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Dissipativity of nonresonant systems with Preisach friction via a canonical example

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Abstract

We prove dissipativity of nonresonant forced non-resonant pendulum with Preisach friction. In a general case we use estimates of the width of hysteresis loop, if the forcing term is quasiperiodic, the equation is always dissipative.

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1. Introduction

The Preisach nonlinearity plays a fundamental role in modelling various phenomena in mechanics, physics, economics etc. In many situations the role of the Preisach model may be explained by the identification theorems [1]. In particular, a canonical ferromagnetic oscillator in a magnetic field is described by the equations

$$x'' + \gamma^2 x = b(t) + \mathfrak{P}x(t). \tag{1}$$

Here $b(t)$ is a forcing, and $\mathfrak{P}$ is a Preisach nonlinearity. The initial value problem for such equation is well defined: for any initial state $\eta_0(\alpha, \beta)$ of the Preisach nonlinearity $\mathfrak{P}x(t)$ and for any initial value $(x_0, x_1) = (x(0), x'(0))$ there exists a unique solution $x(t)$, $t \geqslant t_0$, of Eq. (1), [2].

Eq. (1) is an exiting object for investigation. Extensive study of Eq. (1), see Ref. [3], has demonstrated that, against naive expectations, the Preisach model of the ferromagnetic friction $\mathfrak{P}$ makes the long-term behavior of its solutions extremely rich (contrast to a "dull", beat-type behaviour of the corresponded linear equation $x'' + \gamma^2 x = b(t)$). There were observed multiple stable and unstable periodic solutions of different periods, clusters if invariant tori, and zones of chaotic behaviour.

However, the fundamental question *whether the Preisach operator performs a 'principal role of a friction', suppressing oscillations of large amplitude and making Eq.* (1) *dissipative*, is still open to the best of our knowledge. In this paper, we consider the problem of dissipativity of Eq. (1), that is, the problem whether all its solutions $x(t)$ reach eventually the disk $\{x^2 + (x')^2 \leqslant R_*^2\}$ of a universal radius R_*.

Dissipativity of equations similar to Eq. (1) is due to the friction terms consuming the energy. In Eq. (1) this role is played by the Preisach nonlinearity $\mathfrak{P}$. In this section, we argue that the Preisach-type friction $\mathfrak{P}$ is principally different from many other classical energy consuming terms.

Let the term Fx in the equation

$$x'' + \gamma^2 x = b(t) + Fx(t) \tag{2}$$

describe a kind of friction. The energy consumed within a time interval $[0, T]$ for a particular solution x can be estimated by the quantity

$$E = \int_0^T x'(t) Fx(t)\, dt.$$

It is instructive to illustrate the difference between various types of friction in terms of asymptotic behaviour of the quantity E for a T-periodic function $x(t) = A \sin 2\pi T^{-1} t$ (large solutions of Eq. (2) are close to such functions). For

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the viscous, linear, friction $Fx = kx'$ the consumed energy E is proportional to A^2. For the Coulomb, dry, friction $Fx = k\,\text{sign}(x')$ (or for any friction of the type $Fx = k\,\arctan(x')$) the magnitude of E is proportional to $|A|$. It is well known that a linear or a dry friction is sufficiently strong to guarantee dissipativity of the equation. Let now the friction be hysteretic, in this case E is proportional to the area of the corresponding hysteresis loop. If, in particular, Fx is the plastic friction as described by the classical Prandtl-Besseling-Ishlinskii model, then E is again proportional to the amplitude $|A|$. In all mentioned in this paragraph cases the consumed energy increases rapidly along with the amplitude A, and dissipativity is easy to prove. Moreover, the radius of the corresponded disk is uniformly bounded in a natural sense, and, in contrast to Eq. (1), the long-term behaviour is trivial: each solution converges to a 2π periodic solution of Eq. (1) (note that in the case of plastic friction Eq. (2) may have a family of 2π periodic solutions).

The situation becomes drastically different if the friction is described by the Preisach nonlinearity. In this case the consumed energy is uniformly bounded even when the magnitude $|A|$ increases: it is bounded by the area of the maximal hysteresis loop. Also, in contrast to the Prandtl–-Besseling–Ishlinskii model, the Preisach operators are not maximal monotone. Those principal features of the Preisach nonlinearity makes analysis of dissipativity of pendulums with a Preisach friction interesting and challenging problem. To investigate dissipativity of such systems, we should carefully balance an impact of the energy consuming terms with the impact of inevitable weak resonances destabilizing the system.

2. Main results

Below we suppose that the support of the measure of the Preisach hysteresis is compact, i.e., the output $\mathfrak{P}x$ is equal to its extremal values $\pm H$ for $|x| \geqslant R_0$. Let S be the area of the maximal loop of the Preisach nonlinearity. Let the forcing $b(t)$ be of a general type, it is not supposed to be periodic or almost periodic. Consider the linear nonhomogeneous equation $y'' + y = b(t)$. Let the following nonresonance condition be valid: all solutions of this equation are bounded for $t \geqslant 0$. For instance, this condition is valid if one of the following assumptions holds:

(a1) $b(t)$ is periodic and its period T is π-irrational (the value π/T is irrational);
(a2) $b(t)$ is quasi-periodic and may be represented as a sum of finite number of periodic functions with π-irrational periods;
(a3) $b(t)$ is almost periodic and its Fourier exponents do not approach 1.

Note in passing that assumption (a3) is less restrictive than (a2).

The number

$$r = \inf_{y \in Y} \sup_{t \geqslant 0} |y'(t)| \tag{3}$$

is well-defined.

Theorem 1. *If $S > 2\pi H r$, then equation* (1) *is dissipative.*

The proof will be published in another paper. In Theorem 1 we use minimal assumptions about the function b: the nonresonance condition only. If b is periodic or almost periodic, then the restrictive condition $S > 2\pi H r$, may be sometimes omitted.

Theorem 2. *Let $b(t) = b_1(\gamma_1 t) + \cdots + b_N(\gamma_N t)$, where b_k are 2π-periodic and γ_k are irrational. Then Eq.* (1) *is dissipative.*

We prove Theorem 2 in the next section for the case $N = 1,\ \gamma = \gamma_1$. It would be interesting to obtain an analogue of Theorem 2 for rational γ's and for almost periodic functions of more general type (e.g., for generic quasiperiodic functions). The resonant equation $x'' + x = A \sin t + \mathfrak{P}x$ is not dissipative for large A (even for an arbitrary wide hysteresis loop). This may be proved using linear guiding functions in the same way as in Ref. [4].

3. Proof of Theorem 2

Scheme of the proof: Denote by $y_0(t)$ a unique periodic solution of the linearized equation $y'' + y = b(t)$. Its period equals $2\pi/\gamma$, other solutions of this equation are almost periodic but not periodic.

Let us substitute variable $x(t) := x(t) + y_0(t)$ in Eq. (1) and consider the equation

$$x'' + x = \mathfrak{P}(x + y_0(t)). \tag{4}$$

Since y_0 is bounded, the dissipativity of this equation is equivalent to the dissipativity of initial equation (1). Below we study Eq. (4) only.

Consider initial values $x(t_0) = R \cos \theta,\ x'(t_0) = R \sin \theta$, $\mathfrak{P}x(t_0) = \eta_0$, where η_0 is some admissible state of our Preisach nonlinearity.

As the first step we show that for large T (it depends on R and does not depend of θ and η_0), on any interval $[t_0, t_0 + T]$ the solution x enter some fixed disc $\mathfrak{R}$, its radius does not depend neither of R, nor of θ. Later, as the second step we show that if a solution leaves this disc, then it comes back not later than some fixed time. This will prove the theorem.

Auxiliary lemmas: Let us fix $T > 0$, multiply (4) by $x'(t)$, and then integrate on the interval $[t_0, t_0 + T]$. We obtain an 'energy' equation

$$[x'(t_0 + T)]^2 + [x(t_0 + T)]^2 - R^2 = 2J_T, \tag{5}$$

where

$$J_T = \int_{t_0}^{t_0+T} x'(t)\,\mathfrak{P}(x + y_0(t))\,\mathrm{d}t.$$

The value of T is chosen as follows. Firstly, we construct a number n (denominator of a convergent for γ) in terms of the number S. Secondly, we construct the interval $T_0 = 2n\pi$. Thirdly, we choose $T = T_0$, where K is growing to infinity. Below, will be proved that under conditions of Theorem 2 the value J_T decreases to $-\infty$ as $T \to \infty$, providing that the solution $x(t), t \in [t_0, t_0 + T]$, lies outside of some disc D_*. Therefore Eq. (5) cannot hold for large T. This contradiction proves that our solution not only reaches D_* but also it afterwards visits this disc with uniformly bounded intervals between successive visits.

To estimate J_T we rewrite it as the sum:

$$J_T = \int_{t_0}^{t_0+T} (x'(t) + y_0'(t))\, \mathfrak{P}(x + y_0(t))\,\mathrm{d}t - \int_{t_0}^{t_0+T} y_0'(t)\, \mathfrak{P}(x + y_0(t))\,\mathrm{d}t. \tag{6}$$

The estimate of the first term in Eq. (6) follows from the next lemma.

Lemma 1. *For every $\varepsilon > 0$ there exists an R_ε such that for some large $c_0 = c_0$ the inequality*

$$\int_{t_0}^{t_0+T} (x + y_0(t))'\, \mathfrak{P}(x + y_0(t))\,\mathrm{d}t < -T\,\frac{S - \varepsilon}{2\pi} + c_1,$$

where

$$c_1 = c_0(|x(t_0)| + |x'(t_0)| + |x(t_0 + T)| + |x'(t_0 + T)|),$$

holds for all $t_0, T > 0$, if the trajectory (x, x') lies outside of the disc of the radius R_ε for $t \in [t_0, t_0 + T]$.

The second term in Eq. (6) averages out and is rather small for some special values of T. Let m/n be a convergent of the irrational number γ. For each denominator n the estimate

$$|\varepsilon_n| \leqslant n^{-2}, \quad \varepsilon_n = \gamma - \frac{m}{n} \tag{7}$$

is valid [5]. Consider the interval $[t_1, t_1 + 2n\pi]$ of the length $2n\pi$, here t_1 is an arbitrary initial time.

Lemma 2. *Let us choose an $\varepsilon > 0$. There exists an $R_{\varepsilon,n}$ such that for any t_1 there exist $\psi = \psi(t_1)$ satisfying*

$$\frac{1}{2n\pi}\left|\int_{t_1}^{t_1+2n\pi} y_0'(t)[\,\mathfrak{P}(x + y_0(t)) - H\,\mathrm{sign}(\sin(t + \psi))]\,\mathrm{d}t\right| \leqslant \varepsilon,$$

if (x, x') lies outside of the disc of the radius $R_{\varepsilon,n}$ for $t \in [t_1, t_1 + 2n\pi]$.

The statement of this lemma follows from the asymptotic homogeneity of Preisach operators [6,7].

Lemma 3. *For a positive ε there exists a positive integer n such that for any t_1 and ψ*

$$\frac{1}{2n\pi}\left|\int_{t_1}^{t_1+2n\pi} y_0'(t)\,\mathrm{sign}(\sin(t + \psi))\,\mathrm{d}t\right| \leqslant \varepsilon.$$

Rewrite the expression for J_T as

$$J_T = \int_{t_0}^{t_0+T} (x'(t) + y_0'(t))\, \mathfrak{P}(x + y_0(t))\,\mathrm{d}t - \int_{t_0}^{t_0+T} y_0'(t)\, \mathfrak{P}(x + y(t))\,\mathrm{d}t.$$

From the lemmas above it follows that $J_T \leqslant -T(S - 3\varepsilon)/2\pi + c_1$; if $T \to \infty$. Thus $J_T \to -\infty$ as $T \to \infty$, and the theorem is proved.

Proof of Lemma 1. Let us formulate a simple property of the Preisach nonlinearity.

Statement 1. *Let $x(t)$, $t \in [t_0, t_1]$, be a continuous input of $\mathfrak{P}$, and either $x(t_0) = x(t_1) > R_0$, or $x(t_0) = x(t_1) < -R_0$. Suppose that the preimage of $[-R_0, R_0]$ with respect to $x(t)$ consists of two non-degenerated intervals and $x(t)$ is monotone at each interval. Let $x(t)$ move around the hysteresis loop*[1] *once. Then*

$$\int_{t_0}^{t_1} x'(t)\, \mathfrak{P}x(t)\,\mathrm{d}t = -S. \tag{8}$$

Each large solution $x(t)$ satisfies (within one rotation around the coordinate origin) the assumptions of Statement 1. Now we estimate the minimal number N of rotations for the time T, and we will prove that $N \geqslant [T/(2\pi)] - 2$. To this end we estimate time intervals, which our solution spends inside and outside of the strip $-R_0 \leqslant x \leqslant R_0$. Each crossing of this strip (from left to right or vice versa) requires time which decreases to zero, as norm of the solution increases to infinity. On the other hand, outside of the strip $\pm H$, the motion is governed by the linear non-homogeneous equation $x'' + x = \pm H$. Each solution of this equation is periodic and the period tends to 2π as magnitude increases to ∞. Therefore, each solution $x(t)$ satisfying a sufficiently large initial condition performs a single rotation for time close to 2π. Correspondingly, for the time T it performs about $[T/(2\pi)]$ rotations. Taking into account that the first and the last rotations could be incomplete, we conclude that $N \geqslant [T/(2\pi)] - 2$. The constant c_0 above is due just to the (possibly incomplete) the first and the last rotations. The lemma is proved.

Proof of Lemma 3. Since y_0' is Lipschitz continuous, (7) implies that

$$\left|\int_{t_1}^{t_1+2n\pi} y_0'\left(t\,\frac{m}{n\gamma}\right)\mathrm{sign}(\sin(t + \psi))\,\mathrm{d}t - \int_{t_1}^{t_1+2n\pi} y_0'(t)\,\mathrm{sign}(\sin(t + \psi))\,\mathrm{d}t\right|$$

is uniformly bounded for large n and m. Therefore, to prove the lemma it is sufficient to estimate the value

[1] In the plane $(x, \mathfrak{P}x)$ it is located just above the interval $[-R_0, R_0]$.

$$J_0 = \left| \int_{t_1}^{t_1+2n\pi} y_0'\left(t\frac{m}{n\gamma}\right) \operatorname{sign}(\sin(t+\psi))\,\mathrm{d}t \right| = n\left| \int_0^{2\pi} y_0'(mt/\gamma) \operatorname{sign}(\sin(nt+\psi))\,\mathrm{d}t \right|.$$

To do this, we represent the 2π-periodic functions

$$b(t),\ y_0'(mt/\gamma) \quad \text{and} \quad \operatorname{sign}(\sin(nt+\psi))$$

in the form of Fourier series

$$b(t) = \sum_{k=1}^{\infty} a_k \sin(kt+\varphi_k), \quad y_0'(mt/\gamma) = m\sum_{k=1}^{\infty} \frac{ka_k}{1-\gamma^2k^2} \cos(mkt+\varphi_k),$$

$$\operatorname{sign}(\sin(nt+\psi)) = \frac{4}{\pi} \sum_{j=1,3,\ldots} \frac{\sin(j(nt+\psi))}{j}.$$

After integration the residual part is the product of the harmonics with $k=sn$ and $j=sm$ for odd s and m (because the numbers m and n are coprime). Therefore,

$$J_0 \leqslant 4nm \sum_{s=1,3,\ldots} \left| \frac{ns\,a_{ns}}{ms(1-\gamma^2(ms)^2)} \right| \leqslant 4c_0m^2 \sum_{s=1,3,\ldots} \left| c(ms)^{-2-\delta} \right| \leqslant c_1m^{-\delta}$$

(since $|a_k| \leqslant ck^{-\delta}$), and Lemma 3 is proven. □

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