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Synchronized double-frequency oscillations in a class of weakly resonant systems

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Abstract

Subharmonic oscillations in the weak resonant Hopf bifurcation in control systems is studied. The principal result is that the structure of the set of subharmonics is defined by the main homogeneous part of the nonlinearity *if this main part is not a polynomial*. The analysis is based on topological methods and harmonic linearization.

Keywords: Weak resonances; Hopf bifurcation; Dynamical control systems

1. Introduction

Generating periodic oscillations at a prescribed approximate frequency is important for numerous applications in physics and applied mathematics. From the mathematical point of view, the most important tool to achieve this goal is the phenomenon of Hopf bifurcation. In particular, due to recent progress in fibre optical information transmission systems, there is growing interest in generating oscillations with higher frequency and/or with richer spectrum. Nonlinear methods of frequency mixing are becoming crucial, for example for frequency shifting in wavelength division multiplexed (WDM) communication systems. In this context, we study periodic 'double-frequency' oscillations of the form

$$x(t) = r_1 \sin(wmt) + r_2 \sin(wnt + \varphi), \tag{1.1}$$

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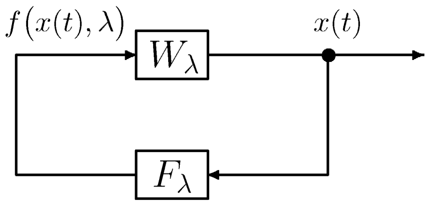

Fig. 1. Block-diagram of a single-loop system.

where $r_1, r_2 > 0, w > 0, \varphi$ are real numbers, whereas n, m are coprime positive integers. The basic frequency w should be close to some prescribed value w_0. Below we discuss an apparently new mathematical scenery, of how this type of oscillation can be generated.

Consider the differential equation

$$L\left(\frac{\mathrm{d}}{\mathrm{d}t}, \lambda\right) x = M\left(\frac{\mathrm{d}}{\mathrm{d}t}, \lambda\right) f(x, \lambda), \tag{1.2}$$

where

$$L(p, \lambda) = p^\ell + a_1(\lambda) p^{\ell-1} + \cdots + a_\ell(\lambda),$$

$$M(p, \lambda) = b_0(\lambda) p^m + b_1(\lambda) p^{m-1} + \cdots + b_m(\lambda)$$

are coprime real polynomials, $\ell = \deg L(p, \lambda) > m = \deg M(p, \lambda)$ whereas $f(x, \lambda)\colon \mathbb{R} \to \mathbb{R}$ is a continuous real function and λ is a parameter. This parameter is a real scalar, unless otherwise is explicitly stated.

This equation describes the dynamics of a single-loop control system, which includes an linear integrating link W with the rational transfer function $W_\lambda(p) = M(p, \lambda)/L(p, \lambda)$ and the nonlinear feedback F_λ given by $x(t) \mapsto f(x(t), \lambda)$. A block-diagram of system (1.2) is shown in Fig. 1. The general theory of such systems is well known (see [15] or almost any textbook in control theory); readers not accustomed to this type of systems could assume that $M(\cdot, \cdot) \equiv 1$, in which case (1.2) becomes an ordinary differential equation of higher order.

Throughout this paper we suppose that the nonlinearity $f(x, \lambda)$ is sublinear at zero:

$$\lim_{x \to 0} \sup_{\lambda} |f(x, \lambda) x^{-1}| = 0. \tag{1.3}$$

In particular, $f(0, \lambda) \equiv 0$, which implies that Eq. (1.2) possesses the trivial solution $x(t) \equiv 0$ for all λ. We are basically interested in small periodic solutions $x(t, \lambda)$ which exist for some small $\lambda - \lambda_0$ and which have a given approximate period. The classical assertion of this kind is the famous Hopf bifurcation theorem [13] and we refer λ_0 as the Hopf bifurcation point with the frequency w_0 if in an arbitrary small neighbourhood $\{\lambda\colon |\lambda - \lambda_0| < \varepsilon\}$ of λ_0 there exists a λ such that Eq. (1.2) has a nonzero periodic solution with magnitude less than ε, whose period differs from $2\pi/w_0$ less than by ε.

A Hopf bifurcation with the frequency w_0 could occur only if $L(p)$ has some roots of the form $nw_0\mathrm{i}$ for a positive integer n; moreover, if there exists exactly one root of such form nwi, it should be equal to iw_0. On the other hand, the Hopf bifurcation theorem guarantees that λ_0 *is the Hopf bifurcation point provided that the polynomial* $L(p,\lambda)$ *has the pair of complex roots* $\sigma(\lambda)\pm w(\lambda)\mathrm{i}$, $\sigma(\lambda_0)=0$, $w(\lambda)\neq 0$; *the numbers* $kw(\lambda_0)\mathrm{i}$ *for* $k=0,2,3,\ldots$ *are not roots of* $L(p,\lambda_0)$; $\sigma'(\lambda)\neq 0$. See [7,10] for details and some sharper results. Of course, this is the main scenery of appearance of small cycles in the vicinity of an equilibrium.

The next possible scenery when the Hopf bifurcation with the frequency w_0 can happen is the existence of exactly two pairs of simple roots $\pm nw_0\mathrm{i}$, $\pm mw_0\mathrm{i}$. In this case one can expect oscillations which are in the first approximation a synchronized superposition of two harmonics with the approximate frequencies nw_0 and mw_0, and this kind of oscillation may be interesting from the point of view of information transmission. We will concentrate on the simplest case when the positive integers m,n are both greater than 1 and are coprime. Naturally, we suppose also that at $\lambda=\lambda_0$ all numbers $\pm kw_0\mathrm{i}$, $k\in\mathbb{Z}, k\neq m, k\neq n$ are not roots of the polynomial $L(p,\lambda_0)$ (rather than only multiples of $mw_0\mathrm{i}$ and $niw_0\mathrm{i}$). For instance, these two pairs of roots may be the only roots on the imaginary axis. (Note in passing that to implement this situation, that is to bring *two* pairs of roots to the imaginary axis simultaneously, we should be able to influence at least two independent parameters of the underlying physical system.)

The situation described in the previous paragraph is called *weak resonance* in the Hopf bifurcation problem [3]. In this situation the polynomial $L(p,\lambda)$ can be represented as

$$L(p;\lambda)=(p^2+\sigma_1(\lambda)p+m^2w_0^2+\tau_1(\lambda))\times(p^2+\sigma_2(\lambda)p+n^2w_0^2+\tau_2(\lambda))L_1(p;\lambda) \tag{1.4}$$

with $\lim_{\lambda\to\lambda_0}\tau_j(\lambda)=\lim_{\lambda\to\lambda_0}\sigma_j(\lambda)=0$. We suppose that $\sigma_1'(\lambda_0),\sigma_2'(\lambda_0)\neq 0$. Then by the previously cited Hopf bifurcation theorem, λ_0 is a Hopf bifurcation point both with the frequencies mw_0 and nw_0. The natural question is whether it is a Hopf bifurcation point with the frequency w_0: that is whether there exist small cycles with the minimal period close to $2\pi/w_0$ and with an approximate representation $x(t)=r_1\sin(wmt)+r_2\sin(wnt+\varphi)$, where both r_1 and r_2 differ from zero.

An answer is defined by the structure of the main homogeneous part $F(x,\lambda)$ of the nonlinearity $f(x,\lambda)$. If $F(x,\lambda)$ is just a positive integer power of $a(\lambda)x^N$ then the situation is well studied. Say, if $N=2$ or $N=3$, then λ_0 can be a Hopf bifurcation point with the frequency w_0 only if the derivatives $\sigma_1'(\lambda_0),\sigma_2'(\lambda_0)$, $\tau_1'(\lambda_0),\tau_2'(\lambda_0)$ satisfy some algebraic equalities. These equalities arise from the properties of the so-called beak of synchronization, it seems that the first paper was [12]. The existence of two- or three-dimensional invariant tori has been well established, but the dynamics on these tori is very intricate. In particular, the effect of the so-called subfurcation [5] is present: for the values of parameter λ approaching λ_0 there arise sporadically some oscillations of unboundedly increasing periods. This interesting effect is unfortunately difficult to exploit due to its rather complicated nature.

The situation could be different if the nonlinearity f is highly degenerate. For instance, the value $\lambda=0$ is the Hopf bifurcation point with the frequency 1 for the degenerate system $(p^2+\lambda p+m^2)(p^2+\lambda p+n^2)x=0$, $p=\mathrm{d}/\mathrm{d}t$, $\lambda\in\mathbb{R}$; indeed any function $a_n \sin(t/n+\phi_n)+a_m \sin(t/m+\phi_m)$ satisfies the equation. However these highly degenerate situations are too difficult to implement.

However in many important situations (see e.g. [2]), especially in control theory, the main homogeneous part of $f(x,\lambda)$ is not just an integer power of x, for instance, $f(x,\lambda)=a(\lambda)x|x|^{\alpha-1}+o(x^\alpha)$ or $f(x,\lambda)=a(\lambda)|x|^\alpha+o(x^\alpha)$ where $\alpha>1$ is not an even positive integer. Such nonlinearities can be introduced into the feedback intentionally, or they could be present due to some small strongly nonlinear, say hysteresis, effects. The gist of the paper is the observation that in this case the situation changes drastically: here often λ_0 is a Hopf bifurcation point with the frequency w_0 for some open set of values of derivatives $\sigma_1'(\lambda_0), \sigma_2'(\lambda_0),\ \tau_1'(\lambda_0), \tau_2'(\lambda_0)$. These sets could be characterized quite explicitly and are often rather large. An important role is played at the oddness or evenness of the numbers m and n as well as the oddness or evenness of the main homogeneous part of the nonlinearity f.

The paper is organized as follows. In Section 2 we formulate the principal result of the paper. In Section 3 we discuss in more detail some corollaries for the simplest case of ordinary differential equations of the fourth-order. In Section 4 some generalizations for delay equations are presented. This topic is important since inevitable, if rather small, delays are always present in the feedback link of the control system shown in Fig. 1. The interaction of such delays with the linear part of the system could lead to some quite unexpected results, see for instance [1,14]. Fortunately, in the problem which we consider in the present paper small delays in the feedback link can be analyzed without difficulties and the results are similar with those presented in Section 2. Section 5 is devoted to an analog of our principal result for the case when subharmonics branch away from infinity, rather than from zero. Section 6 is devoted to the proof of the main theorem. Here we use the method of harmonic linearization[1] and the theory of rotation of vector fields [11] (which contrasts sharply with the method of normal forms [3] as a main tool in the case when the main homogeneous part of f is polynomial). Finally, in the last section we discuss the properties of some specific functions d_γ which are responsible for the appearance of the Hopf bifurcation with the frequency w.

2. Principal result

To avoid some clumsy notation we consider the case when $w_0=1$. Then the polynomial $L(p,\lambda)$ takes the form

$$L(p;\lambda)=(p^2+\sigma_1(\lambda)p+m^2+\tau_1(\lambda))(p^2+\sigma_2(\lambda)p+n^2+\tau_2(\lambda))L_1(p;\lambda) \quad (2.1)$$

with $\lim_{\lambda\to\lambda_0}\tau_j(\lambda)=\lim_{\lambda\to\lambda_0}\sigma_j(\lambda)=0$. Let F denote the main homogeneous part of the nonlinearity f: $f(x,\lambda)=a(\lambda)F(x)+\Phi(x,\lambda)$, where $F(rx)=r^\alpha F(x),\ r>0$ and $\lim_{x\to 0}\sup_\lambda|\Phi(x,\lambda)x^{-\alpha}|=0$. Here and below $\alpha>1$ is a constant. The function $F(x)$ can be odd ($F(x)=a|x|^\alpha$) or even ($F(x)=ax|x|^{\alpha-1}$), it can also be of a more general nature, for instance, $F(x)=2x^2,\ x>0,\ F(x)=-x^2,\ x<0$.

[1] This method is quite usual in control theory for computation of unknown cycles in autonomous systems. The first citation on application of the method to Hopf bifurcation is [6].

We introduce the following functions of the two variables $\rho > 0$ and $\varphi \in [0, 2\pi)$:

$$d_{s,m}(\rho,\varphi) = \int_0^{2\pi} \sin(mt)F(\sin(mt) + \rho\sin(nt+\varphi))\,\mathrm{d}t,$$

$$d_{c,m}(\rho,\varphi) = \int_0^{2\pi} \cos(mt)F(\sin(mt) + \rho\sin(nt+\varphi))\,\mathrm{d}t,$$

$$d_{s,n}(\rho,\varphi) = \int_0^{2\pi} \sin(nt+\varphi)F(\sin(mt) + \rho\sin(nt+\varphi))\,\mathrm{d}t,$$

$$d_{c,n}(\rho,\varphi) = \int_0^{2\pi} \cos(nt+\varphi)F(\sin(mt) + \rho\sin(nt+\varphi))\,\mathrm{d}t.$$

The properties of these functions play an important role in the analysis of subharmonics of small magnitude. Note immediately that these functions could turn to zeros. For instance, this is the case for an even function $F(x)$ and both numbers m and n are odd, or for $F(x)=x^2$ and arbitrary m and n. This is a technical reason why our approach does not work when $F(x,\lambda)$ is just a positive integer power $a(\lambda)x^N$ with $N=2,3$ (and why the method of normal forms does not work in our settings). Sometimes these functions do not depend on φ: for example $F(x)=x^3$ and arbitrary m and n. Generically, all these functions depend on both variables. Further useful analytical, qualitative and graphical information concerning these functions is discussed at the end of the paper.

We suppose that the coefficients of the polynomials $L_1(p,\lambda)$ and $M(p,\lambda)$ are continuous in λ and the coefficients $\sigma_j(\lambda)$ and $\tau_j(\lambda)$ are differentiable with respect to λ at $\lambda=\lambda_0$. We use the notations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\sigma_j(\lambda)\bigg|_{\lambda=\lambda_0} = \sigma_j, \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda}\,\tau_j(\lambda)\bigg|_{\lambda=\lambda_0} = \tau_j \tag{2.2}$$

and denote $W_1(w,\lambda) = M(w\mathrm{i},\lambda)/L_1(w\mathrm{i},\lambda)$ By definition the function $W_1(w,\lambda)$ is well defined and continuous in the vicinity of the points $w=m,\ \lambda=\lambda_0$ and $w=n,\ \lambda=\lambda_0$. Finally we introduce four auxiliary functions of the variables ρ,φ:

$$\Phi_1(\rho,\varphi) = -\Re e\, W_1(m,\lambda_0)d_{s,m}(\rho,\varphi) - \Im m\, W_1(m,\lambda_0)d_{c,m}(\rho,\varphi),$$

$$\Phi_2(\rho,\varphi) = \Im m\, W_1(m,\lambda_0)d_{s,m}(\rho,\varphi) - \Re e\, W_1(m,\lambda_0)d_{c,m}(\rho,\varphi),$$

$$\Phi_3(\rho,\varphi) = \frac{1}{\rho}(\Re e\, W_1(n,\lambda_0)d_{s,n}(\rho,\varphi) + \Im m\, W_1(n,\lambda_0)d_{c,n}(\rho,\varphi)),$$

$$\Phi_4(\rho,\varphi) = \frac{1}{\rho} - (\Im m\, W_1(n,\lambda_0)d_{s,n}(\rho,\varphi) + \Re e\, W_1(n,\lambda_0)d_{c,n}(\rho,\varphi)). \tag{2.3}$$

Consider the following system of equations:

$$2m^2\Delta_w + \tau_1\Delta_\lambda + \Phi_1(\rho,\varphi) = 0, \qquad \sigma_1\Delta_\lambda + \Phi_2(\rho,\varphi) = 0,$$

$$2n^2\Delta_w + \tau_2\Delta_\lambda + \Phi_3(\rho,\varphi) = 0, \qquad \sigma_2\Delta_\lambda + \Phi_4(\rho,\varphi) = 0 \tag{2.4}$$

with unknown $\Delta_w, \Delta_\lambda, \rho, \varphi$. A solution $\Delta_w^*, \Delta_\lambda^*, \rho^* > 0, \varphi^*$ of the system above is said to be *simple* if it is isolated and has a nonzero Kronecker index [11].

Theorem 1. *Let the polynomials $L(p,\lambda_0)$ and $M(p,\lambda_0)$ be coprime. Let $L(p,\lambda)$ be of the form (2.1) and let the numbers ki be not roots of $L_1(p,\lambda_0)$ at integer k. Let, finally, system (2.4) have a simple solution $(\Delta_w^*, \Delta_\lambda^* \varphi^*)$ with $\rho^* > 0$*

Then Eq. (1.2) has a cycle

$$x(t) = r\sin(wmt) + r\rho^* \cos(wnt + \varphi^*) + o(r), \tag{2.5}$$

whose period $2\pi/w$ is close to 2π, for each sufficiently small $r > 0$ at some λ close to λ_0. The period is greater than 2π if $\Delta_w^* a(\lambda_0) > 0$ and is less than 2π if $\Delta_w^* a(\lambda_0) < 0$. The cycle exists for $\lambda < \lambda_0$, if $\Delta_\lambda^* a(\lambda_0) < 0$, and it exists for $\lambda > \lambda_0$ if $\Delta_\lambda^* a(\lambda_0) > 0$. The equalities

$$\lambda = \lambda_0 + r^{\alpha-1}\Delta_\lambda^* \frac{a(\lambda_0)}{\pi(n^2 - m^2)} + o(r^{\alpha-1}),$$

$$w = 1 - r^{\alpha-1}\Delta_w^* \frac{a(\lambda_0)}{\pi(n^2 - m^2)} + o(r^{\alpha-1}) \tag{2.6}$$

hold.

The main term $x(t) = r\sin(wmt) + r\rho^* \cos(wnt + \varphi^*)$ of the solution (2.5) can be rewritten in the form (1.1). We now discuss some corollaries and modifications of the above theorem.

Clearly, a solution of the system can be isolated only if $\sigma_1, \sigma_2 \neq 0$. In this case the unknowns $\Delta_w^*, \Delta_\lambda^*$ can easily be eliminated and the system takes the form

$$\Psi_1(\rho,\varphi) = 0, \quad \Psi_2(\rho,\varphi) = 0, \tag{2.7}$$

where

$$\Psi_1(\rho,\varphi) = \frac{1}{\sigma_1}\Phi_2(\rho,\varphi) - \frac{1}{\sigma_2}\Phi_4(\rho,\varphi),$$
$$\Psi_2(\rho,\varphi) = \frac{1}{m^2}\left(\Phi_1(\rho,\varphi) - \frac{\tau_1}{\sigma_1}\Phi_2(\rho,\varphi)\right) - \frac{1}{n^2}\left(\Phi_3(\rho,\varphi) - \frac{\tau_2}{\sigma_2}\Phi_4(\rho,\varphi)\right).$$

Thus, Theorem 1 implies

Corollary 1. *Let the polynomials $L(p,\lambda_0)$ and $M(p,\lambda_0)$ be coprime and $\sigma_1, \sigma_2 \neq 0$. Let $L(p,\lambda)$ be of the form (2.1) and the numbers ki are not roots of $L_1(p,\lambda_0)$ at integer k. Let, finally, system (2.7) have a simple solution (ρ^*, φ^*) with $\rho^* > 0$.*

Then Eq. (1.2) has a cycle $x(t) = r\sin(wmt) + r\rho^*\cos(wnt + \varphi^*) + o(r)$, whose period $2\pi/w$ is close to 2π, for each sufficiently small $r > 0$ at some λ close to λ_0.

Let us consider the pair (Ψ_1, Ψ_2) as a mapping in two-dimensional space with the coordinates $\{\rho, \varphi\}$. Recall that $\gamma(\Psi, D)$ denotes the rotation of the vector field Ψ at the boundary of an open bounded set D [11]. The following assertion can be proven in the same way as the theorem above:

Proposition 1. *Let the polynomials $L(p,\lambda_0)$ and $M(p,\lambda_0)$ be coprime and $\sigma_1,\sigma_2 \neq 0$. Let $L(p,\lambda)$ be of the form* (2.1) *and the numbers ki are not roots of $L_1(p,\lambda_0)$ for integer k. Let D be a bounded open set in $\mathbb{R}^2$ with the coordinates $\{\rho,\varphi\}$, such that the rotation $\gamma(\Psi,D)$ differs from zero and D belongs to the half-plane $\rho>0$.*

Then Eq. (1.2) *has a cycle $x(t)=r\sin(wmt)+r\rho^*\cos(wnt+\varphi^*)+o(r)$, whose period $2\pi/w$ is close to 2π, for each sufficiently small $r>0$ at some λ close to λ_0.*

3. Differential equations of the fourth order

Let $L_1(p,\lambda)\equiv M(p,\lambda)\equiv 1$, that is (1.2) can be rewritten as

$$(p^2+\sigma_1(\lambda)p+m^2+\tau_1(\lambda))(p^2+\sigma_2(\lambda)p+n^2+\tau_2(\lambda))x=f(x,\lambda), \qquad p=\mathrm{d}/\mathrm{d}t.$$

By definition $\mathfrak{Im}\, W_1(w,\lambda_0)=0$, $\mathfrak{Re}\, W_1(w,\lambda_0)=1$, and system (2.4) takes the form

$$2m^2\Delta_w+\tau_1\Delta_\lambda=d_{s,m}(\rho,\varphi), \qquad \sigma_1\Delta_\lambda=d_{c,m}(\rho,\varphi), \tag{3.1}$$

$$2n^2\Delta_w+\tau_2\Delta_\lambda=-\frac{1}{\rho}d_{s,n}(\rho,\varphi), \qquad \sigma_2\Delta_\lambda=-\frac{1}{\rho}d_{c,n}(\rho,\varphi). \tag{3.2}$$

Let us note now the equality

$$n\rho d_{c,n}(\rho,\varphi)+md_{c,m}(\rho,\varphi)=0. \tag{3.3}$$

It follows from the relationships

$$\rho d_{c,n}(\rho,\varphi)+d_{c,m}(\rho,\varphi)=\int_0^{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\,\Psi(\sin(mt)+\rho\sin(nt+\varphi))\,\mathrm{d}t=0,$$

where $\Psi(u)$ is a primitive of $F(x)$.

Taking into account Equation (3.3) we can simplify the system above. We write $m\sigma_1=n\sigma_2\rho^2$ instead of the last equation (3.1). If $\sigma_1\sigma_2>0$, then we can find immediately the number ρ^*: $\rho^*=\sqrt{m\sigma_1/n\sigma_2}$. If, however, $\sigma_1\sigma_2<0$, then the system has no solutions.

Let $\sigma_1\sigma_2>0$, and moreover we suppose that $\sigma_1>0$, and $\sigma_2>0$. Then the system has simple solutions if and only if the scalar function

$$d(\varphi)=(n^2\tau_1-m^2\tau_2)d_{c,m}(\rho^*,\varphi)-\sigma_1 n^2 d_{s,m}(\rho^*,\varphi)-\sigma_1 m\sqrt{mn\sigma_1\sigma_2}\,d_{s,n}(\rho^*,\varphi)$$

takes both positive and negative values.

Generally speaking, the condition that *the function $d(\varphi)$ takes both positive and negative values* can be satisfied for some numbers σ_j,τ_j and violated for some other numbers. However, in some important cases the verification of this condition is easy.

Theorem 2. *Let the function $F(x)$ be even, m even and n odd. Let the inequality $\sigma_1\sigma_2>0$ hold and the function $d(\varphi)$ not be equal identically zero. Then Eq.* (1.2) *has a subharmonic oscillation $x(t)=r\sin(wmt)+r\rho^*\cos(wnt+\varphi^*)+o(r)$ with the period $2\pi/w\approx 2\pi$ for each sufficiently small $r>0$ at some $\lambda\approx\lambda_0$.*

Proof. It suffices to note that the integral of the function $d(\varphi)$ equals zero by virtue of the lemmas from the last section of the paper. However, the function $d(\varphi)$ is not identically zero. Therefore this function takes both positive and negative values and the assumptions of Proposition 1 are satisfied.

Verification of the new condition that *the function* $d(\varphi)$ *does not equal zero identically* can be simplified in its turn. For instance, this last condition holds if α is not an integer and $n^2\tau_1 \neq m^2\tau_2$. Actually, it seems that it holds always when the number α is not an integer.

Finally we note that usually we cannot calculate the function $d(\varphi)$ exactly. However, it is not necessary to do it. If we investigate a particular system, it suffices to see that a rough graphical representation of the function $d(\varphi)$ takes values of opposite signs. (Although, some estimates of the precision of the calculations are obviously necessary.) □

4. Delay equations

There is often an inevitable, if rather small, delay in the feedback link of the control system presented in Fig. 1. The interaction of such delays with the linear part of the system could lead to some quite unexpected difficulties, see for instance [14]. Fortunately, in the problem which we consider in the present paper, small delays in the feedback link can be analyzed without difficulty and the results are similar to those presented above.

Let the nonlinearity have the form $f(x(t), x(t-\theta), \lambda)$ with

$$f(x, y, \lambda) = a(\lambda)F(x, y) + o((|x| + |y|)^\alpha) \tag{4.1}$$

and $F(x, y)$ is positively homogeneous: $F(rx, ry) = r^\alpha F(x, y),\ r > 0$. Define $u(t; \rho, \varphi) = \sin(mt) + r\sin(nt + \varphi)$ and

$$d_{s,m}(\rho, \varphi) = \int_0^{2\pi} \sin(mt) F(u(t; \rho, \varphi), u(t-\theta; \rho, \varphi))\,\mathrm{d}t,$$

$$d_{c,m}(\rho, \varphi) = \int_0^{2\pi} \cos(mt) F(u(t; \rho, \varphi), u(t-\theta; \rho, \varphi))\,\mathrm{d}t,$$

$$d_{s,n}(\rho, \varphi) = \int_0^{2\pi} \sin(nt + \varphi) F(u(t; \rho, \varphi), u(t-\theta; \rho, \varphi))\,\mathrm{d}t,$$

$$d_{c,n}(\rho, \varphi) = \int_0^{2\pi} \cos(nt + \varphi) F(u(t; \rho, \varphi), u(t-\theta; \rho, \varphi))\,\mathrm{d}t.$$

Naturally the straightforward analog of equality (3.3) does not hold. All four functions here depend on the delay time θ. The functions d_γ possess some special properties that simplify analysis of system (2.4) (periodicity with a relatively small period, oddness or evenness, etc.)

Theorem 3. *Let the polynomials* $L(p, \lambda_0)$ *and* $M(p, \lambda_0)$ *be coprime. Let* $L(p, \lambda)$ *be of the form* (2.1) *and let the numbers ki be not roots of* $L_1(p, \lambda_0)$ *at integer k. Let the* *nonlinearity $f(x, y, \lambda)$ have the form (4.1) and the Lipschitz condition*

$$|f(x_1, y_1) - f(x_2, y_2)| \leqslant \beta(r)(|x_1 - x_2| + |y_1 - y_2|),$$

$$r = \max\{|x_1|, |y_1|, |x_2|, |y_2|\},$$

hold with some $\beta(r) \to 0$ at $r \to 0$. Let, finally, system (2.4) have a simple solution $\Delta_w^* \neq 0, \Delta_\lambda^* \neq 0, \rho^* > 0, \varphi^*$. Then the equation

$$L\left(\frac{\mathrm{d}}{\mathrm{d}t}, \lambda\right) x(t) = M\left(\frac{\mathrm{d}}{\mathrm{d}t}, \lambda\right) f(x(t), x(t-\theta), \lambda)$$

has for each sufficiently small $r > 0$ a solution $x(t) = r\sin(wmt) + r\rho^*\cos(wnt + \varphi^*) + o(r)$, whose period $2\pi/w$ is close to 2π, at some λ close to λ_0. The period of this cycle is greater than 2π if $\Delta_w^* a(\lambda_0) > 0$ and it is less than 2π if $\Delta_w^* a(\lambda_0) < 0$. The cycle exists at $\lambda < \lambda_0$, if $\Delta_\lambda^* a(\lambda_0) < 0$, and it exists at $\lambda > \lambda_0$ if *$\Delta_\lambda^* a(\lambda_0) > 0$. Equalities* (2.6) *hold.*

The theorem above can be supplemented by analogs of Corollary 1 and Proposition 1.

5. Bifurcations at infinity

Our previous construction can be easily modified to embrace the Hopf bifurcation at infinity.

Now the value of the nonlinearity at zero is not important. Instead we assume that the nonlinearity is sublinear at infinity in the sense that the estimate

$$\lim_{|x|\to\infty} \sup_{\lambda} \frac{|f(x, \lambda)|}{|x|} = 0$$

holds. Below we will use, however, a stronger assumption that the nonlinearity is uniformly bounded. The case of unbounded, but sublinear, functions f is technically much more difficult [8] and is beyond the scope of the present paper.

We introduce the functions:

$$d_{s,m}(\rho, \varphi) = \int_0^{2\pi} \sin(mt)\,\mathrm{sign}(\sin(mt) + \rho\sin(nt + \varphi))\,\mathrm{d}t,$$

$$d_{c,m}(\rho, \varphi) = \int_0^{2\pi} \cos(mt)\,\mathrm{sign}(\sin(mt) + \rho\sin(nt + \varphi))\,\mathrm{d}t,$$

$$d_{s,n}(\rho, \varphi) = \int_0^{2\pi} \sin(nt + \varphi)\,\mathrm{sign}(\sin(mt) + \rho\sin(nt + \varphi))\,\mathrm{d}t,$$

$$d_{c,n}(\rho, \varphi) = \int_0^{2\pi} \cos(nt + \varphi)\,\mathrm{sign}(\sin(mt) + \rho\sin(nt + \varphi))\,\mathrm{d}t.$$

These functions are not smooth, but they are continuous. Let us construct an analog of system (2.4). As above we suppose that the polynomial $L(p, \lambda)$ is represented in the form (1.4), the coefficients of the polynomials $L_1(p, \lambda)$ and $M(p, \lambda)$ are continuous in λ, and we will use again notations (2.2). It can be proven that there exist at least two families of cycles of large magnitude, one consisting of cycles with period close to $2\pi/m$, and the other consisting of cycles with periods close to $2\pi/n$. We are interested in existence of subharmonics of large magnitude with the minimal period close to 2π.

Let the nonlinearity $f(x, \lambda)$ be of the form

$$f(x, \lambda) = F(x, \lambda) + \Phi(x, \lambda), \tag{5.1}$$

where $F(x, \lambda)$ satisfies the saturation conditions, i.e. the limits

$$\lim_{\xi \to -\infty} F(\xi, \lambda) = \psi_-(\lambda), \qquad \lim_{\xi \to +\infty} F(\xi, \lambda) = \psi_+(\lambda). \tag{5.2}$$

are well defined. Let the estimate

$$\lim_{x \to \pm\infty} \sup_{\lambda} \left| \frac{1}{x} \int_0^x \Phi(u, \lambda)\, \mathrm{d}u \right| = 0 \tag{5.3}$$

also hold, i.e. the function Φ has sublinear primitives.

Theorem 4. *Let the polynomials $L(p, \lambda_0)$ and $M(p, \lambda_0)$ be coprime. Let $L(p, \lambda)$ be of the form (2.1) and let the numbers ki be not roots of $L_1(p, \lambda_0)$ at integer k. Let, finally, system (2.4) have a simple solution $\Delta_w^* \neq 0$, $\Delta_\lambda^* \neq 0$, $\rho^* > 0$, φ^*. Suppose also that representation (5.1) as well as conditions (5.2), (5.3) hold, and $\psi_-(\lambda_0) \neq \psi_+(\lambda_0)$.*

Then Eq. (1.2) has a cycle $x(t) = r \sin(wmt) + r\rho^* \cos(wnt + \varphi^*) + o(r)$, whose period $2\pi/w$ is close to 2π, for each sufficiently large $r > 0$ at some λ close to λ_0. The period is greater than 2π if $\Delta_w^* a(\lambda_0) > 0$ and it is less than 2π if $\Delta_w^* a(\lambda_0) < 0$. The cycle exists at $\lambda < \lambda_0$, if $\Delta_\lambda^* a(\lambda_0) < 0$, and it exists at $\lambda > \lambda_0$ if $\Delta_\lambda^* a(\lambda_0) > 0$. Equalities (2.6) hold.

The proof is rather similar to that of Theorem 1. Also the following auxiliary statement should be used:

Lemma 1. *Let the function $f(x) = f(x, \lambda)$ satisfy the restrictions listed in the theorem above. Let mes$\{e(t): t \in [a, b], e(t) = 0\} = 0$. Then the equality*

$$\lim_{\xi \to \infty} \sup_{\|h(t)\|_{C^1} \leqslant R} \left| \int_a^b g(t)(f(\xi e(t) + h(t), \lambda) - H(t, \lambda))\, \mathrm{d}t \right| = 0$$

with

$$H(t, \lambda) = \frac{(\psi_+(\lambda) + \psi_-(\lambda))}{2} - \frac{(\psi_+(\lambda) - \psi_-(\lambda))}{2} \operatorname{sign}(e(t)).$$

holds at each positive R.

To conclude this section, we mention that the uniformly bounded functions $\Phi(x)$ quite often have sublinear primitives. In a sense almost all functions F have this property. For instance, all periodic and almost periodic functions with zero average (nonzero constant average is included in $F(x, \lambda)$), functions $\sin(x|x|^\beta)$ for all $\beta > -1$, all functions vanishing at infinity, etc. The function $\sin(\ln(1 + |x|)) \operatorname{sign}(x)$ does not satisfy (5.3).

6. Proof of Theorem 1

6.1. Time substitution

Let us perform a time rescaling $t = w\tau$; now we are investigating 2π-periodic solution of the equation

$$L\left(w\frac{\mathrm{d}}{\mathrm{d}t},\lambda\right)x = M\left(w\frac{\mathrm{d}}{\mathrm{d}t},\lambda\right)f(x,\lambda) \tag{6.1}$$

rather than cycles of unknown period $T = 2\pi/w$, with $w \approx 1$. Instead we consider w as an additional independent variable. We will construct the aforementioned 2π-periodic solution as a real Fourier series with respect to a trigonometric system. Afterwards the principal equation will come to some equalities binding the coefficients at the leading harmonics in the left- and right-hand sides of the equation.

6.2. Linear spaces and operators

Denote by Ω and Λ the small vicinities of the numbers 1 and λ_0 respectively such that the values $\pm wk\mathrm{i}$ do not annihilate neither the polynomial $L(\cdot,\lambda)$ at $k \in \mathbb{Z}, k \neq m, k \neq n$, nor the polynomial $M(\cdot,\lambda)$ at $k = m, n$. Such vicinities exist due to the hypothesis about the structure of the set purely imaginary roots of $L(\cdot,\lambda_0)$, on the one hand, and because $M(\cdot,\lambda)$ is coprime with $L(\cdot,\lambda)$, on the other hand.

Let $w \in \Omega$ and $\lambda \in \Lambda$. Consider[2] the four-dimensional subspace $\Pi \in L^2$, spanned over the functions $\sin(mt), \cos(mt), \sin(nt), \cos(nt)$ and denote by P the orthogonal projector onto Π. Consider also the projector $Q = I - P$ and the subspace $\Pi^* = QL^2$; codim $\Pi^* = 4$.

Introduce the linear operator $A(w,\lambda)$ ($w \in \Omega, \lambda \in \Lambda$) which corresponds to each function $u(t) \in \Pi^*$ the unique solution $x(t) \in \Pi^*$ of the linear equation

$$L\left(w\frac{\mathrm{d}}{\mathrm{d}t},\lambda\right)x = M\left(w\frac{\mathrm{d}}{\mathrm{d}t},\lambda\right)u(t). \tag{6.2}$$

The existence of such solution $x(t)$ follows immediately from the definition of the neighbourhoods Ω and Λ, together with the inclusion $u(t) \in \Pi^*$; this solution should be unique by the inclusion $x(t) \in \Pi^*$. For $w \neq 1$ and $\lambda \neq \lambda_0$ the operators $A(w,\lambda)$ are defined formally onto the whole space L^2, however, their norms increase unboundedly for w approaching 1, and λ approaching k λ_0. It is important that the norms of the restrictions of these operators over the subspace Π^* admit a uniform estimate from above over Ω:

$$\|A(w,\lambda)\|_{\Pi^* \to \Pi^*} \leqslant c_* = \sup_{\lambda \in \Lambda; w \in \Omega} q_*(w,\lambda) < \infty,$$

$$q_*(w,\lambda) \stackrel{\text{def}}{=} \sup_{k \in \mathbb{Z}, k \neq \pm m, k \neq \pm n} \left|\frac{M(wk\mathrm{i},\lambda)}{L(wk\mathrm{i},\lambda)}\right|.$$

[2] All the spaces below consist of scalar functions to be defined onto the segment $[0, 2\pi]$.

Let C_0 denote the space of continuous 2π-periodic functions with the uniform norm. Each operator $A(w,\lambda)$ acts completely continuously from Π^* to C_0, it also acts continuously from $C_0 \cap \Pi^*$ to C^1. In the space Π^* the operator $A(w,\lambda)h$ is completely continuous, moreover the operators $A(w,\lambda)h : \Omega \times \Lambda \times \Pi^* \to \Pi^*$ are also completely continuous, The linear operators $A(w,\lambda)Q$ are defined on the whole L^2; their norms satisfy $\|A(w,\lambda)Q\|_{L^2\to L^2} = q_*(w,\lambda)$, and

$$\|A(w,\lambda)Q\|_{L^2\to C_0} = q^*(w,\lambda)$$

$$\stackrel{\text{def}}{=} \frac{1}{\sqrt{\pi}} \left(\frac{1}{2} \left| \frac{M(0,\lambda)}{L(0,\lambda)} \right|^2 + \sum_{k=2,3,\ldots;\, k\neq n, k\neq m} \left| \frac{M(wk\mathrm{i},\lambda)}{L(wk\mathrm{i},\lambda)} \right|^2 \right)^{1/2}$$

and, therefore, admit the estimate

$$\|A(w,\lambda)Q\|_{C\to C} \leqslant c^* < \infty \tag{6.3}$$

uniformly over $w \in \Omega$, $\lambda \in \Lambda$.

6.3. The equivalent system of equations

To begin with, we formulate a simple assertion, which follows from invariance of the subspaces P_k with respect to differentiation.

Lemma 2. *Let $w \in \Omega, \lambda \in \Lambda$. Then the functions $x(t) = r(\sin(mt) + \rho \sin(nt+\varphi)) + h(t)$, $h \in \Pi^* = QL^2$ and $u(t) \in L^2$ satisfy (6.2), if and only if the equalities $h = A(w,\lambda)Qu$ and*

$$\Re e\, \frac{L(wm\mathrm{i},\lambda)}{M(wm\mathrm{i},\lambda)} = \frac{1}{r\pi} \int_0^{2\pi} \sin(mt)u(t)\,\mathrm{d}t,$$

$$\Re e\, \frac{L(wn\mathrm{i},\lambda)}{M(wn\mathrm{i},\lambda)}\rho = \frac{1}{r\pi} \int_0^{2\pi} \sin(nt+\varphi)u(t)\,\mathrm{d}t,$$

$$\Im m\, \frac{L(wm\mathrm{i},\lambda)}{M(wm\mathrm{i},\lambda)} = \frac{1}{r\pi} \int_0^{2\pi} \cos(mt)u(t)\,\mathrm{d}t,$$

$$\Im m\, \frac{L(wn\mathrm{i},\lambda)}{M(wn\mathrm{i},\lambda)}\rho = \frac{1}{r\pi} \int_0^{2\pi} \cos(nt+\varphi)u(t)\,\mathrm{d}t$$

are valid.

By the above lemma the function

$$x(t) = r(\sin(mt) + \rho\sin(nt+\varphi)) + h(t), \qquad r > 0, \tag{6.4}$$

where $h(t)\in \Pi^*$, represents a 2π-periodic solution of Eq. (6.1) if and only if it satisfies the following system of five equations:

$$\begin{aligned}
\Re e\,\frac{L(wm\mathrm{i},\lambda)}{M(wm\mathrm{i},\lambda)} &= \frac{1}{r\pi}\int_0^{2\pi} \sin(mt) f(x(t),\lambda)\,\mathrm{d}t,\\
\Im m\,\frac{L(wm\mathrm{i},\lambda)}{M(wm\mathrm{i},\lambda)} &= \frac{1}{r\pi}\int_0^{2\pi} \cos(mt) f(x(t),\lambda)\,\mathrm{d}t,\\
\Re e\,\frac{L(wn\mathrm{i},\lambda)}{M(wn\mathrm{i},\lambda)}\rho &= \frac{1}{r\pi}\int_0^{2\pi} \sin(nt+\varphi) f(x(t),\lambda)\,\mathrm{d}t,\\
\Im m\,\frac{L(wn\mathrm{i},\lambda)}{M(wn\mathrm{i},\lambda)}\rho &= \frac{1}{r\pi}\int_0^{2\pi} \cos(nt+\varphi) f(x(t),\lambda)\,\mathrm{d}t, \qquad (6.5)\\
h &= A(w,\lambda)Qf(x,\lambda).
\end{aligned}$$

We emphasize that in representation (6.4) the null-projection on $\cos(mt)$ is fixed as well as the sign of the coefficient at $\sin(mt)$. It did not cause the lack of generality since any shifted function $x(t+\alpha)$ satisfies our system together with $x(t)$. Thus formula (6.4) suppressed the nonuniqueness of the solution. We have simply selected a single convenient representative from the whole set of periodic solutions corresponding to one and the same cycle.

6.4. *Another form of the equivalent system*

Let us rewrite the set of equations (6.5) in a slightly different form. Introduce the notation

$$G(p,\lambda)\stackrel{\text{def}}{=}(p^2+\sigma_1(\lambda)p+m^2+\tau_1(\lambda))(p^2+\sigma_2(\lambda)p+n^2+\tau_2(\lambda)).$$

This polynomial of the degree 4 satisfies the relation $L(p,\lambda)=G(p,\lambda)L_1(p,\lambda)$. The equalities

$$\begin{aligned}
\Re e\, L/M &= \Re e\, G \Re e\, L_1/M - \Im m\, G\, \Im m\, L_1/M,\\
\Im m\, L/M &= \Im m\, G \Re e\, L_1/M + \Re e\, G \Im m\, L_1/M,
\end{aligned}$$

imply easily the following form of the first equation (6.5):

$$\begin{aligned}
\Re e\, G(wm\mathrm{i},\lambda) = \frac{1}{r\pi}\Bigg(&\Re e\, W_1(wm,\lambda)\int_0^{2\pi} \sin(mt) f(x(t),\lambda)\,\mathrm{d}t\\
&+\Im m\, W_1(wm,\lambda)\int_0^{2\pi} \cos(mt) f(x(t),\lambda)\,\mathrm{d}t\Bigg). \qquad (6.6)
\end{aligned}$$

Continuity of the functions $W_1(wm,\lambda)$ and $W_1(wn,\lambda)$ in the variables w and λ nearby the points $w=1,\lambda=\lambda_0$ implies the approximate equalities

$$W_1(wm,\lambda)=W_1(m,\lambda_0)+\delta_1(\lambda-\lambda_0,w-1),$$

$$W_1(wn,\lambda)=W_1(n,\lambda_0)+\delta_2(\lambda-\lambda_0,w-1).$$

Here and below the symbols $\delta_j(\cdot,\cdot)$ and $\delta_j(\cdot)$ denote the functional terms which are infinitesimally small at small values of their arguments.

The function $G(wm\mathrm{i},\lambda)$ is smooth in w and λ at the points $w=1,\lambda=\lambda_0$. Therefore

$$\Re e\, G(wm\mathrm{i},\lambda)=(n^2-m^2)(2m^2(1-w)+\tau_1(\lambda-\lambda_0)) \\ +(\lambda-\lambda_0)\delta_3(\lambda-\lambda_0,1-w)+(1-w)\delta_4(\lambda-\lambda_0,1-w). \quad (6.7)$$

We will try to find λ and w in the form

$$\lambda=\lambda_0+r^{\alpha-1}\Delta_\lambda\,\frac{a(\lambda_0)}{\pi(n^2-m^2)},\quad w=1-r^{\alpha-1}\Delta_w\,\frac{a(\lambda_0)}{\pi(n^2-m^2)}. \quad (6.8)$$

Here Δ_λ and Δ_w are the new unknowns which should be close to Δ_λ^* and Δ_w^* as $r\to 0$. Let us substitute representations (6.8) into Eq. (6.7) and, afterwards into Eq. (6.6). We obtain the new equation

$$a(\lambda_0)r^\alpha(2m^2\Delta_w+\tau_1\Delta_\lambda)=r^\alpha\delta_{11}(r)+\Re e\, W_1(wm,\lambda)\int_0^{2\pi}\sin(mt)f(x(t),\lambda)\,\mathrm{d}t \\ +\Im m\, W_1(wm,\lambda)\int_0^{2\pi}\cos(mt)f(x(t),\lambda)\,\mathrm{d}t. \quad (6.9)$$

We rewrite each integral $\int_0^{2\pi} e(t,\varphi)f(x(t),\lambda)\,\mathrm{d}t$ as

$$\int_0^{2\pi} e(t,\varphi)f(x(t),\lambda)\,\mathrm{d}t=a(\lambda)r^\alpha\int_0^{2\pi} e(t,\varphi)F(\sin(mt)+\rho\sin(nt+\varphi))\,\mathrm{d}t \\ +\int_0^{2\pi} e(t,\varphi)\Phi(x(t),\lambda)\,\mathrm{d}t \\ +a(\lambda)\int_0^{2\pi} e(t,\varphi)(F(x(t),\lambda)-F(x(t)-h(t),\lambda))\,\mathrm{d}t.$$

Finally, the first equation of (6.5) takes the form

$$2m^2\Delta_w+\tau_1\Delta_\lambda=\Re e\, W_1(m,\lambda_0)d_{s,m}(\rho,\varphi)+\Im m\, W_1(m,\lambda_0)d_{c,m}(\rho,\varphi) \\ +\eta_1(r,h,\rho,\varphi,\Delta_\lambda,\Delta_w).$$

Here $e(t,\varphi)$ denotes one of the functions $\sin(mt),\cos(mt),\sin(nt+\varphi),\cos(nt+\varphi)$ and by $\eta_1(r,h,\rho,\varphi,\Delta_\lambda,\Delta_w)$ we denote the rest of components and addends. Analogously, the next three equations (6.5) can be rewritten as

$$\sigma_1 \Delta_\lambda = -\Im m\, W_1(m,\lambda_0) d_{s,m}(\rho,\varphi) + \Re e\, W_1(m,\lambda_0) d_{c,m}(\rho,\varphi) + \eta_2(r,h,\rho,\varphi,\Delta_\lambda,\Delta_w),$$

$$2n^2 \Delta_w + \tau_2 \Delta_\lambda = -\frac{1}{\rho}(\Re e\, W_1(n,\lambda_0) d_{s,n}(\rho,\varphi) + \Im m\, W_1(n,\lambda_0) d_{c,n}(\rho,\varphi)) + \eta_3(r,h,\rho,\varphi,\Delta_\lambda,\Delta_w),$$

$$\sigma_2 \Delta_\lambda = -\frac{1}{\rho}(-\Im m\, W_1(n,\lambda_0) d_{s,n}(\rho,\varphi) + \Re e\, W_1(n,\lambda_0) d_{c,n}(\rho,\varphi)) + \eta_4(r,h,\rho,\varphi,\Delta_\lambda,\Delta_w).$$

Taken into account formulas (2.3) we can summarize our calculations as follows:

Lemma 3. *The system (6.5) is equivalent to the system*

$$\begin{aligned}
&2m^2 \Delta_w + \tau_1 \Delta_\lambda + \Phi_1(\rho,\varphi) + \eta_1(r,h,\rho,\varphi,\Delta_\lambda,\Delta_w) = 0,\\
&\sigma_1 \Delta_\lambda + \Phi_2(\rho,\varphi) + \eta_2(r,h,\rho,\varphi,\Delta_\lambda,\Delta_w) = 0,\\
&2n^2 \Delta_w + \tau_2 \Delta_\lambda + \Phi_3(\rho,\varphi) + \eta_3(r,h,\rho,\varphi,\Delta_\lambda,\Delta_w) = 0,\\
&\sigma_2 \Delta_\lambda + \Phi_4(\rho,\varphi) + \eta_4(r,h,\rho,\varphi,\Delta_\lambda,\Delta_w) = 0,\\
&h - A(w,\lambda) Q f(x,\lambda) = 0.
\end{aligned} \tag{6.10}$$

In this system we consider r as a parameter, whereas $w, \rho, \varphi, \lambda$ and $h(t)$ are the unknowns. To prove the theorem we should establish the solvability of system (6.6) at all sufficiently small values of the parameter $r > 0$.

Regarding the functions $\eta_j(r,h,\rho,\varphi,\Delta_\lambda,\Delta_w)$ we know the following: if

$$\|h(t)\|_C \leqslant K r^\alpha, \tag{6.11}$$

for some constant K, then all functions η_j are uniformly small as $r \to 0$.

6.5. Finalizing the proof

At this stage we can apply some standard topological tools to prove solvability of system (6.10) which would imply solvability of (6.5) by Lemma 3.

Consider the space $\mathbb{E} = \{\mathbb{R}^4 \times \Pi^* \cap C_0\}$, which is treated as the space of the unknown variables $\rho, \varphi, \Delta_\lambda, \Delta_w, h$. Let us choose a ball $B_1 \subset \mathbb{R}^4$ with a sufficiently small radius $\mathbb{R}^4$ centred at a point $\rho^*, \varphi^*, \Delta_\lambda^*, \Delta_w^*$. Introduce also a ball B_2 in the space Π^* with a sufficiently small radius centred at zero and denote by $G \in \mathbb{E}$ the direct product of these two balls.

Let us consider the deformation $\mathscr{F}(\rho,\varphi,\Delta_\lambda,\Delta_w,h,) = \{\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_3,\mathscr{F}_4,\mathscr{F}_5\}$ whose components are defined by the formulae

$$\mathscr{F}_1 = 2m^2 \Delta_w + \tau_1 \Delta_\lambda + \Phi_1(\rho,\varphi) + \xi \eta_1(r,h,\rho,\varphi,\Delta_\lambda,\Delta_w),$$

$$\mathscr{F}_2 = \sigma_1 \Delta_\lambda + \Phi_2(\rho, \varphi) + \xi \eta_2(r, h, \rho, \varphi, \Delta_\lambda, \Delta_w),$$

$$\mathscr{F}_3 = 2n^2 \Delta_w + \tau_2 \Delta_\lambda + \Phi_3(\rho, \varphi) + \xi \eta_3(r, h, \rho, \varphi, \Delta_\lambda, \Delta_w),$$

$$\mathscr{F}_4 = \sigma_2 \Delta_\lambda + \Phi_2(\rho, \varphi) + \xi \eta_4(r, h, \rho, \varphi, \Delta_\lambda, \Delta_w),$$

$$\mathscr{F}_5 = h - \xi A(w, \lambda) Q f(x, \lambda)$$

at the boundary of the set G. Here all the terms η_j depend on $r, h, \rho, \varphi, \Delta_\lambda, \Delta_w$ and $\xi \in [0, 1]$ is the deformation parameter.

We will show in the next subsection that this deformation is non-singular for sufficiently small positive $r > 0$. This fact implies that the rotation of the vector field $\mathscr{F}$ at the boundary ∂G is well defined and assumes one and the same value at all $\xi \in [0, 1]$. In particular the rotation at $\xi = 0$ equal that at $\xi = 1$. Clearly, at $\xi = 1$ the deformation just coincides with the mapping for which zeros provide the solutions of the system in question. On the other hand, at $\xi = 0$ the rotation differs from zero: this follows from the standard product formula [11]. Thus to finalize the proof we should prove the following assertion:

Lemma 4. *The deformation* $\mathscr{F}$ *is nonsingular.*

Proof. Let us establish an a priori estimate (6.11) of all zeros of the deformation that belong to G. Suppose that $h = \xi A(w, \lambda) Q f(x, \lambda)$ and $r > 0$ is sufficiently small. The function $f(x, \lambda)$ admits the estimate $f(x, \lambda) \leqslant \varepsilon |x|$ at each $\varepsilon > 0$ for all sufficiently small r. This estimate together with (6.3) implies the inequality $\|h\|_C \leqslant c_0 \varepsilon \|A(w, \lambda)\|_{C \to C}$ $(r + \|h\|_C)$. Therefore $\|h\|_C$ satisfies $\|h\|_C \leqslant c_1 r$ and, further, $\|x\|_C \leqslant c_2 r$. On the other hand, $|f(x, \lambda)| \leqslant c_3 |x|^\alpha$ by the assumptions of the theorem and inequality (6.11) holds.

The nondegeneracy of the deformation is clear. The set G is common for all r, for small r there are no zeros on its boundary ∂G: on the part $\{h \in \partial B_2\}$ of ∂G the infinite dimensional component h in nondegenerate, on the part $\{(\rho, \varphi, \Delta_\lambda, \Delta_w) \in \partial B_2\}$ of ∂G one of the first four components in nondegenerate according to isolated character of simple solutions. □

7. Properties of the functions $d_{s,m}$, $d_{c,m}$, $d_{s,n}$, $d_{c,n}$

The symbol γ below denotes one of the four indices (s, m); (c, m); (s, n); (c, n).

Proposition 2. *The functions* $d_\gamma(\rho, \varphi)$ *have the following properties*:

(a) ($2\pi/m$-*periodicity*) $d_\gamma(\rho, 2\pi/m + \varphi) = d_\gamma(\rho, \varphi)$,
(b) (*Evenness and oddness*) *If the function* $F(x)$ *is even*: $F(-x) = F(x)$, *then*

$$d_{s,m}(\rho, -\varphi) = -d_{s,m}(\rho, \varphi), \quad d_{s,n}(\rho, -\varphi) = -d_{s,n}(\rho, \varphi),$$

$$d_{c,m}(\rho, -\varphi) = d_{c,m}(\rho, \varphi), \quad d_{c,n}(\rho, -\varphi) = d_{c,n}(\rho, \varphi).$$

If the function be odd: $F(-x) = -F(x)$, *then*

$$d_{s,m}(\rho,-\varphi) = d_{s,m}(\rho,\varphi), \quad d_{s,n}(\rho,-\varphi) = d_{s,n}(\rho,\varphi),$$

$$d_{c,m}(\rho,-\varphi) = -d_{c,m}(\rho,\varphi), \quad d_{c,n}(\rho,-\varphi) = -d_{c,n}(\rho,\varphi).$$

(c) (*Symmetries with respect to* π/m) *If the function* $F(x)$ *is even, then*

$$d_{s,m}(\rho,\pi/m-\varphi) = -d_{s,m}(\rho,\pi/m+\varphi),$$

$$d_{s,n}(\rho,\pi/m-\varphi) = -d_{s,n}(\rho,\pi/m+\varphi),$$

$$d_{c,m}(\rho,\pi/m-\varphi) = d_{c,m}(\rho,\pi/m+\varphi),$$

$$d_{c,n}(\rho,\pi/m-\varphi) = d_{c,n}(\rho,\pi/m+\varphi).$$

If the function $F(x)$ *is odd, then*

$$d_{s,m}(\rho,\pi/m-\varphi) = d_{s,m}(\rho,\pi/m+\varphi), \quad d_{s,n}(\rho,\pi/m-\varphi) = d_{s,n}(\rho,\pi/m+\varphi),$$

$$d_{c,m}(\rho,\pi/m-\varphi) = -d_{c,m}(\rho,\pi/m+\varphi), \quad d_{c,n}(\rho,\pi/m-\varphi) = -d_{c,n}(\rho,\pi/m+\varphi).$$

Proof. We will prove only Assertion (a); other two assertions can be proved similarly. Let us consider the chain of equalities

$$\begin{aligned} d_{s,m}(\rho,\varphi) &= \int_0^{2\pi} \sin(mt)F(\sin(mt)+\rho\sin(nt+\varphi))\,\mathrm{d}t \\ &= \int_0^{2\pi} \sin(m(t+2\pi/m))F(\sin(m(t+2\pi/m)) \\ &\quad +\rho\sin(n(t+2\pi/m)+\varphi))\,\mathrm{d}t \\ &= \int_0^{2\pi} \sin(mt)F(\sin(mt+\rho\sin(nt+2n\pi/m+\varphi))\,\mathrm{d}t \\ &= d_{s,m}(\rho,2n\pi/m+\varphi). \end{aligned}$$

By virtue of these equalities the function $d_{s,m}(\rho,\varphi)$ is $2n\pi/m$-periodic. Analogously one can establish $2n\pi/m$-periodicity of three other functions $d_\gamma(\rho,\varphi)$. On the other hand, these functions are by definition 2π-periodic. Since m is coprime with n the equality $km+sn=1$ holds for some integers k,s. Thus

$$\begin{aligned} d_\gamma(\rho,\varphi+2\pi/m) &= d_\gamma(\rho,\varphi+2\pi(km+sn)/m) \\ &= d_\gamma(\rho,\varphi+2k\pi+s2n\pi/m) = d_\gamma(\rho,\varphi), \end{aligned}$$

and the proof is completed. □

Some graphical information concerning the functions $d_\gamma(\rho,\phi)$ is provided in Figs. 2 and 3. In Fig. 2 we give the graphs of our functions for the case $m=2, n=5$, $F(x)=|x|^{1.6}$. In Fig. 3 we give the graphs of the functions for the case $m=2, n=5$, $F(x)=\operatorname{sign} x$, as used in Section 5.

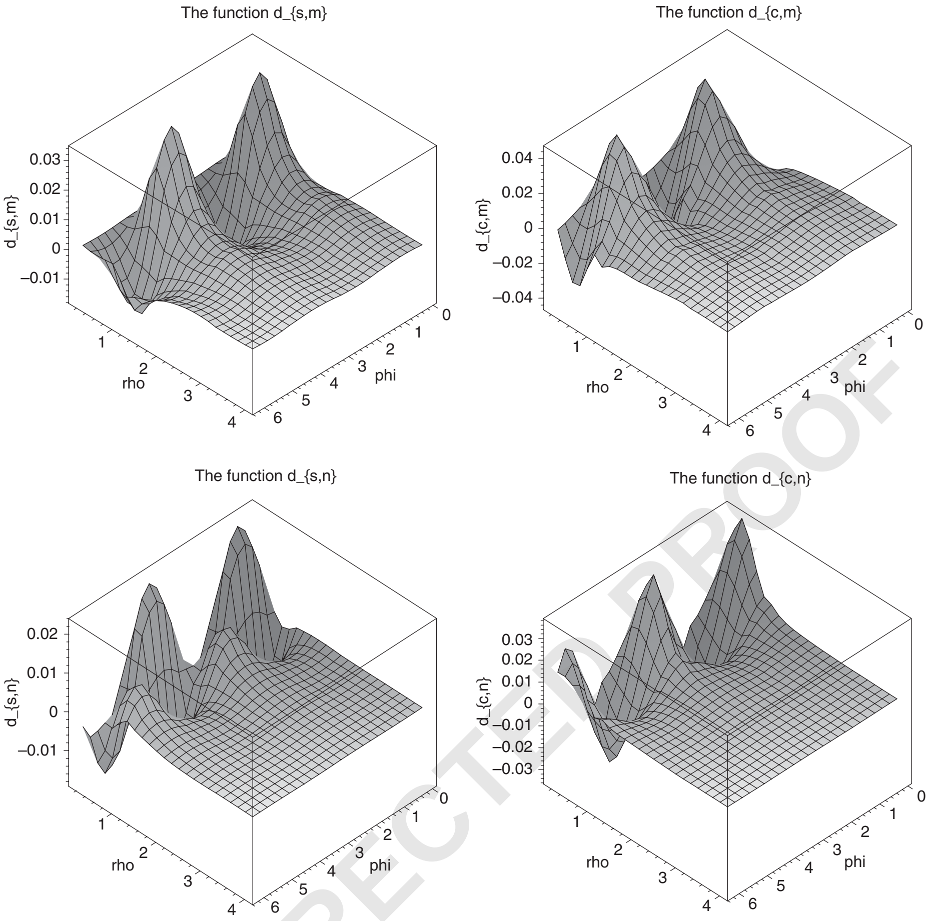

Fig. 2. The functions $d_\gamma(\rho,\varphi)$ of $\varphi \in [0,2\pi] \times \rho \in [.1,4.1]$ for $F(x)=|x|^{1.6}$.

8. Conclusions and discussions

We have considered the problem of generating periodic 'double frequency oscillations', similar to the functions

$$x(t) = r_1 \sin(wmt) + r_2 \sin(wnt+\varphi) \tag{8.1}$$

in the situation of weak Hopf resonance. Single-loop control systems described by equations

$$L\left(\frac{\mathrm{d}}{\mathrm{d}t},\lambda\right)x = M\left(\frac{\mathrm{d}}{\mathrm{d}t},\lambda\right) f(x,\lambda) \tag{8.2}$$

with sublinear feedback f have been analyzed. The principal result is that such oscillations often exist (and are reasonably robust) if the main homogeneous part of the

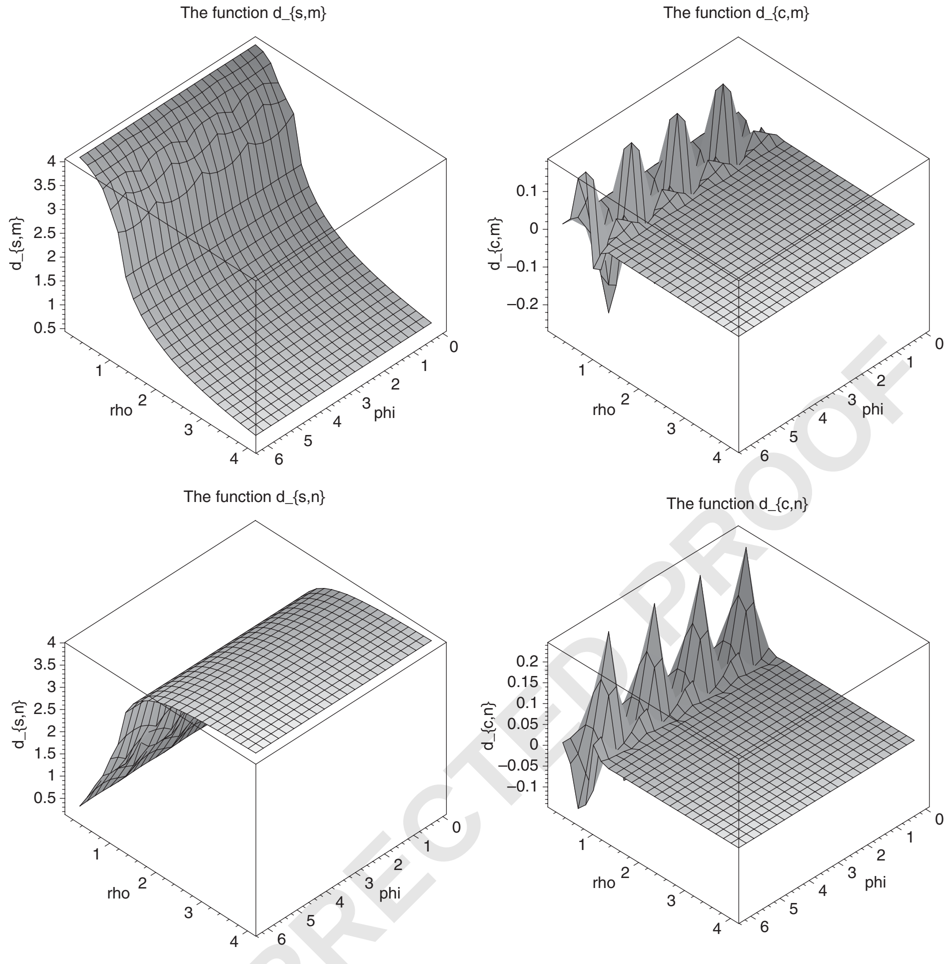

Fig. 3. The functions $d_\gamma(\rho,\varphi)$ of $\varphi \in [0, 2\pi] \times \rho \in [.1, 4.1]$ for $F(x) = \operatorname{sign} x$.

nonlinearity is not a positive integer power of x, for example, if

$$f(x,\lambda) = a(\lambda)x|x|^{\alpha-1} + o(x^\alpha) \quad \text{or} \quad f(x,\lambda) = a(\lambda)|x|^\alpha + o(x^\alpha), \tag{8.3}$$

where α is not an even positive integer. From the technical side, analysis of systems with the aforementioned nonlinearities (8.3) reduces to investigation of a system of two auxiliary scalar equation with two variables. The latter system can easily be analyzed on a computer with a sufficient accuracy.

The existence of solutions of type (8.1) is rare if a nonlinearity f has a quadratic or cubic leading term. So, in general terms, we have demonstrated that the behaviour of system (8.2) becomes substantially simpler and more robust, when some 'nonpolynomial' terms are used as the main homogeneous part of the nonlinearity. It seems that this observation, that the dynamics can be simplified by introducing nonpolynomial nonlinearities, is of a rather general nature. Below we list some situations when analogs of our methods can be used.

Resonances $n:1$: Let $n > m = 1$. Then, by the Hopf bifurcation theorem, for λ close to λ_0 there exists a family of oscillations with periods close to $2\pi/n$. The question is whether there are any oscillations with a minimal period close to 2π. If $\alpha=2$ and $F(x)=x^2+ax^3$ then the answer depends on n. The case $n=2$ is discussed in detail in [9], see also [4]. The point is that for $n>3$ the situation becomes similar to the case of a weak $m:n$ resonance, as considered in the present paper, and oscillations with approximate minimal period close to 2π exist for many nonlinearities with non-polynomial main homogeneous parts.

Weak resonance in systems of general type: Our methods could be adjusted for the weakly resonant systems $x' = A(\lambda)x + f(x,\lambda)$, $x \in \mathbb{R}^N$. Roughly speaking, it is the case when the matrix $A(\lambda_0)$ has two pairs of eigenvalues $\pm wm\mathrm{i}$ and $\pm nw\mathrm{i}$. The construction similar to described in the paper can be performed in the case when the main homogeneous part of f is not a homogeneous polynomial in N variables.

Hopf bifurcation for mappings: Consider the mapping $F:\mathbb{R}^N \to \mathbb{R}^N$. Suppose that $f(0)=0$ and the linearization A of this mapping at zero has just one pair of eigenvalues on the unit circle in the complex plane. Suppose further that these eigenvalues are $\mathrm{e}^{\pm \mathrm{i}q/p}$ with a prime positive integer $p>4$ and a positive integer $0<q<p/2$. The question is whether the mapping has some p-periodic orbits close to zero. Our methods are workable in this situation if the main homogeneous part of f is not a homogeneous polynomial in N variables.

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