Nonlocal branches of cycles, bistability, and topologically persistent mixed mode oscillations

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(Received 26 January 2007; accepted 30 May 2007)

A possible mechanism for generating mixed mode oscillations is based on an appropriate S-shaped structure, which graphs the relation between the parameter and the collection of periodic oscillations existing for a particular parameter value in the product of parameter and phase spaces. This natural scenario should be supplemented by simple and constructive criteria of existence, and methods of localization, of such S-shaped structures. These criteria are the main focus of the paper. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2779847]

During the past decade, significant attention has been paid to mixed mode oscillations (MMO), whose characteristic feature is a regular alternation of large- and small-magnitude oscillations in the observed time series. This phenomenon plays an important role in chemical, biological, and industrial applications. Identification and a thorough investigation of general scenarios leading to this phenomenon is important from both theoretical and practical perspectives. One natural scenario may be informally described as follows. The system is treated as a parametric control system with an object and a feedback loop. The object is a dynamical system with a finite-dimensional state containing one parameter; the object's dynamics, for a given value of the parameter, are described by a differential equation. The feedback adjusts the value of the parameter in terms of the current value of the state of the object. An essential feature of the object is coexistence (for a range of parameter values) of two different stable oscillatory modes; this situation is often referred to as bi- or multistability. The role of the feedback is to ensure a regular, nearly periodic switching between the aforementioned periodic modes. The simplest mechanism here is based on an appropriate S-shaped structure, which graphs the relation between the parameter and the collection of periodic oscillations existing for a particular parameter value in the product of parameter and phase spaces. This scenario is natural and theoretically satisfactory. To be useful in practice, it should be supplemented by simple and constructive criteria of existence, and methods of localization, of such S-shaped structures. These criteria are the main focus of the paper.

I. INTRODUCTION

In this paper, we make methodological remarks concerning the existence of mixed mode oscillations (MMO). Our starting point is a well known analogy between MMO and relaxation oscillations. It is instructive to keep in mind a specific example,

$$\dot{x} = y, \quad \varepsilon \dot{y} = g(x,y),$$

where g has the totality of zeros as shown by the solid line in Fig. 1, also indicating the sign of g. The solid line is thus the slow manifold of the system.

This system exhibits a nearly periodic series of switchings between two horizontal branches of the slow manifold. The dynamics has thus two distinct phases: during one the energy is stored up slowly; during the other the energy is discharged much more quickly when one of the critical thresholds, $x=\alpha$ or $x=\beta$, is attained. If switching between two steady states, as in this example, is replaced by switchings between two or more modes of stable periodic (or nearly periodic) oscillations, then one observes the MMO-like behavior: this simple mechanism is described, for example, in Ref. 1.

The key feature of relaxation oscillations is the existence of a nonlocal S-shaped slow manifold. It is therefore tempting to link MMO to the existence of a nonlocal S-shaped "slow branch of self-oscillations." To be more definite, let us consider an autonomous equation with the scalar parameter $\lambda \in (\lambda_-, \lambda_+)$ of the form

$$L(d/dt)x = F(x,\lambda), \tag{1}$$

with a polynomial $L(p)=a_0p^\ell + a_1p^{\ell-1} + \cdots + a_\ell$ of degree $\ell \geq 3$. Suppose that this equation has isolated cycles $x(t)$ de-

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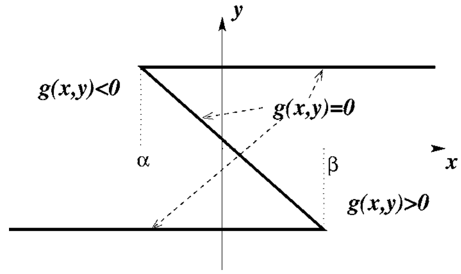

FIG. 1. Relaxation oscillations in a singularly perturbed ordinary differential equation.

pending on λ, which we visualize as a curve in the plane $(\lambda, \|x\|_C)$ with $\|x\|_C = \max|x(t)|$. Moreover, suppose that Eq. (1) possesses an *S-shaped branch of cycles*, that is, the curve obtained has the shape presented in Fig. 2. This curve consists of three parts: the lower and the upper branches contain stable cycles (they are drawn bold in Fig. 2), and the intermediate part contains unstable cycles. Let the parameter λ oscillate slowly between λ_- and λ_+ [say, put $2\lambda = (\lambda_- + \lambda_+) + (\lambda_+ - \lambda_-)\sin(\varepsilon t)$, ε is small] and consider a solution x of the resulting nonautonomous equation. Since cycles on the lower branch of the curve Γ are stable, the solution x should follow closely the cycle of the autonomous system, lying on this branch, on the time scale t. On the time scale εt, the attracting cycle will vary slowly, following the change of the parameter λ. As $\lambda = \lambda(t)$ reaches the value λ_r, the solution x switches to the stable cycle on the upper branch of the curve Γ. Then it slowly follows this branch until λ reaches the value λ_ℓ, where it switches back to the lower stable branch of Γ, etc. The switches between the two stable branches of Γ account for the switches between the two oscillation regimes with the sudden (on the εt time scale) change of frequency and amplitude. The slow forcing of λ can be replaced in this scheme by a feedback, which couples Eq. (1) with another equation, say of the form $\dot{\lambda} = \varepsilon g(x, \lambda)$, ensuring that the parameter λ oscillates slowly between λ_- and λ_+. The actual form of g does not matter in the context of this paper. It is enough to ensure that

$$\int_0^{T_-(\lambda)} g(x_-(t;\lambda),\lambda)dt > 0, \quad \int_0^{T_+(\lambda)} g(x_+(t;\lambda),\lambda)dt < 0,$$

where $x_-(t,\lambda)$ is the periodic solution of Eq. (1) on the lower stable branch of Γ, $x_+(t,\lambda)$ is the periodic solution on the upper stable branch, and $T_-(\lambda), T_+(\lambda)$ are periods of these solutions.

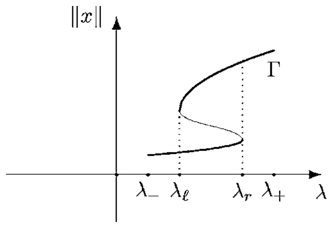

FIG. 2. S-shaped continuous branch of cycles with twofold bifurcations at $\lambda = \lambda_\ell$ and $\lambda = \lambda_r$; $\|x\| = \|x\|_C$ is the amplitude of the cycle; stable parts of the branch are shown in bold.

The above link between MMO and S-shaped curves of cycles is fruitful only if supplemented with robust and constructive criteria for the existence of those curves. Such criteria are the focus of this paper. We combine a proper extension of our former results on continuous branches of cycles born via Hopf bifurcations (the global branches connecting an equilibrium and infinity[2]) with theorems on the existence of multiple cycles for a given parameter value.

II. S-SHAPED BRANCHES OF CYCLES

A simple picture underpinning and illustrating the results of this section is the following. Suppose that for $\lambda < \lambda_0$, Eq. (1) has a globally stable equilibrium at zero, which, at $\lambda = \lambda_0$, loses stability via the supercritical Hopf bifurcation. Hence, there is a branch of small stable cycles for $\lambda > \lambda_0$. Let, for some $\lambda_\infty > \lambda_0$, the Hopf bifurcation at infinity occur, and let the system be globally unstable for $\lambda > \lambda_\infty$, i.e., any nonzero solutions tend to infinity. Under appropriate conditions, in this situation, there is a continuous branch of cycles connecting the Hopf bifurcation points at the zero equilibrium and infinity for $\lambda_0 < \lambda < \lambda_\infty$. If, for some $\lambda_* \in (\lambda_0, \lambda_\infty)$, the equation has three cycles, then we may expect that this branch is S-shaped.

We consider equations of the form

$$L(d/dt)x = qf(x) + \lambda x. \tag{2}$$

We consider Eq. (2) for a fixed value of the parameter $q > 0$, while λ ranges over an interval $[\lambda_-, \lambda_+]$. Assume that f satisfies $f(0) = 0$, hence Eq. (2) has a zero solution for all λ. Furthermore, suppose that f is globally Lipschitz continuous,

$$|f(x_1) - f(x_2)| \le K|x_1 - x_2|, \quad x_1, x_2 \in \mathbb{R}, \tag{3}$$

and has finite derivatives at zero and at infinity, which are different,

$$f'(0) = \alpha_0, \quad f'(\infty) := \lim_{x \to \pm\infty} f(x)/x = \alpha_\infty, \quad \alpha_0 \ne \alpha_\infty. \tag{4}$$

We assume that $\alpha_0 > \alpha_\infty$ and $\lambda_- < 0 < \lambda_+$, $[-\alpha_0 q, -\alpha_\infty q] \subset (\lambda_-, \lambda_+)$, and that the polynomial L has a pair of pure imaginary eigenvalues $\pm i w_0$ with $w_0 > 0$. Hence L may be factorized as $L(p) = (p^2 + w_0^2)M(p)$, where M is a polynomial of degree $\ell - 2$. Let us also suppose that the nonresonance and transversality conditions

$$M(inw_0) \ne 0, \quad n \in \mathbb{Z}; \quad \operatorname{Im} M(iw_0) \ne 0 \tag{5}$$

hold. Relations (4) and (5) imply that $\lambda_0 = -\alpha_0 q$ is a point of the *Hopf bifurcation from the zero* for Eq. (2), and $\lambda_\infty = -\alpha_\infty q$ is a point of the *Hopf bifurcation from infinity* as in Refs. 3 and 4. If $\operatorname{Im} M(iw) \ne 0$ for all $w > 0$, then λ_0 and λ_∞ are the only Hopf bifurcation points from zero and from infinity, respectively. The main case of interest for us is when $M(p)$ is a Hurwitz polynomial and $\operatorname{Im} M(iw_0) < 0$, which implies that the zero equilibrium loses stability via the Hopf bifurcation at the point $\lambda = -\alpha_0 q$ while λ increases.

Set

$$\Phi(r) = \frac{1}{\pi r}\int_0^{2\pi} f(r \sin t)\sin t \, dt, \quad r \ge 0, \tag{6}$$

$$\mu(\lambda,w) = \min_{n\in\mathbb{Z},n\neq\pm1} |L(nwi) - \lambda|. \tag{7}$$

Relations (4) imply $\Phi(0) := \lim_{r\to0}\Phi(r) = \alpha_0$, $\Phi(\infty) := \lim_{r\to\infty}\Phi(r) = \alpha_\infty$, hence $\Phi(0) > \Phi(\infty)$.

Theorem 1. *Let for some* $\lambda \in (\lambda_-,\lambda_+)$ *and some* $w_+ > w_- > 0$ $[w_0 \in (w_-,w_+)]$ *the relations*

$$\min_{w\in[w_-,w_+]} \mu(\lambda,w) > qK, \tag{8}$$

$$|w_0^2 - w^2||\mathrm{Im}\, M(iw)| > \frac{q^2K^2}{\sqrt{\mu^2(\lambda,w) - q^2K^2}} \quad \text{for } w = w_-,w_+, \tag{9}$$

$$\mathrm{Im}\, M(iw) \neq 0 \quad \text{for } w \in [w_-,w_+]. \tag{10}$$

hold. Let, in addition,

$$|q^{-1}\lambda + \Phi(r)| > \max_{w\in[w_-,w_+]} \left(1 + \frac{|\mathrm{Re}\, M(iw)|}{|\mathrm{Im}\, M(iw)|}\right) \times \frac{qK^2}{\sqrt{\mu^2(\lambda,w) - q^2K^2}} \quad \text{for } r = r_-,r_+ \tag{11}$$

for some $r_+ > r_- > 0$, *and*

$$[q^{-1}\lambda + \Phi(r_-)][q^{-1}\lambda + \Phi(r_+)] < 0. \tag{12}$$

Then for this particular value of λ , *Eq.* (2) *has a cycle* $x = x(t)$ *of a period* $2\pi/w$ *with* $w \in (w_-,w_+)$ *satisfying*

$$r_- < \left| \frac{w}{\pi}\int_0^{2\pi/w} x(t)e^{iwt}dt \right| < r_+.$$

The next corollary ensures the coexistence of multiple cycles for a fixed λ.

Corollary 1. *Suppose that there exist numbers* $r_M > r_m > 0$ *such that*

$$\Phi(r_m) = \min_{r\in[0,r_M]} \Phi(r), \quad \Phi(r_M) = \sup_{r\geq r_m}\Phi(r) \tag{13}$$

and that the values $\alpha_m = \Phi(r_m)$, $\alpha_M = \Phi(r_M)$ *of function* (6) *satisfy*

$$\alpha_0 > \alpha_M > \alpha_m > \alpha_\infty. \tag{14}$$

Let for some interval $(w_-,w_+) \ni w_0$ *with* $w_- > 0$ *and for* $\lambda = \lambda_c := -q(\alpha_m + \alpha_M)/2$ *relations* (8)–(10) *and*

$$\frac{\alpha_M - \alpha_m}{2} > \max_{w\in[w_-,w_+]} \left(1 + \frac{|\mathrm{Re}\, M(iw)|}{|\mathrm{Im}\, M(iw)|}\right) \frac{qK^2}{\sqrt{\mu^2(\lambda_c,w) - q^2K^2}} \tag{15}$$

hold. Then for each λ *sufficiently close to* λ_c, *Eq.* (2) *has at least three cycles* x_k: *these cycles and their periods* $2\pi/w_k$ *satisfy* $w_k \in (w_-,w_+)$ *and*

$$\begin{aligned}\tilde{r}_m &< \left| \frac{w_1}{\pi}\int_0^{2\pi/w_1} x_1(t)e^{iw_1t}dt \right| < r_m \\ &< \left| \frac{w_2}{\pi}\int_0^{2\pi/w_2} x_2(t)e^{iw_2t}dt \right| < r_M \\ &< \left| \frac{w_3}{\pi}\int_0^{2\pi/w_3} x_3(t)e^{iw_3t}dt \right| < \tilde{r}_M,\end{aligned} \tag{16}$$

where the bounds $\tilde{r}_M > \tilde{r}_m > 0$ *are defined by*

$$\tilde{r}_M > r_M, \quad \Phi(\tilde{r}_M) = \alpha_m; \quad \Phi(r) \geq \alpha_m \quad \text{for } r \leq \tilde{r}_M, \tag{17}$$

$$\tilde{r}_m < r_m, \quad \Phi(\tilde{r}_m) = \alpha_M; \quad \Phi(r) \leq \alpha_M \quad \text{for } r \geq \tilde{r}_m. \tag{18}$$

If there exist λ_m, λ_M *such that*

$$\alpha_0 > -q^{-1}\lambda_m > \alpha_M > \alpha_m > -q^{-1}\lambda_M > \alpha_\infty, \tag{19}$$

and for some interval $[w',w''] \supset [w_-,w_+]$

$$\mu(\lambda,w) > qK \quad \text{for } \lambda = \lambda_m,\lambda_M, \quad w \in [w',w''], \tag{20}$$

$$-(q^{-1}\lambda_m + \alpha_M) > \max_{w\in[w',w'']} \left(1 + \frac{|\mathrm{Re}\, M(iw)|}{|\mathrm{Im}\, M(iw)|}\right) \times \frac{qK^2}{\sqrt{\mu^2(\lambda_m,w) - q^2K^2}}, \tag{21}$$

$$q^{-1}\lambda_M + \alpha_m > \max_{w\in[w',w'']} \left(1 + \frac{|\mathrm{Re}\, M(iw)|}{|\mathrm{Im}\, M(iw)|}\right) \times \frac{qK^2}{\sqrt{\mu^2(\lambda_M,w) - q^2K^2}}, \tag{22}$$

then for $\lambda = \lambda_m$ Eq. (2) *does not have* $2\pi/w$ *-periodic cycles with*

$$w \in [w',w''], \quad \left| \frac{w}{\pi}\int_0^{2\pi/w} x(t)e^{iwt}dt \right| \geq \tilde{r}_m \tag{23}$$

and for $\lambda = \lambda_M$ *it has no* $2\pi/w$-periodic cycles with

$$w \in [w',w''], \quad \left| \frac{w}{\pi}\int_0^{2\pi/w} x(t)e^{iwt}dt \right| \leq \tilde{r}_M. \tag{24}$$

The existence of numbers $\tilde{r}_M, \tilde{r}_m$ satisfying Eqs. (17) and (18) follows from continuity of the function Φ and relations (14).

We say that Eq. (2) *has a continuous curve of cycles* if a segment $[w',w''] \subset (0,\infty)$ and continuous functions $\lambda = \lambda(r)$, $w = w(r)$ of a parameter $r > 0$ with values in the intervals $[\lambda_-,\lambda_+]$, $[w',w'']$ exist, such that for each $r > 0$, Eq. (2) with $\lambda = \lambda(r)$ has a nonstationary periodic solution $x_r = x_r(t)$ with the period $2\pi/w(r)$, the function $x_r(t/w(r))$ depends continuously on r in the space $C(0,2\pi)$, and

$$\lim_{r\to0} \|x_r(t/w(r))\|_{C(0,2\pi)} = 0, \quad \lim_{r\to\infty} \|x_r(t/w(r))\|_{C(0,2\pi)} = \infty.$$

We say that a continuous curve of cycles is S-shaped if there are numbers $0 < r_0 < r^0$ such that

$$\lambda(r_0) < \lambda(r) \quad \text{for } r > r_0, \quad \lambda(r) < \lambda(r^0) \quad \text{for } r < r^0 \tag{25}$$

and the function $\lambda(r)$ is *not monotone* on the segment $r_0 \le r \le r^0$. If $\lambda_{\rm in}=\lambda(r_{\rm in})$, $\lambda_{\rm end}=\lambda(r_{\rm end})$, and $[\lambda(r_0),\lambda(r^0)] \subset [\lambda_{\rm in},\lambda_{\rm end}]$, then relations (25) imply that $[r_0,r^0] \subset [r_{\rm in},r_{\rm end}]$. Therefore, if λ changes monotonically from $\lambda_{\rm in}$ to $\lambda_{\rm end}$ (or from $\lambda_{\rm end}$ to $\lambda_{\rm in}$) and the point (r,λ) is always on the graph of the continuous curve $\lambda(r)$, then r must have jumps, because $\lambda(r)$ is nonmonotone on $[r_0,r^0]$. These jumps account for switching between oscillation modes.

Set

$$\chi(\xi,\eta) = 1 + \xi^2 + \eta^2 + \sqrt{(1-\xi^2-\eta^2)^2 + 4\xi^2}, \tag{26}$$

$$\nu(n,w) = \frac{1}{2}\chi\left(\frac{\operatorname{Im} L'(iw) - n \operatorname{Im} L'(inw)}{\operatorname{Re} L'(iw)}, \frac{n \operatorname{Re} L'(inw)}{\operatorname{Re} L'(iw)}\right), \tag{27}$$

where $L'=L'(p)$ is the derivative of the polynomial $L=L(p)$. For $0<w'<w''$, define

$$\rho_1 = \min_{\lambda\in\{\lambda_-,\lambda_+\},w\in[w',w'']} |L(wi)-\lambda|,$$
$$\rho_2 = \min_{\lambda\in[\lambda_-,\lambda_+],w\in\{w',w''\}} |L(wi)-\lambda|, \quad \rho=\min\{\rho_1,\rho_2\}, \tag{28}$$

$$\Omega_\rho = \{(\lambda,\omega): |L(wi)-\lambda| \le \rho, \lambda\in[\lambda_-,\lambda_+], w\in[w',w'']\}. \tag{29}$$

Theorem 2. *Let the function $\varphi(w)=\operatorname{Im} L(wi)$ have a nonzero derivative on some interval $[w',w'']$ with $w''>w_0>w'>0$,*

$$\operatorname{Re} L'(iw) \neq 0, \quad w\in[w',w''], \tag{30}$$

and let the function (7) satisfy the estimate

$$\mu(\lambda,w) > qK \quad \text{for all } (\lambda,w)\in\Omega_\rho, \tag{31}$$

where the set Ω_ρ is defined by Eqs. (28) and (29). Suppose that

$$q^2K^2\left(\max_{(\lambda,w)\in\Omega_\rho}\frac{1}{\mu^2(\lambda,w)} + \max_{(\lambda,w)\in\Omega_\rho}\frac{\pi q^2K^2\mu^2(\lambda,w)}{\mu^2(\lambda,w)-q^2K^2}\right.$$
$$\left.\times \max_{n\neq\pm1,(\lambda,w)\in\Omega_\rho}\frac{\nu^2(n,w)}{|L(nwi)-\lambda|^4}\right) < 1, \tag{32}$$

$$q^2K^2 \max_{(\lambda,w)\in\Omega_\rho}\frac{\mu^2(\lambda,w)}{\mu^2(\lambda,w)-q^2K^2} \le \rho^2 \tag{33}$$

with $\nu(\cdot,\cdot)$ defined by Eqs. (26) and (27). Then Eq. (2) has a continuous curve of cycles with $(\lambda(r),w(r))\in\Omega_\rho$ for all $r>0$ and with

$$\lambda(0) := \lim_{r\to+0}\lambda(r) = -\alpha_0 q, \quad \lambda(\infty) := \lim_{r\to+\infty}\lambda(r) = -\alpha_\infty q. \tag{34}$$

Moreover, all cycles of Eq. (2) with $(\lambda,w)\in\Omega_\rho$ belong to this curve.

Relations $\lambda_-<0<\lambda_+$, $w'<w_0<w''$ and Eq. (30) imply that $\rho>0$, and the set Ω_ρ is nonempty. Therefore, combining Corollary 1 and Theorem 2, we obtain the following result.

Theorem 3. *Suppose that all the assumptions of Corollary 1 and Theorem 2 are satisfied, i.e., relations (13)–(15), (19)–(22), and (30)–(33), and*

$$|w_0^2 - w^2||\operatorname{Im} M(iw)| > \frac{q^2K^2}{\sqrt{\mu^2(\lambda_c,w)-q^2K^2}} \quad \text{for } w=w_-,w_+ \tag{35}$$

with $\lambda_c=-q(\alpha_m+\alpha_M)/2$ and $[w_-,w_+]\subset[w',w'']$ hold. Let

$$|L(wi)-\lambda| < \rho \quad \text{for } \lambda\in[\lambda_m,\lambda_M], \quad w\in[w_-,w_+]. \tag{36}$$

Then Eq. (2) *has an S-shaped continuous curve of cycles with $(\lambda(r),w(r))\in\Omega_\rho$.*

Condition (36) implies $[\lambda_m,\lambda_M]\times[w_-,w_+]\subset \operatorname{Int}\Omega_\rho$, hence, from Eq. (31) it follows that Eq. (8) holds in some neighborhood of the segment $[\lambda_m,\lambda_M]\ni\lambda_c$. Relation (30) and the second of relations (5) imply $\operatorname{Im} M(wi)\neq 0$ on $[w',w'']$ and, hence, Eq. (10). Figure 3 illustrates the result and extends Fig. 2. If relations (13) and (14) hold for the function Φ, then the conditions of Theorem 3 are satisfied for any sufficiently small q. This implies the next corollary.

Corollary 2. *Relations (13) and (14) imply that Eq. (2) has an S-shaped continuous curve of cycles for each sufficiently small $q>0$.*

Theorem 3 provides one with an algorithm to obtain a lower bound for the range $0<q\le q_0$ of the values of the parameter q for which an S-shaped curve of cycles exists. In examples, such a bound is of the same order as coefficients of the polynomial L and the value of the Lipschitz coefficient K of the nonlinearity f.

An example of an equation to which Theorem 3 can be applied is $-x'''-x''-x'-x=qf(x)+\lambda x$ with $w_0=1$ and $M(p)=-p-1$. Figure 4 shows a typical graph of the function Φ satisfying conditions (13) and (14) of Theorem 3. This particular function Φ is generated by the nonlinearity $f(x)=x(1-|x|)(3-|x|)(20-|x|)/(40+|x|^3)$ with $\alpha_0=1.5$ and

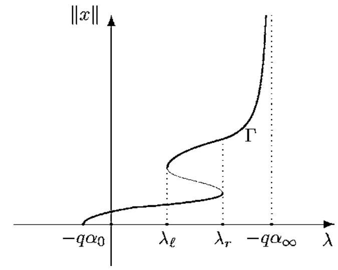

FIG. 3. S-shaped curve of cycles, connecting Hopf bifurcations at zero and at infinity.

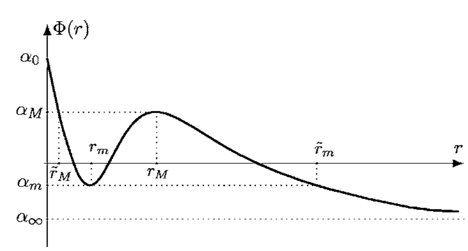

FIG. 4. Function Φ generated by $f(x)=x(1-|x|)(3-|x|)(20-|x|)/(40+|x|^3)$ with $\alpha_0=1.5$, $\alpha_M\approx 0.74$, $\alpha_m\approx -0.31$, and $\alpha_\infty=-1$.

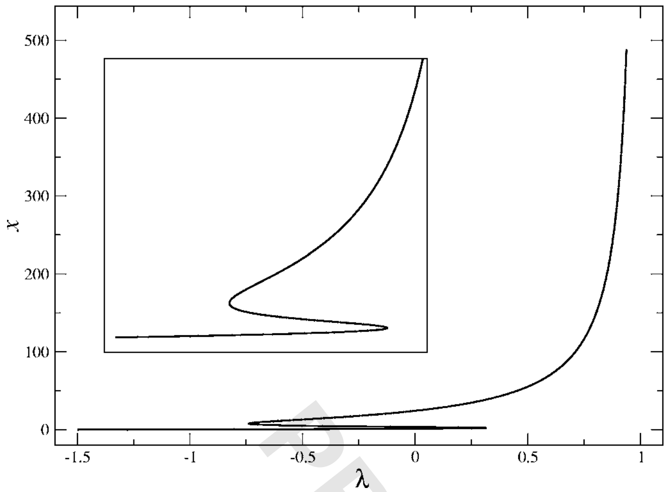

FIG. 5. S-shaped curve of cycles for the equation $-x'''-x''-x'-x=f(x)+\lambda x$ with f as in Fig. 4 and λ ranging over $[-\alpha_0,-\alpha_\infty]=[-1.5,1]$. The vertical coordinate shows the maximum of x. The lower part of the picture is zoomed.

$\alpha_\infty=-1$. Figure 5 draws the corresponding curve of cycles obtained numerically for the equation $-x'''-x''-x'-x=f(x)+\lambda x$ with this f and $q=1$. Figure 6 presents stable oscillations for the equation $-x'''-x''-x'-x=f(x)+\lambda(\varepsilon t)x$, where the parameter λ varies slowly back and forth between the folds of the curve of cycles shown in Fig. 5 [the range of $\lambda(\varepsilon t)$ is a little bit larger than the interval between the projections of the fold points on the λ axis]. The solution follows closely the stable branches of the curve of cycles and switches from one branch to another at the fold points, generating the MMO-type pattern.

Generically, the function Φ is S-shaped whenever the function $f(x)/x$ is, provided that the two humps of $f(x)/x$ are wide and large enough; then this shape is inherited by the curve of cycles, for some range of q at least, according to Theorem 3 and Corollary 2. Natural simple examples are delivered by piecewise linear continuous functions f.

The graphs of f and Φ can have more than two "U-turns," in which case the coexistence of more than two stable cycles of Eq. (2) is possible for some range of λ: this can lead to switching between multiple oscillation modes when λ changes slowly to and fro as a function of t or x, as discussed in the Introduction. We consider the simplest mechanism of such switching, requiring further assumptions to make it work, which basically means that the dynamics of the system is simple. In this way, we assume that the fold bifurcation is the only scenario responsible for the change of stability of the cycle on the S-shaped curve and that basins of attraction of the stable branches of this curve stretch to the fold points to ensure switching between these branches (for example, the cycle on the stable branches is globally stable to the left from λ_ℓ and to the right from λ_r in Fig. 3). The above theorems do not imply this: the cycle can possibly change stability via period-doubling bifurcation, Neimark-Sacker bifurcation, etc. If a stable object, say an invariant torus, is born in such a bifurcation, then the system may switch to it. Alternatively, the system can switch to a stable cycle separated from the S-shaped curve, or to another attracting object in the phase space, or behave in a more complicated manner. However, the above simple scenario is easily observed in examples.

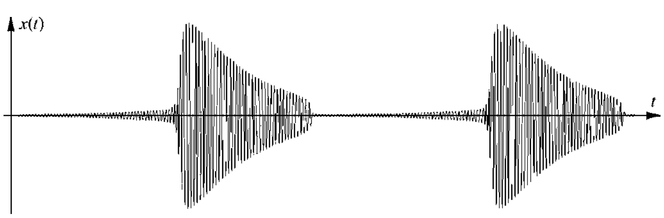

FIG. 6. A solution of the equation $-x'''-x''-x'-x=f(x)-[a\cos(\varepsilon t)+b]x$ with f as in Fig. 4, $a=0.6$, $b=0.25$, and $\varepsilon=0.01$. Blocks of small oscillations (stipe of the "mushroom") and large oscillations (the "mushroom" cap) correspond to the motion along the two stable branches of cycles shown in Fig. 5.

The results of this section can be extended to equations of the type (2) with the left-hand part, the nonlinearity f, and the equilibrium depending on the parameter λ, nonlinearities containing derivatives of x, and more general systems of differential equations.

III. PROOFS

A. Proof of Theorem 1

We scale the time in the system using the transformation $t\mapsto wt$ to obtain

$$L\left(w\frac{d}{dt}\right)x = qf(x)+\lambda x, \tag{37}$$

where the new parameter $w>0$ is the unknown frequency of the cycle. We now look for 2π-periodic solutions $x(t)$ of Eq. (37): if such a solution exists for some $w>0$, then Eq. (2) has a cycle of the period $2\pi/w$. Because in our setting the linear term of Eq. (37) dominates the nonlinearity, the first harmonics of x play a special role. Consequently, we consider the orthogonal projections of a solution x of Eq. (37) on $\sin t$, $\cos t$, and on the orthogonal complement

$$\mathbb{E}=\{h\in L^2(0,2\pi):\langle h,\sin t\rangle_{\mathbb{L}^2}=\langle h,\cos t\rangle_{\mathbb{L}^2}=0\}$$

to these functions in the space $\mathbb{L}^2=\mathbb{L}^2(0,2\pi)$,

$$x(t)=r\sin t+\tilde{r}\cos t+h(t),\quad h\in\mathbb{E}. \tag{38}$$

Here $\langle\cdot,\cdot\rangle_{\mathbb{L}^2}$ is the usual scalar product in $\mathbb{L}^2$. Since any time shift $x(t+\varphi)$ of a solution $x(t)$ is a solution of Eq. (37) too, the phase can be chosen arbitrarily. Hence, we set $\tilde{r}$ to zero in representation (38), thus extracting one particular periodic solution from the continuum of time shifts. Thus, given λ, we are going to prove the existence of a 2π-periodic solution of the form $x(t)=r\sin t+h(t)$ with $r>0$, $h\in\mathbb{E}$ for at least one $w>0$.

Consider the orthogonal projections of Eq. (37) on $\sin t$, $\cos t$, and $\mathbb{E}$ in $\mathbb{L}^2$,

$$\pi r[(w_0^2-w^2)M_{\text{even}}(iw)-\lambda]=q\langle\sin t,f(r\sin t+h(t))\rangle_{\mathbb{L}^2}, \tag{39}$$

$$-\pi r(w_0^2-w^2)iM_{\text{odd}}(iw)=q\langle\cos t,f(r\sin t+h(t))\rangle_{\mathbb{L}^2}, \tag{40}$$

$$h = q(L_w - \lambda)^{-1} P f(r \sin t + h(t)), \tag{41}$$

where

$$2M_{\text{even}}(p) = M(p) + M(-p), \quad 2M_{\text{odd}}(p) = M(p) - M(-p),$$

by P we denote the orthogonal projector onto the subspace $\mathbb{E}$ in $\mathbb{L}^2$, and $(L_w - \lambda)^{-1}$ denotes the inverse of the differential operator $L(wd/dt) - \lambda$ with the 2π-periodic boundary conditions in $\mathbb{E}$. Condition (8) implies $\mu(\lambda, w) \neq 0$ and thus ensures that the operator $(L_w - \lambda)^{-1}$, which sends any function $u \in \mathbb{E}$ to a unique 2π-periodic solution h of the equation $L(wd/dt)h - \lambda h = u$ satisfying $h \in \mathbb{E}$, is bounded on the whole subspace $\mathbb{E}$ of $\mathbb{L}^2$, and moreover its norm

$$\begin{aligned}\|(L_w - \lambda)^{-1}\|_{\mathbb{E}\to\mathbb{E}} &= \|(L_w - \lambda)^{-1} P\|_{\mathbb{L}^2\to\mathbb{L}^2} \\ &= \max_{n\in\mathbb{Z}, n\neq\pm 1} |L(nwi) - \lambda|^{-1}\end{aligned} \tag{42}$$

is uniformly bounded for all $w \in [w_-, w_+]$ [however, the norm of the operator $(L_w - \lambda)^{-1}$ on the whole space $\mathbb{L}^2$ goes to infinity as $\lambda \to 0$ and $w \to w_0$, because $L(iw_0) = 0$]. Consequently, the system of Eqs. (39)–(41) with the unknowns r, $w > 0$, and $h \in \mathbb{E}$ is equivalent to the 2π-periodic problem for Eq. (37). We note that both the functions $M_{\text{even}}(iw)$ and $iM_{\text{odd}}(iw)$ that enter this system are real-valued polynomials of w.

Consider an *a priori* bound of solutions (r, w, h) of Eqs. (39)–(41). From the relations $f(0) = 0$ and Eq. (3) it follows that $|f(x)| \leq K|x|$, hence

$$\begin{aligned}\|f(r \sin t + h(t))\|_{\mathbb{L}^2} &\leq K\|r \sin t + h(t)\|_{\mathbb{L}^2} \\ &= K\sqrt{\pi r^2 + \|h\|^2_{\mathbb{L}^2}}.\end{aligned} \tag{43}$$

Consequently, Eq. (41) implies

$$\|h\|_{\mathbb{L}^2} \leq qK\|(L_w - \lambda)^{-1} P\|_{\mathbb{L}^2\to\mathbb{L}^2} \sqrt{\pi r^2 + \|h\|^2_{\mathbb{L}^2}},$$

which, due to Eq. (42), is equivalent to the following *a priori* bound for h:

$$\|h\|_{\mathbb{L}^2} \leq \frac{rqK\sqrt{\pi}}{\sqrt{\mu^2(\lambda, w) - q^2K^2}} \tag{44}$$

with μ defined by Eq. (7). Now, because $\langle \cos t, f(r \sin t)\rangle_{\mathbb{L}^2} = 0$, Eq. (40) can be rewritten as

$$\begin{aligned}&-\pi r(w_0^2 - w^2) iM_{\text{odd}}(iw) \\ &\quad = q\langle \cos t, f(r \sin t + h(t)) - f(r \sin t)\rangle_{\mathbb{L}^2}.\end{aligned}$$

Combining this with the estimate

$$\|f(r \sin t + h(t)) - f(r \sin t)\|_{\mathbb{L}^2} \leq K\|h\|_{\mathbb{L}^2}, \tag{45}$$

which follows from the Lipschitz condition (3), we obtain

$$\begin{aligned}\pi r|w_0^2 - w^2||M_{\text{odd}}(iw)| &\leq q\sqrt{\pi}\|f(r \sin t + h(t)) - f(r \sin t)\|_{\mathbb{L}^2} \\ &\leq qK\sqrt{\pi}\|h\|_{\mathbb{L}^2}.\end{aligned}$$

Together with Eq. (44), this implies a bound for $w - w_0$,

$$|w_0^2 - w^2||M_{\text{odd}}(iw)| \leq \frac{q^2K^2}{\sqrt{\mu^2(\lambda, w) - q^2K^2}}. \tag{46}$$

Finally, from Eq. (39) it follows that

$$\begin{aligned}&\pi r[(w_0^2 - w^2) M_{\text{even}}(iw) - \lambda - q\Phi(r)] \\ &\quad = q\langle \sin t, f(r \sin t + h(t)) - f(r \sin t)\rangle_{\mathbb{L}^2},\end{aligned}$$

hence Eq. (45) implies $\pi r|\lambda + q\Phi(r)| \leq qK\sqrt{\pi}\|h\|_{\mathbb{L}^2} + \pi r|(w_0^2 - w^2) M_{\text{even}}(iw)|$, and relations (44) and (46) yield an estimate for r,

$$|q^{-1}\lambda + \Phi(r)| \leq \left(1 + \frac{|M_{\text{even}}(iw)|}{|M_{\text{odd}}(iw)|}\right) \frac{qK^2}{\sqrt{\mu^2(\lambda, w) - q^2K^2}}. \tag{47}$$

Let us consider the continuous deformation

$$\begin{aligned}&\pi r[\xi(w_0^2 - w^2) M_{\text{even}}(iw) - \lambda - q\Phi(r)] \\ &\quad = \xi q\langle \sin t, f(r \sin t + h(t)) - f(r \sin t)\rangle_{\mathbb{L}^2},\end{aligned} \tag{48}$$

$$-\pi r(w_0^2 - w^2) iM_{\text{odd}}(iw) = \xi q\langle \cos t, f(r \sin t + h(t))\rangle_{\mathbb{L}^2}, \tag{49}$$

$$h = \xi q(L_w - \lambda)^{-1} P f(r \sin t + h(t)) \tag{50}$$

that transforms Eqs. (39)–(41) to the equations

$$\begin{aligned}&-\pi r q(\lambda q^{-1} + \Phi(r)) = 0, \\ &-\pi r(w_0^2 - w^2) iM_{\text{odd}}(iw) = 0, \quad h = 0\end{aligned} \tag{51}$$

as the parameter ξ ranges over the segment $[0,1]$ (from 1 to 0). The same argument as above shows that the *a priori* bounds (44), (46), and (47) we obtained for solutions of system (39)–(41) hold for all the solutions of system (48) and (49) for all $0 \leq \xi \leq 1$. Therefore, relations (9) and (11), where $\operatorname{Im} M(iw) = -iM_{\text{odd}}(iw)$, $\operatorname{Re} M(iw) = M_{\text{even}}(iw)$ ensure that system (48)–(50) does not have solutions (r, w, h) on the boundary of the domain,

$$\begin{aligned}G &= \{(r, w, h) : r \in [r_-, r_+], w \in [w_-, w_+], \|h\|_{\mathbb{L}^2} \leq d\} \\ &\subset \mathbb{R} \times \mathbb{R} \times \mathbb{E}\end{aligned}$$

with a sufficiently large $d > 0$. Consequently, from the topological degree theory it follows that system (39)–(41) has a solution in the domain G if the rotation $\gamma(\Psi, G)$ of the vector field $\Psi(r, w, h) = (-\pi r q(\lambda q^{-1} + \Phi(r)), -\pi r(w_0^2 - w^2) iM_{\text{odd}}(iw), h)$ on the boundary of G is nonzero: here the components of Ψ are the left-hand parts of Eqs. (51). The rotation product formula (see, e.g., Ref. 5) implies the relation $\gamma(\Psi, G) = \gamma_r \gamma_w \gamma_h$, where γ_r and γ_w are the rotations of the first and the second scalar components of the vector field Ψ on the boundaries of the segments $[r_-, r_+] \ni r$ and $[w_-, w_+] \ni w$, respectively, and γ_h is the rotation of the last component h of Ψ on the sphere $\|h\|_{\mathbb{L}^2} = d$, which equals 1 by definition. Relation (12) implies that the first component of Ψ has different signs at the ends of the segment $[r_-, r_+]$; similarly, relations (10) and $w_- < w_0 < w_+$ imply that the second component of Ψ has different signs at the ends of the segment $[w_-, w_+]$ for each $r > 0$. Hence $|\gamma_r| = |\gamma_w| = 1$ and thus $|\gamma(\Psi, G)| = 1$, which completes the proof.

B. Proof of Corollary 1

Since relations (8)–(10) hold for $\lambda_c = -q(\alpha_m+\alpha_M)/2$, by continuity they also hold for all the nearby values of λ. Furthermore, from Eq. (15) it follows that relations (11) and (12) hold at the ends of each of the segments $[\tilde r_m, r_m]$, $[r_m, r_M]$, and $[r_M, \tilde r_M]$ for any λ close to λ_c. Hence, Theorem 1 implies the existence of three cycles satisfying estimates (16) for such λ.

The last statement of the corollary follows from the *a priori* estimate (47) of the cycles: for $\lambda=\lambda_m,\lambda_M$, this estimate holds for all cycles with $w\in[w',w'']$ due to Eq. (20). Indeed, combining Eq. (47) with $\lambda=\lambda_M$, $w\in[w',w'']$ and relation (22), we obtain

$$|q^{-1}\lambda_M + \Phi(r)| < q^{-1}\lambda_M + \alpha_m. \tag{52}$$

But for $r\le\tilde r_M$, relations (17) imply $q^{-1}\lambda_M+\Phi(r) \ge q^{-1}\lambda_M+\alpha_m>0$, which is opposite to Eq. (52). Consequently, the bound $r>\tilde r_M$ holds for all $2\pi/w$-periodic cycles of Eq. (2) with $\lambda=\lambda_M$, $w\in[w',w'']$. Similarly, estimate (21) combined with the *a priori* bound (47) implies $|q^{-1}\lambda_m +\Phi(r)|<-(q^{-1}\lambda_m+\alpha_M)$ for each cycle of Eq. (2) with $w\in[w',w'']$, $\lambda=\lambda_m$, while from relations (18) it follows that if $r\ge\tilde r_m$, then $-[q^{-1}\lambda_m+\Phi(r)]\ge-(q^{-1}\lambda_m+\alpha_M)>0$. Consequently, $r<\tilde r_m$ for all such cycles, hence the proof is complete.

C. Proof of Theorem 2

By assumption, the function $\varphi(w):=\operatorname{Im} L(iw)$ has a nonzero derivative on the interval $[w',w'']$. Hence, the inverse smooth function φ^{-1} is defined on the segment $J=\varphi([w',w''])$. Furthermore, the real planar map

$$Q:(\lambda,w)\mapsto(\operatorname{Re} L(wi)-\lambda, \operatorname{Im} L(wi))=:(u_1,u_2) \tag{53}$$

from the rectangle $\lambda\in[\lambda_-,\lambda_+]$, $w\in[w',w'']$ to the domain

$$D=\{(u_1,u_2):u_2\in J, \operatorname{Re} L[i\varphi^{-1}(u_2)]-u_1\in[\lambda_-,\lambda_+]\}$$

of the plane (u_1,u_2) is a diffeomorphism. This diffeomorphism maps the set Ω_ρ onto the disk $\mathbb{D}=\{(u_1,u_2):u_1^2+u_2^2\le\rho^2\}\subset D$. Now, we introduce the new variables $u=(u_1,u_2)\in\mathbb{D}$ and $y=y(t)\in\mathbb{E}$ related to λ, w, and h by the one-to-one relations (53) and $[L(wd/dt)-\lambda]h(t)=ry(t)$ or, equivalently,

$$(\lambda,w)=Q^{-1}(u_1,u_2),\quad h=r(L_w-\lambda)^{-1}y, \tag{54}$$

where the existence of the bounded operator $(L_w-\lambda)^{-1}$ for any $(u_1,u_2)\in\mathbb{D}$ follows from assumption (31), and Q^{-1} denotes the inverse of map (53). With this notation, system (39)–(41) can be rewritten equivalently as

$$\begin{aligned}(u_1,u_2,y)&=\frac{q}{r}(\pi^{-1}\langle\sin t, f(x(t))\rangle_{\mathbb{L}^2},\pi^{-1}\\ &\qquad\langle\cos t, f(x(t))\rangle_{\mathbb{L}^2}, Pf(x(t)))\\ &=:A_r(u_1,u_2,y),\end{aligned}$$

where $x(t)=r\sin t+h(t)$ is assigned to u_1, u_2, and y by formulas (54) and the operator A_r for every $r>0$ acts in the space $\mathbb{R}\times\mathbb{R}\times\mathbb{E}$ of triples $z=(u_1,u_2,y)$ with the norm $\|z\|_0=\sqrt{\pi u_1^2+\pi u_2^2+\|y\|^2_{\mathbb{L}^2}}$ and is defined on the domain $\mathbb{D}\times\mathbb{E}$. We shall show that the cylinder $\mathbb{B}=\{z=(u_1,u_2,y):(u_1,u_2)\in\mathbb{D},\|y\|_{\mathbb{L}^2}\le b\}$ with

$$b^2=\max_{(\lambda,w)\in\Omega_\rho}\frac{\pi q^2K^2\mu^2(\lambda,w)}{\mu^2(\lambda,w)-q^2K^2}$$

is invariant for the operator A_r, and A_r is a contraction on this cylinder.

1. Contracting property of A_r

Consider two points $z_j=(u_{j1},u_{j2},y_j)\in\mathbb{B}$, $j=1,2$, $z_1\ne z_2$. Let us estimate the norm $\|\cdot\|_0$ of the difference $\Delta=A_r(u_{11},u_{12},y_1)-A_r(u_{21},u_{22},y_2)$. Set

$$(\lambda_j,w_j)=Q^{-1}(u_{j1},u_{j2}),\quad h_j=r(L_{w_j}-\lambda_j)^{-1}y_j,$$

$$x_j(t)=r\sin t+h_j(t).$$

From the definition of A_r and the equality

$$\|(\pi^{-1}\langle\sin t,v\rangle_{\mathbb{L}^2},\pi^{-1}\langle\cos t,v\rangle_{\mathbb{L}^2},Pv)\|_0=\|v\|_{\mathbb{L}^2}, \tag{55}$$

it follows that $\|\Delta\|_0=qr^{-1}\|f(x_1(t))-f(x_2(t))\|_{\mathbb{L}^2}$. The Lipschitz condition (3) implies

$$\|f(x_1(t))-f(x_2(t))\|_{\mathbb{L}^2}\le K\|x_1-x_2\|_{\mathbb{L}^2}=K\|h_1-h_2\|_{\mathbb{L}^2},$$

hence $\|\Delta\|_0\le qKr^{-1}\|h_1-h_2\|_{\mathbb{L}^2}$. Consequently, using the representation

$$\begin{aligned}h_1-h_2&=r(L_{w_2}-\lambda_2)^{-1}(y_1-y_2)\\ &\quad+r[(L_{w_1}-\lambda_1)^{-1}-(L_{w_2}-\lambda_2)^{-1}]y_1,\end{aligned}$$

the explicit expressions for the norms of the operators $(L_{w_2}-\lambda_2)^{-1}$ and $(L_{w_1}-\lambda_1)^{-1}-(L_{w_2}-\lambda_2)^{-1}$ that act in the subspace $\mathbb{E}$ of $\mathbb{L}^2$,

$$\begin{aligned}\|(L_{w_2}-\lambda_2)^{-1}\|_{\mathbb{L}^2\to\mathbb{L}^2}&=\max_{n\in\mathbb{Z},n\ne\pm1}|L(nw_2i)-\lambda_2|^{-1}\\ &=1/\mu(\lambda_2,w_2),\end{aligned} \tag{56}$$

$$\begin{aligned}&\|(L_{w_1}-\lambda_1)^{-1}-(L_{w_2}-\lambda_2)^{-1}\|_{\mathbb{L}^2\to\mathbb{L}^2}\\ &\quad=\max_{n\in\mathbb{Z},n\ne\pm1}|[L(nw_1i)-\lambda_1]^{-1}-[L(nw_2i)-\lambda_2]^{-1}|,\end{aligned}$$

and the estimate $\|y_j\|_{\mathbb{L}^2}\le b$, we obtain

$$\begin{aligned}\|\Delta\|_0&\le\frac{qK\|y_1-y_2\|_{\mathbb{L}^2}}{\mu(\lambda_2,w_2)}\\ &\quad+\max_{n\in\mathbb{Z},n\ne\pm1}\frac{qKb|L(nw_1i)-L(nw_2i)-(\lambda_1-\lambda_2)|}{|L(nw_1i)-\lambda_1||L(nw_2i)-\lambda_2|}.\end{aligned} \tag{57}$$

Here the quantity $|L(nw_1i)-L(nw_2i)-(\lambda_1-\lambda_2)|$ is the Euclidean distance $|\cdot|_e$ between the points $Q(\lambda_1,nw_1)$ and $Q(\lambda_2,nw_2)$. Because the Jacobi matrix of the composition of the maps $(u_1,u_2)\mapsto Q^{-1}(u_1,u_2)=(\lambda,w)$ and $(\lambda,w)\mapsto Q(\lambda,nw)$ equals $DQ(\lambda,nw)I_n(DQ)^{-1}(\lambda,w)$, where

$DQ(\cdot,\cdot)$ is the Jacobi matrix of map (53) and $I_n=\text{diag}\{1,n\}$, it follows that

$$|Q(\lambda_1,nw_1)-Q(\lambda_2,nw_2)|_e \leq \delta \max_{(\lambda,w)\in Q^{-1}(\Gamma)} |DQ(\lambda,nw)I_n(DQ)^{-1}(\lambda,w)|_e,$$

where $|\cdot|_e$ on the right-hand side denotes the Euclidean norm of the matrix,

$$\delta=\sqrt{(u_{11}-u_{21})^2+(u_{12}-u_{22})^2},$$

and $Q^{-1}(\Gamma)$ is the image of the segment $\Gamma=\{(u_1,u_2)=s(u_{11},u_{12})+(1-s)(u_{21},u_{22}):0\leq s\leq 1\}$ under the inverse of map (53). By direct calculation, we see that

$$DQ(\lambda,w)=\begin{pmatrix} -q & -\text{Im } L'(iw) \\ 0 & \text{Re } L'(iw)\end{pmatrix},$$

$$DQ(\lambda,nw)I_n(DQ)^{-1}(\lambda,w) = \begin{pmatrix} 1 & \dfrac{\text{Im } L'(iw)-n\text{ Im } L'(inw)}{\text{Re } L'(iw)} \\ 0 & \dfrac{n\text{ Re } L'(inw)}{\text{Re } L'(iw)}\end{pmatrix},$$

and $|DQ(\lambda,nw)I_n(DQ)^{-1}(\lambda,w)|_e=\nu(n,w)$ with ν defined by Eqs. (26) and (27). Consequently,

$$|L(nw_1i)-L(nw_2i)-(\lambda_1-\lambda_2)| = |Q(\lambda_1,nw_1)-Q(\lambda_2,nw_2)|_e \text{ f}\leq \delta \max_{(\lambda,w)\in Q^{-1}(\Gamma)} \nu(n,w).$$

Combining these relations with Eq. (57), we arrive at the bound

$$\|\Delta\|_0 \leq \frac{qK\|y_1-y_2\|_{\mathbb{L}^2}}{\mu(\lambda_2,w_2)} + \max_{n\neq\pm1}\max_{(\lambda,w)\in Q^{-1}(\Gamma)} \frac{\delta qKb\nu(n,w)}{|L(nw_1i)-\lambda_1||L(nw_2i)-\lambda_2|},$$

which, due to $\|z_1-z_2\|_0=\sqrt{\pi\delta^2+\|y_1-y_2\|^2_{\mathbb{L}^2}}$, implies

$$\frac{\|\Delta\|_0}{\|z_1-z_2\|_0} \leq qK\left(\frac{1}{\mu^2(\lambda_2,w_2)}+\max_{n\neq\pm1}\max_{(\lambda,w)\in Q^{-1}(\Gamma)}\frac{b^2\nu^2(n,w)}{|L(nw_1i)-\lambda_1|^2|L(nw_2i)-\lambda_2|^2}\right)^{1/2}.$$

If we consider any partition of the segment connecting the points z_1 and z_2, then a similar bound holds for any element of the partition. Hence, sending the partition mesh to zero and using the fact that $Q^{-1}(\Gamma)\subset\Omega_\rho$, we obtain

$$\frac{\|\Delta\|_0}{\|z_1-z_2\|_0} \leq qK\left(\max_{(\lambda,w)\in\Omega_\rho}\frac{1}{\mu^2(\lambda,w)} + b^2\max_{n\neq\pm1,(\lambda,w)\in\Omega_\rho}\frac{\nu^2(n,w)}{|L(nwi)-\lambda|^4}\right)^{1/2}.$$

This bound, the definition of b, and relation (32) imply that the operator A_r is a contraction on the cylinder $\mathbb{B}$ with a contraction coefficient $a<1$ independent of r.

2. Invariance of the cylinder $\mathbb{B}$

Consider a point $z=(u_1,u_2,y)\in\mathbb{B}$. Let $w,\lambda,h=h(t)$ and $x=x(t)=r\sin t+h(t)$ be defined by Eq. (54). From the definition of A_r and relation (55), it follows that $\|A_r(u_1,u_2,y)\|_0 \leq qr^{-1}\|f(x(t))\|_{\mathbb{L}^2}$. This, when combined with Eq. (43), implies

$$\|A_r(u_1,u_2,y)\|_0 \leq r^{-1}qK\sqrt{\pi r^2+\|h\|^2_{\mathbb{L}^2}} = qK\sqrt{\pi+\|(L_w-\lambda)^{-1}y\|^2_{\mathbb{L}^2}}, \qquad (58)$$

and with the use of Eq. (56) and $\|y\|_{\mathbb{L}^2}\leq b$,

$$\|A_r(u_1,u_2,y)\|_0 \leq qK\sqrt{\pi+\frac{b^2}{\mu^2(\lambda,w)}}.$$

Since $(\lambda,w)\in\Omega_\rho$ for each $z\in\mathbb{B}$, it follows that

$$\|A_r(u_1,u_2,y)\|_0 \leq qK\max_{(\lambda,w)\in\Omega_\rho}\sqrt{\pi+\frac{b^2}{\mu^2(\lambda,w)}},$$

where the right-hand part equals b, as the definition of b implies. Hence $\|A_r(u_1,u_2,y)\|_0\leq b$. Relation (33) ensures that $b\leq\sqrt{\pi}\rho$, consequently the ball $\|z\|_0\leq b$ is contained in the cylinder $\mathbb{B}$, and thus $A_r(u_1,u_2,y)\in\mathbb{B}$ for each $(u_1,u_2,y)\in\mathbb{B}$, i.e., the cylinder $\mathbb{B}$ is invariant for the operator A_r for each $r>0$. Therefore, from the contraction mapping principle, it follows that A_r has a unique fixed point $z_r^*=(u_1^*(r),u_2^*(r),y_r^*)$ in $\mathbb{B}$ for every $r>0$. Hence, for each positive r, Eq. (2) has a cycle $x_r^*=x_r^*(t)=r\sin t+h_r^*(t)$ of the frequency w_r^* for $\lambda=\lambda_r^*$ with $(\lambda_r^*,w_r^*)\in\Omega_\rho$, where λ_r^*,w_r^*,h_r^* are related with the components of z_r^* by formulas (54).

3. Lipschitz continuity of the branch of cycles

The local Lipschitz continuity of the curve z_r^*, $0<r<\infty$, and consequently of the branch of cycles, follows from Eq. (3) by the standard argument. Namely, consider the fixed points $z_r^*,z_s^*\in\mathbb{B}$ of A_r,A_s for any $r>s>0$. Since A_r is a contraction, $\|A_r(z_r^*)-A_r(z_s^*)\|_0\leq a\|z_r^*-z_s^*\|_0$ with $a<1$, hence

$$\|z_r^* - z_s^*\|_0 = \|A_r(z_r^*) - A_s(z_s^*)\|_0$$
$$\leq a\|z_r^* - z_s^*\|_0 + \|A_r(z_s^*) - A_s(z_s^*)\|_0$$

and thus

$$\|z_r^* - z_s^*\|_0 \leq (1-a)^{-1}\|A_r(z_s^*) - A_s(z_s^*)\|_0. \tag{59}$$

The definition of A_r and relation (55) imply

$$\|rA_r(z_s^*) - sA_s(z_s^*)\|_0 = q\|f(r[\sin t + (L_{w_s^*} - \lambda_s^*)^{-1}y_s^*])$$
$$- f(s[\sin t + (L_{w_s^*} - \lambda_s^*)^{-1}y_s^*])\|_{\mathbb{L}^2},$$

hence we see from Eq. (3) that

$$\|rA_r(z_s^*) - sA_s(z_s^*)\|_0 \leq qK(r-s)\sqrt{\pi + \|(L_{w_s^*} - \lambda_s^*)^{-1}y_s^*\|_{\mathbb{L}^2}^2}$$

and therefore

$$r\|A_r(z_s^*) - A_s(z_s^*)\|_0 \leq (r-s)\|A_s(z_s^*)\|_0 + qK(r$$
$$- s)\sqrt{\pi + \|(L_{w_s^*} - \lambda_s^*)^{-1}y_s^*\|_{\mathbb{L}^2}^2}.$$

Here $A_s(z_s^*) = z_s^*$, $\|z_s^*\|_0 \leq \sqrt{\pi\rho^2 + b^2}$ and, due to Eq. (56),

$$\pi + \|(L_{w_s^*} - \lambda_s^*)^{-1}y_s^*\|_{\mathbb{L}^2}^2 \leq \pi + \frac{b^2}{\mu^2(\lambda_s^*, w_s^*)}$$
$$\leq \pi + b^2q^{-2}K^{-2}.$$

Consequently, $r\|A_r(z_s^*) - A_s(z_s^*)\|_0 \leq (r-s)c_0$ and $c_0 := \sqrt{\pi\rho^2 + b^2} + \sqrt{\pi q^2K^2 + b^2}$. This estimate and estimate (59) imply

$$\|z_r^* - z_s^*\|_0 \leq \frac{(r-s)c_0}{r(1-a)}, \quad r > s > 0,$$

which proves local Lipschitz continuity of the curve z_r^*, $r > 0$, and completes the proof of the existence of a continuous branch of cycles with $(\lambda, w) \in \Omega_\rho$ for Eq. (2).

Now, linearizing Eq. (2) at zero, we obtain

$$L(d/dt)x - (q\alpha_0 + \lambda)x = 0. \tag{60}$$

The assumption that function $\varphi(w) = \operatorname{Im} L(iw)$ is strictly monotone on the segment $[w', w'']$ ensures that equation $L(iw) - (q\alpha_0 + \lambda) = 0$ has the only solution (w, λ) in the rectangle $[\lambda_-, \lambda_+] \times [w', w''] \supset \Omega_\rho$, namely $w = w_0$, $\lambda = -\alpha_0 q$. In other words, the characteristic equation $L(p) - (q\alpha_0 + \lambda) = 0$ of Eq. (60) has an imaginary root $p = iw$ with $w \in [w', w'']$ only for $\lambda = -\alpha_0 q$. Since the presence of an imaginary root is a necessary condition for the Hopf bifurcation, we conclude that the first of relations (34) holds for our branch of cycles. Similarly, the fact that the characteristic equation $L(p) - (q\alpha_\infty + \lambda) = 0$ of the linearization of Eq. (2) at infinity has a root $p = iw$ with $w \in [w', w'']$ for a unique $\lambda = -\alpha_\infty q$ implies the second relation of (34).

Finally, if $(u_1, u_2) \in \mathbb{D}$, i.e., $Q^{-1}(u_1, u_2) = (\lambda, w) \in \Omega_\rho$, and $z = (u_1, u_2, y)$ is a fixed point of A_r, then Eq. (58) implies the relation

$$\|y\|_{\mathbb{L}^2}^2 \leq q^2K^2[\pi + \|y\|_{\mathbb{L}^2}^2/\mu^2(\lambda, w)]$$

and hence $\|y\|_{\mathbb{L}^2} \leq b$. Therefore, the fixed point z of A_r lies in the cylinder $\mathbb{B}$ where A_r is a contraction. Thus A_r has a unique fixed point with $(u_1, u_2) \in \mathbb{D}$ and consequently Eq. (2) has a unique periodic solution $x = r \sin t + h(t)$ with $(\lambda, w) \in \Omega_\rho$, $h \in \mathbb{E}$ for each r, i.e., all the cycles with $(\lambda, w) \in \Omega_\rho$ are included in the above continuous curve. This completes the proof.

D. Proof of Theorem 3

Consider the continuous curve of cycles with $(\lambda(r), w(r)) \in \Omega_\rho \subset [\lambda_-, \lambda_+] \times [w', w'']$, which exists by Theorem 2. Consider numbers r_0, r^0 such that $\lambda(r_0) = \lambda_m$, $\lambda(r^0) = \lambda_M$, and relations (25) hold. The existence of such numbers follows from the continuity of $\lambda(r)$ and the relations $\lambda(0) = -\alpha_0 q < \lambda_m$ and $\lambda(\infty) = -\alpha_\infty q > \lambda_M$. According to the last conclusion of Corollary 1, Eq. (2) does not have cycles satisfying Eq. (23) for $\lambda = \lambda_m$ and Eq. (24) for $\lambda = \lambda_M$, consequently the equalities $\lambda(r_0) = \lambda_m$ and $\lambda(r^0) = \lambda_M$ imply $r_0 < \tilde{r}_m$ and $\tilde{r}_M < r^0$, i.e.,

$$[\tilde{r}_m, \tilde{r}_M] \subset (r_0, r^0). \tag{61}$$

Also, the corollary states that for each λ sufficiently close to $\lambda_c = -q(\alpha_m + \alpha_M)/2 \in (\lambda_m, \lambda_M)$, Eq. (2) has two different cycles: one with $r \in (\tilde{r}_m, r_m)$, the other with $r \in (r_M, \tilde{r}_M)$, and both with $w \in [w_-, w_+]$. Because $[\lambda_m, \lambda_M] \times [w_-, w_+] \subset \Omega_\rho$ according to condition (36) and all cycles with $(\lambda, w) \in \Omega_\rho$ belong to the continuous curve by Theorem 2, we conclude that the function $\lambda(r)$ takes all values from some nonempty interval $(\lambda_c - \delta, \lambda_c + \delta)$ on each of the nonintersecting intervals $(\tilde{r}_m, r_m)$ and $(r_M, \tilde{r}_M)$. Hence, $\lambda(r)$ is nonmonotone on the segment (61), which completes the proof.

ACKNOWLEDGMENTS

The authors thank Oleg Rasskazov and Andrei Zhezherun for valuable discussion of numerical examples. This publication has emanated from research conducted with the financial support of the Science Foundation Ireland, Enterprise Ireland, IRCSET, and the Russian Foundation for Basic Research (Grant Nos. 04-01-00330 and 06-01-72552).

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