# ON A NUMBER OF BRANCHES OF PERIODIC SOLUTIONS

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We present estimates for the number of unbounded branches of periodic solutions in a problem on bifurcation from infinity with degeneration of the linear part of order two and suggest sufficient conditions for the existence of many barnches.

## 1. Problem statement

Consider the scalar equation

$$x'' + \lambda x = b(t) + f(x) \tag{1}$$

with a continuous function $b(t) \equiv b(t+2\pi)$, a continuous bounded function $f$ and the scalar parameter $\lambda$. We say that Eq. (1) has an *unbounded branch of periodic solutions with the limit* $\xi = \xi(t)$ *as* $\lambda \to 1$ if for every sufficiently large $r > 0$ Eq. (1) with some $\lambda = \lambda_r$ has a solution $x_r(t) \equiv x_r(t+2\pi)$ such that $\|x_r\|_{L^2} = r$ and $\lambda_r \to 1$, $\|x_r/r - \xi\|_{L^2} \to 0$ as $r \to \infty$, where $L^2 = L^2(0, 2\pi)$. If such a branch exists, then $\lambda = 1$ is a bifurcation point in the problem on bifurcation of $2\pi$-periodic solutions of Eq. (1) from infinity. It is important to note that due to the boundedness of $f$ the linearization at infinity brings Eq. (1) with $\lambda = 1$ to the equation $x'' + x = 0$, which has the two-dimensional set of $2\pi$-periodic solutions $r\sin(t+\varphi)$, i.e. we have a bifurcation problem with degeneration of the main linear part of order two.

Let $f = f_0 + f_1$ and the continuous bounded functions $f_0$, $f_1$ satisfy

$$\lim_{x\to-\infty} f_0(x) = f_-, \quad \lim_{x\to+\infty} f_0(x) = f_+; \quad \lim_{x\to\infty} \frac{1}{x}\int_0^x f_1(s)\,ds = 0 \tag{2}$$

with $f_- \neq f_+$. Set $\bar{f} = 2|f_- - f_+|$ and consider the Fourier expansion

$$b(t) = B_0 + \sum_{k=1}^{\infty} B_k \sin(kt + \beta_k), \qquad B_k \geq 0,\ k \in \mathbb{N}, \tag{3}$$

of the forcing $b$. If $B_1 > 0$, $B_1 \neq \bar{f}$, then Eq. (1) has 2 unbounded branches of periodic solutions as $\lambda \to 1$; this follows, e.g. from the results of [1].

Here we study the case $B_1 = 0$, which turns out to be "richer".

## 2. Main result

Using the Fourier expansion (3), define the continuous $2\pi$-periodic function

$$\chi(\varphi) = \sum_{k=3,5,7,\ldots} (1-k^2)^{-1} B_k \sin(k\varphi - \beta_k). \tag{4}$$

An isolated zero $\varphi_0$ of the function $\chi$ is called *regular* if in some neighborhood of $\varphi_0$ this function has different signs for $\varphi < \varphi_0$ and for $\varphi > \varphi_0$.

**Theorem 2.1.** *Let $B_1 = 0$. Let $f = f_0 + f_1$ and relations* (2) *hold with $f_- \neq f_+$. Let function* (4) *have $K > 0$ regular zeros $\varphi_1, \ldots, \varphi_K$ on the segment* $[0, 2\pi)$. *Then Eq.* (1) *has at least $K$ unbounded branches of periodic solutions with the limits $\xi_k(t) = \pi^{-1/2} \sin(t + \varphi_k)$, $k = 1, \ldots, K$, as $\lambda \to 1$.*

From the Sturm – Hurwitz Theorem it follows that any real cotinuous $2\pi$-periodic function (3) has at least $2n$ zeros on the segment $I = [0, 2\pi)$ if $B_0 = \cdots = B_{n-1} = 0$. Therefore function (4) has at least 6 zeros on $I$.

If all zeros of function (4) are isolated, then it has at least 6 regular zeros on $I$ and by Theorem 2.1 Eq. (1) has at least 6 unbounded branches of $2\pi$-periodic solutions as $\lambda \to 1$. For $b(t) = \cos(2t) + \sin(nt)$ with any odd $n > 1$ formula (4) implies $\chi(\varphi) = (1-n^2)^{-1} \sin(n\varphi)$; here there are $2n$ regular zeros on $I$ and $2n$ unbounded branches of periodic solutions. The number $K$ of regular zeros of function (4) on $I$ is even for any function $b$.

In Theorem 2.1 all the branches are continuous in the sense of [3].

## 3. Remarks

**a.** Starting from the pioneering work [2], interest of many authors is attracted to the equation

$$x'' + x = b(t) + f(x), \tag{5}$$

i.e. Eq. (1) with the fixed $\lambda = 1$; this is the so-called *resonance* case. The relation $B_1 \neq \bar{f}$ implies an *a priori* estimate $\|x\|_{C^1} \leq const$ for the norms of all $2\pi$-periodic solutions (if any) of Eq. (5). If $B_1 < \bar{f}$, then Eq. (5) has at least one such solution; the topological index at infinity of the completely continuous vector fields related to the periodic problem is non-zero. If $B_1 > \bar{f}$, then Eq. (5) may have no $2\pi$-periodic solutions; the topological index at infinity is zero. In particular, there are no $2\pi$-periodic solutions if $B_1 > \bar{f}$ and $f_+ < f(x) < f_-$ for all $x \in \mathbb{R}$.

**b.** We do not use the variational structure of the periodic problem for Eq. (1). The conclusion of Theorem 2.1 is valid for the perturbed equation of the form, e.g. $x'' + x = b(t) + f(x) + (\lambda - 1)g(t, x, x', x(t-\tau), \lambda)$ with any continuous function $g(t, \cdots) = g(t + 2\pi, \cdots)$ satisfying the global estimate $|g(\cdots)| \leq \delta$ for a sufficiently small $\delta > 0$. Generically, this perturbation is non-potential and destroys the variational structure for $\lambda \neq 1$.

Analogs of Theorem 2.1 are valid for other problems, e.g. periodic problems for vector equations, problems on subharmonics, two-point problems etc. For vector equations the problems are generically non-potential.

**c.** The proof of Theorem 2.1 is by topological methods. The main point is to reduce the problem to operator equations, which are topologically non-degenerate. The number of unbounded branches of periodic solutions of (1) is defined by a scalar branching equation. In case $B_1 > 0$, one obtains this equation just by multiplying Eq. (1) with $x'$, integrating over the period and passing to the limit as $\|x\|_C \to \infty$ in the resulting equation $\langle x', b \rangle = 0$; here $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(0, 2\pi)$. One arrives at $X_1 B_1 \sin(\varphi - \beta_1) = 0$, where $X_1 \sin(t + \varphi)$ and $B_1 \sin(t + \beta_1)$ are the first harmonics in the Fourier expansions of $x$ and $b$. Here there are 2 solutions $\varphi = \beta_1$ and $\varphi = \beta_1 + \pi$, which define 2 unbounded branches of $2\pi$-periodic solutions of (1).

If $B_1 = 0$ as in Theorem 2.1, then the limit of $\langle x', b \rangle = 0$ as $\|x\|_C \to \infty$ is $0 = 0$ and the above scheme does not work. The trick is to use in place of $\langle x', b \rangle = 0$ the equivalent equation $\langle u', f(x) + (1 - \lambda)x \rangle = 0$, where $u$ is a unique $2\pi$-periodic solution of $u'' + u = b(t)$ satisfying $\langle u, \sin t \rangle = \langle u, \cos t \rangle = 0$. Passing to the limit in $\langle u', f(x) + (1 - \lambda)x \rangle = 0$ as $\|x\|_C \to \infty$, $\lambda \to 1$, one obtains the branching equation $\chi(\varphi) = 0$.

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