Remark on the Hopf Bifurcation Theorem

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A simple generalization of the Hopf Bifurcation Theorem for scalar higher order ordinary differential equations is suggested. We study the degenerate case where several roots of the characteristic polynomial cross the imaginary axis at the same point for some value λ_0 of the parameter λ. The main result is that if N_1 roots cross the imaginary axis from the left to the right and N_2 roots cross it from the right to the left, then generically $|N_1 - N_2|$ branches of small cycles exist in some neighborhood of the zero equilibrium. In particular, in the classical Hopf Bifurcation Theorem the numbers N_j are 0 and 1.

1 Introduction

Consider the equation

$$L\left(\frac{d}{dt},\lambda\right)x = f(x, x', \dots, x^{(\ell-1)}, \lambda) \tag{1}$$

with the scalar parameter $\lambda \in \Lambda = (\lambda_1, \lambda_2)$. Here

$$L(p,\lambda) = p^\ell + a_1(\lambda)p^{\ell-1} + \cdots + a_\ell(\lambda);$$

the coefficients of this polynomial depend continuously on λ; the nonlinearity

$$f(x_0, x_1, \dots, x_{\ell-1}, \lambda) : \mathbb{R}^\ell \times \Lambda \to \mathbb{R}$$

is continuous with respect to the set of its arguments.

Definition 1.1 A value λ_0 of the parameter is called a Hopf bifurcation point with the frequency $w_0 > 0$ for equation (1) if for every sufficiently small $r > 0$ there exists a $\lambda(r)$ such that equation (1) with $\lambda = \lambda(r)$ has a non-stationary periodic solution $x(t;r)$ with a period $T(r)$ and $\lambda(r) \to \lambda_0$, $T(r) \to 2\pi/w_0$, $\|x(\cdot;r)\|_{C^{\ell-1}} \to 0$ as $r \to 0$.

The use of an auxiliary parameter different from λ is rather standard for Hopf bifurcation problems (see, e.g., [1] and Hopf's original paper therein).

Let the polynomial $L(p,\lambda)$ have *simple* conjugate roots $\mu(\lambda)$ and $\overline{\mu}(\lambda)$ which depend continuously on λ. We say that these roots cross the imaginary axis at the points $\pm w_0 i$ for $\lambda = \lambda_0$ if $L(\pm w_0 i, \lambda_0) = 0$ and the function $(\lambda - \lambda_0)\,\Re e\,\mu(\lambda)$ has the same signature for all $\lambda \neq \lambda_0$ sufficiently close to λ_0. No transversality conditions of the type $d/d\lambda\big(\Re e\,\mu(\lambda)\big) \neq 0$ are supposed.

Proposition 1.2 *Let the polynomial $L(p,\lambda)$ have a pair of* simple *conjugate roots $\mu(\lambda)$ and $\overline{\mu}(\lambda)$ which cross the imaginary axis at the points $\pm w_0 i$ $(w_0 > 0)$ for $\lambda = \lambda_0$ and let $L(kw_0 i, \lambda_0) \neq 0$ for all $k \neq \pm 1, k \in \mathbb{Z}$. Let the nonlinearity be sublinear:*

$$\lim_{|x_0|+\cdots+|x_{\ell-1}|\to 0} \ \sup_{\lambda\in\Lambda} \frac{|f(x_0, x_1, \dots, x_{\ell-1}, \lambda)|}{|x_0| + \cdots + |x_{\ell-1}|} = 0. \tag{2}$$

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Then λ_0 is a Hopf bifurcation point with the frequency w_0 for equation (1).

This proposition (it is a reformulation of a more general result for vector ODE) was announced in [2] and proved in [3] by the special technique of parameter functionalization.

In this paper we suggest some similar statements without the assumption that the roots $\mu(\lambda)$ and $\overline{\mu}(\lambda)$ are simple. Instead, we suppose that the roots $\pm w_0 i$ of the polynomial $L(p, \lambda_0)$ have a multiplicity $N > 1$. Generically, in this case N pairs of roots of the polynomial $L(p, \lambda)$ cross the imaginary axis at the points $\pm w_0 i$ for $\lambda = \lambda_0$.

By $\mathbb{R}^\ell$ we denote the phase space of equation (1). Cycles of this equation are closed curves in $\mathbb{R}^\ell$ defined (in the standard way) by non-stationary periodic solutions $x(t)$.

Below the notion of *the rotation of vector fields* ([4]) on the boundaries of some domains in Banach spaces plays an important role. This notion is equivalent to the notion of the degree of mapping.

2 Main result

2.1 Assumptions

Our main assumptions on the principal linear part of equation (1) are as follows.

- For $\lambda = \lambda_0$ the polynomial $L(p, \lambda_0)$ has the roots $\pm w_0 i$ ($w_0 > 0$) of multiplicity $N > 1$.
- For $\lambda = \lambda_0$ the polynomial $L(p, \lambda_0)$ has no other roots of the form $k w_0 i$, i.e., $L(k w_0 i, \lambda_0) \neq 0$ for all integer $k \neq \pm 1$.
- For any $\lambda \neq \lambda_0$ sufficiently close to λ_0 the polynomial $L(p, \lambda)$ has no roots on the imaginary axis close to the points $\pm w_0 i$.

The first assumption implies that the polynomial $L(p, \lambda)$ has exactly $N > 1$ roots in some small vicinity of the point $w_0 i$ on the complex plane $\mathbb{C}$ for each $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ with a sufficiently small $\varepsilon > 0$ (we count n times each root of multiplicity n). Let N_-^ℓ of those N roots lie in the open left half-plane $\Re e\, z < 0$ and N_-^r of them lie in the open right half-plane $\Re e\, z > 0$ of $\mathbb{C}$ for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$. From the third assumption it follows that the numbers N_-^ℓ, N_-^r are the same for all $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$ and $N_-^\ell + N_-^r = N$. Similarly, we denote by N_+^ℓ the number of the roots in the left half-plane $\Re e\, z < 0$ and by N_+^r their number in the right half-plane $\Re e\, z > 0$ for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$; here also $N_+^\ell + N_+^r = N$. For $\lambda = \lambda_0$ all the N roots equal $w_0 i$.

In the simplest case, we have either $N_-^\ell = N_+^r = N$ or $N_+^\ell = N_-^r = N$ which means that all the N roots cross the imaginary axis at the point $w_0 i$ in the same direction for $\lambda = \lambda_0$ (either from the left to the right or from the right to the left as λ increases, respectively). Otherwise, the situation may be described in different ways depending on how the roots are identified. We may say that some roots stay in the same half-plane $\Re e\, z > 0$ or $\Re e\, z < 0$ for all values of $\lambda \neq \lambda_0$ or that there are roots which cross the imaginary axis in opposite directions.

2.2 Main theorem

Define the integer number

$$M = N_-^\ell - N_+^\ell = N_+^r - N_-^r.$$

Theorem 2.1 *Let all assumptions of Subsection* 2.1 *be valid. Let $M \neq 0$. Let the nonlinearity be sublinear, i.e., equality* (2) *hold. Then λ_0 is a Hopf bifurcation point with the frequency w_0 for equation* (1).

Theorem 2.1 is a simple extension of Proposition 1.2 from the case $N = 1$ to the case $N > 1$.

The auxiliary parameter r used in Definition 1.1 of a Hopf bifurcation point has a natural sense under the assumptions of Theorem 2.1. Let $T = 2\pi/w$ be the period of a cycle of equation (1) for some λ. Then there exists a unique periodic solution of the form $x(t) = r \sin wt + h(wt)$ that defines this cycle; here the Fourier series of the 2π-periodic function $h(\cdot)$ does not contain the first harmonics. It is the amplitude $r > 0$ of the first harmonics of $x(t)$ that we use as the auxiliary parameter in the proof of Theorem 2.1 below.

Generically, under the assumptions of Theorem 2.1 for any sufficiently small $r > 0$ equation (1) has $|M|$ different periodic solutions $x(t) = r \sin wt + h(wt)$ that exist for different λ and have different periods $2\pi/w$.

3 Remarks

3.1 Counterexample

If $M = 0$, then the assertion of Theorem 2.1 may be not valid. For example, the equation

$$x^{(\mathrm{IV})} + 2(1-\lambda^2)x'' + (1+\lambda^2)^2 x = (x')^3$$

has no non-trivial periodic solutions. To prove this fact, it suffices to multiply the equation by x' and integrate over a period. The left-hand side vanishes, we obtain $\int_0^T \left(x'(t)\right)^4 dt = 0$ and therefore $x \equiv const = 0$. Here the polynomial $L(p,\lambda) = p^4 + 2(1-\lambda^2)p^2 + (1+\lambda^2)^2$ has four roots $\pm\lambda \pm i$, hence $N_-^\ell = N_-^r = N_+^\ell = N_+^r = 1$ and $M = 0$. For $\lambda = \lambda_0 = 0$ the roots are $\pm i$, each of multiplicity $N = 2$.

3.2 Equations in $\mathbb{R}^m$

Consider the vector equation

$$z' = \mathcal{A}(\lambda)z + f(z,\lambda), \qquad z \in \mathbb{R}^m, \tag{3}$$

with the continuous right-hand part, where the vector-valued function f is sublinear, i.e., $|z|^{-1}\sup_{\lambda\in\Lambda}|f(z,\lambda)| \to 0$ as $z \to 0$. It is proved in [2, 3] that if *simple* conjugate eigenvalues $\mu(\lambda), \bar{\mu}(\lambda)$ of the matrix $\mathcal{A}(\lambda)$ cross the imaginary axis at the points $\pm w_0 i$ for $\lambda = \lambda_0$ and if the numbers $kw_0 i$ do not belong to the spectrum of the matrix $\mathcal{A}(\lambda_0)$ for all integer $k \neq \pm 1$, then λ_0 is a Hopf bifurcation point with the frequency w_0 for equation (3). The authors do not know if similar results are valid without the assumption that the eigenvalues $\mu(\lambda), \bar{\mu}(\lambda)$ are simple (in other words, the question is how to generalize this statement for equation (3), like Theorem 2.1 generalizes Proposition 1.2).

3.3 Continuous branches of cycles

We say that a set of cycles of equation (1) is a local *continuous branch of cycles* (a *CBC*) for $\lambda \in \Lambda$ if on the boundary of any open bounded set G ($0 \in G \subset \mathbb{R}^\ell$) of a sufficiently small diameter there is at least one point that belongs to a cycle $\Gamma \subset \overline{G}$ of equation (1) for some $\lambda \in \Lambda$.

One can show that the assumptions of Proposition 1.2 and Theorem 2.1 guarantee the existence of a local CBC. Generically, under the assumptions of Proposition 1.2 there is one local CBC and under the assumptions of Theorem 2.1 there exist at least $|M|$ local CBCs which do not intersect (note that the join of CBCs is a CBC).

In smooth cases cycles of a CBC form a 2 dimensional surface in $\mathbb{R}^\ell$.

3.4 Equations with delays

Simple analogs of Theorem 2.1 are valid for systems with delays; they may be obtained by straightforward modification of the proof presented in the next section. For example, consider the equation

$$L\left(\frac{d}{dt},\lambda\right)x = f(x, x(t-\theta), \lambda). \tag{4}$$

Let the nonlinearity $f(x,y,\lambda)$ be sublinear. Suppose that all the assumptions of Subsection 2.1 are valid and $M \neq 0$. Then λ_0 is a Hopf bifurcation point with the frequency w_0 for equation (4).

Similar results are valid for equations with several delays, distributed delays, delays depending on the parameter λ, equations with delays in the linear part and with the nonlinearities that depend on the derivatives $x', x'', \dots$ and $x'(t-\theta), x''(t-\theta), \dots$, etc.

3.5 Non-sublinear nonlinearities

Instead of sublinear nonlinearities, it is possible to consider the functions f satisfying the sector estimates

$$|f(x_0, x_1, \dots, x_{\ell-1}, \lambda)| \le q\sqrt{\mu_0 x_0^2 + \dots + \mu_{\ell-1}x_{\ell-1}^2}, \qquad |x_j| \le r_0, \quad \mu_j \ge 0$$

with a sufficiently small $q > 0$. Theorems on Hopf bifurcations for equations with such nonlinearities and an algorithm to estimate the admissible coefficient q are presented in [5].

3.6 Bifurcations at infinity

A value λ_0 of the parameter is called a *Hopf bifurcation point at infinity* with the frequency w_0 for equation (1) if for every sufficiently large $r > 0$ there exists a $\lambda(r)$ such that equation (1) with $\lambda = \lambda(r)$ has a non-stationary periodic solution $x(t;r)$ with a period $T(r)$ and $\lambda(r) \to \lambda_0$, $T(r) \to 2\pi/w_0$, $\|x(\cdot;r)\|_{C^{\ell-1}} \to \infty$ as $r \to \infty$. For the case of simple roots (simple eigenvalues for systems (3)) crossing the imaginary axis, this type of Hopf bifurcations was considered in [6].

Theorem 3.1 *Let all assumptions of Subsection* 2.1 *be valid. Let* $M \neq 0$. *Let the nonlinearity be sublinear at infinity, i.e.,*

$$\lim_{|x_0|+\cdots+|x_{\ell-1}|\to\infty} \sup_{\lambda\in\Lambda} \frac{|f(x_0, x_1, \ldots, x_{\ell-1}, \lambda)|}{|x_0| + \cdots + |x_{\ell-1}|} = 0.$$

Then λ_0 *is a Hopf bifurcation point at infinity with the frequency* w_0 *for equation* (1).

Under the assumptions of Theorem 3.1 the cycles of equation (1) form a *CBC at infinity*, which means that on the boundary of any open bounded set G that contains a sufficiently large ball $\{x = (x_0, \ldots, x_{\ell-1}) \in \mathbb{R}^\ell : |x| \leq \rho\}$ in the phase space $\mathbb{R}^\ell$ there is at least one point that belongs to a cycle $\Gamma \subset \overline{G}$ of equation (1) for some $\lambda \in \Lambda$.

3.7 Equations of control theory

Without essential changes, Theorem 2.1 may be extended to equations

$$L\left(\frac{d}{dt}, \lambda\right) x = M\left(\frac{d}{dt}, \lambda\right) f(x, x', \ldots, x^{(k)}, \lambda). \tag{5}$$

Here the real polynomials

$$L(p, \lambda) = p^\ell + a_1(\lambda)p^{\ell-1} + \cdots + a_\ell(\lambda), \qquad M(p, \lambda) = b_0(\lambda)p^m + \cdots + b_m(\lambda)$$

with $b_0(\lambda) \neq 0$ depend continuously on λ and are coprime for each λ; their degrees ℓ and m satisfy $\ell > m + k$. The definition of solutions of equation (5) may be found in most books on control theory (see, e.g., [7] and [8]). Any equation (5) is equivalent to a system of the form

$$z' = \mathcal{A}(\lambda)z + c(\lambda)f(x, x', \ldots, x^{(k)}, \lambda), \quad x = \langle b, z\rangle$$

with $z, b, c = c(\lambda) \in \mathbb{R}^\ell$, where $\langle\cdot,\cdot\rangle$ denotes a scalar product in $\mathbb{R}^\ell$. The assumptions of Theorem 2.1 imply that λ_0 is a Hopf bifurcation point with the frequency w_0 for equation (5).

3.8 Final remark

The remarks above may be applied in various combinations. Consider only one example. Let for all x_j and λ the estimate

$$|f(x_0, x_1, \ldots, x_{\ell-1}, \lambda)| \leq c + q\sqrt{\mu_0 x_0^2 + \cdots + \mu_{\ell-1}x_{\ell-1}^2}, \qquad \mu_j \geq 0 \tag{6}$$

hold. Let all assumptions of Subsection 2.1 be valid and $M \neq 0$. Then there exists a $q > 0$ (determined by the polynomial $L(p, \lambda)$ and the numbers ρ and μ_j) such that estimate (6) implies that λ_0 is a Hopf bifurcation point at infinity with the frequency w_0 for equation (1). An algorithm to estimate q from below may be found in [5].

4 Proof of Theorem 2.1

4.1 Preliminaries

Let us change the time in (1) and consider the equation

$$L\left(w\frac{d}{dt}, \lambda\right) x = f(x, wx', \ldots, w^{\ell-1}x^{(\ell-1)}, \lambda). \tag{7}$$

Every 2π-periodic solution $x(t)$ of (7) defines the $(2\pi/w)$-periodic solution $x(wt)$ of (1). We look for 2π-periodic solutions of equation (7) in the form

$$x(t) = r\sin t + h(t), \tag{8}$$

where the Fourier expansion of the function $h = h(t)$ does not contain the harmonics $\sin t$ and $\cos t$. We are going to prove that for any sufficiently small $r > 0$ there exist numbers $\lambda = \lambda(r)$ and $w = w(r) > 0$ and a 2π-periodic function $h = h(t) = h(t;r)$ such that formula (8) defines a classical solution of equation (7) and the relations $\lambda(r) \to \lambda_0$, $w(r) \to w_0$, $\|h(\cdot;r)\|_{C^{\ell-1}} \to 0$ are valid as $r \to 0$. This implies the conclusion of Theorem 2.1 and moreover, it means that in our setting an auxiliary parameter $r > 0$ used in Definition 1.1 is the amplitude of the first harmonics of periodic solutions. A special role of those harmonics is natural, since periodic solutions of the linearized equation (1) with $f \equiv 0$ are $x(t) = r\sin(w_0 t + \phi)$.

Thus, for each small value of the parameter $r > 0$ our unknowns are the numbers $\lambda = \lambda(r)$, $w = w(r)$ and the component $h = h(t) = h(t;r)$ of the 2π-periodic solution (8) of equation (7). Remark that every non-stationary 2π-periodic solution $x(t)$ of this autonomous equation is included in the continuum $x(t+\phi)$ of such solutions (which define the same cycle in the phase space $\mathbb{R}^\ell$ of equation (7) for all $\phi \in \mathbb{R}$), but at most one of the solutions $x(t+\phi)$ has the form (8) with $r > 0$.

4.2 Auxiliary constructions

We use the spaces C, C^k, L^2 and $W^{k,2}$ of functions $x = x(t) : [0, 2\pi] \to \mathbb{R}$ with the usual norms and scalar products. Denote by $E \subset L^2$ the linear span of the functions $\sin t$ and $\cos t$ and by $E^\perp \subset L^2$ the orthogonal complement of the plane E. Then

$$\mathcal{P}x(t) = \frac{1}{\pi}\int_0^{2\pi} \cos(t-s)\, x(s)\, ds$$

and $\mathcal{Q} = I - \mathcal{P}$ are orthogonal projectors onto the subspaces E and $E^\perp$ of L^2.

The assumptions of Subsection 2.1 imply that for any sufficiently small vicinity $\mathcal{D}$ of the point (w_0, λ_0) on the plane (w, λ) the point (w_0, λ_0) is a unique zero of the complex-valued function $L(wi, \lambda)$ in the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ and $L(kwi, \lambda) \neq 0$ for all integer $k \neq \pm 1$ and all $(w, \lambda) \in \overline{\mathcal{D}}$. Everywhere below we denote by $\mathcal{D}$ a vicinity with these properties, also satisfying $\overline{\mathcal{D}} \subset \{(w, \lambda) : w > 0, \lambda \in \Lambda\}$.

Consider the differential operator $\mathcal{L} = L(w\, d/dt, \lambda)$ with the 2π-periodic boundary conditions. ¿From the definition of $\mathcal{D}$ it follows that the kernel of this differential operator is the plane E for $\lambda = \lambda_0$, $w = w_0$ and is zero for any other point $(w, \lambda) \in \overline{\mathcal{D}}$. For each $(w, \lambda) \in \overline{\mathcal{D}}$ denote by $H = H(w, \lambda)$ the linear operator that maps any function $y \in E^\perp \subset L^2$ to a unique solution $x = Hy \in E^\perp \bigcap W^{\ell,2}$ of the equation $\mathcal{L}x = y$. The existence of the solution $x = x(t)$ follows from the relations $y \in E^\perp$ and $L(kwi, \lambda) \neq 0$ for all $k \neq \pm 1$, $(w, \lambda) \in \overline{\mathcal{D}}$, the uniqueness follows from $x \in E^\perp$.

Operators $H(w, \lambda) : E^\perp \to E^\perp \bigcap W^{\ell,2}$ are continuous; their norms and the norms of the operators

$$H^{(n)} = H^{(n)}(w, \lambda) = \frac{d^n}{dt^n} H$$

acting in the subspace $E^\perp$ of L^2 are defined by

$$\|H\|_{L^2 \to L^2} = \max_{k=0,2,3,\ldots} |L(kwi, \lambda)|^{-1}, \qquad \|H^{(n)}\|_{L^2 \to L^2} = \max_{k=0,2,3,\ldots} k^n |L(kwi, \lambda)|^{-1}$$

for all $n = 1, \ldots, \ell$. Therefore the uniform estimate

$$\|H(w, \lambda)\|_{L^2 \to C^{\ell-1}} \le \nu < \infty \qquad \text{for all} \qquad (w, \lambda) \in \overline{\mathcal{D}} \tag{9}$$

holds. Moreover, the map $(w, \lambda, y) \mapsto H(w, \lambda)y$ of the product $\overline{\mathcal{D}} \times E^\perp$ to $C^{\ell-1}$ is completely continuous. Also, each operator $H(w, \lambda)$ maps continuously $E^\perp \bigcap C$ to C^ℓ.

4.3 Homotopy

We look for solutions of equation (7) of the form $x(t) = r\sin t + H(w,\lambda)y(t)$ with $y = y(t) \in E^\perp$ and $(w,\lambda) \in \overline{\mathcal{D}}$. By definition of the operator $H = H(w,\lambda)$, the function $Hy \in W^{\ell,2}$ satisfies the 2π-periodic boundary conditions, hence the same is true for the function $x \in W^{\ell,2}$, i.e., $x(0) = x(2\pi), \dots, x^{(\ell-1)}(0) = x^{(\ell-1)}(2\pi)$.

For each $r > 0$ set

$$\mathcal{Z}y = \mathcal{Z}_r(w,\lambda)y = f\big(r\sin t + Hy, w(r\sin t + Hy)', \dots, w^{\ell-1}(r\sin t + Hy)^{(\ell-1)}, \lambda\big).$$

Since the map $(w,\lambda,y) \mapsto H(w,\lambda)y$ of the product $\overline{\mathcal{D}} \times E^\perp$ to $C^{\ell-1}$ is completely continuous and the function f is continuous, it follows that the map $(w,\lambda,y) \mapsto \mathcal{Z}_r(w,\lambda)y$ from $\overline{\mathcal{D}} \times E^\perp$ to C is completely continuous for each r. Consider in the space $\mathbb{R} \times \mathbb{R} \times E^\perp$ the completely continuous deformation $\Theta = \Theta(w,\lambda,y,\xi) = \big(\Theta_w, \Theta_\lambda, \Theta_y\big)$ with the components

$$\begin{aligned} \Theta_w(w,\lambda,y,\xi) &= r\,\Re e\, L(wi,\lambda) - \frac{\xi}{\pi}\int_0^{2\pi} \mathcal{Z}y(t)\sin t\,dt \in \mathbb{R}, \\ \Theta_\lambda(w,\lambda,y,\xi) &= r\,\Im m\, L(wi,\lambda) - \frac{\xi}{\pi}\int_0^{2\pi} \mathcal{Z}y(t)\cos t\,dt \in \mathbb{R}, \\ \Theta_y(w,\lambda,y,\xi) &= y(t) - \xi\mathcal{Q}\mathcal{Z}y(t) \in E^\perp \subset L^2, \end{aligned} \tag{10}$$

where $(w,\lambda,y) \in \overline{\mathcal{D}} \times E^\perp$ and $\xi \in [0,1]$ is the deformation parameter.

Lemma 4.1 *For any $r > 0$ and $\xi \in [0,1]$ each zero (w,λ,y) of the deformation Θ defines a classical 2π-periodic solution $x(t) = r\sin t + H(w,\lambda)y(t)$ of the equation*

$$L\left(w\frac{d}{dt},\lambda\right)x = \xi f(x, wx', \dots, w^{\ell-1}x^{(\ell-1)}, \lambda). \tag{11}$$

To prove Lemma 4.1, it suffices to substitute the formula $x(t) = r\sin t + Hy(t)$ in (11) and to apply the projectors $\mathcal{P}$ and $\mathcal{Q}$ to the resulting equation. Then using the definition of the operator $H = H(w,\lambda)$ one obtains $\Theta = 0$ (the reader will easily obtain the omitted details). To conclude, it remains to note that due to $\mathcal{Z}y \in C$ the equation $\Theta_y = 0$ implies $y \in C$ and therefore $Hy \in C^\ell$ and $x \in C^\ell$, which means that $x(t) = r\sin t + H(w,\lambda)y(t)$ is a classical 2π-periodic solution of equation (7) iff $\Theta(w,\lambda,y,\xi) = 0$.

Lemma 4.2 *There exist $\varepsilon_y > 0$ and $r_0 > 0$ such that for any $r < r_0$ each zero (w,λ,y) of the deformation Θ with $\|y\|_{L^2} \le \varepsilon_y$ satifies*

$$\|y\|_{L^2} \le r, \qquad \|Hy\|_{C^{\ell-1}} \le r. \tag{12}$$

The conclusion of Lemma 4.2 follows from the assumption (2). Indeed, relations (2) and (9) imply $\|\mathcal{Z}y\|_C = o(r + \|y\|_{L^2})$ ar $r \to 0$, $\|y\|_{L^2} \to 0$ and therefore the equality $\Theta_y = 0$ yields $\|y\|_{L^2} = o(r)$ whenever r and $\|y\|_{L^2}$ are sufficiently small. Combining this with (9), one obtains (12), which proves the lemma.

Consider any sufficiently small $\varepsilon_w, \varepsilon_\lambda > 0$ such that the rectangle $Q = \{(w,\lambda) : |w - w_0| \le \varepsilon_w, |\lambda - \lambda_0| \le \varepsilon_\lambda\}$ belongs to the set $\mathcal{D}$. Set

$$\alpha = \min_{(w,\lambda)\in\partial Q} |L(wi,\lambda)| > 0,$$

where ∂Q is the boundary of Q. ¿From (2) and (12) it follows that for any $q > 0$ there exists a $r_q > 0$ such that for any $r < r_q$ each zero (w,λ,y) of the deformation Θ with $\|y\|_{L^2} \le \varepsilon_y$ satisfies the estimate

$$\|\mathcal{Z}y\|_{L^2} \le qr. \tag{13}$$

Let us fix some positive $q < \alpha\sqrt{\pi}$. Let $r < \min\{r_0, r_q, \varepsilon_y/2\}$. We look for zeros of the deformation Θ in the set

$$\Omega = \big\{(w,\lambda,y) \in \mathbb{R} \times \mathbb{R} \times E^\perp : (w,\lambda) \in Q,\ \|y\|_{L^2} \le 2r\big\}. \tag{14}$$

The boundary $\partial\Omega$ of Ω is the join $\partial\Omega = \Omega_y \cup \Omega_Q$ of the sets

$$\Omega_y = \{(w,\lambda,y) : (w,\lambda) \in Q, \|y\|_{L^2} = 2r\}, \quad \Omega_Q = \{(w,\lambda,y) : (w,\lambda) \in \partial Q, \|y\|_{L^2} \le 2r\}.$$

Lemma 4.2 implies that the deformation Θ is non-zero on the set Ω_y. Furthermore, if $\Theta_w = \Theta_y = 0$, then

$$r^2|L(wi,\lambda)|^2 = \frac{\xi^2}{\pi^2}\left[\left(\int_0^{2\pi} \mathcal{Z}y(t)\sin t\,dt\right)^2 + \left(\int_0^{2\pi} \mathcal{Z}y(t)\cos t\,dt\right)^2\right] \le \frac{1}{\pi}\,\|\mathcal{Z}y\|_{L^2}^2$$

and (13) implies $|L(wi,\lambda)| \le q/\sqrt{\pi} < \alpha$. By definition of α, this means that $\Theta_w^2 + \Theta_y^2 \ne 0$ on the set Ω_Q. Thus, we have the following lemma.

Lemma 4.3 *There are no zeros of the deformation* (10) *on the boundary of the set* (14).

From Lemma 4.3 it follows that the rotation $\gamma^* = \gamma(\Theta, \partial\Omega)$ of the vector field $\Theta = \Theta(\cdot,\cdot,\cdot,\xi)$ on the boundary $\partial\Omega$ of the set Ω is well-defined for any $\xi \in [0,1]$ and that γ^* is the same for all the ξ. For $\xi = 0$ the components of Θ have the form

$$\Theta_w = r\,\Re e\,L(wi,\lambda), \quad \Theta_\lambda = r\,\Im m\,L(wi,\lambda), \quad \Theta_y = y.$$

Here the scalar components Θ_w and Θ_λ depend on w and λ only. Due to the fact that the last component is y, this implies (according to the standard rotation product formula, see e.g. [4]) that the rotation γ^* is equal to the topological index of the planar vector field $\Phi(w,\lambda) = \big(\Re e\,L(wi,\lambda),\ \Im m\,L(wi,\lambda)\big)$ at the point (w_0,λ_0).

Lemma 4.4 *The topological index of the planar vector field* $\Phi = \Phi(w,\lambda)$ *at the point* (w_0,λ_0) *equals* $M \ne 0$.

This lemma is proved in the next subsection.

From Lemma 4.4 it follows that $\gamma_* = \gamma(\Theta,\partial\Omega) \ne 0$. According to the general principle of the degree theory, this relation implies that the deformation Θ ha,s at least one zero in the set Ω for each ξ. By Lemma 4.1, for $\xi = 1$ such a zero $(w_r,\lambda_r,y_r) \in \Omega$ defines the classical 2π-periodic solution $x(t) = r\sin t + H(w_r,\lambda_r)y_r(t)$ of equation (7) with $\lambda = \lambda_r$, which implies that $x(w_r t)$ is a non-stationary $(2\pi/w_r)$-periodic solution of equation (1).

The last step of the proof is to show that the zeros (w_r,λ_r,y_r) satisfy $w_r \to w_0$, $\lambda_r \to \lambda_0$, $\|Hy_r\|_{C^{\ell-1}} \to 0$ as $r \to 0$. This follows from the second of estimates (12) of Lemma 4.2 and from the inclusion $(w_r,\lambda_r) \in Q$ (since ε_w and ε_λ are arbitrarily small).

4.4 Proof of Lemma 4.4

Let $P_j = P_j(w,\lambda) : \mathbb{R}^2 \to \mathbb{C}$ be continuos functions such that $P_0 = P_1P_2$. Consider the real planar vector fields

$$\Xi_j = \Xi_j(w,\lambda) = \big(\Re e\,P_j(w,\lambda),\ \Im m\,P_j(w,\lambda)\big)$$

associated with the functions P_j. We use a simple formula

$$\gamma(\Xi_0,\Gamma) = \gamma(\Xi_1,\Gamma) + \gamma(\Xi_2,\Gamma) \tag{15}$$

to calculate the rotation (called also the *winding number*) of the vector field Ξ_0 along a simple closed curve Γ. This formula follows from the definition of the winding numbers $\gamma(\Xi_j,\Gamma)$ and from the relation $\mathrm{Arg}(z_1z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_1)$, $z_1, z_2 \in \mathbb{C}$. It is supposed that $P_0(w,\lambda) \ne 0$ for all $(w,\lambda) \in \Gamma$, which gurantees that the vector fields Ξ_j are non-zero on Γ and therefore the winding numbers $\gamma(\Xi_j,\Gamma)$ are well-defined. Formula (15) is valid if Γ is the boundary of any bounded open set (not necessarily a simple closed curve).

By assumption, $w_0 i$ is a root of multiplicity N of the polynomial $L(p,\lambda_0)$. Hence, if $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ are sufficiently small, then for every λ from the interval $|\lambda - \lambda_0| \le \delta$ the polynomial $L(p,\lambda)$ has exactly N roots $\mu_1(\lambda),\ldots,\mu_N(\lambda)$ satisfying $|\mu_j(\lambda) - w_0 i| \le \varepsilon$. Here $\mu_{j_1}(\lambda) = \cdots = \mu_{j_n}(\lambda)$ for multiple roots of multiplicity n, particularly $\mu_j(\lambda_0) = w_0 i$ for all $j = 1,\ldots,N$. We define the N functions $\mu_j = \mu_j(\lambda)$ in such a way that they are continuous on the interval $|\lambda - \lambda_0| \le \delta$ (the choise of the continuous branches $\mu_j = \mu_j(\lambda)$ is non-unique, that is we fix one of those choices) and factorize the polynomial $L(p,\lambda)$ as

$$L(p,\lambda) = L_*(p,\lambda)\prod_{j=1}^{N}(p - \mu_j(\lambda)).$$

By construction, $L_*(wi,\lambda) \neq 0$ in the rectangle $Q = \{(w,\lambda) : |w-w_0| \leq \varepsilon, |\lambda-\lambda_0| \leq \delta\}$, therefore the real vector field $\Xi_* = \Xi_*(w,\lambda) = \big(\Re e(-i)^N L_*(w,\lambda),\ \Im m(-i)^N L_*(w,\lambda)\big)$ assotiated with the complex-valued function $(-i)^N L_*(wi,\lambda)$ has no zeros in Q, which implies $\gamma(\Xi_*,\Gamma)=0$ for every closed curve $\Gamma \subset Q$. Consequently, relations (15) and

$$L(wi,\lambda) = (-i)^N L_*(wi,\lambda) \prod_{j=1}^{N} (-w - i\mu_j(\lambda))$$

imply for every closed curve $\Gamma \subset Q$ such that $L(wi,\lambda) \neq 0$ on Γ

$$\gamma(\Phi,\Gamma) = \sum_{j=1}^{N} \gamma(\Xi_j,\Gamma), \tag{16}$$

where $\Phi(w,\lambda) = \big(\Re e\, L(wi,\lambda), \Im m\, L(wi,\lambda)\big)$ and $\Xi_j(w,\lambda) = (\Im m\, \mu(\lambda) - w, -\Re e\, \mu_j(\lambda))$; these real vector fields are associted with the complex-valued functions $L(wi,\lambda)$ and $-w - i\mu_j(\lambda)$.

From the assumptions of Subsection 2.1 it follows that for every sufficiently small $\varepsilon_1 > 0$ the relation $\Re e\, \mu_j(\lambda) \neq 0$ is valid whenever $\lambda \neq \lambda_0$, $|\lambda - \lambda_0| \leq \varepsilon_1$ for all $j = 1,\dots,N$ and that (w_0,λ_0) is a unique zero of the vector field Φ in the square $|w-w_0| \leq \varepsilon_1$, $|\lambda-\lambda_0| \leq \varepsilon_1$. We fix such a $\varepsilon_1 \leq \varepsilon$ and then define a $\delta_1 = \delta_1(\varepsilon_1) > 0$ satisfying $\delta_1 \leq \delta$, $\delta_1 \leq \varepsilon_1$ such that $|\Im m\, \mu_j(\lambda) - w_0| < \varepsilon_1$ whenever $|\lambda - \lambda_0| \leq \delta_1$ for all $j = 1,\dots,N$ (this holds for each sufficiently small δ_1, since $\Im m\, \mu_j(\lambda_0) = w_0$ for all j).

Denote by Γ the boundary of the rectangle $Q^* = \{(w,\lambda) : |w-w_0| \leq \varepsilon_1, |\lambda-\lambda_0| \leq \delta_1\} \subset Q$ and set $\sigma_-^j = \operatorname{sign} \Re e\, \mu_j(\lambda_0 - \delta_1)$, $\sigma_+^j = \operatorname{sign} \Re e\, \mu_j(\lambda_0 + \delta_1)$. Since λ_0 is a unique zero of the function $\Re e\, \mu_j(\lambda)$ in the segment $|\lambda - \lambda_0| \leq \delta_1$, it follows that each of the signutures σ_-^j, σ_+^j is either 1 or -1 and that for $\lambda \neq \lambda_0$ from this segment the relations

$$\operatorname{sign} \Re e\, \mu_j(\lambda) = \sigma_-^j \quad \text{if} \quad \lambda_0 - \delta_1 \leq \lambda < \lambda_0, \qquad \operatorname{sign} \Re e\, \mu_j(\lambda) = \sigma_+^j \quad \text{if} \quad \lambda_0 < \lambda \leq \lambda_0 + \delta_1$$

hold. The relation $|\Im m\, \mu_j(\lambda) - w_0| < \varepsilon_1$ with $|\lambda - \lambda_0| \leq \delta_1$ implies

$$(\Im m\, \mu_j(\lambda) - w)(w - w_0) < 0 \quad \text{for} \quad |w - w_0| = \varepsilon_1,\ |\lambda - \lambda_0| \leq \delta_1.$$

From this estimate and the relation $\sigma_-^j \sigma_+^j \neq 0$ it follows that the rotation $\gamma_j = \gamma(\Xi_j,\Gamma)$ of the vector field $\Xi_j(w,\lambda) = (\Im m\, \mu(\lambda) - w, -\Re e\, \mu_j(\lambda))$ on Γ is defined by the formula

$$\gamma(\Xi_j,\Gamma) = \frac{\sigma_+^j - \sigma_-^j}{2}, \tag{17}$$

i.e. $\gamma_j = 0$ if $\sigma_-^j = \sigma_+^j$, $\gamma_j = 1$ if $\sigma_-^j = -\sigma_+^j = -1$, and $\gamma_j = -1$ if $\sigma_-^j = -\sigma_+^j = 1$. Fig. 1 shows the vector fields Ξ_j on the boundary Γ of the rectangle Q^* with different rotations $\gamma_j = \gamma(\Xi_j,\Gamma)$ depending on the signatures $\sigma_- = \sigma_-^j$ and $\sigma_+ = \sigma_+^j$.

Fig. 1 Vector field Ξ_j and its rotation γ_j.

Equalities (16), (17) imply

$$\gamma(\Phi,\Gamma) = \frac{1}{2}\sum_{j=1}^{N} \sigma_+^j - \frac{1}{2}\sum_{j=1}^{N} \sigma_-^j = \frac{N_+^r - N_+^\ell}{2} - \frac{N_-^r - N_-^\ell}{2},$$

where N_+^r is the number of the roots $\mu_j(\lambda)$ satisfying $\Re e\, \mu_j(\lambda) > 0$ and N_+^ℓ is the number of the roots $\mu_j(\lambda)$ satisfying $\Re e\, \mu_j(\lambda) < 0$ for $\lambda_0 < \lambda \leq \lambda_0 + \delta_1$; the numbers N_-^r and N_-^ℓ are defined in the same way for $\lambda_0 - \delta_1 \leq \lambda < \lambda_0$. The equalities $N_+^r + N_+^\ell = N_-^r + N_-^\ell = N$ imply $\gamma(\Phi,\Gamma) = N_+^r - N_+^\ell = M$. Since the vector field Φ has a unique zero (w_0,λ_0) in the rectangle Q^*, it follows that the topological index of this field at the point (w_0,λ_0) equals $\gamma(\Phi,\Gamma) = M$. This completes the proof of Lemma 4.4 and Theorem 2.1.

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