m z^m, \quad |z| < 1,$$

and the generating function

$$\Pi(z) = \sum_{i=0}^{\infty} \pi_i z^i = \Pi_1(z) + \Pi_2(z).$$

Theorem 1. *The generating function* $\Pi(z)$ *for the number of customers in our system is*

$$\Pi(z) = \frac{\pi_0}{(z-\beta_2^*(z))\left(1-\rho_2-H_2-H\sum\limits_{i=0}^{j}\beta_i^{(1)}\right)}$$

$$\times\left[\beta_1^*(z)\Bigl(1-\rho_2-H_2-H\sum_{i=0}^{j}\beta_i^{(1)}\Bigr)(z-1)+(\beta_1^*(z)-\beta_2^*(z))\right.$$

$$\times\left(\sum_{i=0}^{j}\alpha_i^{(1)}z^i(1-\rho_2-H_2)-H\sum_{i=0}^{j}\sum_{k=0}^{j}\beta_i^{(1)}\alpha_k^{(1)}(z^k-z^i)\right.$$

$$\left.\left.+H_2(1-H_1)\sum_{i=0}^{j}\beta_i^{(1)}z^i\right)+z\left(\sum_{i=0}^{j}\alpha_i^{(1)}H+H_2(1-H_1)\right)\right], \tag{2}$$

where

$$\beta_r^*(z) = B_r^*(\lambda_r(1-z)+\varphi_r), \quad r=\overline{1,2},$$

$$\rho_r = \lambda_r\int\limits_0^{+\infty} te^{-\varphi_r t}\,dB_r(t), \quad r=\overline{1,2},$$

$$H = H_1(1-\rho_2)+H_2(\rho_1-1). \tag{3}$$

Here $\alpha_i^{(1)}$ $(i=\overline{0,j}\,)$ *is computed by the recurrent relation*

$$\alpha_0^{(1)}=1, \quad \alpha_i^{(1)} = f_i^{(1)}\nu_0+\sum_{m=1}^{i}f_{i-m}^{(1)}(\nu_m-\nu_{m-1}), \quad i=\overline{1,j}, \tag{4}$$

$$\nu_0=\frac{1}{f_0^{(1)}}, \quad \nu_{i+1}=\frac{1}{f_0^{(1)}}\left(\nu_i-\sum_{k=0}^{i}\nu_k f_{i-k+1}^{(1)}\right), \quad i=\overline{0,j-1},$$

whereas $\beta_i^{(1)}$ $(i=\overline{0,j}\,)$ *is computed by the formula*

$$\beta_0^{(1)}=0, \quad \beta_i^{(1)}=-\sum_{l=0}^{i}\alpha_l^{(1)}, \quad i=\overline{1,j}. \tag{5}$$

The proof of Theorem 1 is given in the Appendix.

Corollary. *The stationary probability* π_i $(i=\overline{0,\infty}\,)$ *for the system to contain* i *customers at the instant of departure of a customer is*

$$\pi_0=\left[1-\rho_2-H_2-H\sum_{i=0}^{j}\beta_i^{(1)}\right]\left[(\rho_1-\rho_2-H_2+H_1)\sum_{i=0}^{j}\alpha_i^{(1)}\right.$$

$$\left.+(H_2-H_1)(1-H_1)\sum_{i=0}^{j}\beta_i^{(1)}+1-H_1\right]^{-1}, \tag{6}$$

$$\pi_i=\pi_0\left(\alpha_i^{(1)}+\gamma\beta_i^{(1)}\right), \quad i=\overline{1,j}, \tag{7}$$

$$\pi_k=\pi_0\left(\alpha_k^{(2)}+\gamma\beta_k^{(2)}\right), \quad k=\overline{j+1,\infty}, \tag{8}$$

where

$$\gamma = \frac{H\sum\limits_{i=0}^{j}\alpha_i^{(1)} + H_2(1-H_1)}{1-\rho_2-H_2-H\sum\limits_{i=0}^{j}\beta_i^{(1)}}, \tag{9}$$

$$\alpha_k^{(2)} = \vartheta_k - \sum_{i=0}^{j}\alpha_i^{(1)}\Delta_{k-i}, \quad \beta_k^{(2)} = \xi_k - \sum_{i=0}^{j}\beta_i^{(1)}\Delta_{k-i}, \quad k=\overline{j+1,\infty}. \tag{10}$$

Here H is determined by formula (3), $\alpha_i^{(1)}$ and $\beta_i^{(1)}$ $(i=\overline{1,j})$ are computed by formulas (4) and (5), respectively, and ϑ_m and Δ_m are computed by the recurrent formulas

$$\Delta_0 = \frac{f_0^{(1)}}{f_0^{(2)}}, \quad \Delta_1 = \frac{\Delta_0(1-f_1^{(2)})-1+f_1^{(1)}}{f_0^{(2)}}, \tag{11}$$

$$\Delta_m = \frac{\Delta_{m-1} - \sum\limits_{k=0}^{m-1}\Delta_k f_{m-k}^{(2)} + f_m^{(1)}}{f_0^{(2)}}, \quad m \ge 2,$$

$$\vartheta_m = f_m^{(1)} + \sum_{i=0}^{m} f_i^{(1)}\psi_{m-i+1}, \quad m \ge 0, \tag{12}$$

where

$$\psi_0 = 1, \quad \psi_{m+1} = \frac{1}{f_0^{(2)}}\left(\psi_m - \sum_{k=1}^{m}\psi_k f_{m-k+1}^{(2)} - f_m^{(2)}\right), \quad m \ge 0,$$

and ξ_k $(k=\overline{j+1,\infty})$ is defined by the expression

$$\xi_k = \Delta_k - \sum_{i=0}^{k}\vartheta_i, \quad k=\overline{j+1,\infty}. \tag{13}$$

The proof of the corollary is given in the Appendix.

4. PROBABILITY OF SUCCESSFUL SERVICE OF A CUSTOMER AND MEAN INTER-DEPARTURE INTERVAL

Let t_n be the nth departure instant. Let $\{i_n, c_n\}$, $n \ge 1$, be a two-dimensional Markov chain. The process c_n, $n \ge 1$, is defined as follows:

- $c_n = 1$ if t_n is the instant of completion of service of a customer and
- $c_n = 0$ if t_n is the instant of occurrence of a disaster.

If $c_n = 1$, then i_n is regarded as the number of customers in the system at instant $t_n + 0$, $i_n \ge 0$. If $c_n = 0$, then i_n is regarded as the number of customers departing from the system due to a disaster.

Unlike the stationary probabilities π_i, $i \ge 0$ examined earlier, let

$$p_i = \lim_{n\to\infty} P\{i_n = i,\ c_n = 1\}$$

be the stationary probability that the service of a customer is successfully completed at the instant when a customer quits and there remain i, $i \ge 0$, customers in the system and let

$$k_i = \lim_{n\to\infty} P\{i_n = i,\ c_n = 0\}$$

be the stationary probability that i, $i \ge 1$, customers quit the system at the instant when a disaster occurs.

Let us consider the generating functions

$$P_1(z) = \sum_{i=0}^{j} p_i z^i, \quad P_2(z) = \sum_{i=j+1}^{\infty} p_i z^i, \quad K(z) = \sum_{i=1}^{\infty} k_i z^i, \quad |z| < 1.$$

The stationary probabilities p_i ($i \ge 0$) and k_i ($i \ge 1$) satisfy the linear algebraic equations

$$\begin{gathered}
p_i = \left(p_0 + \sum_{m=1}^{\infty} k_m\right) f_i^{(1)} + \sum_{n=1}^{i+1} p_n f_{i-n+1}^{(1)}, \quad i = \overline{0, j-1}, \\
p_{j+l} = \left(p_0 + \sum_{m=1}^{\infty} k_m\right) f_{j+l}^{(1)} + \sum_{k=1}^{j} p_k f_{j+l-k+1}^{(1)} + \sum_{k=1}^{l+1} p_{j+k} f_{l-k+1}^{(2)}, \quad l \ge 0, \\
k_i = \left(p_0 + \sum_{m=1}^{\infty} k_m\right) h_i^{(1)} + \sum_{m=1}^{i} p_m h_{i-m}^{(1)}, \quad i = \overline{1, j}, \\
k_{j+l} = \left(p_0 + \sum_{m=1}^{\infty} k_m\right) h_{j+l}^{(1)} + \sum_{m=1}^{j} p_m h_{j+l-m}^{(1)} + \sum_{m=1}^{l} p_{j+m} h_{l-m}^{(2)}, \quad l \ge 1,
\end{gathered} \tag{14}$$

where

$$h_i^{(r)} = \int_0^{\infty} \frac{(\lambda_r t)^i}{i!} e^{-\lambda_r t} \varphi_r e^{-\varphi_r t}(1 - B_r(t))\, dt, \quad r = \overline{1, 2}, \quad i \ge 1.$$

Multiplying (14) by suitable powers of z and then summing, we obtain the following relations for the generating functions $P_1(z)$, $P_2(z)$, and $K(z)$:

$$\begin{gathered}
P_1(z)(z - \beta_1^*(z)) + P_2(z)(z - \beta_2^*(z)) = -p_0\beta_1^*(z) + (p_0 + K(1))z\beta_1^*(z), \\
K(z) = -p_0 S_1(z) + (p_0 + K(1))z S_1(z) + P_1(z) S_1(z) + P_2(z) S_2(z),
\end{gathered} \tag{15}$$

where

$$S_i(z) = \int_0^{\infty} e^{-\lambda_i(1-z)t} \varphi_i e^{-\varphi_i t}(1 - B_i(t))\, dt = \frac{\varphi_i}{\lambda_i(1 - z) + \varphi_i}(1 - \beta_i^*(z)), \quad i = \overline{1, 2}.$$

Using the relation $\pi_0 = p_0 + K(1)$, normalization condition $P_1(1) + P_2(1) + K(1) = 1$, and equalities (15), we can obtain the relations

$$\begin{gathered}
p_0 = \pi_0(1 - \gamma), \\
P_1(1) = \pi_0 \left(-\gamma + \sum_{i=0}^{j} \alpha_i^{(1)} + \gamma \sum_{i=0}^{j} \beta_i^{(1)}\right), \\
P_2(1) = \pi_0 \left(\gamma \frac{1 - H_1 + H_2}{H_2} - \sum_{i=0}^{j} \alpha_i^{(1)} - \gamma \sum_{i=0}^{j} \beta_i^{(1)}\right), \\
K(1) = \pi_0 \gamma, \\
K'(1) = \pi_0 + \pi_0\gamma \left(\frac{\lambda_2}{\varphi_2} + \frac{H_2 - 1}{H_2}\right) + \pi_0 \left[H_1 \left(\frac{\lambda_1}{\varphi_1} - \frac{\lambda_2}{\varphi_2}\right) + \frac{H_1 - H_2}{H_2}\right] \left(\sum_{i=0}^{j} \alpha_i^{(1)} + \gamma \sum_{i=0}^{j} \beta_i^{(1)}\right).
\end{gathered} \tag{16}$$

Here $P_1(1) + P_2(1)$ is the probability of successful completion of service at a departure instant and $K'(1)$ is the mean number of customers arriving at the instant of occurrence of a disaster.

Let P_+ denote the probability of successful service of a customer. Applying the ergodic theorem for functionals defined on Markov chains (see, for example, [10]), we obtain the following theorem.

Theorem 2. *The probability P_+ of successful completion service of a customer is*

$$P_+ = \frac{P_1(1) + P_2(1)}{P_1(1) + P_2(1) + K'(1)}. \tag{17}$$

Let Λ^{-1} denote the mean inter-departure interval. For an $M/G/1$ system without disasters,

$$\Lambda = \lambda, \tag{18}$$

where λ is the input arrival intensity.

Since a batch of unserved customers may quit our system with disasters, equality (18) does not hold. Applying renewal theory, we can easily prove Theorem 3.

Theorem 3. *The mean inter-departure interval Λ^{-1} is given by the equality*

$$\Lambda = \frac{\lambda_1}{P_1(1) + P_2(1)\left(1 + \overline{b}_1^{(2)}(\lambda_1 - \lambda_2)\right) + K'(1)}, \tag{19}$$

where

$$\overline{b}_1^{(2)} = \int\limits_0^\infty te^{-\varphi_2 t}\, dB_2(t) + \int\limits_0^\infty t(1 - B_2(t))\varphi_2 e^{-\varphi_2 t}\, dt = \frac{H_2}{\varphi_2} \tag{20}$$

is the mean service time in operation mode II.

5. A QUALITY CRITERION

Theorem 4. *The quality criterion* (1) *depends on the threshold j:*

$$\begin{aligned} I(j) = \frac{a}{H_2} \Bigg[&-(j^2 + j)m_1 \sum_{i=0}^{j} \alpha_i^{(1)} + m_2 \sum_{i=0}^{j} i\alpha_i^{(1)} + m_1 \sum_{i=0}^{j} i^2 \alpha_i^{(1)} \\ &+ \frac{H}{2}(H_2 - H_1) \sum_{i=0}^{j}\sum_{k=0}^{j} ((j+1)(2i - j) + k(k - 2i - 1))\alpha_i^{(1)}\alpha_k^{(1)} \Bigg] \\ &\times \left[1 - \rho_2 - H_2 + H(j+1)\sum_{i=0}^{j} \alpha_i^{(1)} - H \sum_{i=0}^{j} i\alpha_i^{(1)} \right]^{-1} \\ &+ \left[m_3 + (m_4 + jm_5)\sum_{i=0}^{j} \alpha_i^{(1)} - m_5 \sum_{i=0}^{j} i\alpha_i^{(1)} \right] \\ &\times \left[m_6 + (m_7 + jm_8)\sum_{i=0}^{j} \alpha_i^{(1)} - m_8 \sum_{i=0}^{j} i\alpha_i^{(1)} \right]^{-1}, \end{aligned} \tag{21}$$

where

$$\begin{gathered} m_1 = \frac{1}{2}H_2(H_2 - H_1)(1 - H_1), \quad m_2 = (H_2 - H_1)\left(1 - \rho_2 - H_2 - \frac{1}{2}H_2(1 - H_1)\right), \\ m_3 = (c_1 + g\lambda_1)(1 - \rho_2 - H_2) + H_2 m'(1 - H_1), \quad m_4 = Hm' + (1 - \rho_2 - H_1 H_2)m'', \\ m_5 = -H(c_1 + g\lambda_1) + H_2(1 - H_1)m'', \quad m_6 = 1 - \rho_2 - H_2 + H_2\frac{\lambda_1}{\varphi_2}(1 - H_1), \\ m_7 = \frac{H_2}{\varphi_2}(\rho_1 - 1) + \frac{H_1}{\varphi_1}(1 - \rho_2), \quad m_8 = H + H_1 H_2 \lambda_1 (1 - H_1)\left(\frac{1}{\varphi_2} - \frac{1}{\varphi_1}\right), \end{gathered} \tag{22}$$

and m' and m'' are defined by the equalities

$$m' = \frac{\lambda_1}{\varphi_2}(c_2 + g\lambda_2) + g\lambda_1\frac{H_2 - 1}{H_2},$$

$$m'' = H_1\lambda_1\left(\frac{c_1}{\varphi_1} - \frac{c_2}{\varphi_2}\right) + H_1 g\lambda_1\left(\frac{\lambda_1}{\varphi_1} - \frac{\lambda_2}{\varphi_2}\right) + g\lambda_1\frac{H_1 - H_2}{H_2}.$$

The proof of Theorem 4 is given in the Appendix.

6. A NUMERICAL EXAMPLE

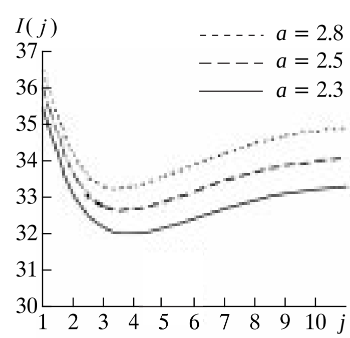
Let the intervals between the instants of arrival of customers in a queueing system in operation modes I and II be exponentially distributed with parameters $\lambda_1 = 5$ and $\lambda_2 = 3.5$, and let the intervals between the instants of occurrence of disasters be exponentially distributed with parameters $\varphi_1 = 0.12$ and $\varphi_2 = 0.06$, let the customer service time have the distribution function $B_1(t)$ and $B_2(t)$ with the Laplace–Stieltjes transform $B_1^*(s) = \frac{1}{2}\left(e^{-0.17s} + e^{-0.22s}\right)$ and $B_2^*(s) = \frac{1}{2}\left(e^{-0.15s} + e^{-0.22s}\right)$, respectively. The cost indexes are $c_1 = 15$, $c_2 = 40$, and $g = 10$. Then $\rho_1 = 0.952$, $\rho_2 = 0.64$, $H_1 = 0.023$, and $H_2 = 0.011$.

Table

a	$I(j^*)$	I_1	I_2
2.3	32.035	61.97	44.45
2.5	32.691	66.01	44.82
2.8	33.341	72.05	45.36

Let I_r denote the value of the quality criterion in the rth operation mode, $r = \overline{1,2}$. The minimal values of the quality criterion $I(j^*)$ for $0 \le j \le 10$ and the values of I_1 and I_2 for different penalties a are listed in table.

Figure shows the quality criterion (1) as a function of the threshold j for different penalties a.



Quality criterion $I(j)$ versus threshold j.

7. CONCLUSIONS

The paper is devoted to a controllable queueing system in which the operation mode may change, depending on the queue length at an instant and an additional flow of disasters arrives, thereby resulting in the evacuation of the system. The stationary state probability distribution of the system and the values of the quality criterion for fixed thresholds are determined. The results are illustrated by a numerical example.

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APPENDIX

Proof of Theorem 1. The stationary probabilities π_i $(i \geq 0)$ satisfy the linear algebraic equations

$$\begin{aligned} \pi_0 &= \pi_0 f_0^{(1)} + \pi_1 f_0^{(1)} + H_1 \sum_{i=0}^{j} \pi_i + H_2 \sum_{i=j+1}^{\infty} \pi_i, \\ \pi_i &= \pi_0 f_i^{(1)} + \sum_{k=1}^{i+1} \pi_k f_{i-k+1}^{(1)}, \quad i = \overline{1, j-1}, \\ \pi_{j+l} &= \pi_0 f_{j+l}^{(1)} + \sum_{k=1}^{j} \pi_k f_{j+l-k+1}^{(1)} + \sum_{k=1}^{l+1} \pi_{j+k} f_{l-k+1}^{(2)}, \quad l = \overline{0, \infty}. \end{aligned} \tag{A.1}$$

Multiplying the equations in system (A.1) by suitable powers of z and then summing, we obtain the following relation for the partial generating functions $\Pi_i(z)$ $(i = \overline{1,2})$:

$$\pi_0 \beta_1^*(z)(1-z) + \Pi_1(z)(z - \beta_1^*(z)) + \Pi_2(z)(z - \beta_2^*(z)) - z(\Pi_1(1)H_1 + \Pi_2(1)H_2) = 0. \tag{A.2}$$

Thus, we have obtained one equation for two unknown functions $\Pi_1(z)$ and $\Pi_2(z)$. Therefore, we must find some additional relations for determining these functions.

To find $\Pi_1(z)$, let us determine a relationship between π_0 and π_i $(i = \overline{1, j})$. Expressing $\Pi_1(z)$ in terms of (A.2), we obtain

$$\Pi_1(z) = \pi_0 \frac{\beta_1^*(z)(z-1)}{z - \beta_1^*(z)} + \Pi_2(z) \frac{\beta_2^*(z) - z}{z - \beta_1^*(z)} + z \frac{\Pi_1(1)H_1 + \Pi_2(1)H_2}{z - \beta_1^*(z)}. \tag{A.3}$$

Since

$$\frac{\Pi_1^{(i)}(0)}{i!} = \pi_i \quad (i = \overline{0, j}),$$

from (A.3) we obtain the expression for π_i $(i = \overline{0, j})$

$$\begin{aligned} \pi_i = \pi_0 \frac{1}{i!} \frac{\partial^i}{\partial z^i} \frac{\beta_1^*(z)(z-1)}{z - \beta_1^*(z)}\Big|_{z=0} + \frac{1}{i!} \frac{\partial^i}{\partial z^i} \left[\Pi_2(z) \frac{\beta_2^*(z) - z}{z - \beta_1^*(z)} \right]\Big|_{z=0} \\ + \Big(\Pi_1(1)H_1 + \Pi_2(1)H_2 \Big) \frac{1}{i!} \frac{\partial^i}{\partial z^i} \frac{z}{z - \beta_1^*(z)}\Big|_{z=0}, \quad i = \overline{0, j}. \end{aligned} \tag{A.4}$$

Using the definition of $\Pi_2(z)$, we can show that the second term in (A.4) is 0. Using the notation

$$\alpha_i^{(1)} = \frac{1}{i!} \frac{\partial^i}{\partial z^i} \frac{\beta_1^*(z)(1-z)}{\beta_1^*(z) - z}\Big|_{z=0}, \tag{A.5}$$

$$\beta_i^{(1)} = \frac{1}{i!} \frac{\partial^i}{\partial z^i} \frac{z}{z - \beta_1^*(z)}\Big|_{z=0}, \quad i = \overline{0, j}, \tag{A.6}$$

$$\theta = \Pi_1(1)H_1 + \Pi_2(1)H_2, \tag{A.7}$$

we can express (A.4) for π_i $\left(i=\overline{0,j}\,\right)$ as

$$\pi_i = \alpha_i^{(1)}\pi_0 + \theta\beta_i^{(1)}, \quad i=\overline{0,j}. \tag{A.8}$$

It is easy to show that the coefficients $\alpha_i^{(1)}$ $\left(i=\overline{0,j}\,\right)$ satisfy the recurrent relation (4). We can also verify that (A.5) and (A.6) imply the validity of (5).

Using equality (A.8) for π_i $\left(i=\overline{0,j}\,\right)$, we obtain the expression for the function $\Pi_1(z)$:

$$\Pi_1(z) = \sum_{i=0}^{j}\left(\alpha_i^{(1)}\pi_0 + \theta\beta_i^{(1)}\right) z^i, \tag{A.9}$$

where $\alpha_i^{(1)}$ and $\beta_i^{(1)}$ $\left(i=\overline{0,j}\,\right)$ are known coefficients. The function $\Pi_1(z)$ is known within to π_0 and θ. Let us find the relationship between θ and π_0. For this, let us rewrite (A.7) as

$$\theta = \left(\Pi_1(z)H_1 + \Pi_2(z)H_2\right)\Big|_{z=1} \tag{A.10}$$

and express the function $\Pi_2(z)$ in terms of (A.2). By virtue of (A.9), we obtain

$$\Pi_2(z) = \pi_0\left[\frac{\beta_1^*(z)(z-1)}{z-\beta_2^*(z)} + \sum_{i=0}^{j}\alpha_i^{(1)}\frac{\beta_1^*(z)-z}{z-\beta_2^*(z)}z^i\right] + \theta\left[\frac{z}{z-\beta_2^*(z)} + \sum_{i=0}^{j}\beta_i^{(1)}\frac{\beta_1^*(z)-z}{z-\beta_2^*(z)}z^i\right]. \tag{A.11}$$

Substituting (A.9) and (A.11) into (A.10), we obtain

$$\theta = \sum_{i=0}^{j}(\alpha_i^{(1)}\pi_0 + \theta\beta_i^{(1)})\left(H_1 + H_2\frac{\beta_1^*(z)-z}{z-\beta_2^*(z)}\right)z^i\Bigg|_{z=1} + \left(\theta\frac{zH_2}{z-\beta_2^*(z)} + \pi_0 H_2\frac{\beta_1^*(z)(z-1)}{z-\beta_2^*(z)}\right)\Bigg|_{z=1}.$$

Expressing θ in terms of the last expression, we obtain

$$\theta = \pi_0\frac{\Phi(z)\sum\limits_{i=0}^{j}\alpha_i^{(1)}z^i + H_2\beta_1^*(z)(z-1)}{z-\beta_2^*(z) - zH_2 - \Phi(z)\sum\limits_{i=0}^{j}\beta_i^{(1)}z^i}\Bigg|_{z=1}, \tag{A.12}$$

where $\Phi(z) = H_1(z-\beta_2^*(z)) + H_2(\beta_1^*(z)-z)$.

To find the expression in the right side of (A.12), let us compute

$$\beta_i^*(z)\Big|_{z=1} = \int_0^\infty e^{-\varphi_i t}\,dB_i(t) = 1 - H_i, \quad i=\overline{1,2}. \tag{A.13}$$

By virtue of (A.13), an uncertainty of type $\dfrac{0}{0}$ arises in the right side of (A.12) for $z=1$. Therefore, let us take the limit as $z\to 1$ and apply the Hospital rule. For this, first we find

$$\frac{\partial}{\partial z}\beta_i^*(z)\Big|_{z=1} = \int_0^\infty \lambda_i t e^{-\varphi_i t}\,dB_i(t) = \rho_i, \quad i=\overline{1,2}. \tag{A.14}$$

By virtue of (A.13) and (A.14), we finally obtain

$$\theta = \pi_0\gamma, \tag{A.15}$$

where γ is determined by formula (9) under the conditions of the corollary and H is determined by equality (3) under the conditions of Theorem 1.

We now find the generating function $\Pi(z)$ for the number of customers in the system. Since

$$\Pi_2(z) = \Pi(z) - \Pi_1(z),$$

we can find the function $\Pi(z)$ from equality (A.2), using (A.9):

$$\Pi(z) = \frac{\pi_0}{z - \beta_2^*(z)} \left[\beta_1^*(z)(z-1) + \left(\beta_1^*(z) - \beta_2^*(z)\right) \times \left(\sum_{i=0}^{j} \alpha_i^{(1)} + \gamma \sum_{i=0}^{j} \beta_i^{(1)} \right) z^i + \gamma z \right]. \quad (A.16)$$

Substituting γ from (9) into (A.16), we obtain the expression for the generating function $\Pi(z)$ (2).

Proof of the corollary. Let us find the value of the probability π_0 from the normalization condition $\Pi(1) = 1$. Substituting $z = 1$ into (2), we obtain expression (6) for π_0.

Since $\theta = \pi_0 \gamma$, substituting this expression into (A.8), we obtain expression (7) for π_i $(i = \overline{1, j}\,)$. To find the probabilities π_k $(k = \overline{j+1, \infty}\,)$, let us determine $\Pi_2(z)$, using relation (A.11). Using the equality

$$\frac{\Pi_2^{(k)}(0)}{k!} = \pi_k \qquad (k = \overline{j+1, \infty}\,)$$

we find from (A.11) the expression for π_k:

$$\pi_k = \pi_0 \left[\frac{1}{k!} \frac{\partial^k}{\partial z^k} \frac{\beta_1^*(z)(z-1)}{z - \beta_2^*(z)} - \sum_{i=0}^{j} \alpha_i^{(1)} \sum_{m=0}^{k} \frac{C_k^m}{k!} \frac{\partial^m}{\partial z^m}(z^i) \frac{\partial^{k-m}}{\partial z^{k-m}} \frac{\beta_1^*(z) - z}{\beta_2^*(z) - z} \right] \Bigg|_{z=0}$$

$$+ \theta \left[\frac{1}{k!} \frac{\partial^k}{\partial z^k} \frac{z}{z - \beta_2^*(z)} - \sum_{i=0}^{j} \beta_i^{(1)} \sum_{m=0}^{k} \frac{C_k^m}{k!} \frac{\partial^m}{\partial z^m}(z^i) \frac{\partial^{k-m}}{\partial z^{k-m}} \frac{\beta_1^*(z) - z}{\beta_2^*(z) - z} \right] \Bigg|_{z=0}, \quad k = \overline{j+1, \infty}. \quad (A.17)$$

Let us find the factors at $\alpha_i^{(1)}$ and $\beta_i^{(1)}$

$$\sum_{m=0}^{k} \frac{C_k^m}{k!} \frac{\partial^m}{\partial z^m}(z^i) \frac{\partial^{k-m}}{\partial z^{k-m}} \frac{\beta_1^*(z) - z}{\beta_2^*(z) - z} \bigg|_{z=0} = \frac{1}{(k-i)!} \frac{\partial^{k-i}}{\partial z^{k-i}} \frac{\beta_1^*(z) - z}{\beta_2^*(z) - z} \bigg|_{z=0}.$$

By virtue of (A.15), we find that (A.17) for π_k $(k = \overline{j+1, \infty}\,)$ assumes the form (8), where $\alpha_k^{(2)}$ and $\beta_k^{(2)}$ $(k = \overline{j+1, \infty}\,)$ are defined by formulas (10), and ϑ_k, Δ_k, and ξ_k $(k = \overline{j+1, \infty}\,)$ are defined by the expressions

$$\vartheta_k = \frac{1}{k!} \frac{\partial^k}{\partial z^k} \frac{\beta_1^*(z)(z-1)}{z - \beta_2^*(z)} \bigg|_{z=0},$$

$$\Delta_k = \frac{1}{k!} \frac{\partial^k}{\partial z^k} \frac{\beta_1^*(z) - z}{\beta_2^*(z) - z} \bigg|_{z=0},$$

$$\xi_k = \frac{1}{k!} \frac{\partial^k}{\partial z^k} \frac{z}{z - \beta_2^*(z)} \bigg|_{z=0}, \quad k = \overline{j+1, \infty}.$$

We can show that Δ_k, ϑ_k, and ξ_k $(k = \overline{j+1, \infty}\,)$ are given by the recurrent formulas (11), (12), and (13), respectively. This completes the proof of formula (8).

Proof of Theorem 4. The mean number L of customers at the instant of departure of a customer from the system is

$$L = \Pi'(1).$$

Here $\Pi^{'}(1)$ is defined by equality (A.16)

$$\Pi'(1) = \pi_0 \left[\frac{1 - H_1}{H_2} + \frac{\gamma}{H_2^2}(H_2 + \rho_2 - 1) + \frac{H}{H_2^2} \left(\sum_{i=0}^{j} \alpha_i^{(1)} + \gamma \sum_{i=0}^{j} \beta_i^{(1)} \right) \right.$$
$$\left. + \frac{H_2 - H_1}{H_2} \left(\sum_{i=0}^{j} i\alpha_i^{(1)} + \gamma \sum_{i=0}^{j} i\beta_i^{(1)} \right) \right]. \qquad \text{(A.18)}$$

The probabilities P_1 and P_2, and R are given by the expressions

$$P_2 = \Lambda \overline{b}_1^{(2)} P_2(1),$$
$$P_1 = 1 - P_2,$$
$$R = (\lambda_1 P_1 + \lambda_2 P_2)(1 - P_+).$$

By virtue of (17) and (19), we obtain

$$\begin{aligned} P_2 &= \frac{\lambda_1 \overline{b}_1^{(2)} P_2(1)}{P_1(1) + P_2(1)(1 + \overline{b}_1^{(2)}(\lambda_1 - \lambda_2)) + K'(1)}, \\ P_1 &= 1 - P_2, \\ R &= \frac{\lambda_1 K'(1)}{P_1(1) + P_2(1)(1 + \overline{b}_1^{(2)}(\lambda_1 - \lambda_2)) + K'(1)}. \end{aligned} \qquad \text{(A.19)}$$

Using equalities (5), we obtain the relations

$$\sum_{i=0}^{j} \beta_i^{(1)} = -(j+1) \sum_{i=0}^{j} \alpha_i^{(1)} + \sum_{i=0}^{j} i\alpha_i^{(1)}, \qquad \text{(A.20)}$$
$$\sum_{i=0}^{j} i\beta_i^{(1)} = -\frac{1}{2}(j^2 + j) \sum_{i=0}^{j} \alpha_i^{(1)} - \frac{1}{2} \sum_{i=0}^{j} i\alpha_i^{(1)} + \frac{1}{2} \sum_{i=0}^{j} i^2 \alpha_i^{(1)}.$$

Substituting the expressions for L, P_1, P_2, and R from (A.18) and (A.19) into (1), by virtue of relations (16) for $P_1(1)$, $P_2(1)$, and $K'(1)$, equality (20) for $\overline{b}_1^{(2)}$, and relations (A.20), we find that the relationship between the quality criterion (1) and threshold j takes the form (21), where the constants m_i $\left(i = \overline{1, 8}\right)$ are given by formulas (22).

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