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J. Differential Equations 226 (2006) 30–53
Journal of Differential Equations
www.elsevier.com/locate/jde

Subharmonic bifurcation from infinity ☆

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Received 31 January 2005; revised 26 August 2005

Available online 23 November 2005

Abstract

We are concerned with a subharmonic bifurcation from infinity for scalar higher order ordinary differential equations. The equations contain principal linear parts depending on a scalar parameter, 2π-periodic forcing terms, and continuous nonlinearities with saturation. We suggest sufficient conditions for the existence of subharmonics (i.e., periodic solutions of multiple periods $2\pi n$) with arbitrarily large amplitudes and periods. We prove that this type of the subharmonic bifurcation occurs whenever a pair of simple roots of the characteristic polynomial crosses the imaginary axis at the points $\pm\alpha i$ with an irrational α. Under some further assumptions, we estimate asymptotically the parameter intervals, where large subharmonics of periods $2\pi n$ exist. These assumptions relate the quality of the Diophantine approximations of α, the rate of convergence of the nonlinearity to its limits at infinity, and the smoothness of the forcing term.

MSC: 34B15; 37G15

Keywords: Periodic solutions; Subharmonics; Bifurcation; Synchronization; Nonlinearities with saturation

☆ This work was partially done at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin. The authors were supported by Grants 03-01-00258 and 04-01-00330 of the Russian Foundation for Basic Researches and the President of Russia Grant NS-1532.2003.1.

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[1] Supported by the Enterprise Ireland, Grant SC/2003/376.
[2] Supported by the Russian Science Support Foundation and the President of Russia Grant MD-87.2003.01.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2005.09.011

1. Introduction

In this paper we study the existence of periodic solutions with large periods and amplitudes for scalar nonautonomous equations that contain a 2π-periodic forcing term and depend on a scalar parameter. If the linearization of the equation at infinity is degenerate (has periodic solutions) for some parameter value $\lambda = \lambda_0$, then for close values of the parameter the nonlinear equation may have periodic solutions with arbitrarily large amplitudes and periods $2\pi n_k$, n_k are positive integers. More precisely, we show that there is typically a sequence of parameter intervals Λ_k accumulating near the bifurcation parameter value λ_0 such that the nonlinear equation has a branch of subharmonic solutions (which is homeomorphic to a circle) of a period $2\pi n_k$ on Λ_k and the amplitudes and the periods $2\pi n_k$ of these solutions go to infinity as $\lambda \to \lambda_0$.

The similar situation is well known for local problems near the origin (or any equilibrium point). Here, in the smooth case, the bifurcation of invariant tori from the origin and classical facts on dynamics on those tori account for the existence of the synchronization intervals Λ_k (see, e.g., [1]). For a nonsmooth local situation (where, possibly, there are no invariant tori), the existence of sporadic intervals Λ_k, where the system has subharmonics with periods increasing to infinity near the bifurcation parameter value λ_0, was first proved in [2], still under the assumption that nonlinearities admit the Taylor expansion at the origin. The latter situation is more relevant to the context of the present paper, since we consider equations nonsmooth at infinity. Following the terminology introduced in [2] for local problems, we call the bifurcation of subharmonics with unbounded periods and amplitudes a *subfurcation from infinity*.

Natural assumptions on sufficient smoothness (or analyticity) in local problems near the origin lead to equations with polynomial principal nonlinear terms. These assumptions are essential in various bifurcation problems on periodic solutions, because the results may be rather different for the smooth case and for equations with nonsmooth nonlinearities such as, e.g., $x|x|$. One of the reasons is that in problems with nonsmooth nonlinearities principal nonlinear terms of bifurcation equations generically define the behavior of solutions, while in problems with polynomial nonlinearities the principal terms are typically degenerate, which means that smaller order terms play a role. This can make the analysis of nonsmooth problems simpler than that of the smooth ones, although the latter are better studied.

A natural counterpart of the smoothness assumptions at the origin in problems at infinity is that the principal nonlinear terms at infinity have the form x^{-N}, $N = 0, 1, 2, \ldots,$ since for both types of problems similar phenomena are observed. However, this does not cover the class of nonlinearities with saturation behavior at infinity, which is important from the point of view of applications and is traditionally considered in theoretical studies. In this sense, such nonlinearities with different saturation limits at $+\infty$ and $-\infty$ are nonsmooth at infinity, because they have the nonconstant principal nonlinear term, and indeed, results on bifurcation problems at infinity with such nonlinearities do not have direct counterparts in bifurcation problems at the origin. Particularly, in the present paper we study subfurcation from infinity for equations with this type of nonlinearities. Let us emphasize that a nonlinearity with saturation is nonsmooth at the infinite point, whereas it can be analytic in any finite domain like, e.g., $\arctan(\cdot)$.

We consider the equation

$$L\left(\frac{d}{dt},\lambda\right)x = f(x) + b(t) \tag{1}$$

with the scalar parameter λ. Here

$$L(p,\lambda) = p^\ell + a_1(\lambda)p^{\ell-1} + \cdots + a_\ell(\lambda) \tag{2}$$

is a real polynomial of degree $\ell \geqslant 2$, continuous in λ; the functions b and f are bounded, f is continuous, b is measurable, and $b(t) \equiv b(t+2\pi)$. We assume that the polynomial $L(p,\lambda)$ has a pair of simple roots (depending on λ) that for some $\lambda = \lambda_0$ cross transversally the imaginary axis of the complex plane at points $\pm i\alpha$ with an irrational $\alpha > 0$, and that the polynomial $L(p,\lambda_0)$ has no roots different from $\pm i\alpha$ on the imaginary axis. This immediately implies that for any parameter value λ from some vicinity of λ_0 Eq. (1) has at least one 2π-periodic solution and the set of all such solutions is bounded. Moreover, if Eq. (1) has periodic solutions of multiple periods $2\pi n$ (subharmonics), then the set of all subharmonics of any fixed period is also bounded uniformly w.r.t λ. Therefore if a sequence of subharmonics $x_k = x_k(t)$ (x_k are solutions of (1) for $\lambda = \lambda_k$) satisfies $\|x_k\|_C \to \infty$, then their periods satisfy $2\pi n_k \to \infty$ and $\lambda_k \to \lambda_0$.

The existence of subharmonics is explained by the classical picture suggested by Arnold, which can be adapted to our setting of the subfurcation from infinity as follows. Consider a real polynomial $\tilde{L}(p,\lambda_1,\lambda_2)$ that depends on some two scalar parameters λ_1, λ_2 and the differential equation

$$\tilde{L}\left(\frac{d}{dt},\lambda_1,\lambda_2\right)x = f(x) + b(t). \tag{3}$$

Assuming that $\tilde{L}$ has a pair of simple roots $\eta \pm i\xi$, let us use their real and imaginary parts η, ξ as new natural parameters in place of the original ones λ_1, λ_2. Let us consider Eq. (3) in a small parameter domain near the point $\eta = 0$, $\xi = \alpha$ and look for the sets Ω_q in this domain, defined for all rationals $q = m/n$ sufficiently close to α by the condition that for $(\eta,\xi) \in \Omega_q$ Eq. (3) has at least one periodic solution with the minimal period $2\pi n$ and with a sufficiently large amplitude. One can show that the sets Ω_q have roughly the shape of the Arnold tongues (this terms is more standard for local synchronization problems) with the vertex of the tongue Ω_q at the point $\eta = 0$, $\xi = q$, see Fig. 1, and that amplitudes of $2\pi n$-periodic subharmonics go to infinity as the point $(\eta,\xi) \in \Omega_q$ approaches the vertex. Now, if we allow only one scalar parameter to vary, like the parameter λ in Eq. (1), then the set of possible (η,ξ) is a curve $\Gamma = \{(\eta,\xi)\colon\ \eta = \eta(\lambda), \xi = \xi(\lambda)\}$ passing through the point $(0,\alpha)$ for $\lambda = \lambda_0$. Therefore the equation has subharmonics of a period $2\pi n$ whenever this curve intersects a tongue Ω_q with some $q = m/n$, which is the case if the tongue length is at least of order $|\alpha - q|$, since the curve Γ and the tongue borders are locally almost straight lines. Actually, as it follows from the proofs presented below, the length of Ω_q is estimated from below by $n^{-1-\varepsilon}$ (more precisely, for any $\varepsilon > 0$ there exists a n_0 such that for each $q = m/n$ with $n > n_0$ the length of the tongue Ω_q is greater than $n^{-1-\varepsilon}$). For

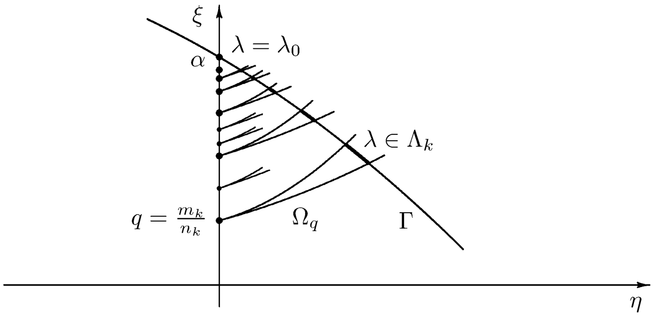

Fig. 1. Intersections of the Arnold tongues with the curve Γ.

the most tongues Ω_q this is not enough to guarantee that Ω_q intersects with Γ, because $|\alpha - q|$ is of order n^{-1} for the most rationals q, i.e. the tongues Ω_q with such q may be too short to reach Γ. However, for some rationals q, like in particular for the convergents of α, the difference $|\alpha - q|$ is of order n^{-2} and therefore for these q the tongues Ω_q are large enough to intersect Γ. As conclusion, there is a sequence of the parameter intervals Λ_k, converging to λ_0, and a sequence of positive integers $n_k \to \infty$ such that for $\lambda \in \Lambda_k$ the equation has a periodic solution of the minimal period $2\pi n_k$. The estimation of the lengths $|\Lambda_k|$ of these parameter intervals involves more work, because $|\Lambda_k|$ is infinitesimally small compared to the distance from Λ_k to λ_0. The reason is that the angle between the borders of the tongue Ω_q at its vertex vanishes as q approaches α.

All the above interpretation, although not exactly matching our further consideration of Eq. (1) on the formal level, gives however a well-enough explanation and the main underlying idea of what happens.

Below in this paper, we do not consider the Arnold tongues. Using another approach, we prove that generically Eq. (1) with one parameter λ has a sequence of subharmonics x_k with unbounded amplitudes and periods for a sequence of parameter values $\lambda_k \to \lambda_0$ if the nonlinearity f has different saturation limits at plus and minus infinity (and thus, in that case, subfurcation from infinity occurs at the parameter value λ_0). Then we give explicit asymptotic formulas for the lengths $|\Lambda_k|$ of the synchronization intervals Λ_k for large k under some further assumptions. The main of these assumptions are, first, that the Fourier series of the forcing term b contains infinite number of harmonics (in particular, b cannot be a trigonometric polynomial) and the Fourier coefficients of b go to zero not too quickly. Secondly, the irrational number α is assumed to be very well approximable by rationals, namely, the estimate

$$\left|\alpha - \frac{m_k}{n_k}\right| \leqslant \frac{c}{n_k^s}, \tag{4}$$

with a proper $s > 2$ has to hold for the convergents of α. In this connection, we recall that every α satisfying this estimate is transcendental and for every $s > 2$ the set of all such α is a zero measure everywhere dense set [6] (while $|\alpha - m_k/n_k| \leqslant 1/n_k^2$ holds for every irrational α). The conditions of our theorem on the lengths of the intervals Λ_k relate the rate of convergence of the Fourier coefficients of the forcing term b to zero, the exponent s

characterizing the quality of the approximation of α by its convergents, and the exponent β introduced below to characterize the rate of convergence of the nonlinearity f to its limits at infinity.

The results are in the next section, other sections contain proofs.

2. Main results

2.1. Subfurcation from infinity

Let for all λ from some neighborhood of the point λ_0 the representation

$$L(p,\lambda)=\big(p^2+(\lambda-\lambda_0)\sigma(\lambda)p+\alpha^2+(\lambda-\lambda_0)w(\lambda)\big)M(p,\lambda) \tag{5}$$

be valid with continuous $\sigma=\sigma(\lambda)$ and $w=w(\lambda)$, where $M(p,\lambda)$ is a polynomial of the degree $\ell-2$, also continuous in λ. Set

$$\sigma_0=\sigma(\lambda_0),\qquad w_0=w(\lambda_0),\qquad R_0=\Re e\, M(i\alpha,\lambda_0),\qquad J_0=\Im m\, M(i\alpha,\lambda_0). \tag{6}$$

Assume that the nonlinearity f in Eq. (1) has finite limits

$$f_-=\lim_{x\to-\infty} f(x),\qquad f_+=\lim_{x\to+\infty} f(x) \tag{7}$$

and $f_-\neq f_+$. In order to simplify the notation, we suppose everywhere

$$f_+=-f_-\neq 0,$$

which is not a restriction, since we can add opposite constants to the nonlinearity and the forcing term b.

Let for some $\beta>1$ and $\tilde K, x_0>0$

$$\big|f(x)-f_+\operatorname{sign}x\big|\leqslant \tilde K|x|^{-\beta},\quad |x|\geqslant x_0. \tag{8}$$

Theorem 2.1. *Let the transversality condition $\sigma_0\neq 0$ be valid and let*

$$\alpha\sigma_0R_0+w_0J_0\neq 0. \tag{9}$$

Then there exist a sequence $\lambda_k\to\lambda_0$ and a sequence of positive integers $n_k\to\infty$ such that for $\lambda=\lambda_k$ Eq. (1) *has a periodic solution $x_k=x_k(t)$ of the minimal period $2\pi n_k$ and $\|x_k\|_C\to\infty$ as $k\to\infty$, i.e. $\lambda=\lambda_0$ is a point of subfurcation from infinity for Eq.* (1).

Conditions (8) are important for Theorem 2.2 of the next subsection. We include them in Theorem 2.1 in order to simplify some parts of the proof and to join them with the corresponding parts of the proof of Theorem 2.2. In fact, Theorem 2.1 is valid without these conditions, it is enough to assume that f satisfies (7).

In Theorem 2.1 the positive integers n_k are the denominators of Diophantine approximations $q_k = m_k/n_k$ of the irrational α. One can use any sequence of rational approximations such that estimate (4) with some exponent $s > 1$ is valid and for all $q_k = m_k/n_k$ the relation

$$\operatorname{sign}(\alpha - m_k/n_k) = \operatorname{sign}\big(f_+(\alpha\sigma_0 R_0 + w_0 J_0)\sigma_0\big) \tag{10}$$

holds, which means that we use one-sided approximations of α. For example, for the sequence of all convergents q_k of α estimate (4) is valid with $s = 2$, $c = 1$ and (10) holds either for all odd or for all even k. We do not know any facts about approximations with $1 < s < 2$.

Denote

$$\varepsilon_k = \alpha - m_k/n_k,$$

then (4) takes the form $\varepsilon_k = O(n_k^{-s})$ as $k \to \infty$. Define the constants

$$K' = \frac{2f_+(\alpha\sigma_0 R_0 + w_0 J_0)}{\pi\alpha^2\sigma_0(R_0^2 + J_0^2)}, \qquad K'' = \frac{-2\alpha J_0}{\alpha\sigma_0 R_0 + w_0 J_0}. \tag{11}$$

It follows from the proof of Theorem 2.1 presented below that the unbounded sequence of subharmonics x_k satisfies the asymptotic relations

$$\|x_k\|_C = \frac{K'}{\varepsilon_k} + o\big(\varepsilon_k^{-1}\big), \qquad \lambda_k - \lambda_0 = K''\varepsilon_k + o(\varepsilon_k), \quad k \to \infty. \tag{12}$$

In particular, this implies that $\lambda_k \neq \lambda_0$ and the signature of $\lambda_k - \lambda_0$ is the same for all sufficiently large k if $\ell \geqslant 3$ and $J_0 \neq 0$.

Theorem 2.1 can be generalized in various directions. Its conclusions hold if in place of the nonlinearity f with saturation one considers a nonlinearity $f + g$, where f satisfies (7) and g has sublinear primitives, i.e.

$$\frac{1}{x}\int_0^x g(u)\,du \to 0, \quad |x| \to \infty.$$

The nonlinearity and the forcing term b may also depend on λ.

Similar results can be proved for vector systems

$$\frac{dx}{dt} = A(\lambda)x + f(x) + b(t), \quad x \in \mathbb{R}^N,$$

with $b(t) \equiv b(t + 2\pi)$. Here the main assumption about the linear part is that for some $\lambda = \lambda_0$ a pair of simple complex conjugate eigenvalues of the real matrix $A(\lambda)$ crosses the imaginary axis on the complex plain at the points $\pm i\alpha$ with an irrational α. As a counterpart of the scalar saturation condition (7), one can suppose that the vector function $f : \mathbb{R}^N \to \mathbb{R}^N$ has radial limits at infinity. These limits should satisfy proper nondegeneracy conditions (which have the form $f_+ \neq f_-$ in the scalar case above).

2.2. Asymptotic estimates of synchronization intervals

By a continuity argument, under the assumptions of Theorem 2.1 there exists generically a sequence of intervals $\Lambda_k \ni \lambda_k$ of positive lengths $|\Lambda_k|$ such that Eq. (1) has at least one subharmonic of the minimal period $2\pi n_k$ for every $\lambda \in \Lambda_k$. The amplitudes of these subharmonics tend to infinity as $k \to \infty$ uniformly w.r.t. λ, while $\lambda_k \to \lambda_0$ and $|\Lambda_k| \to 0$. In this section we derive asymptotic formulas for $|\Lambda_k|$ for large k.

Our proofs provide some further information about the synchronization intervals Λ_k and subharmonics. We fix a sequence of rational approximations $q_k = m_k/n_k$ to α and look for $2\pi n_k$-periodic subharmonics of the special form $x(t) = r\sin(q_k t + \varphi) + h(t)$, where the functions h are uniformly bounded for all k, while $r \to \infty$ as $k \to \infty$. It is proved that for each sufficiently large k and every phase $\varphi \in S^1$ Eq. (1) has such subharmonics $x = x(t)$ for some value of λ. Moreover, the pairs (x, λ) form a continuous (w.r.t. the parameter φ) branch in the sense of Mark Krasnosel'skii [5]. In simple cases (if solutions are isolated), it means that the set of large-amplitude subharmonics x of the minimal period $2\pi n_k$ consists of a number of homeomorphic images of a circle. For each image the values of λ cover an interval Λ_k twice. However, the structure of the continuous branch (x, λ) can be more complicated.

Let the polynomial $L(p, \lambda)$ be Lipschitz continuous in λ in some vicinity of the point λ_0. Let all the conditions of Theorem 2.1 be valid and let

$$\Re e\, L(\alpha i + \eta i, \lambda_0 + \zeta) = A\eta + B\zeta + O(\eta^2 + \zeta^2), \tag{13}$$

$$\Im m\, L(\alpha i + \eta i, \lambda_0 + \zeta) = C\eta + D\zeta + E\eta^2 + F\eta\zeta + G\zeta^2 + O\big((|\eta| + |\zeta|)^3\big) \tag{14}$$

as $|\eta| + |\zeta| \to 0$. From representation (5) it follows that the coefficients of the linear terms here are related with values (6) by

$$A = -2\alpha R_0, \qquad B = w_0 R_0 - \alpha\sigma_0 J_0, \qquad C = -2\alpha J_0, \qquad D = \alpha\sigma_0 R_0 + w_0 J_0.$$

Therefore condition (9) takes the form $D \neq 0$ and the second of values (11) equals $K'' = C/D$.

In order to formulate the next theorem, we introduce some further notations. Set

$$K_0 = -\frac{16 f_+^2}{\pi^2} \sum_{k=3,5,7,\ldots} \frac{\Im m\, L(ik\alpha, \lambda_0)}{k|L(ik\alpha, \lambda_0)|^2} \tag{15}$$

and

$$K_1 = \frac{\pi^3 (BK'' - A)^2}{16 f_+^2 D} = \frac{4\pi^3 \alpha^4 \sigma_0^2 (R_0^2 + J_0^2)^2}{16 f_+^2 D^3}. \tag{16}$$

Remark that the transversality condition $\sigma_0 \neq 0$ implies $BK'' - A \neq 0$ and consequently $K_1 \neq 0$. Using the complex Fourier series

$$b(t) = \sum_{k=-\infty}^{\infty} X_k e^{ikt} \qquad (X_{-k} = \bar{X}_k \in \mathbb{C})$$

of the function b, define the real 2π-periodic function (see [4])

$$\Phi_{m,n}(\varphi) = \frac{4f_+}{\pi} \sum_{k=\pm1,\pm3,\dots} \frac{X_{km} e^{-ikn\varphi}}{L(ikm, \lambda_0)} \tag{17}$$

for positive integers m, n and set

$$\Sigma_{m,n} = \max_{\varphi\in[0,2\pi]} \Phi_{m,n}(\varphi) - \min_{\varphi\in[0,2\pi]} \Phi_{m,n}(\varphi).$$

We consider odd numbers n only, which implies that the function $\Phi_{m,n} = \Phi_{m,n}(\varphi)$ is antisymmetric and therefore $\Sigma_{m,n} = 2\|\Phi_{m,n}\|_C$.

Denote by $\mathfrak{K}$ the set of all indexes k such that the kth convergent m_k/n_k of the number α satisfies relation (10) and has an *odd* denominator n_k.

Theorem 2.2. *Let the set $\mathfrak{K}$ be infinite. Let*

$$|\varepsilon_k|^{-\beta/(1+\beta)} \Sigma_{m_k,n_k} \to \infty, \qquad n_k^{-1} |\varepsilon_k|^{-2\beta/(1+2\beta)} \Sigma_{m_k,n_k} \to \infty \tag{18}$$

as $k \to \infty$ with $\varepsilon_k = \alpha - m_k/n_k$. Then for every sufficiently large $k \in \mathfrak{K}$ there exists an interval Λ_k such that for any $\lambda \in \Lambda_k$ Eq. (1) *has at least one periodic solution of the minimal period $2\pi n_k$, the amplitudes of these solutions go to infinity uniformly w.r.t. $\lambda \in \Lambda_k$ as $k \to \infty$, $k \in \mathfrak{K}$, the intervals Λ_k converge to the point λ_0, and their lengths $|\Lambda_k|$ satisfy*

$$\frac{|\Lambda_k|}{\varepsilon_k^2 \Sigma_{m_k,n_k}} \to |K_1|, \quad k \to \infty,\ k \in \mathfrak{K}. \tag{19}$$

Since $\Sigma_{m_k,n_k} > c|X_{m_k}| m_k^{-\ell}$, conditions (18) are valid, for example, if

$$|X_{m_k}||\varepsilon_k|^{-\beta/(1+\beta)} m_k^{-\ell} \to \infty, \qquad |X_{m_k}||\varepsilon_k|^{-2\beta/(1+2\beta)} m_k^{-\ell-1} \to \infty.$$

In particular, these relations hold if the Fourier coefficients X_{m_k} of the function b satisfy $|X_{m_k}| \geqslant c m_k^{-\gamma} > 0$ and estimate (4) for the rational approximations m_k/n_k of α holds for

$$s > \max\left\{ \frac{(\gamma+\ell)(1+\beta)}{\beta}, \frac{(\gamma+\ell+1)(1+2\beta)}{2\beta} \right\}. \tag{20}$$

Estimate (20) implies $s > \ell + 1 \geqslant 3$; for large β it means that $s > \gamma + \ell + 1$.

If $|X_n| \sim n^{-\gamma}$ (i.e., $c_1 n^{-\gamma} \geqslant |X_n| \geqslant c_2 n^{-\gamma} > 0$ for all n) and $\varepsilon_k \sim n_k^{-s}$, then relations (18) are equivalent to (20) and relation (19) implies $|\Lambda_k| \sim n_k^{-2s-\gamma-\ell}$. The relation $|X_n| \sim n^{-\gamma}$ means that the function b is $\gamma - 1/2$ times differentiable.

From the proofs presented below, it follows that under the assumptions of Theorem 2.2 for all $\lambda \in \Lambda_k$ the asymptotic formula (more exact than (12))

$$\lambda - \lambda_0 = K''\varepsilon_k + K\varepsilon_k^2 + o\big(\varepsilon_k^2\big)$$

holds with

$$K = \frac{\pi^3(BK'' - A)^2 K_0}{16 f_+^2 D} - \frac{E - FK'' + GK''^2}{D}. \tag{21}$$

3. Proof of Theorem 2.1

3.1. Operator equations

Let us fix a sequence of rational approximations $q_k = m_k/n_k$ to α (the positive integers m_k and n_k are coprime) such that for all k estimate (4) with $s = 2$ holds, i.e.

$$\left|\alpha - \frac{m_k}{n_k}\right| \leqslant \frac{c}{n_k^2}, \tag{22}$$

and relation (10) is valid. We prove the conclusion of Theorem 2.1 for this sequence, more precisely, we show that for any sufficiently large k (equivalently, any large n_k) Eq. (1) has a solution $x = x_k(t)$ with the minimal period $2\pi n_k$ for some $\lambda = \lambda_k$ and asymptotic formulas (12) hold. In what follows, we omit the indexes and write m, n in place of m_k, n_k, always meaning that m and n are sufficiently large and m is uniquely determined by (22) for a given n.

Rescaling the time $t := nt$ in (1), we arrive at the equation

$$L\left(n^{-1}\frac{d}{dt}, \lambda\right)x = f(x) + b(nt). \tag{23}$$

Every 2π-periodic solution $x = x(t)$ of (23) defines the $2n\pi$-periodic subharmonic $x(t/n)$ of Eq. (1). Let $\mathbb{E}_m \subset L^2 = L^2[0, 2\pi]$ be the orthogonal subspace to the plane $\mathbb{E}_m^\perp = \{x(t) = \xi \sin mt + \eta \cos mt\colon\ \xi, \eta \in \mathbb{R}\}$ in L^2 and let Q_m be the orthogonal projector onto $\mathbb{E}_m$ (here and henceforth, functions $x \in L^2$ are identified with their 2π-periodic extensions). We look for a 2π-periodic solution of (23) in the form

$$x(t) = r\sin(mt + \varphi) + h(t), \qquad h = h(t) \in \mathbb{E}_m, \quad r > 0, \tag{24}$$

where we mark out the principal harmonics of the order m and the Fourier series of the function h contains the other harmonics of x. Moreover, we consider the phase φ as a parameter, and λ as an unknown (instead of φ). Thus, our purpose is to prove that for each φ there exist numbers r, λ and a function $h \in \mathbb{E}_m$ such that (24) is a 2π-periodic solution of Eq. (23). From now on, a φ is fixed up to the end of the proof of Theorem 2.1.

Let us consider the orthogonal projections of Eq. (23) onto the plane $\mathbb{E}_m^\perp$ and onto the subspace $\mathbb{E}_m$ of codimension 2 in L^2. Projecting onto $\mathbb{E}_m^\perp$, we obtain the system of the two scalar equations

$$r\pi \Re e\, L(im/n, \lambda) = \int_0^{2\pi} f\bigl(r\sin(mt+\varphi)+h(t)\bigr)\sin(mt+\varphi)\,dt, \tag{25}$$

$$r\pi \Im m\, L(im/n, \lambda) = \int_0^{2\pi} f\bigl(r\sin(mt+\varphi)+h(t)\bigr)\cos(mt+\varphi)\,dt, \tag{26}$$

where the function $b(nt)$ is not present, because it is orthogonal to $\mathbb{E}_m^\perp$. The projection of (23) onto $\mathbb{E}_m$ has the form

$$L\left(n^{-1}\frac{d}{dt}, \lambda\right)h = b(nt) + Q_m f\bigl(r\sin(mt+\varphi)+h(t)\bigr). \tag{27}$$

In this equation we invert the linear differential operator in the left-hand side with the 2π-periodic boundary conditions. The inverse integral operator $H_n = H_n(\lambda)$ that maps a function $u = u(t)$ to a unique 2π-periodic solution $x = H_n u$ of the linear equation $L(n^{-1}\frac{d}{dt}, \lambda)x = u(t)$ is defined on the whole space L^1 since $L(iq, \lambda) \neq 0$ for any rational q. Moreover, H_n maps continuously L^1 to $C^{\ell-1}$ and any L^p to $W^{\ell,p}$, it is a completely continuous operator from each L^p with $p > 1$ to $C^{\ell-1}$, and it is normal in L^2, i.e. $H_n^* H_n = H_n H_n^*$ (all the functional spaces consist of functions defined on the segment $[0, 2\pi]$). This allows to rewrite Eq. (27) as

$$h = \mathfrak{f}h, \qquad (\mathfrak{f}h)(t) := H_n\Bigl(b(nt) + Q_m f\bigl(r\sin(mt+\varphi)+h(t)\bigr)\Bigr), \tag{28}$$

where the nonlinear operator $\mathfrak{f} : \mathbb{E}_m \to \mathbb{E}_m$ depends on n and the scalar parameters r, λ, φ. Consequently, the 2π-periodic problem for Eq. (23) is equivalent to the system of Eqs. (25), (26), and (28).

Set $H_{m,n} = H_n Q_m$. For our purposes, it is important that $\mathbb{E}_m$ is an invariant subspace of the operator H_n and that the norm of the restriction of H_n to the subspace $\mathbb{E}_m \subset L^2$ is much less than the norm of H_n on the whole space L^2. More precisely,

$$\|H_{m,n}\|_{L^2 \to L^2} \leqslant cn \tag{29}$$

with c independent of n, while $\|H_n\|_{L^2\to L^2} \sim |1/\varepsilon|$ with

$$\varepsilon = \alpha - m/n = O\bigl(n^{-2}\bigr).$$

Estimate (29) follows from the relation

$$L(ik/n,\lambda)B_k = A_k \tag{30}$$

between the Fourier coefficients of functions

$$u(t) = \sum_{k\in\mathbb{Z}} A_k e^{ikt}, \qquad h(t) = (H_{m,n}u)(t) = \sum_{k\in\mathbb{Z},\ k\neq\pm m} B_k e^{ikt},$$

where $A_{-k} = \bar{A}_k$, $B_{-k} = \bar{B}_k \in \mathbb{C}$ and $L(ik/n,\lambda)$ are the eigenvalues of H_n, and from the estimate

$$\bigl|M(iw,\lambda)\bigr| \geqslant \mu > 0. \tag{31}$$

Estimate (31) is valid, because the polynomial M does not vanish on the imaginary axis and its highest term is $p^{\ell-2}$ due to (2).

3.2. Auxiliary lemmas

Our next step is to find an invariant ball for the operator $\mathfrak{f}$ defined in (28). First, we formulate an estimate for the norms of the functions

$$\Delta_f = \Delta_f(r,\varphi,h)(t) := f\bigl(r\sin(mt+\varphi)+h(t)\bigr) - f_+ \operatorname{sign}\bigl(\sin(mt+\varphi)\bigr).$$

Lemma 3.1. *For any $\rho > 0$ there exists a $r_0 = r_0(\rho) > 0$ such that*

$$\|\Delta_f\|_{L^p} \leqslant c^* r^{-\beta/(1+p\beta)} \tag{32}$$

for each $r \geqslant r_0$, each $h \in C$ from the ball $\|h\|_C \leqslant \rho$, and each $\varphi \in \mathbb{R}$, with some $c^* > 0$ independent of ρ and $p \geqslant 1$.

Define the continuous 2π-periodic functions

$$u_1(t) = \sum_{k=-\infty}^{\infty} \frac{X_k e^{ikt}}{L(ik,\lambda_0)}, \qquad u_2(t) = \frac{4f_+}{\pi} \sum_{k=3,5,7,\ldots} \Re \frac{e^{ikt}}{ikL(ik\alpha,\lambda_0)}, \tag{33}$$

where X_k are the Fourier coefficients of the function b (all the denominators in these series are nonzero, because $\pm i\alpha$ are the only roots of the polynomial $L(p,\lambda_0)$ on the imaginary axis, by assumption). Set

$$\rho_1 = \|u_1\|_C, \qquad \rho_2 = \|u_2\|_C. \tag{34}$$

Lemma 3.2. *Let $\rho > \rho_1 + \rho_2$ and $\bar{c}, \bar{\bar{c}} > 0$. Then for every sufficiently large n and every $r \geqslant \bar{c}n^2$, $|\lambda - \lambda_0| \leqslant \bar{\bar{c}}n^{-2}$, and $\varphi \in \mathbb{R}$ the operator $\mathfrak{f}$ maps the ball $\{h \in \mathbb{E}_m \cap C\colon \|h\|_C \leqslant \rho\}$ into the interior of this ball.*

In the next subsection, we complete the proof of Theorem 2.1. The proofs of the above auxiliary lemmas are placed further at the end of this section. Lemma 3.1 is used to prove Lemma 3.2.

3.3. Equivalent system

Define the nonlinear functionals $F_S = F_S(r,\varphi,h)$, $F_C = F_C(r,\varphi,h)$ by

$$F_S = \frac{1}{\pi}\int_0^{2\pi} f\big(r\sin(mt+\varphi)+h(t)\big)\sin(mt+\varphi)\,dt,$$

$$F_C = \frac{1}{\pi}\int_0^{2\pi} f\big(r\sin(mt+\varphi)+h(t)\big)\cos(mt+\varphi)\,dt. \tag{35}$$

Instead of (25), (26), consider the equivalent equations

$$r = \frac{F_S}{\Re e\, L(im/n,\lambda)}, \qquad F_S \Im m\, L(im/n,\lambda) = F_C \Re e\, L(im/n,\lambda).$$

Furthermore, using representation (5), rewrite these equations as

$$r = \frac{F_S}{(\alpha^2 - m^2/n^2)R + (\lambda-\lambda_0)(wR - \sigma Jm/n)}, \tag{36}$$

$$\lambda - \lambda_0 = \frac{(\alpha^2 - m^2/n^2)(RF_C - JF_S)}{(\sigma Rm/n + wJ)F_S + (\sigma Jm/n - wR)F_C}, \tag{37}$$

where

$$R = R(\lambda) = \Re e\, M(im/n,\lambda), \qquad J = J(\lambda) = \Im m\, M(im/n,\lambda),$$

and then substitute $(\lambda-\lambda_0)$ in (36) to obtain

$$r = \frac{F_S(\sigma Rm/n + wJ) + F_C(\sigma Jm/n - wR)}{(\alpha^2 - m^2/n^2)(R^2+J^2)\sigma m/n}. \tag{38}$$

Now, let us show that system (28), (37), (38), equivalent to (25)–(27), has a solution (r,λ,h) for any sufficiently large n due to the Schauder principle. The construction below implies that $F_C \to 0$, $F_S \to \text{const} \neq 0$ (see (40)), and $\sigma Rm/n + wJ \to \sigma_0 R_0 \alpha + w_0 J_0 \neq 0$ (see (9)) as $n \to \infty$, and consequently the denominator in (37) is nonzero.

Denote $\mathbb{B}_\rho = \{h \in \mathbb{E}_m \cap C\colon\ \|h\|_C \leqslant \rho\}$; this is the ball of the codimension 2 subspace $\mathbb{E}_m \cap C$ of the space C, defined in Lemma 3.2. For any fixed δ satisfying $0 < \delta < |K'|$ with

K' defined by (11) ($K' \neq 0$ due to (9)) consider the Cartesian product Ω_n of the ball $\mathbb{B}_\rho$ and the segments

$$|r - K'/\varepsilon| \leqslant \delta/|\varepsilon|, \qquad |\lambda - \lambda_0 - K''\varepsilon| \leqslant \delta|\varepsilon|. \tag{39}$$

Estimates (39) and relation (10) imply $r \geqslant \bar{c}n^2$, $|\lambda - \lambda_0| \leqslant \bar{\bar{c}}n^{-2}$, since $\varepsilon = \alpha - m/n = O(n^{-2})$. Therefore $r \to +\infty$, $\lambda \to \lambda_0$ and $R \to R_0$, $J \to J_0$, $\sigma \to \sigma_0$, $w \to w_0$ as $n \to \infty$. Moreover, from Lemma 3.1 it follows that $\sup_{(r,\lambda,h)\in\Omega_n} \|\Delta_f\|_{L^p} \to 0$ and hence

$$F_S \to \frac{f_+}{\pi} \int_0^{2\pi} \left|\sin(mt + \varphi)\right| dt = \frac{4f_+}{\pi},$$

$$F_C \to \frac{f_+}{\pi} \int_0^{2\pi} \cos(mt + \varphi)\,\mathrm{sign}\bigl(\sin(mt + \varphi)\bigr)\, dt = 0 \tag{40}$$

as $n \to \infty$, where the convergence is uniform w.r.t $(r, \lambda, h) \in \Omega_n$. Therefore the right-hand side of (38) equals $(K' + o(1))/\varepsilon$, and the right-hand side of (37) equals $(K'' + o(1))\varepsilon$. For sufficiently large n these values belong to the interior of intervals (39). At the same time, Lemma 3.2 implies that the right-hand side $\mathfrak{f}h$ of (28) belongs to the ball $\mathbb{B}_\rho$ for all $(r, \lambda, h) \in \Omega_n$ with large enough n. Thus, system (28), (37), (38) has a solution (r, λ, h) in Ω_n by the Schauder principle. By construction, this solution defines a 2π-periodic solution (24) of Eq. (23) and consequently a $2\pi n$-periodic solution x of (1). Finally, since $\delta > 0$ can be chosen arbitrarily small, estimates (39) imply relations (12). Theorem 2.1 is proved.

3.4. Proof of Lemma 3.1

Denote $t_j = (j\pi - \varphi)/m$ for $j = 0, 1, \ldots, 2m - 1$ and define the sets

$$A = \bigcup_{j=0}^{2m-1} [t_j - \delta, t_j + \delta], \qquad B = [t_0 - \delta, 2\pi + t_0 - \delta] \setminus A \tag{41}$$

with $\delta = m^{-1}r^{-p\beta/(1+p\beta)}$. We assume that r is large, therefore the intervals $[t_j - \delta, t_j + \delta]$ do not intersect and the Lebesgue measures of A and B are

$$\mathrm{mes}\, A = 4m\delta = 4r^{-p\beta/(1+p\beta)}, \qquad \mathrm{mes}\, B = 2\pi - 4r^{-p\beta/(1+p\beta)}. \tag{42}$$

If $t \in B$, then $|\sin(mt + \varphi)| \geqslant |\sin(m\delta)| \geqslant 2m\delta/\pi$. Since $rm\delta = r^{1/(1+p\beta)}$, these estimates imply for any h from a fixed ball $\|h\|_C \leqslant \rho$ and any large enough r

$$r\left|\sin(mt + \varphi)\right| - \left|h(t)\right| \geqslant \pi^{-1}r^{1/(1+p\beta)}.$$

Consequently, the functions $x(t) = r\sin(mt+\varphi) + h(t)$ and $\sin(mt+\varphi)$ have the same signatures at each point of the set B, hence $\Delta_f(t) = f(x(t)) - f_+ \operatorname{sign} x(t)$ on this set and (8) implies

$$\left|\Delta_f(t)\right| \leqslant \tilde{K}\left|x(t)\right|^{-\beta} \leqslant \tilde{K}\left(r\left|\sin(mt+\varphi)\right| - |h|\right)^{-\beta} \leqslant \tilde{K}\pi^\beta r^{-\beta/(1+p\beta)} \tag{43}$$

for $t \in B$. Therefore

$$\int\limits_{t_0-\delta}^{2\pi+t_0-\delta} \left|\Delta_f(t)\right|^p dt \leqslant \left(\sup|f| + |f_+|\right)^p \operatorname{mes} A + \tilde{K}^p \pi^{p\beta} r^{-p\beta/(1+p\beta)} \operatorname{mes} B.$$

Combining this with (42), we obtain (32), which completes the proof. □

3.5. Proof of Lemma 3.2

Let us write the function $\mathfrak{f}h$ in the form

$$\mathfrak{f}h(t) = H_{m,n}b(nt) + f_+ H_{m,n}\left(\operatorname{sign}\sin(mt+\varphi)\right) + H_{m,n}\Delta_f(t) \tag{44}$$

and estimate the norm of each term in the right-hand side separately. From the equalities

$$b(nt) = \sum_{k=-\infty}^{\infty} X_k e^{iknt}, \qquad \operatorname{sign}\sin(mt+\varphi) = \frac{4}{\pi}\sum_{k=1,3,5,\ldots} \frac{\sin(k(mt+\varphi))}{k},$$

and (30), it follows that the first and the second terms in (44) equal

$$H_{m,n}b(nt) = \sum_{k=-\infty}^{\infty} \frac{X_k e^{iknt}}{L(ik,\lambda)},$$

$$H_{m,n}\left(\operatorname{sign}\sin(mt+\varphi)\right) = \frac{4}{\pi}\sum_{k=3,5,7,\ldots} \Re\frac{e^{ik(mt+\varphi)}}{ikL(ikm/n,\lambda)}. \tag{45}$$

Since $|\alpha - m/n| \leqslant cn^{-2}$, $|\lambda - \lambda_0| \leqslant \bar{\bar{c}}n^{-2}$, these equalities imply by the continuity arguments

$$\left\|H_{m,n}b(nt) - u_1(nt)\right\|_C \to 0, \qquad \left\|f_+H_{m,n}\left(\operatorname{sign}\sin(mt+\varphi)\right) - u_2(mt+\varphi)\right\|_C \to 0$$

as $n \to \infty$, where u_1, u_2 are functions (33). Taking into account the relations $\|u_1(nt)\|_C = \rho_1$, $\|u_2(mt+\varphi)\|_C = \rho_2$, and $\rho > \rho_1 + \rho_2$, we conclude that for all sufficiently large n

$$\left\|H_{m,n}b(nt)\right\|_C + \left\|f_+H_{m,n}\left(\operatorname{sign}\sin(mt+\varphi)\right)\right\|_C < \rho.$$

Consequently, to complete the proof, it suffices to show that the last term in (44) satisfies $\|H_{m,n}\Delta_f\|_C \to 0$ as $n \to \infty$ uniformly w.r.t. all $h \in \mathbb{E}_m \cap C$ from the ball $\|h\|_C \leqslant \rho$ and all $r \geqslant \bar{c}n^2$, $|\lambda - \lambda_0| \leqslant \bar{\bar{c}}n^{-2}$, $\varphi \in \mathbb{R}$. For this purpose, we use Lemma 3.1.

Let us fix an arbitrary p from the interval $1 < p < 2 - 1/\beta$, define the conjugate q of p by $p^{-1} + q^{-1} = 1$, and consider the real Fourier series

$$\Delta_f(t) = \sum_{k=0}^{\infty} D_k \sin(kt + \phi_k), \qquad D_k \geqslant 0,$$

of the function Δ_f. According to Hausdorff–Young theorem (see, e.g., [7, p. 190]), the inclusion $\Delta_f \in L^p$ implies that the sequence $D = \{D_k\}$ belongs to the space ℓ_q and

$$\|D\|_{\ell_q} = \left(\sum_{k=0}^{\infty} D_k^q\right)^{1/q} \leqslant \|\Delta_f\|_{L^p}. \tag{46}$$

From relations (30), it follows that the real Fourier series of the function $H_{m,n}\Delta_f$ has the form

$$H_{m,n}\Delta_f(t) = \sum_{k\geqslant 0,\ k\neq m} \frac{D_k \sin(kt + \psi_k)}{|L(ik/n, \lambda)|},$$

consequently

$$\|H_{m,n}\Delta_f\|_C \leqslant \sum_{k\geqslant 0,\ k\neq m} D_k \bigl|L(ik/n, \lambda)\bigr|^{-1}$$

and by the Gölder inequality $\|H_{m,n}\Delta_f\|_C \leqslant \|D\|_{\ell_q} S^{1/p}$ with

$$S = \sum_{k\geqslant 0,\ k\neq m} \bigl|L(ik/n, \lambda)\bigr|^{-p}.$$

Therefore (46) implies

$$\|H_{m,n}\Delta_f\|_C \leqslant \|\Delta_f\|_{L^p} S^{1/p}. \tag{47}$$

An estimate of the first multiplier in the right-hand side of (47) follows from Lemma 3.1. To estimate S, consider the factorization (5) of the polynomial L. Relation (31) implies

$$\begin{aligned}\bigl|L(ik/n, \lambda)\bigr| &\geqslant c_1\bigl|-n^{-2}k^2 + \alpha^2 + (\lambda - \lambda_0)w(\lambda)\bigr| \\ &\geqslant c_1 n^{-2}\bigl|m^2 - k^2\bigr| - c_1\bigl|\alpha^2 - n^{-2}m^2 + (\lambda - \lambda_0)w(\lambda)\bigr|\end{aligned}$$

and from $|\alpha - m/n| \leqslant cn^{-2}$, $|\lambda - \lambda_0| \leqslant \bar{\bar{c}}n^{-2}$ it follows that

$$\bigl|L(ik/n, \lambda)\bigr| \geqslant c_1 n^{-2}\bigl|m^2 - k^2\bigr| - c_2 n^{-2}$$

with some $c_2 > 0$. For any sufficiently large m and all $k \neq m$ this implies

$$\left|L(ik/n,\lambda)\right|^{-p} \leqslant \frac{c_1^{-p} n^{2p}}{(|m^2-k^2| - c_2/c_1)^p}.$$

Summing this for $k \geqslant 0$, $k \neq m$ and taking into account that for large m

$$\sum_{k=0}^{m-1} \frac{1}{(m^2-k^2-c_2/c_1)^p} = \sum_{k=1}^{m} \frac{1}{(m^2-(m-k)^2-c_2/c_1)^p} \leqslant \sum_{k=1}^{m} \left(\frac{2}{mk}\right)^p$$

and

$$\sum_{k=m+1}^{\infty} \frac{1}{(k^2-m^2-c_2/c_1)^p} = \sum_{k=1}^{\infty} \frac{1}{((k+m)^2-m^2-c_2/c_1)^p} \leqslant \sum_{k=1}^{\infty} \left(\frac{1}{mk}\right)^p,$$

we obtain

$$S \leqslant c_3 n^{2p} m^{-p} \leqslant 2c_3 n^p.$$

Finally, for all $h \in \mathbb{E}_m \cap C$ satisfying $\|h\|_C \leqslant \rho$, all $\varphi \in \mathbb{R}$, and all sufficiently large n estimates (32) and $r \geqslant \bar{c}n^2$ imply $\|\Delta_f\|_{L^p} \leqslant c_4 n^{-2\beta/(1+p\beta)}$ and thus (47) gives

$$\|H_{m,n}\Delta_f\|_C \leqslant \|\Delta_f\|_{L^p} S^{1/p} \leqslant c_5 n^{1-2\beta/(1+p\beta)},$$

where the exponent of n is negative due to our choice of p. This implies $\|H_{m,n}\Delta_f\|_C \to 0$ as $n \to \infty$ and consequently completes the proof. □

4. Proof of Theorem 2.2

4.1. Scheme of the proof

From the proof of Theorem 2.1 it follows that for every sufficiently large $k \in \mathfrak{K}$ and every φ equation (23) with $n = n_k$ has a 2π-periodic solution $x(t) = r\sin(mt+\varphi) + h(t)$ with $m = m_k$ for some $\lambda = \lambda_k(\varphi)$ and that relations (12) hold uniformly in φ. The existence of each solution was proved by the Schauder principle applied to a completely continuous operator that acts in the space of triples (r,λ,h) and maps the Cartesian product $\Omega_n = \Omega_{n_k}$ of the ball $\mathbb{B}_\rho = \{h \in \mathbb{E}_{m_k} \cap C\colon \|h\|_C \leqslant \rho\}$ (of a fixed radius ρ) and the segments (39) depending on k into itself for any φ. Let us denote the set of all the fixed points (r,λ,h) of this operator lying in Ω_{n_k} for a given φ by $G^k(\varphi)$ and the projection $(r,\lambda,h) \mapsto \lambda$ of the set $G^k(\varphi)$ onto the λ-axis by $\Pi^k(\varphi)$; consequently, $\Pi^k(\varphi)$ is a nonempty bounded closed set.

In the next subsections, we show that for each φ and all $\lambda \in \Pi^k(\varphi)$

$$\lambda - \lambda_0 = \varepsilon K'' + \varepsilon^2 K + \varepsilon^2 K_1 \Phi_{m,n}(\varphi) + \varepsilon^2 O\left(|\varepsilon|^{\beta/(1+\beta)} + n|\varepsilon|^{2\beta/(1+2\beta)}\right) \qquad (48)$$

with $\varepsilon = \varepsilon_k$ and let the parameter φ vary over the segment $[0, 2\pi]$ for a fixed k (remark that φ was fixed in the above proof of Theorem 2.1). If the function $\Phi_{m,n} = \Phi_{m,n}(\varphi)$ reaches its minimum and maximum at the points $\varphi = \varphi_1$ and $\varphi = \varphi_2$, respectively, then for every $\lambda_1 \in \Pi^k(\varphi_1)$ and $\lambda_2 \in \Pi^k(\varphi_2)$ formula (48) implies

$$\lambda_2 - \lambda_1 = \varepsilon^2 |K_1| \Sigma_{m,n} + \varepsilon^2 O\big(|\varepsilon|^{\beta/(1+\beta)} + n|\varepsilon|^{2\beta/(1+2\beta)}\big)$$

and taking into account condition (18), we arrive at

$$\lambda_2 - \lambda_1 = \varepsilon^2 |K_1| \Sigma_{m,n}\big(1 + o(1)\big), \quad \lambda_1 \in \Pi^k(\varphi_1),\ \lambda_2 \in \Pi^k(\varphi_2).$$

In particular, this implies asymptotics (19) for the length $|\Lambda_k|$ of the segment $\Lambda_k = [\lambda^k_{\min}, \lambda^k_{\max}]$, where $\lambda^k_{\min} = \max \Pi^k(\varphi_1)$, $\lambda^k_{\max} = \min \Pi^k(\varphi_2)$. Moreover, from a standard topological argument it follows that for every $\lambda \in (\lambda^k_{\min}, \lambda^k_{\max})$ one finds a φ such that $\lambda \in \Pi^k(\varphi)$ and therefore the values of λ such that Eq. (23) has at least one solution (24) with $(r, \lambda, h) \in \Omega_{n_k}$ fill in the whole segment Λ_k, which implies all the conclusions of Theorem 2.2. This topological argument proceeds as follows.

Let Ω be a bounded convex closed solid set in a Banach space E. Let for each value of a scalar parameter φ from a segment $J = [\varphi_{\min}, \varphi_{\max}]$ a completely continuous operator $F(\cdot, \varphi) : \Omega \to E$ be defined on the set Ω and let $F(x, \varphi)$ depend continuously on φ uniformly w.r.t. $x \in \Omega$. Suppose that the operator $F(\cdot, \varphi)$ does not have zeros on the boundary $\partial\Omega$ of the set Ω for all $\varphi \in J$ and consequently the rotation $\gamma = \gamma(I - F(\cdot, \varphi), \partial\Omega)$ of the vector field $x - F(x, \varphi) \in E$ on the boundary $\partial\Omega$ of Ω is defined and has the same value for all φ. Let this rotation be nonzero (for example, in our case $\gamma = 1$, because the operator $F(\cdot; \varphi)$ maps the set Ω into its interior $\Omega \setminus \partial\Omega$). Then the set $G(\varphi)$ of fixed points of $F(\cdot; \varphi)$ is nonempty for each $\varphi \in J$ and moreover (see, e.g., [3]), the set $\Gamma = \{(\varphi, x) \in J \times \Omega:\ \varphi \in J,\ x \in G(\varphi)\}$ is a continuous branch w.r.t. φ, which means that Γ has a nonempty intersection with the boundary ∂U of any open domain U of the space $J \times E$ such that U contains the cross-section $\varphi = \varphi_{\min}$, $x \in G(\varphi_{\min})$ of Γ and U does not intersect the cross-section $\varphi = \varphi_{\max}$, $x \in G(\varphi_{\max})$ of Γ.

In particular, this implies that if $P : E \to \mathbb{R}$ is a linear continuous functional and $\max P(G(\varphi_1)) < \min P(G(\varphi_2))$ for some $\varphi_1, \varphi_2 \in J$, then each number from the segment $[\max P(G(\varphi_1)), \min P(G(\varphi_2))]$ belongs to the set $P(G(\varphi))$ for at least one $\varphi \in J$. In our case, $P(r, \lambda, h) = \lambda$ and $P(G^k(\varphi)) = \Pi^k(\varphi)$. Thus, to complete the proof, it suffices to obtain formula (48).

4.2. Estimates

In the further proof, we determine the principal terms of integrals (35).

4.2.1. The main terms of the integral F_C

Consider a 2π-periodic solution (24) of Eq. (23). Since $x'(t) = rm\cos(mt+\varphi) + h'(t)$, then

$$F_C = \frac{1}{\pi rm}\int_0^{2\pi} f\bigl(x(t)\bigr)\bigl(x'(t) - h'(t)\bigr)\,dt;$$

the integral of $f(x)x'$ over the period is zero for any periodic x, therefore

$$F_C = -\frac{1}{\pi rm}\int_0^{2\pi} h'(t) f\bigl(x(t)\bigr)\,dt.$$

Substituting here in place of h the right-hand side $\mathfrak{f}h$ of (28), we arrive at

$$F_C = -\frac{1}{\pi rm}\int_0^{2\pi} f\bigl(x(t)\bigr)\frac{d}{dt}H_{m,n}b(nt)\,dt - \frac{1}{\pi rm}\int_0^{2\pi} f\bigl(x(t)\bigr)\frac{d}{dt}H_{m,n}f\bigl(x(t)\bigr)\,dt. \tag{49}$$

Define the functions

$$\Psi_{m,n}(\varphi,\lambda) = -\frac{f_+}{\pi m}\int_0^{2\pi} \operatorname{sign}\bigl(\sin(mt+\varphi)\bigr)\frac{d}{dt}H_{m,n}b(nt)\,dt,$$

$$K_{m,n}(\lambda) = -\frac{f_+^2}{\pi m}\int_0^{2\pi} \operatorname{sign}\bigl(\sin(mt+\varphi)\bigr)\frac{d}{dt}H_{m,n}\operatorname{sign}\bigl(\sin(mt+\varphi)\bigr)\,dt.$$

Let us show that

$$rF_C = \Psi_{m,n}(\varphi,\lambda) + K_{m,n}(\lambda) + \rho, \tag{50}$$

where ρ is asymptotically small compared to the first and the second terms of the right-hand side, and that these terms satisfy

$$\Psi_{m,n}(\varphi,\lambda) = \Phi_{m,n}(\varphi) + O(\varepsilon), \qquad K_{m,n}(\lambda) = K_0 + O(\varepsilon) \tag{51}$$

as $n \to \infty$, where K_0 and $\Phi_{m,n}$ are defined by formulas (15), (17).

4.2.2. Proof of relations (51)

For any locally integrable 2π-periodic function u, we have

$$\int_0^{2\pi} \operatorname{sign}\bigl(\sin(mt+\varphi)\bigr)\frac{d}{dt}H_{m,n}u(t)\,dt$$

$$= \sum_{j=1}^{2m} \int_{(j-1)\pi/m-\varphi/m}^{j\pi/m-\varphi/m} \operatorname{sign}\bigl(\sin(mt+\varphi)\bigr)\frac{d}{dt}H_{m,n}u(t)\,dt$$

$$= \sum_{j=1}^{2m}(-1)^{j-1}\bigl(H_{m,n}u(t)|_{t=j\pi/m-\varphi/m} - H_{m,n}u(t)|_{t=(j-1)\pi/m-\varphi/m}\bigr)$$

$$= 2\sum_{j=0}^{2m-1}(-1)^{j-1}H_{m,n}u(t)|_{t=j\pi/m-\varphi/m}.$$

Therefore

$$\Psi_{m,n}(\varphi,\lambda) = -\frac{2f_+}{\pi m}\sum_{j=0}^{2m-1}(-1)^{j-1}H_{m,n}b(nt)|_{t=j\pi/m-\varphi/m},$$

$$K_{m,n}(\lambda) = -\frac{2f_+^2}{\pi m}\sum_{j=0}^{2m-1}(-1)^{j-1}H_{m,n}\operatorname{sign}\bigl(\sin(mt+\varphi)\bigr)|_{t=j\pi/m-\varphi/m}$$

and (45) implies

$$\Psi_{m,n}(\varphi,\lambda) = \frac{2f_+}{\pi m}\sum_{j=0}^{2m-1}(-1)^j\sum_{k=-\infty}^{\infty}\frac{X_k e^{iknt}}{L(ik,\lambda)}\bigg|_{j\pi/m-\varphi/m}$$

$$= \frac{2f_+}{\pi m}\sum_{k=-\infty}^{\infty}\frac{X_k}{L(ik,\lambda)}\sum_{j=0}^{2m-1}(-1)^j e^{ikn(j\pi/m-\varphi/m)}$$

$$= \frac{2f_+}{\pi m}\sum_{k=-\infty}^{\infty}\frac{X_k e^{-ikn\varphi/m}}{L(ik,\lambda)}\sum_{j=0}^{2m-1}(-1)^j e^{iknj\pi/m}$$

$$= \frac{2f_+}{\pi m}\sum_{k=-\infty}^{\infty}\frac{X_k e^{-ikn\varphi/m}}{L(ik,\lambda)}\cdot\begin{cases}0 & \text{for } e^{ikn\pi/m}\neq -1,\\ 2m & \text{for } e^{ikn\pi/m} = -1\end{cases}$$

$$= \frac{2f_+}{\pi m}\sum_{k=-\infty}^{\infty}\frac{X_k e^{-ikn\varphi/m}}{L(ik,\lambda)}\cdot\begin{cases}0 & \text{if } k \text{ is not a multiple of } m,\\ 0 & \text{if } kn/m \text{ is even},\\ 2m & \text{if both } k/m \text{ and } n \text{ are odd;}\end{cases}$$

and

$$K_{m,n}(\lambda) = \frac{8f_+^2}{\pi^2 m} \sum_{j=0}^{2m-1} (-1)^j \sum_{k=3,5,7,\ldots} \Re \frac{e^{ikj\pi}}{ikL(ikm/n,\lambda)}$$
$$= -\frac{16f_+^2}{\pi^2} \sum_{k=3,5,7,\ldots} \frac{\Im m\, L(ikm/n,\lambda)}{k|L(ikm/n,\lambda)|^2}.$$

Equivalently,

$$\Psi_{m,n}(\varphi,\lambda) = \frac{4f_+}{\pi} \sum_{k=\pm1,\pm3,\ldots} \frac{X_{km}e^{-ikn\varphi}}{L(ikm,\lambda)},$$
$$K_{m,n}(\lambda) = -\frac{16f_+^2}{\pi^2} \sum_{k=3,5,7,\ldots} \frac{\Im m\, L(ik(\alpha-\varepsilon),\lambda)}{k|L(ik(\alpha-\varepsilon),\lambda)|^2},$$

where we use the fact that n is odd (for any even n, one gets $\Psi_{n,m} \equiv 0$). Now, from the Lipschitz continuity of L w.r.t λ it follows

$$\Psi_{m,n}(\varphi,\lambda) = \Phi_{m,n}(\varphi) + O(\lambda-\lambda_0), \qquad K_{m,n}(\lambda) = K_0 + O\big(|\varepsilon| + |\lambda-\lambda_0|\big).$$

Due to (12), $\lambda - \lambda_0 = O(\varepsilon)$, hence (51) holds.

4.2.3. Estimation of smaller terms of the integral F_C

Here we estimate the term ρ in (50). To be concise, let us introduce the notation $v_*(t,\varphi) = f_+\mathrm{sign}(\sin(mt+\varphi))$, then by definition

$$\Delta_f(t) = f\big(x(t)\big) - v_*(t,\varphi),$$
$$\Psi_{m,n}(\varphi,\lambda) = -\frac{1}{\pi m}\int_0^{2\pi} v_*(t,\varphi)\frac{d}{dt}H_{m,n}b(nt)\,dt,$$
$$K_{m,n}(\lambda) = -\frac{1}{\pi m}\int_0^{2\pi} v_*(t,\varphi)\frac{d}{dt}H_{m,n}v_*(t,\varphi)\,dt.$$

Relations (49) and (50) imply

$$\rho = -\frac{1}{\pi m}\int_0^{2\pi} \Delta_f(t)\frac{d}{dt}H_{m,n}b(nt)\,dt + \frac{1}{\pi m}\int_0^{2\pi} v_*(t,\varphi)\frac{d}{dt}H_{m,n}v_*(t,\varphi)\,dt$$
$$- \frac{1}{\pi m}\int_0^{2\pi} f\big(x(t)\big)\frac{d}{dt}H_{m,n}f\big(x(t)\big)\,dt.$$

Therefore

$$\pi m|\rho| \leqslant \|\Delta_f\|_{L^1}\left\|\frac{d}{dt}H_{m,n}b(nt)\right\|_C + \|\Delta_f\|_{L^1}\left\|\frac{d}{dt}H_{m,n}v_*(t,\varphi)\right\|_C$$
$$+ \|\Delta_f\|_{L^1}\left\|\left(\frac{d}{dt}H_{m,n}\right)^* v_*(t,\varphi)\right\|_C + \|\Delta_f\|_{L^2}^2\left\|\frac{d}{dt}H_{m,n}\right\|_{L^2\to L^2}, \qquad (52)$$

where $(d/dt\,H_{m,n})^*$ denotes the conjugate operator for $d/dt\,H_{m,n}$.

From the formulas

$$\frac{d}{dt}H_{m,n}b(nt) = \sum_{k=-\infty}^{\infty}\frac{inkX_k e^{iknt}}{L(ik,\lambda)},$$
$$\frac{d}{dt}H_{m,n}v_*(t,\varphi) = \frac{2f_+}{\pi}\sum_{k=\pm3,\pm5,\pm7,\dots}\frac{me^{ik(mt+\varphi)}}{L(ikm/n,\lambda)},$$
$$\left(\frac{d}{dt}H_{m,n}\right)^* v_*(t,\varphi) = \frac{2f_+}{\pi}\sum_{k=\pm3,\pm5,\pm7,\dots}\frac{-me^{ik(mt+\varphi)}}{L(-ikm/n,\lambda)},$$

it follows

$$\left\|\frac{d}{dt}H_{m,n}b(nt)\right\|_C,\quad \left\|\frac{d}{dt}H_{m,n}v_*\right\|_C,\quad \left\|\left(\frac{d}{dt}H_{m,n}\right)^* v_*\right\|_C \leqslant cn. \qquad (53)$$

Furthermore,

$$\left\|\frac{d}{dt}H_{m,n}\right\|_{L^2\to L^2} \leqslant \max_{k\neq m}\frac{k}{|L(ik/n,\lambda)|} \leqslant \max_{k\neq m}\frac{k}{\mu|-k^2/n^2+\alpha^2+(\lambda-\lambda_0)w(\lambda)|},$$

where $\mu>0$ is a lower bound for $|M(i\omega,\lambda)|$ with $\omega\in\mathbb{R}$. For any sufficiently large n, the relations $\lambda-\lambda_0 = O(\varepsilon) = O(n^{-2})$ imply that if $k>2\alpha n$, then

$$\frac{k}{|-k^2/n^2+\alpha^2+(\lambda-\lambda_0)w(\lambda)|} \leqslant \frac{k}{k^2/n^2-k^2/(2n^2)} = \frac{2n^2}{k} \leqslant \frac{n}{\alpha};$$

if $k\leqslant 2\alpha n$, $k\neq m$, then

$$\frac{k}{|-k^2/n^2+\alpha^2+(\lambda-\lambda_0)w(\lambda)|} \leqslant \frac{2nk}{\alpha|k-m|} \leqslant \frac{2nk}{\alpha} \leqslant 4n^2.$$

Consequently,

$$\left\|\frac{d}{dt}H_{m,n}\right\|_{L^2\to L^2} \leqslant \frac{4n^2}{\mu},$$

which together with (52) and (53) implies $|\rho| \leqslant c_1\|\Delta_f\|_{L^1} + c_2 n\|\Delta_f\|^2_{L^2}$. By Lemma 3.1, $\|\Delta_f\|_{L^1} \leqslant c^* r^{-\beta/(1+\beta)}$, $\|\Delta_f\|_{L^2} \leqslant c^* r^{-\beta/(1+2\beta)}$, where $r \sim 1/|\varepsilon|$ due to (12) (recall that the norm $\|h\|_C$ is uniformly bounded for all n), therefore

$$|\rho| \leqslant c\big(|\varepsilon|^{\beta/(1+\beta)} + n|\varepsilon|^{2\beta/(1+2\beta)}\big).$$

From (50) and (51) it follows that $rF_C = \Phi_{m,n}(\varphi) + K_0 + \rho + O(\varepsilon)$. Because $|\varepsilon|^{\beta/(1+\beta)} > |\varepsilon|$, we, finally, obtain

$$rF_C = \Phi_{m,n}(\varphi) + K_0 + O\big(|\varepsilon|^{\beta/(1+\beta)} + n|\varepsilon|^{2\beta/(1+2\beta)}\big). \tag{54}$$

4.2.4. Smaller terms of the integral F_S

To complete the proof, we need a more exact estimate of the difference $F_S - 4f_+/\pi$ than that in (40). Let us write F_S in the form

$$F_S = \frac{1}{\pi}\int_0^{2\pi} v_*(t,\varphi)\sin(mt+\varphi)\,dt + \mathcal{I}_3 = \frac{4f_+}{\pi} + \mathcal{I}_3$$

with

$$\mathcal{I}_3 = \frac{1}{\pi}\int_0^{2\pi} \Delta_f(t)\sin(mt+\varphi)\,dt.$$

To estimate the integral $\mathcal{I}_3$ we use the argument similar to that of the proof of Lemma 3.1. The difference is that here we are able to benefit from the smallness of the integrand on the set A, because

$$\big|\sin(mt+\varphi)\big| \leqslant m\delta, \quad t \in A.$$

Put $\delta = m^{-1}r^{-\beta/(2+\beta)}$ and consider sets (41) with this δ. Repeating the proof of Lemma 3.1 with the new δ, we obtain for all $t \in B$ the estimate

$$\big|\Delta_f(t)\big| \leqslant \tilde{K}\pi^\beta r^{-2\beta/(2+\beta)},$$

analogous to (43). Consequently,

$$\pi|\mathcal{I}_3| \leqslant m\delta\|\Delta_f\|_{L_\infty}\,\mathrm{mes}\,A + \tilde{K}\pi^\beta r^{-2\beta/(2+\beta)}\,\mathrm{mes}\,B.$$

Here $\mathrm{mes}\,A = 4m\delta$, $\mathrm{mes}\,B \leqslant 2\pi$, and $\|\Delta_f\|_{L_\infty} \leqslant \sup|f| + |f_+|$, hence $|\mathcal{I}_3| \leqslant c(m^2\delta^2 + r^{-2\beta/(2+\beta)}) = 2cr^{-2\beta/(2+\beta)}$ and $r \sim 1/|\varepsilon|$ implies

$$F_S = \frac{4f_+}{\pi} + O\big(|\varepsilon|^{2\beta/(2+\beta)}\big). \tag{55}$$

4.3. End of the proof

Let us multiply (26) by r and let us square equality (25) to obtain

$$r^2\pi\Im m\,L(im/n,\lambda)=rF_C,\qquad r^2\pi^2\big(\Re e\,L(im/n,\lambda)\big)^2=F_S^2.$$

Dividing the first of these relations by the second one, we arrive at

$$\frac{\Im m\,L(im/n,\lambda)}{\pi(\Re e\,L(im/n,\lambda))^2}=\frac{rF_C}{F_S^2}.$$

Formulas (54) and (55) imply that the right-hand side here equals

$$\frac{rF_C}{F_S^2}=\frac{\Phi_{m,n}(\varphi)+K_0+O(|\varepsilon|^{\beta/(1+\beta)}+n|\varepsilon|^{2\beta/(1+2\beta)})}{16f_+^2/\pi^2+O(|\varepsilon|^{2\beta/(2+\beta)})}$$

and consequently

$$\frac{\Im m\,L(im/n,\lambda)}{(\Re e\,L(im/n,\lambda))^2}=\frac{\pi^3(\Phi_{m,n}(\varphi)+K_0)}{16f_+^2}+O\big(|\varepsilon|^{\beta/(1+\beta)}+n|\varepsilon|^{2\beta/(1+2\beta)}\big),\qquad(56)$$

where we take into account that $2\beta/(2+\beta)>\beta/(1+\beta)$ and $\Phi_{m,n}=o(1)$.

Let us replace λ with the new variable ξ defined by the relation $\lambda-\lambda_0=K''\varepsilon+\xi\varepsilon^2$, which allows to rewrite the left-hand side of (56) as

$$\frac{\Im m\,L(im/n,\lambda)}{(\Re e\,L(im/n,\lambda))^2}=\frac{\Im m\,L(i\alpha-i\varepsilon,\lambda_0+K''\varepsilon+\xi\varepsilon^2)}{(\Re e\,L(i\alpha-i\varepsilon,\lambda_0+K''\varepsilon+\xi\varepsilon^2))^2}.$$

Here K'' is the second of numbers (11) and (12) implies $\xi\varepsilon=o(1)$. From assumptions (13) and (14) it follows

$$\begin{aligned}\Re e\,L\big(i\alpha-i\varepsilon,\lambda_0+K''\varepsilon+\xi\varepsilon^2\big)&=(BK''-A)\varepsilon+\xi O\big(\varepsilon^2\big)+O\big(\varepsilon^2\big),\\ \Im m\,L\big(i\alpha-i\varepsilon,\lambda_0+K''\varepsilon+\xi\varepsilon^2\big)&=(DK''-C)\varepsilon+\big(\xi D+E-FK''+GK'^{\prime 2}\big)\varepsilon^2\\ &\quad+\xi O\big(\varepsilon^3\big)+O\big(\varepsilon^3\big)\end{aligned}$$

and, since $K''=C/D$,

$$\frac{\Im m\,L(im/n,\lambda)}{(\Re e\,L(im/n,\lambda))^2}=\bigg(\frac{D}{(BK''-A)^2}+O(\xi\varepsilon)+O(\varepsilon)\bigg)\xi+\frac{E-FK''+GK''^2}{(BK''-A)^2}+O(\varepsilon).$$

Combining this with (56), we obtain

$$\xi = \frac{\pi^3(BK'' - A)^2(\Phi_{m,n}(\varphi) + K_0)}{16 f_+^2 D} - \frac{E - FK'' + GK''^2}{D} + O\big(|\varepsilon|^{\beta/(1+\beta)} + n|\varepsilon|^{2\beta/(1+2\beta)}\big)$$

and equivalently,

$$\xi = K + K_1\Phi_{m,n}(\varphi) + O\big(|\varepsilon|^{\beta/(1+\beta)} + n|\varepsilon|^{2\beta/(1+2\beta)}\big)$$

with K_1 and K defined by (16) and (21). Therefore (48) holds and the proof is complete. □

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