LARGE SUBHARMONICS OF PENDULUM-LIKE EQUATIONS

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Dedicated to the memory of our friend Nikolai Bobylev

We formulate criteria on existence of sequences of periodic solutions (subharmonics) for equations $x'' + \alpha^2 x = b(t) + g(x, x')$ (irrational α is fixed) with increasing to infinity periods and amplitudes. Irrationality of the number α plays an essential role in our theorems.

1. Introduction

Consider the equation

$$x'' + \alpha^2 x = b(t) + f(x); \tag{1}$$

here $b(t)$ is a continuous 2π-periodic function, $f(x)$ is a continuous bounded nonlinearity. We study periodic solutions of Eq. (1), their periods may be equal to 2π or to some multiples of 2π. Periodic solutions of the period $2n\pi$ are called *subharmonics* if $n > 1$. The existence of subharmonics was studied by many authors, for different situations and by various methods. For example, in [1] the authors prove existence of subharmonics for vector equations, the essential assumptions in this paper are the gradient form of the nonlinearity and the convexity of the potential.

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Below we study the question of existence of subharmonics of large amplitudes for an irrational α. Clearly, if α is irrational, then the set of $2n\pi$-periodic solutions of Eq. (1) is bounded for any fixed n, the estimate depends on n. Thus, the amplitudes a_k of a sequences $x_k(t)$ of subharmonics for Eq. (1) may increase to infinity, only if the corresponded minimal periods $2n_k\pi$ also increase to infinity.

Let us emphasize that since α is irrational there is no resonance, and the existence of some $2n\pi$-periodic solution can be easily proven for any integer n. The problem is to find solutions for which $2n\pi$, $n > 1$, is its *minimal* period. Moreover, such solutions exist rather rare and under essential additional assumptions. The existence of a 2π-periodic solution (it is also $2n\pi$-periodic for any integer n) obscure the situation. The accumulated Kronecker index [2] of all $2n\pi$-periodic solutions equals to 1 for any n and the total index of all periodic solutions with the least period $2n\pi$, $n > 1$, is equal to 0 (see the remark immediately after Theorem 1). Therefore, to apply the topological degree theory [2] to prove the existence of subharmonics it is necessary to localize preliminary some of them. Such localization is an important stage in the proofs below.

It is useful to survey briefly the general situation with non-emptiness and boundedness of the set of all periodic solutions of Eq. (1).

If α is non-integer (nonresonant case), then the set of all 2π-periodic solutions $x(t)$ is non-empty and *a priori* bounded: $\|x\| \leq C|n_0^2 - \alpha^2|^{-1}$. Here n_0 is the nearest to α integer, the constant C depends on b and f. If f is smooth, $|f'(x)| \leq q$, and $q < |n_0^2 - \alpha^2|$, then there exists a unique 2π-periodic solution.

If $\alpha = m \in \mathbb{N}$ is integer (resonant case) the existence of 2π-periodic solutions $x(t)$ is defined by the properties of f and by the value

$$\bar{b} = \Big| \int_{-\pi}^{\pi} e^{imt} b(t)\, dt \Big|. \tag{2}$$

First results in this direction were obtained by Lazer and Leach [3], further these results were continued by various authors (see e.g. [4−6]) and by different methods (topological, variational etc). The usual assumption on f is the existence of the limits $f^{\pm}$ of $f(x)$ as $x \to \pm\infty$.

If such limits exist, then for $2|f^+ - f^-| \neq \bar{b}$ the set of 2π-periodic solutions $x(t)$ is bounded; if, moreover, $2|f^+ - f^-| > \bar{b}$, then this set is non-empty. The twice resonant case ($\alpha \in \mathbb{N}$ and $2|f^+ - f^-| = \bar{b}$) was partially studied in [7].

If $\alpha = m \in \mathbb{N}$ and if the limits of $f_{\pm}$ of $f(x)$ as $x \to \pm\infty$ do not exist, then a sequence of 2π-periodic solutions with arbitrarily large amplitudes may exist (see [6]). Let, for instance, $f : \mathbb{R} \to \mathbb{R}$ be continuous and bounded. We consider the odd function

$$\Psi(R) = \int_{-\pi}^{\pi} \sin t \, f(R \sin t) \, dt \tag{3}$$

The main assumptions on the function f in (1) are

$$\psi^* \stackrel{\text{def}}{=} \limsup_{\xi \to +\infty} |\Psi(\xi)| > \psi_* \stackrel{\text{def}}{=} \liminf_{\xi \to +\infty} |\Psi(\xi)|. \tag{4}$$

Techniques and explanations how to compute the numbers ψ^* and ψ_* and the following result may be found in [6]. *If $\psi_* < |\bar{b}| < \psi^*$, then Eq.* (1) *has an unbounded sequence of 2π-periodic solutions. If $|\bar{b}| < \psi_*$, then Eq.* (1) *has at least one 2π-periodic solution, and the set of such solutions is bounded. If $|\bar{b}| > \psi^*$, then Eq.* (1) *may have or have not 2π-periodic solutions, the set of them is bounded again.* It seems, however, that the case $\psi^* = \psi_*$ is more natural than $\psi^* \neq \psi_*$.

This result on unbounded sequences of periodic solutions cannot be adapted for the case of rational non-integer α. The equation

$$x'' + m^2 n^{-2} x = f(x) + b(t)$$

(m and n are coprime integers) after the rescaling of time has the form

$$x'' + m^2 x = n^2 [f(x) + b(nt)].$$

The value (2) computed for this equation is always zero, and the mentioned above result from [6] does not guarantee the existence of unbounded sequences of $2n\pi$-periodic solutions. If $\psi_* > 0$, then at least one $2n\pi$-periodic solution exists; the set of such solutions is *a priori* bounded.

Equations (1) (resonant or nonresonant) may have subharmonics. For example, let $f(x) = k \,\mathrm{sign}(x)$ for $|x| > 1$ and $f(x) = kx$ for $x \leq 1$; let $b(t) = \varepsilon \sin t$. If $4\alpha^2 = 4k + 1$, then Eq. (1) has for $|x| < 1$ the form $x'' + x/4 = \varepsilon \sin t$; it has the unique 2π-periodic solution $x_0(t) = -4/3 \, \varepsilon \sin t$ and the two-dimensional continuum of subharmonics $x_0(t) + a \sin(t/2 + \phi)$. The only condition is that a and ε are so small that $\max |x_0 + a \sin(t/2 + \phi)| \leq 1$.

If α is irrational, then the set of $2n\pi$-periodic solutions of Eq. (1) is bounded for any fixed n, the estimate depends on n. If α is rational, then this set may be unbounded as well as in the resonant case.

The question on the boundedness of the set of all periodic solutions in C for resonant equations seems to be rather difficult (e.g., see papers by R. Ortega and coauthors in 90's or [8] and citations therein).

2. Main results

2.1. *Assumptions on nonlinearities*

We study the equation

$$x'' + \alpha^2 x = g(t, x, x') \tag{5}$$

with a bounded continuous nonlinearity $g(t, x, x')$. The main assumption of Theorems 1 and 2 below is that g has the following special form:

$$g(t, x, y) = b(t) + f(x) + f_0(y) + f_1(t, x, y), \tag{6}$$

i.e., it contains the principal (controllable) part $b(t) + f(x) + f_0(y)$ and the smaller (uncontrollable) term f_1. The odd function (3) and the corresponding limit $\psi = \liminf_{R\to\infty} |\Psi(R)|$ plays the principal role in our constructions. The main condition is $\psi > 0$ or, equivalently, $\psi \neq 0$. This condition is valid for functions f that satisfy

$$f_+ = \lim_{x\to+\infty} f(x) \neq f_- = \lim_{x\to-\infty} f(x). \tag{7}$$

In this case $\psi = 2|f_+ - f_-|$. The condition $\psi > 0$ holds for various other functions ([6]). If the primitive of $\tilde{f}$ has sublinear growth at infinity, then

$$\int_{-\pi}^{\pi} \sin t\, \tilde{f}(R \sin t)\, dt \to 0 \quad \text{as} \quad R \to \infty.$$

We can choose as $\tilde{f}$ any periodic or almost periodic functions with zero average, functions of the type $\sin(x^3)$ and $\sin(\sqrt[3]{x})$, their combinations and many others. For an even function $\tilde{f}$ such integral is zero. Therefore, if the function f is the sum of a function satisfying (7), a function with sublinear primitive, and an even function, then again $\psi = 2|f_+ - f_-| \neq 0$.

In Sec. 3.1 we discuss more general types of admissible nonlinearities.

2.2. *Main theorems*

Theorem 1. *Let the number α be irrational and suppose that the equations $b(-t) = -b(t)$, $f_1(-t, -x, y) = -f_1(t, x, y)$ and $f(-x) = -f(x)$ hold. Let $\psi > 0$ and*

$$\Phi = \sup |f_1(t, x, y)| < \frac{1}{4}\psi. \tag{8}$$

Then Eq. (5) with $f_0 \equiv 0$ has an infinite sequence of odd periodic solutions, whose minimal periods and amplitudes tend to infinity.

Under the assumptions of this theorem the function $t \mapsto g\big(t, x(t), x'(t)\big)$ is odd for any odd differentiable function $x(t)$.

In the proof below we find the required odd subharmonics of the period $2n\pi$ in the form $x_n(t) = R\sin(\frac{m}{n}t) + h(t)$, where $R > 0$ is large, $\|h\| = o(R)$. For the same n there exists another odd subharmonic $\tilde{x}_n = -R\sin(\frac{m}{n}t) + \tilde{h}(t)$, generically $h(t) \neq -\tilde{h}(t)$. In an appropriate Banach space of the pairs (R, h) the topological indices of these subharmonics are $+1$ and -1.

Theorem 2. *Let α be irrational, and suppose that $f_1(-t, x, -y) = f_1(t, x, y)$ and $b(-t) = b(t)$. Let $\psi > 0$ and furthermore suppose that* (8) *holds. Then Eq.* (5) *with an arbitrary even bounded f_0 has an infinite sequence of even periodic solutions, their minimal periods and amplitudes tend to infinity.*

The proof of Theorem 2 is conceptually close to that of Theorem 1 (see Sec. 4) and we omit it. The difference is that the unknown solution instead of (17) has the form $x(t) = R\cos mt + h(t)$ with an even h.

Let us stress that in Theorem 2 we do not suppose any symmetry of the function $f(x)$; from $\psi > 0$ it follows that its odd part $[f(x) - f(-x)]/2$ is essentially nonzero. Under the assumptions of Theorem 2 the function $t \mapsto g\big(t, x(t), x'(t)\big)$ is even for any even function $x(t)$.

Both Theorems 1 and 2 are applicable for equations with non-odd and non-even $b(t)$, if a shift $b(t + \phi)$ is either odd or even. For the equation $x'' + \alpha^2 x = \sin t + f(x)$ with odd f both theorems are applicable (Theorem 2 after an appropriate shifting the time). Every periodic solution of the least period $2n\pi$ is embedded in the family $\mathfrak{N}$ of n shifts $x(t + 2k\pi)$, $k = 0, 1, \ldots, n-1$. In this case the family $\mathfrak{N}$ may contain both even and odd solutions.

Under the assumptions of Theorems 1 and 2 the amplitudes A_n of large subharmonics of the period $2n\pi$ satisfy the conditions $A_n \geq c\,n^2$ $(c > 0)$. It is possible that there exist also subharmonics with smaller amplitudes.

Generically, the situation under the conditions of Theorems 1 and 2 is as follows. The accumulated index of all periodic solutions of the least period $2n\pi$ in $L^2\big((0, 2n\pi), \mathbb{R}\big)$ is zero. Only the possibility to consider the corresponding operators in more narrow spaces of odd or even functions allows us to find a solution of a non-zero index. The situation may be illustrated by the following example. If we perturb the equation $z\bar{z} = 1$ in the complex plane with a small term $\varepsilon i z\bar{z}$, then all solutions of the non-perturbed equation (there exist plenty of them!) disappear for any arbitrary small ε. In contrast, solutions of the real equation $x^2 = 1$ can not be destroyed by small real perturbations.

3. Possible generalizations

3.1. *Nonautonomous controllable nonlinearities*

It is possible to include in the controllable part of the system some other nonlinearities. Let us give some examples.

Almost without changing the proof it is possible in Theorem 1 to include terms of the form $a(t)f_2(x')$ with arbitrary f_2 and odd $a(t)$ and terms of the form $a(t)f_2(x)$ with even (odd) $a(t)$ and odd (even) $f_2(x)$. The only assumption is that $a(t)$ is finite trigonometrical polynomial with zero average. The proof uses the equality

$$\int_{-\pi}^{\pi} \sin mt\, \sin(nkt + \phi)\; f_2(R \sin mt)\, dt = 0,$$

it is valid for any fixed $k \neq 0$ for sufficiently large coprime m and n.

Analogously, it is possible to include the terms $a(t)f_2(x)$ and $a(t)f_2(x')$ in the formulation of Theorem 2.

It is also possible to generalize all theorems to the case of unbounded sublinear nonlinearities using constructions from [9].

3.2. *Equations of higher order*

Theorems 1 and 2 without additional difficulties may be generalized for equations

$$L\Big(\frac{d}{dt}\Big)x = g(t, x, x')$$

of higher order with an even polynomial $L(p)$, and for more general equations

$$L\Big(\frac{d}{dt}\Big)x = M\Big(\frac{d}{dt}\Big)g(t, x, x')$$

from the control theory (here M is also some even polynomial, $\deg M < \deg L$). The only condition is that $L(p) = (p^2 + \alpha^2)L_1(p)$ where the polynomial L_1 has no roots on the imaginary axis.

It is also possible to obtain analogs of Theorem 1 and 2 for vector systems $z'' + Az = b(t) + f(t, z, z')$ with odd or even 2π-periodic forcing term $b(t)$ and the corresponding symmetric vector nonlinearity f, if the matrix A has a simple positive eigenvalue that is not square of any rational.

3.3. *On a quantity of subharmonics*

The set of numbers n such that there exist subharmonics of the period $2n\pi$ does not coincide with sufficiently large denominators of convergents.

Let us choose $\gamma \in (0,1)$ and $C_\gamma > 0$. Let the number α may be approximated by the rational number m/n, i.e., let

$$\left|\alpha - \frac{m}{n}\right| \leq C_\gamma n^{-1-\gamma}. \tag{9}$$

As it follows from the proof given below, for sufficiently large such n subharmonics of the period $2n\pi$ also exist. Estimate (9) seems to be valid for essentially 'dense' set of integer numbers n than the set of denominators of convergents.

3.4. *Result for equations without symmetries*

Here we formulate without proof a theorem on existence of subharmonics that are neither even nor odd. This theorem is applicable only for the case where the forcing term has infinitely many nonzero terms in its Fourier series. Authors do not know similar results for $b(t)$ having a finite number harmonics only (e.g. for $b(t) = \sin t$)).

Consider Eq. (1). Let limits (7) exist, without loss of generality we suppose that $f_+ = -f_- \neq 0$. Furthermore, let us suppose that for some $\beta > 0$ the estimate

$$\limsup_{|x|\to\infty} |x|^\beta \left|f(x) - f_{\operatorname{sign} x}\right| < \infty \tag{10}$$

is valid; $f_{\operatorname{sign} x} = f_+$ for $x > 0$ and $f_{\operatorname{sign} x} = f_-$ for $x < 0$.

We denote n and $m = m(n)$ the numerators and the denominators of convergents of irrational α. Let $\mathfrak{N}$ be an infinite sequence of the denominators n such that $\varepsilon_n \operatorname{sign}(f_+) = \alpha^2 - m^2/n^2 > 0$. (i.e., we consider either even or odd convergents only). Suppose the sequence $\mathfrak{N}$ contains an infinite number of convergents with odd denominators n.

Let the function b be absolutely continuous and have a piece-wise continuous derivative. Consider the Fourier series

$$b(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \sin(kt + \varphi_k), \qquad a_k > 0,\ k = 1, 2, \ldots \tag{11}$$

of the function b and put

$$\Delta_m = \max_{t\in\mathbb{R}} |d_m|, \qquad d_m(t) = \sum_{k=1,3,5,\ldots} \frac{a_{km}}{m^2k^2 - \alpha^2} \sin(\varphi_{km} - kt). \tag{12}$$

The function d_m has the period 2π and satisfies $d_m(\pi + t) = -d_m(t)$.

The principal conditions for subharmonics existence are given in terms of the rate coefficient β, of the asymptotics of Δ_m as $m \to \infty$, and of the numbers ε_n.

Theorem 3. *Let $f_+ \neq 0$. Let the number α be transcendental and either $|\varepsilon_n| n^{1+\beta} \leq c < \infty$ and*

$$\lim_{n\to\infty} \frac{\Delta_{m(n)}^{1+\beta^{-1}}}{n^{2(1+\beta^{-1})}|\varepsilon_n|} = \infty, \tag{13}$$

or $|\varepsilon_n| n^{1+\beta} \to \infty$ and

$$\lim_{n\to\infty} \frac{\Delta_{m(n)}}{n^3|\varepsilon_n|} = \infty. \tag{14}$$

Then Eq. (1) has an infinite sequence of $2n\pi$-periodic solutions, their amplitudes A_n tend to infinity as $|\varepsilon_n|^{-1}$.

The condition $|\varepsilon_n| n^{1+\beta} \to \infty$ may be valid for $\beta > 1$ only.

Let $|\varepsilon_n| \sim n^{-1-\beta+\varepsilon}$, $\varepsilon > 0$ and $a_m \geq m^{-\gamma}$. Then the assumptions of this theorem are valid if $\beta > \gamma + 2 - \varepsilon$.

Equalities (13) and (14) and usual estimates on the rate of convergence of Fourier coefficients to zero imply the condition $\varepsilon_n n^{2+\gamma} \to 0$ as $n \to \infty$ for some $\gamma > 0$. This condition may be valid only for transcendental α (Roth Theorem). Moreover, the set of such very well approximated α has the Lebesgue measure zero.

4. The proof of Theorem 1

4.1. *Introduction to the proof*

We will use below the continued fraction of the irrational number α ([10]). We use the estimate

$$\left|\alpha - \frac{m}{n}\right| \leq \frac{1}{n^2} \tag{15}$$

that is valid for any convergent m/n of the continued fraction of the irrational number α ([10]). According to (15), the estimate $|\varepsilon_n| \leq (2\alpha + 1)\, n^{-2}$ holds. The numerator $m \sim \alpha n$ tends to infinity almost proportionally to n.

We fix an *a priori* unknown positive integer n and find $2n\pi$-periodic solution of Eq. (5). In the proof n is a sufficiently large denominator of convergent of α. We denote $\varepsilon_n = \alpha^2 - m^2 n^{-2}$; this value may be positive or negative.

Let us rescale time in (5). Each $2n\pi$-periodic solution of (5) may be generated by a corresponding 2π-periodic solution of the equation

$$x'' + \alpha^2 n^2 x = n^2 g(nt, x, x' n^{-1}), \tag{16}$$

where $g(t,x,x') = b(t) + f(x) + f_1(t,x,x')$. We search the required odd 2π-periodic solutions of (16) in the form

$$x(t) = R \sin mt + h(t). \tag{17}$$

If $h(t)$ is odd and 2π-periodic, then $x(t)$ is also odd and 2π-periodic.

The integer number n is considered as a large parameter, the value $R > 0$ and the function h are unknowns; they depend on n; the value of R is large, the norm $\|h\|_{C^1}$ is $o(R)$.

Under the main condition $\psi > 0$ for large R the function Ψ has the constant signature $\sigma(f)$. In the proof below we consider only 'a half' of the values of n: such that $\sigma(f)\varepsilon_n > 0$. We put $R \sim |\varepsilon_n|^{-1}$, therefore for algebraic irrational α the value R has the order n^2, for transcendental α it may tend to infinity arbitrary fast.

Below we show that for sufficiently large n Eq. (16) has an odd 2π-periodic solution (17).

4.2. *Spaces and operators*

Consider the space $L^2_{odd} = L^2_{odd}(-\pi, \pi)$ of odd square integrable functions with the usual norm. Any $x(t) \in L^2_{odd}$ may be represented in the form

$$x(t) = \sum_{k=1}^{\infty} a_k \sin kt,$$

the convergence here is in L^2.

Denote by $\Gamma = \Gamma_n$ the linear operator in L^2_{odd} that is inverse to the differential operator $x'' + \alpha^2 n^2 x$ with the 2π-periodic boundary conditions. Let $Q_m = E - P_m$, where P_m is the orthogonal projector onto the line $\lambda \sin mt$. The operator Γ_n is completely continuous and self-adjoint in L^2_{odd}, its spectrum consists from the sequence of simple eigenvalues $(\alpha^2 n^2 - k^2)^{-1}$, $k = 1, 2, \ldots$, and from zero. For any $x \in L^2_{odd}$ the function Γx is differentiable. The operator $\Gamma' = \Gamma'_n = \dfrac{d}{dt}\Gamma$ does not act in L^2_{odd}. It is defined on L^2_{odd} and maps odd functions to even ones; it is completely continuous as an operator from L^2_{odd} to L^2. Under the assumptions of Theorem 1 the operator $x \mapsto g(t, \Gamma_n x, n^{-1}\Gamma'_n x)$ acts in the space L^2_{odd} and is completely continuous.

The norm of Γ in L^2_{odd} equals $n^{-2}|\varepsilon_n|^{-1}$, it is attained on the function $\sin mt$. The norm of the operator ΓQ_m in L^2_{odd} admits the estimate $c\,n^{-1}$. Here and below we use the same notation c for all constants, their exact values do not play any role.

4.3. *Equivalent system*

Function (17) is an odd solution of Eq. (16) iff the pair (R, y), $R \in \mathbb{R}^+$, $y(t) \in L^2_{odd}$ is a solution of the system

$$\pi\varepsilon_n R = \int_{-\pi}^{\pi} \sin mt\, Y_1(t)\, dt, \qquad y = n^2 Q_m Y_1(t) \tag{18}$$

with two unknowns R and y. For $\xi \in [0,1]$ we use the denominations

$$Y_\xi(t) = g_\xi\Big(nt, R\sin mt + \xi\Gamma y(t), \big(mR\sin mt + \xi\Gamma' y(t)\big)n^{-1}\Big),$$
$$g_\xi(t,x,y) = f_0(x) + \xi\big[b(t) + f_1(t,x,y)\big].$$

The mapping $(R, y(t), \xi) \mapsto Y_\xi(t)$ is completely continuous with respect to its variables for any n. Consider for $\xi \in [0,1]$ the deformation

$$\Xi = \Xi(R, y; \xi) = \big(\Xi_R(R, y; \xi), \Xi_h(R, y; \xi)\big)$$

with the components

$$\begin{aligned} \Xi_R(R, h; \xi) &= R - [\pi\varepsilon_n]^{-1} \int_{-\pi}^{\pi} \sin mt\, Y_\xi(t)\, dt, \\ \Xi_h(R, h; \xi) &= y - \xi n^2 Q_m Y_\xi(t). \end{aligned} \tag{19}$$

Singular points (R, y) of the deformation for $\xi = 1$ generate necessary periodic solutions (17) with $h = \Gamma y$.

Let us prove that for a proper choice of constants $r_j > 0$ on the boundary of the cylinder

$$Z_n = \Big\{R:\, R \in [r_1\varepsilon_n^{-1}\sigma(f), r_2\varepsilon_n^{-1}\sigma(f)]\Big\} \times \Big\{y \in L^2_{odd}:\, \|y\|_{L^2} \le r_3 n^2\Big\}$$

this deformation has no singular points for $\xi \in [0,1]$ (see Sec. 4.5). Then we prove (in Sec. 4.6) that for $\xi = 0$ the rotation on ∂Z_n of the vector field $\Xi(R, y; 0)$ is non-zero. This proves Theorem 1 according to the general degree theory [2].

4.4. *Auxiliary statements*

Let positive ρ belong to some unbounded set $\mathfrak{R} \in \mathbb{R}^+$, and let $R(\rho),\ r(\rho) : \mathfrak{R} \to \mathbb{R}$ be some functions, and $R(\rho) \to \infty$ as $\rho \to \infty$.

Lemma 1. *Let $q(t)$ be Lipschitz continuous, and suppose that $e(t)$ is twice continuously differentiable and satisfies $mes\{t \in [-\pi, \pi] : e'(t) = 0\} = 0$. Let, furthermore,*

$$\lim_{\rho\to\infty} \frac{r(\rho)}{R(\rho)} = 0. \tag{20}$$

Then the following equality is valid:

$$\sup_{\|h\|_{C^1} \le r(\rho)} \int_{-\pi}^{\pi} q(t)\Big(f\big(R(\rho)e(t) + h(t)\big) - f\big(R(\rho)e(t)\big)\Big)\,dt = 0. \tag{21}$$

¿From this Lemma it follows that equality (21) implies the relation

$$\sup_{\substack{\|h\|_C \le c\,r;\\ m^{-1}\|h'\|_C \le c\,r}} \int_{-\pi}^{\pi} \sin mt\Big(f\big(R\sin mt + h(t)\big) - f\big(R\sin mt\big)\Big)\,dt = 0 \tag{22}$$

for any integer m (in particular, for $m = m(\rho) \to \infty$). After the change of variables $t := mt$ we get the average of $m \to \infty$ infinitesimal terms.

The proof of Lemma 1 is given in Sec. 4.7, the proof for $r = const$ was given in [6].

In the proof of Theorems 1 and 2 we use estimates of norms in C^1 of functions $h = \Gamma y$.

Lemma 2. *The inequalities*

$$\|\Gamma y\|_C \le c\,n, \qquad \|\Gamma' y\|_C \le c\,n^2, \qquad \|\big(\Gamma y\big)''\|_C \le c\,n^3 \tag{23}$$

are valid for the component y of all singular points of $\Xi(R, y; \xi)$.

Two first estimates (23) follow from the estimates

$$\|\Gamma_n Q_m\|_{L^2\to C} \le c\,n^{-1} \quad \text{and} \quad \|\Gamma'_n Q_m\|_{L^2\to C} \le c,$$

from the uniform boundedness of Y_ξ. The last estimate follows from the equality $\big(\Gamma y\big)'' + \alpha^2 n^2 \Gamma y = n^2 g_\xi$ and the first estimate (23).

¿From estimates (23) for zeros (R, y) of the deformation $\Xi(R, y; \xi)$ it follows the relationship

$$\Psi(R) - \int_{-\pi}^{\pi} \sin mt\, f\big(R\sin mt + \xi\Gamma y(t)\big)\,dt = o(1). \tag{24}$$

Remark. *We do not use estimates of the second derivatives* $(\Gamma y)''$ *in the proof of Theorem* 1 *since there is no term* $f_0(x')$*. However, in the proof of Theorem* 2 *(and in the proofs of nonformulated generalizations of both Theorems* 1 *and* 2*, mentioned in Sec.* 3.1*) the relationships*

$$\int_{-\pi}^{\pi} q(t)\left[f_0\big(R\frac{m}{n}\cos mt+\xi n^{-1}\Gamma' y(t)\big)-f_0\big(R\frac{m}{n}\cos mt\big)\right]dt\to 0$$

are necessary; they can be easily obtained.

4.5. *Nondegeneracy of the deformation*

It is necessary to prove that the deformation Ξ is a homotopy of the boundary ∂Z_n of the cylinder Z_n for a proper choice of the numbers r_j. This boundary consists from two parts: from the set

$$\partial Z_n^R=\Big\{R\varepsilon_n\sigma(f)=r_j,\ j=1,2;\ \|y\|_{L^2}\le r_3n^2\Big\}$$

and from the set

$$\partial Z_n^y=\Big\{R\in[r_1\varepsilon_n^{-1}\sigma(f),r_2\varepsilon_n^{-1}\sigma(f)];\ \|y\|_{L^2}=r_3n^2\Big\}.$$

Nondegeneracy on the set ∂Z_n^R is valid for sufficiently small $r_1>0$ and sufficiently large $r_2>0$. Let our deformation is degenerated for some $\xi\in[0,1]$. Suppose without loss of generality that $\sigma(f)=1,\varepsilon_n>0$ and $R\varepsilon_n=r_1$. Let us take the limits in the equality

$$R=[\pi\varepsilon_n]^{-1}\int_{-\pi}^{\pi}\sin mt\,Y_\xi(t).$$

For large n due to (24) it implies the inequality

$$\pi r_1\ge\psi-\int_{-\pi}^{\pi}|\sin mt|\Phi\,dt,$$

it is impossible according to condition (8) for $r_1<\frac{1}{\pi}(\psi-4\Phi)$.

Nondegeneracy of the deformation on the set ∂Z_n^R for any large enough $r_3>0$ follows from Lemma 1.

4.6. *Index calculation*

For $\xi=0$ the homotopy Ξ takes the form

$$\Xi_R(R,h;0)=R-[\pi\varepsilon_n]^{-1}\int_{-\pi}^{\pi}\sin mt\,f(R\sin mt)\,dt,$$

$$\Xi_h(R,h;0)=y.$$

The rotation of this field on ∂Z_n according to the standard Rotation Product Formula coincides with the non-zero rotation of the scalar field

$$R - [\pi\varepsilon_n]^{-1}\int_{-\pi}^{\pi} \sin mt\, f(R\sin mt)\, dt = R - [\pi\varepsilon_n]^{-1}\Psi(R)$$

on the interval $[r_1\varepsilon_n^{-1}\sigma(f), r_2\varepsilon_n^{-1}\sigma(f)]$. It is easy to check that this field take the values of various signature at the ends of the interval.

4.7. *Proof of Lemma 1*

Let us choose a number $\varepsilon > 0$ and cover the set $\{t \in [-\pi,\pi] : e'(t) = 0\}$ by a finite family of open intervals with the total length less than $\varepsilon/(4\sup|q|\sup|f|)$. Fix these intervals, they are independent from ρ. Since the functions f and q are bounded, the corresponding integral is not greater than $\varepsilon/2$.

The uncovered part of the interval $[-\pi,\pi]$ is another family of intervals $[a_j, b_j]$. On any such interval the function $e(t)$ is strictly monotone and its derivative is separated from zero:

$$|e'(t)| > \delta > 0, \qquad t \in \bigcup [a_j, b_j].$$

For sufficiently large values of ρ the function $R(\rho)e(t) + h(t)$ is also strictly monotone on any $[a_j, b_j]$ and

$$|R(\rho)e'(t) + h'(t)| > \delta/2, \qquad t \in \bigcup [a_j, b_j].$$

To prove the lemma it is sufficient to establish that for any j

$$\lim_{\rho\to\infty} \sup_{\|h\|_{C^1}\le r(\rho)} \int_{a_j}^{b_j} q(t)\Big(f\big(R(\rho)e(t)+h(t)\big) - f\big(R(\rho)e(t)\big)\Big)\, dt = 0.$$

Now j is fixed up to the end of the proof of the lemma.

Let us make in the integral

$$\int_{a_j}^{b_j} q(t)\, f\big(R(\rho)e(t) + h(t)\big)\, dt$$

the monotone substitution of time $t = t(\tau)$, $\tau = \tau(t)$ defined by the formula $R(\rho)e(\tau) = R(\rho)e(t) + h(t)$; this substitution is well defined by the construction. In the obtained integral

$$\int_{\tau(a_j)}^{\tau(b_j)} q\big(t(\tau)\big)\tau'(t)\, f\big(R(\rho)e(\tau)\big)\, d\tau$$

we should prove that

$$|a_j - \tau(a_j)|,\ |b_j - \tau(b_j)|,\ |\tau - t(\tau)|,\ |\tau'(t) - 1| \to 0. \tag{25}$$

The functions $e(t)$ and $e(t) + [R(\rho)]^{-1}h(t)$ are strictly monotone for large ρ. Therefore on the corresponding set the inverse function e^{-1} is well-defined. By construction, it is Lipschitz continuous. Hence from

$$\tau = e^{-1}\big(e(t) + [R(\rho)]^{-1}h(t)\big) = t + e^{-1}\big(e(t) + [R(\rho)]^{-1}h(t)\big) - e^{-1}\big(e(t)\big)$$

it follows that

$$|\tau - t| = \Big|e^{-1}\big(e(t) + [R(\rho)]^{-1}h(t)\big) - e^{-1}\big(e(t)\big)\Big| \le L\big|[R(\rho)]^{-1}h(t)\big| = o(1)$$

($L = L(\delta)$ is a Lipschitz constant of e^{-1}). This guarantees that three first terms in (25) tend to zero. Now let us differentiate the identity $R(\rho)e(\tau) = R(\rho)e(t) + h(t)$ with respect to τ: we obtain $R(\rho)e'(\tau) = [R(\rho)e'(t) + h'(t)]t'(\tau)$. The last value from (25) tends to zero according to

$$|t'(\tau) - 1| = \Big|\frac{e'(\tau)}{e'(t) + [R(\rho)]^{-1}h'(t)} - 1\Big| = \Big|\frac{e'(\tau) - e'(t) - [R(\rho)]^{-1}h'(t)}{e'(t) + [R(\rho)]^{-1}h'(t)}\Big|.$$

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