Bifurcation of forced periodic oscillations for equations with Preisach hysteresis

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Abstract. We study oscillations in resonant systems under periodic forcing. The systems depend on a scalar parameter and have the form of simple pendulum type equations with ferromagnetic friction represented by the Preisach hysteresis nonlinearity. If for some parameter value the period of free oscillations of the principal linear part of the system coincides with the period of the forcing term, then one may expect the existence of unbounded branches of periodic solutions for nearby parameter values. We present conditions for the existence and nonexistence of such branches and estimates of their number.

1. Introduction

Consider the equation

$$x'' + \lambda x = b(t) + F(x). \tag{1}$$

Here x is an unknown scalar function, b is a 2π-periodic forcing, λ is a scalar parameter, and F is a bounded nonlinearity of hysteresis or functional type. We are interested in the existence of unbounded branches of 2π-periodic solutions in the case that the parameter λ varies in a vicinity Λ of the point $\lambda_0 = 1$. More exactly, we study the questions whether the set of all 2π-periodic solutions of equation (1) for all $\lambda \in \Lambda$ is bounded (say, in C), how to estimate the number of unbounded branches of 2π-periodic solutions if those branches exist and what is their structure.

Even in simple cases, the answers to the above questions may be not straightforward for several reasons. One of the difficulties is of a topological nature. The principal part of equation (1) at infinity is the linear operator $\mathcal{L}x = x'' + \lambda x$ that under the 2π-periodic boundary conditions has the two-dimensional kernel spanned by the functions $\sin t$ and $\cos t$ for $\lambda = \lambda_0$. In problems with odd degeneracy of the linear part (e.g., in case of simple degeneracy), the existence of unbounded branches of solutions follows from the classical theorems of Mark Krasnosel'skii [1] on asymptotic bifurcation points and thus is based on linear analysis where all bounded terms are neglected. The analysis of problems with even degeneracy of the linear part (particularly, with degeneracy 2 as in our case) is more complicated and requires additional constructions and assumptions for each concrete problem.

Another difficulty is a consequence of the non-potential character of hysteresis nonlinearities. In potential problems the parity of the degeneracy may be not essential (see [1] for bifurcations at zero and [2] for bifurcations at infinity). For example, if F is a functional nonlinearity of the form $(F(x))(t) = f(x(t))$, then the 2π-periodic problem for equation (1) is potential and consequently

the set of all its solutions is unbounded, which means that $\lambda = 1$ is an asymptotic bifurcation point [1]. Moreover, one can estimate the number of unbounded branches of periodic solutions for important classes of nonlinearities F defined from generic natural properties. In particular, equation (1) with a functional nonlinearity of saturation type has exactly 2 unbounded branches of 2π-periodic solutions in the general case and at least 6 unbounded branches of such solutions in a special case of codimension one [3]. The latter special case is determined by the condition

$$\int_0^{2\pi} b(t)e^{it}\, dt = 0. \tag{2}$$

This condition is also important for answering the questions raised above for equations with a Preisach hysteresis nonlinearity F. From the saturation property of Preisach hysteresis it follows that if (2) is not valid, then equation (1) with the Preisach nonlinearity has exactly two unbounded branches of 2π-periodic solutions as in the case of a functional nonlinearity of saturation type.

In this paper, we consider equation (1) with the Preisach nonlinearity under the assumption that (2) holds and show that in this case the number of unbounded branches of periodic solutions, which is generically even, ranges from zero to six or more, depending on the 'width' of the maximal hysteresis loop. More precisely, the existence (absence) and the number of unbounded branches depends on the relation between the hysteresis loop width and the amplitude of some function related to the forcing b. If the hysteresis loop is wide enough, then all 2π-periodic solutions for all parameter values belong to some common ball in C, which implies the absence of unbounded branches (no resonance case). If the loop is sufficiently narrow, then there exist unbounded branches of periodic solutions and we give estimates for their number from below.

We present our approach mainly to the simplest substantial situation, restricting ourselves to short remarks on further applications of the approach to more general classes of equations.

2. Preisach nonlinearity

Denote by

$$\eta(t) = \mathcal{R}_{\alpha,\beta}[t_0, \eta_0]x(t), \qquad t \geq t_0, \tag{3}$$

the variable state of the *non-ideal relay* with threshold values α, β ($\alpha < \beta$), an input $x(t)$ ($t \geq t_0$), and an initial state η_0. Here the input is an arbitrary continuous scalar function; η_0 takes the values 1 and -1; the scalar function $\eta(t)$ satisfies $\eta(t_0) = \eta_0$ and $|\eta(t)| = 1$ for any t and has at most a finite number of jumps on any finite interval $t_0 \leq t \leq t_1$. The values of the operator (3) are defined in a standard way for $\eta_0 = -1$ if $x(t_0) \leq \alpha$, for $\eta_0 = 1$ if $x(t_0) \geq \beta$, and both for $\eta_0 = -1$ and $\eta_0 = 1$ if $\beta > x(t_0) > \alpha$. The equality $\eta(t) = 1$ holds whenever $x(t) \geq \beta$ and $\eta(t) = -1$ holds whenever $x(t) \leq \alpha$ for all $t \geq t_0$.

For various applications it is convenient to define operator (3) for any initial state (e.g., for $\eta_0 = 1$ if $x(t_0) \leq \alpha$) by the equality $\mathcal{R}_{\alpha,\beta}[t_0, \eta_0]x(t) = \mathcal{R}_{\alpha,\beta}[t_0, \eta_1]x(t)$, where

$$\eta_1 = \begin{cases} -1 & \text{if } x(t_0) \leq \alpha, \\ 1 & \text{if } x(t_0) \geq \beta, \\ \eta_0 & \text{if } \alpha < x(t_0) < \beta, \end{cases} \tag{4}$$

which extends the domain of (3) to the set of all pairs (x, η_0) with any continuous $x = x(t)$ ($t \geq t_0$) and any of the two values $\eta_0 = \pm 1$.

Consider a compact set Ω on the half-plane $\{(\alpha, \beta) : \beta > \alpha\}$ and suppose that Ω is endowed with a finite continuous measure μ. We call any measurable function $\eta_0 = \eta_0(\alpha, \beta) : \Omega \to \{-1, 1\}$ an initial state of the *Preisach model* (equivalently, the Preisach nonlinearity). For any initial

state $\eta_0 = \eta_0(\alpha, \beta)$ and any continuous input $x(t)$ $(t \geq t_0)$ the formula

$$\xi(t) = \mathcal{P}[t_0, \eta_0]x(t) := \int_\Omega \mathcal{R}_{\alpha,\beta}[t_0, \eta_0(\alpha, \beta)]x(t)\, d\mu, \qquad t \geq t_0, \tag{5}$$

defines a continuous scalar output of the Preisach model. Further details can be found, e.g. in [4]. Here we formulate some particular properties of the Preisach nonlinearities that are important for our considerations. Along with (5), we use a shorter notation $\xi = \mathcal{P}x$ for the outputs, where the arguments t_0 and η_0 are omitted.

Denote by $\mathcal{F}(r, K)$ the set of all 2π-periodic continuous inputs $x = x(t)$ of the form $x(t) = r\sin(t + \varphi) + h(t)$, where $r > 0$ is sufficiently large, $\varphi \in \mathbb{R}$, and $\|h\|_C \leq K$. In the following statement C_0 is the space of all continuous 2π-periodic functions with the norm $\|x\|_C = \max|x(t)|$.

Proposition 1 *Fix any $K > 0$. If $r > 0$ is sufficiently large, then for any input $x \in \mathcal{F}(r, K) \subset C_0$, there exists a unique initial state $\eta_0 = \eta_0(x)$ of the Preisach model such that the output (5) is 2π-periodic. Thus, the relation $(\mathcal{P}_{per}x)(t) := \mathcal{P}[t_0, \eta_0(x)]x(t)$ defines a single-valued operator $\mathcal{P}_{per}$ in C_0 with the domain $\mathcal{F}(r, K)$. This operator is continuous.*

The next property is called *asymptotic homogeneity* [5] of the Preisach nonlinearity. We formulate it for the operator $\mathcal{P}_{per}$, which sends periodic inputs to periodic outputs.

Proposition 2 *For any $K > 0$ and any continuous bounded function g*

$$\lim_{r \to \infty} \sup_{x \in \mathcal{F}(r,K)} \left| \int_0^{2\pi} g(t)\Big((\mathcal{P}_{per}x)(t) - H\,\mathrm{sign}\,(\sin(t + \varphi))\Big)\, dt \right| = 0, \quad H = \int_\Omega d\mu. \tag{6}$$

We also use the area S of the maximal hysteresis loop of the Preisach nonlinearity. Let C_0^1 be the space of all 2π-periodic continuously differentiable functions with the C^1-norm and $\mathcal{F}'(r, K)$ be a subset of C_0^1 consisting of the inputs $x(t) = r\sin(t + \varphi) + h(t)$ with $\|h\|_{C^1} \leq K$, $\varphi \in \mathbb{R}$.

Proposition 3 *Fix any $K > 0$. If $r > 0$ is sufficiently large, then for any input $x \in \mathcal{F}'(r, K)$*

$$\int_0^{2\pi} x'(t)\,(\mathcal{P}_{per}x)(t)\, dt = -S. \tag{7}$$

The statements of Propositions 1 and 3 follow from the general properties of the Preisach hysteresis [4]. The minus sign in the right-hand part of (7) means the energy dissipation ($S > 0$) in consistence with the counterclockwise orientation of the hysteresis loop.

3. Main results

Consider the equation
$$x'' + \lambda x = b(t) + \mathcal{P}x \tag{8}$$
with a continuous 2π-periodic forcing b and $\lambda \in \Lambda = (1 - \varepsilon_0, 1 + \varepsilon_0)$, where $0 < \varepsilon_0 < 1$.

Let $Z_\rho = \{(x, \lambda) : \|x\|_C < \rho,\ \lambda \in \Lambda\}$ be a cylinder in the space $C \times \Lambda$. Following [1], let us call a set $\mathfrak{N} \subset C \times \Lambda$ an *unbounded continuous branch* (or simply *unbounded branch*) of 2π-periodic solutions of (8) if $(x_*, \lambda_*) \in \mathfrak{N}$ implies that $x = x_*$ is a 2π-periodic solution of (8) for $\lambda = \lambda_*$ and for any sufficiently large $\rho > 0$ on the boundary of any bounded open set $\Pi \subset C \times \Lambda$, $\Pi \supset Z_\rho$ there is at least one point $(x_*, \lambda_*) \in \mathfrak{N} \setminus Z_\rho$. By definition, if $\mathfrak{N}$ is an unbounded branch, then

$$\lim_{\rho \to \infty} \sup_{(x,\lambda) \in \mathfrak{N} \setminus Z_\rho} |\lambda - 1| = 0,$$

and for every sufficiently large $\rho > 0$ there is at least one point $(x, \lambda) \in \mathfrak{N}$ with $\|x\|_C = \rho$. We remark that an unbounded branch is not necessarily a continuous curve in the space $C \times \Lambda$, it may have a more complicated structure. In its definition the space C can be replaced by other spaces, e.g. C_0 or L^p.

Consider the Fourier series

$$b(t) = B_0 + \sum_{k=2}^{\infty} B_k \sin(kt + \varphi_k), \qquad B_k \geq 0, \qquad (9)$$

of the forcing b (we suppose that $B_1 = 0$, which is equivalent to condition (2)). Define

$$\chi(\varphi) = 4 \sum_{k=3,5,7,\ldots} \frac{B_k}{1 - k^2} \sin(k\varphi - \varphi_k) \qquad (10)$$

and set

$$\Delta = \max \chi(\varphi).$$

The function χ is continuous and antiperiodic, i.e. $\chi(\varphi + \pi) \equiv -\chi(\varphi)$, therefore $\Delta > 0$ (if $B_k = 0$ for all odd k, then b is π-periodic). This function may be defined by the equivalent formula

$$\chi(\varphi) = \int_0^{2\pi} u(t) \operatorname{sign} \sin(t + \varphi) \, dt,$$

where u is a unique 2π-periodic solution of the linear equation $u'' + u = b'(t)$, having no first harmonics in its Fourier series.

Set $\psi(\varphi) = S + H\chi(\varphi)$. We say that the number of sign changes of the 2π-periodic function ψ is N (or that ψ changes its sign N times) if there are points $\phi_0 < \phi_1 < \ldots < \phi_N = \phi_0 + 2\pi$ such that $\psi(\phi_{k-1})\psi(\phi_k) < 0$ for all $k = 1, \ldots, N$ and there are no $N' > N$ points with the same properties; relation $\psi(\varphi + 2\pi) \equiv \psi(\varphi)$ implies that the number N is even. Similarly, we say that the number of sign changes of ψ is infinite (on the period) if there is a monotone sequence ϕ_k such that $\psi(\phi_{k-1})\psi(\phi_k) < 0$ and $0 \leq \phi_k \leq 2\pi$ for all $k \in \mathbb{N}$.

Theorem 1 *If $S < H\Delta$, then the number of unbounded branches of 2π-periodic solutions of equation (8) is not less than the number of sign changes of the function $\psi(\varphi) = S + H\chi(\varphi)$.*

The condition $S < H\Delta$ implies that ψ changes its sign at least twice and consequently equation (1) has at least 2 unbounded branches of 2π-periodic solutions. If the value S/H (the width of the hysteresis loop) is sufficiently small for a given b and χ, then there exist at least 6 unbounded branches as in the case of functional nonlinearities with saturation. Indeed, for small S/H the number N of sign changes of the function $\psi = S + H\chi$ satisfies $N \geq 6$, because χ changes its sign at least 6 times. The latter fact follows from the Sturm–Hurwitz theorem (see, e.g. comments by S. Kuksin to Problem 1996-5 in [6]), which states that the number of sign changes of any real continuous periodic function on its period is not less than $2k$, where k is the degree of the first nonzero harmonic in the Fourier series of the function. By definition, $k \geq 3$ for χ.

If the function ψ changes its sign infinitely many times on the period, then according to Theorem 1 equation (1) has infinitely many unbounded branches of 2π-periodic solutions.

We say that an isolated zero of a scalar function is *proper* if the function takes both positive and negative values in any vicinity of this zero. If ψ has a finite number of zeros on the interval $0 \leq \varphi < 2\pi$, then ψ changes its sign N times, where N is the number of proper zeros of ψ on this interval. We say that the unbounded branch $\mathfrak{N}$ is *directed* if

$$\lim_{\rho \to \infty} \sup_{(x,\lambda) \in \mathfrak{N} \setminus Z_\rho} \|x/\|x\|_C - e\|_C = 0 \qquad (11)$$

for some $e \in C_0$; the function $e = e(t)$ is then called an *asymptotic limit* for $\mathfrak{N}$.

Theorem 2 *If $S < H\Delta$, then the number of directed unbounded branches of 2π-periodic solutions of equation* (8) *is not less than the number N of proper zeros of the function $\psi(\varphi) = S + H\chi(\varphi)$ on the interval $0 \le \varphi < 2\pi$. For each proper zero φ_* there exists a directed unbounded branch with the asymptotic limit $e_{\varphi_*}(t) = \sin(t + \varphi_*)$.*

From the proofs presented below, it follows that under the conditions of Theorems 1 and 2 the estimate $\lambda > 1$ holds for all 2π-periodic solutions x with a sufficiently large norm $\|x\|_C$. Moreover, $(\lambda - 1)\|x\|_C \to 4H/\pi$ as $\|x\|_C \to \infty$.

Theorem 3 *If $S > H\Delta$, then all the 2π-periodic solutions of equation* (8) *for all $\lambda \in \Lambda$ satisfy an* a priori *estimate $\|x\|_C \le c$, where c is independent of λ.*

If $S > H\Delta$, then $\min \psi(\varphi) > 0$. Theorems 1–3 say nothing about the critical case $S = H\Delta$.

Consider an example. Let $b(t) = B_0 + B_2 \sin(2t + \varphi_2) + B_3 \sin(3t + \varphi_3)$; then $\chi(\varphi) = (B_3/2)\sin(\varphi_3 - 3\varphi)$ and $\Delta = B_3/2$. If $2S > HB_3$, then the set of all 2π-periodic solutions of equation (8) for all λ sufficiently close to $\lambda_0 = 1$ is bounded in C. If $2S < HB_3$, then equation (8) has 6 directed unbounded branches of 2π-periodic solutions with the asymptotic limits $e_k(t) = \sin(t + \varphi_*^k)$, where $\varphi_*^k = ((-1)^k \arcsin(2S/(HB_3)) + \pi k + \varphi_3)/3$ and $0 \le k \le 5$.

4. Remarks

4.1. Bounded branches of periodic solutions

Equation (1) may also have bounded branches of 2π-periodic solutions for values of λ close to $\lambda_0 = 1$. In particular, such a bounded branch exists whenever equation (1) has a 2π-periodic solution of non-zero topological index for $\lambda = 1$.

From the asymptotic homogeneity of the Preisach nonlinearity it follows that the equation $x'' + x = b(t) + \mathcal{P}x$ has 2π-periodic solutions if the Lazer condition

$$\left| \int_0^{2\pi} b(t)e^{it}\, dt \right| < 4H$$

is satisfied [5, 7]. In our case, this estimate follows from (2) and consequently equation (1) has at least one 2π-periodic solution for $\lambda = 1$.

4.2. Generalizations of the equation

Theorems of the previous section may be generalized in various ways.

1. The right-hand part (both the Preisach nonlinearity and the forcing) of (1) may depend on the parameter λ. Similar theorems are valid if λ is close to m^2 with any integer $m > 1$.

For example, let us consider equation (1) with b depending on λ. For λ close to $\lambda_0 = 1$, let

$$b(t, \lambda) = B_0(\lambda) + \sum_{k=1}^{\infty} B_k(\lambda)\sin(kt + \varphi_k(\lambda)).$$

Assume that $B_1(\lambda_0) = 0$ and $B_1(\lambda) = (\lambda - \lambda_0)B_1' + o(\lambda - \lambda_0)$. Set

$$\chi^*(\varphi) = 4B_1'\sin(\varphi - \varphi_1(\lambda_0)) + 4\sum_{k=3,5,7,\ldots} \frac{B_k(\lambda_0)}{1 - k^2}\sin(k\varphi - \varphi_k(\lambda_0)), \quad \text{and} \quad \psi^*(\varphi) = S + H\chi^*(\varphi).$$

These functions play the same role as the functions χ and ψ in Theorems 1–3. One obtains the counterparts of those theorems for the case of forcings b depending on λ by replacing the functions χ, ψ, and the norm $\Delta = \max \chi$ of χ with functions χ^*, ψ^*, and the norm $\Delta^* = \max \chi^*$ of χ^* in all the formulations.

2. With straightforward minor modification, the proofs presented below may be applied to higher order ODEs

$$L\Big(\frac{d}{dt},\lambda\Big)x = b(t) + \mathcal{P}x$$

and to more general control theory equations

$$L\Big(\frac{d}{dt},\lambda\Big)x = M\Big(\frac{d}{dt},\lambda\Big)(b(t) + \mathcal{P}x), \tag{12}$$

where the real polynomials $L(p,\lambda)$ and $M(p,\lambda)$ in the variable p contain only even degrees of p for all λ, i.e. $L(p,\lambda) \equiv L(-p,\lambda)$, $M(p,\lambda) \equiv M(-p,\lambda)$. We assume that the polynomials L and M are coprime for every λ and their coefficients depend continuously on λ, while their degrees ℓ and m are independent of λ and satisfy $\ell > m$.

Let the polynomial $L(p,\lambda)$ have a pair of simple roots $\pm\mu(\lambda)i$ for every $\lambda \in \Lambda = (\lambda_-, \lambda_+)$, where the continuous function $\mu = \mu(\lambda)$ is strictly monotone and $\mu(\lambda_0) = 1$ for some $\lambda_0 \in \Lambda$. Let $L(ki,\lambda) \neq 0$ for every integer $k \neq \pm 1$ and every $\lambda \in \Lambda$. Let $L(i,\lambda) \neq 0$ for $\lambda \neq \lambda_0$, $\lambda \in \Lambda$. Using the Fourier series (9) of b, define

$$\chi_{L,M}(\varphi) = 4 \sum_{k=3,5,7,\dots} \frac{B_k\,M(ki,\lambda_0)}{L(ki,\lambda_0)}\sin(k\varphi - \varphi_k), \qquad \Delta_{L,M} = \max \chi_{L,M}(\varphi).$$

The function $\chi_{L,M}$ coincides with (10) for $L(p,\lambda) = p^2 + \lambda^2$, $M \equiv 1$, and $\lambda_0 = 1$. The following statement extends Theorems 1 and 3 to equations (12). We present it without proof.

Theorem 4 *If $S < H\Delta_{L,M}$, then the number of unbounded branches of 2π-periodic solutions of equation* (12) *is at least two and is not less than the number of sign changes of the function $S + H\chi_{L,M}(\varphi)$. If $S > H\Delta_{L,M}$, then all the 2π-periodic solutions of* (12) *for all $\lambda \in \Lambda$ satisfy an* a priori *estimate $\|x\|_C \leq c$ with c independent of λ.*

4.3. Principle of the change of index
Topological principles based on the change of topological index [1] are not applicable to our situation, because under the assumptions of Theorems 1–3 the topological index at infinity of the periodic problem considered is the same for all $\lambda \in \Lambda$. For $\lambda \neq 1$ the index at infinity is defined by the linear approximation, consequently the fact that for $\lambda = 1$ this approximation is degenerate with multiplicity 2 implies that the index is the same for $\lambda < 1$ and $\lambda > 1$. Computation of the index for $\lambda = 1$ may be found in [5], where asymptotic homogeneity of the Preisach model plays the main role for the application of standard constructions (see, e.g. [7, 8, 9]).

4.4. Preisach nonlinearities with unbounded domains Ω
Our results can be extended to equations with Preisach nonlinearities with unbounded domains Ω endowed with a finite measure μ. For such models Proposition 1 must be revised, because $\mathcal{P}_{per}$ becomes a set-valued operator whose values may be parameterized in a natural way by a scalar parameter. In this case, unbounded branches of 2π-periodic solutions look generically like bands with their width tending to zero at infinity (this type of branch in problems on bifurcations at infinity was studied in [10] for equations with the Prandtl–Ishlinskii hysteresis nonlinearities), while in the case of a bounded domain Ω considered in this paper the unbounded branches are generically curves in the space $C \times \Lambda$.

Although we assume above that the measure μ is positive (as in the classical Preisach model), this is not important. With a natural minor change of formulation, Theorems 1–3 are valid if a signed measure $\mu = \mu_1 - \mu_2$ with bounded total variation is used in place of the positive measure μ. The assumption that μ is continuous implies continuity of the Preisach nonlinearity. It excludes models formed by a finite number of relays.

5. Proofs

5.1. Linear operators

First, we introduce some necessary linear functional spaces and linear operators. Define

$$(Px)(t) = \frac{1}{\pi} \int_0^{2\pi} \cos(t - s)\, x(s)\, ds, \qquad Qx = x - Px.$$

These projectors are orthogonal in L^2 (all the functional spaces consist of functions defined on the segment $0 \le t \le 2\pi$; each 2π-periodic function is identified with its restriction to this segment). Set $E_0 = PL^2$, $E^0 = QL^2$. Let Γ_λ be the linear operator that maps a function $y \in E^0$ to a unique 2π-periodic solution $h \in E^0$ of the linear equation $h'' + \lambda h = y(t)$. The operators Γ_λ are well-defined for all $\lambda \in \Lambda$, each Γ_λ is a self-adjoint completely continuous operator in L^2. Moreover, Γ_λ is also completely continuous as an operator from L^2 to C_0^1 and continuous as an operator from L^2 to $W^{2,2}$. The operators $\Gamma_\lambda : L^2 \to C_0^1$ depend continuously on λ and their norms are uniformly bounded for all $\lambda \in \Lambda$.

We shall consider some other operators generated by Γ_λ, particularly the operators $\Gamma_\lambda Q : L^2 \to E^0$ defined on the whole space L^2 and the operators $\Gamma_\lambda' : E^0 \to E^0$ defined by

$$(\Gamma_\lambda' y)(t) = \frac{d}{dt}(\Gamma_\lambda y)(t).$$

The latter operators satisfy the identity

$$\int_0^{2\pi} u(t)\, (\Gamma_\lambda' u)(t)\, dt = 0, \qquad u \in E^0. \tag{13}$$

5.2. Equivalent system

We look for 2π-periodic solutions of equation (8) in the form

$$x(t) = r \sin(t + \theta) + h(t), \qquad h \in E^0 \cap C_0,\ r \ge 0,\ \theta \in \mathbb{R}. \tag{14}$$

Substituting this formula in (8) and considering the orthogonal projections of the resulting equation onto $\sin t$, $\cos t$, and the subspace E^0 in L^2, one obtains the system

$$\pi(\lambda - 1)r = \int_0^{2\pi} \sin(t + \theta)\, (\mathcal{P}_{per} x)(t)\, dt, \quad 0 = \int_0^{2\pi} \cos(t + \theta)\, (\mathcal{P}_{per} x)(t)\, dt, \quad h = \Gamma_\lambda(b + Q\mathcal{P}_{per} x). \tag{15}$$

The last equation of this system implies an *a priori* estimate $\|h\|_{C^1} < K_0$ for h with

$$K_0 = \sup_{\lambda \in \Lambda} \|\Gamma_\lambda Q\|_{C \to C^1}\, (\|b\|_C + H) + 1 < \infty.$$

Consequently $h \in \mathcal{F}'(r, K_0) \subset \mathcal{F}(r, K_0)$ for all r, and $\|x\|_C \to \infty$ if and only if $r \to \infty$.

Let us multiply the second equation of (15) by $r > 0$ and rewrite it as

$$\int_0^{2\pi} h'(t)\, (\mathcal{P}_{per} x)(t)\, dt = \int_0^{2\pi} x'(t)\, (\mathcal{P}_{per} x)(t)\, dt.$$

According to Proposition 3, if r is sufficiently large, then this equation is equivalent to

$$S + \int_0^{2\pi} h'(t)\, (\mathcal{P}_{per} x)(t)\, dt = 0. \tag{16}$$

Furthermore, from the third equation of system (15) it follows that $h' = \Gamma'_\lambda(b + Q\mathcal{P}_{per}x)$. Therefore identity (13) implies that one can rewrite (16) as

$$S + \int_0^{2\pi}(\Gamma'_\lambda b \cdot \mathcal{P}_{per}x)(t)\,dt = 0.$$

Replacing the second equation of (15) with this equation, one arrives at the system

$$\lambda - 1 = \frac{1}{\pi r}\int_0^{2\pi}\sin(t+\theta)\,(\mathcal{P}_{per}x)(t)\,dt, \quad S + \int_0^{2\pi}(\Gamma'_\lambda b \cdot \mathcal{P}_{per}x)(t)\,dt = 0, \quad h = \Gamma_\lambda(b + Q\mathcal{P}_{per}x).$$
(17)

We conclude that system (17) is equivalent to the problem of the existence of large-amplitude 2π-periodic solutions of equation (8). More precisely, x is a 2π-periodic solution of (8) with a sufficiently large norm $\|x\|_C$ if and only if r is sufficiently large in representation (14) of x and r, λ, θ, h satisfy (17).

5.3. Additional equation

In order to prove the existence of unbounded continuous branches of solutions, we complete the system of 3 equations (17) containing 4 arguments λ, θ, h, r with an additional scalar equation.

Let Π be an arbitrary open bounded set in the space $C \times \Lambda$ of pairs (x, λ). Define the nonlinear continuous functional $\Phi_\Pi : C \times \Lambda \to \mathbb{R}$ by the formula

$$\Phi_\Pi(x, \lambda) = \begin{cases} -\inf_{(y,\eta)\in\partial\Pi}(\|x-y\|_C + |\lambda - \eta|) & \text{if } (x,\lambda) \in \overline{\Pi}, \\[2mm] \inf_{(y,\eta)\in\partial\Pi}(\|x-y\|_C + |\lambda - \eta|) & \text{if } (x,\lambda) \notin \overline{\Pi}, \end{cases}$$

where $\overline{\Pi}$ is the closure and $\partial\Pi$ is the boundary of Π. This functional is negative inside Π, positive outside $\overline{\Pi}$, and vanishes on $\partial\Pi$. We consider system (17) completed with the condition

$$\Phi_\Pi(x, \lambda) = 0,$$
(18)

which guarantees that the solutions lie on the boundary $\partial\Pi$ of the set Π.

5.4. Homotopy and proof of Theorem 1

Let $\psi(\theta_-)\psi(\theta_+) < 0$. Consider the cylinder

$$G = \{(r, \lambda, \theta, h) \in \mathbb{E} : \ r_- \le r \le r_+; \ |\lambda - 1| \le \varepsilon; \ \theta_- \le \theta \le \theta_+; \ \|h\|_C \le K_0\}$$

in the space $\mathbb{E} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (C_0 \cap E^0)$ of vectors (r, λ, θ, h) with $r_+ > r_- > 0$ and a sufficiently small $\varepsilon > 0$. Assume that $\Pi \subset C \times \Lambda$ is an open bounded set such that for $(r, \lambda, \theta, h) \in G$ and x defined by (14)

$$\Phi_\Pi(x, \lambda) < 0 \quad \text{if} \quad r = r_-; \qquad \Phi_\Pi(x, \lambda) > 0 \quad \text{if} \quad r = r_+.$$
(19)

To prove Theorem 1 it suffices to show that if r_- is sufficiently large, then the system of equations (17), (18) has at least one solution in G.

Let us introduce the notation

$$(\mathcal{P}_\xi x)(t) = \xi(\mathcal{P}_{per}x)(t) + (1-\xi)H \operatorname{sign} \sin(t+\theta).$$

This formula implies $\mathcal{P}_\xi = \mathcal{P}_{per}$ for $\xi = 1$ and $(\mathcal{P}_\xi x)(t) = H \operatorname{sign} \sin(t + \theta)$ for $\xi = 0$. Consider, for all $0 \leq \xi \leq 1$, the system of equations

$$\lambda - 1 = \frac{\xi}{\pi r} \int_0^{2\pi} \sin(t+\theta) \, (\mathcal{P}_{per}x)(t) \, dt, \quad S + \int_0^{2\pi} (\Gamma'_{\xi\lambda+1-\xi} b \cdot \mathcal{P}_\xi x)(t) \, dt = 0, \quad h = \xi \Gamma_\lambda(b + Q\mathcal{P}_{per}x),$$
(20)

which for $\xi = 1$ coincides with (17), and for $\xi = 0$ has the trivial form

$$\lambda - 1 = 0, \qquad \psi(\theta) = 0, \qquad h = 0. \tag{21}$$

As a consequence of the simple geometry of the domain G, from the facts that the first component of the vector field $(\Phi_\Pi(\cdot), \lambda - 1, \psi(\theta), h)$ (defined by the left-hand parts of equations (18) and (21)) satisfies (19), each of the second and the third components has different signs at the ends of the intervals $|\lambda - 1| \leq \varepsilon$ and $\theta_- \leq \theta \leq \theta_+$, respectively, and the last component is the identity, it follows that the rotation of this vector field on the boundary of the domain G is non-zero.

Now general degree theory implies that the proof will be complete if we show that on the boundary of G there are no solutions of the system (18), (20) for all $0 \leq \xi \leq 1$ whenever r_- is sufficiently large. The boundary of G is the union of the following four sets: $G_r = \{(r, \lambda, \theta, h) \in G : r = r_\pm\}$, $G_\lambda = \{(r, \lambda, \theta, h) \in G : |\lambda - 1| = \varepsilon\}$, $G_\theta = \{(r, \lambda, \theta, h) \in G : \theta = \theta_\pm\}$, and $G_h = \{(r, \lambda, \theta, h) \in G : \|h\|_C = K_0\}$. Relations (19) imply the absence of solutions of equation (18) on the set G_r. From the estimate $\|\mathcal{P}_{per}x\|_C \leq H$ (valid for all x) and from $r \geq r_-$ it follows that if r_- is sufficiently large, then there are no solutions of the first of equations (20) on G_λ, because $\lambda - 1 = O(1/r)$ for all such solutions. By Proposition 2,

$$\lim_{r \to \infty} \sup_{x \in \mathcal{F}(r, K_0)} \left| \int_0^{2\pi} (\Gamma'_1 b \cdot \mathcal{P}_\xi x)(t) \, dt - \int_0^{2\pi} H(\Gamma'_1 b)(t) \, \operatorname{sign} \sin(t + \theta) \, dt \right| = 0,$$

where the second integral equals $H\chi(\theta)$; therefore the left-hand part of the second equation of system (20) uniformly approaches the function $\psi(\theta) = S + H\chi(\theta)$ as $r \to \infty$, $\lambda \to 1$ for all $\|h\|_C \leq K_0$, and consequently the relation $\psi(\theta_\pm) \neq 0$ implies that for large r_- this system does not have solutions on the set G_θ. Finally, the *a priori* estimate $\|h\|_C < K_0$ valid for all solutions of the last equation of (20) implies their absence on G_h. Theorem 1 is thus completely proved.

5.5. Proof of Theorems 2 and 3

Assume that there is a sequence x_n of 2π-periodic solutions of (8) for some $\lambda_n \in \Lambda$ with $\|x_n\|_C \to \infty$. As we have seen, for all sufficiently large n these solutions must satisfy the system (17), which implies $\|h_n\|_C \leq K_0$, $r_n \to \infty$, and $\lambda_n \to 1$, where we use the representation $x_n(t) = r_n \sin(t + \theta_n) + h_n(t)$ with $h_n \in E^0$, $0 \leq \theta_n \leq 2\pi$. Therefore, passing to the limit in the equation

$$S + \int_0^{2\pi} (\Gamma'_{\lambda_n} b \cdot \mathcal{P}_{per} x_n)(t) \, dt = 0$$

(this is the second equation of (17) written for the solution x_n) as $n \to \infty$ and using Proposition 2, we obtain $\psi(\theta_n) \to 0$ for any sequence of solutions x_n with $\|x_n\|_C \to \infty$.

Under the conditions of Theorem 2, for every proper zero φ_* of ψ there is an interval $(\theta_-, \theta_+) \ni \varphi_*$ such that φ_* is a unique zero of ψ in (θ_-, θ_+) and $\psi(\theta_-)\psi(\theta_+) < 0$. Consequently, from the proof of Theorem 1 above, it follows that there is an unbounded branch $\mathfrak{N}$ of 2π-periodic solutions $x(t) = r \sin(t + \theta) + h(t)$ of (8) such that $\theta \in (\theta_-, \theta_+)$ for every $(x, \lambda) \in \mathfrak{N}$. For any sequence $(x_n, \lambda_n) \in \mathfrak{N}$ with $\|x_n\|_C \to \infty$ the relation $\psi(\theta_n) \to 0$ combined with the inclusion $\theta_n \in (\theta_-, \theta_+)$ implies that $\theta_n \to \varphi_*$, which is equivalent to $\|x_n/\|x_n\|_C - e\|_C \to 0$ for $e(t) = \sin(t + \varphi_*)$. This implies (11) and proves Theorem 2.

Under the condition $S > H\Delta$ of Theorem 3, the continuous function ψ has no zeros, therefore the relation $\psi(\theta_n) \to 0$ is impossible, and hence there is no sequence x_n of 2π-periodic solutions of (8) with $\|x_n\|_C \to \infty$. This proves Theorem 3.

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