ON CONTINUOUS BRANCHES OF TWICE PERIODIC SOLUTIONS OF PDE *

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Abstract. Theorems on the existence of a continuum of twice periodic solutions with all amplitudes from zero to infinity are presented for autonomous PDEs. The results are applicable to functional differential equations.

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1. Twice periodic solutions. Let us start with an example. Consider the equation

$$u_{xx} + u_{yy} + 2u_x - u_y + 5u = f(u, u_x, u_y) \tag{1}$$

with twice periodic boundary conditions

$$u(x + T_x, y + T_y) \equiv u(x, y), \qquad x, y \in \mathbb{R}. \tag{2}$$

Here both the periods T_x, T_y are *a priori* unknown.

Any solution $u(x, y)$ of (1) – (2) defines the continuum of shifted solutions $u(x + \phi_x, y + \phi_y)$. Generically, this continuum is 2-parametric. We shall consider non-stationary solutions of the form $u(x, y) = v(w_1 x + w_2 y)$; here the corresponding continuum $v(w_1 x + w_2 y + \phi)$ is 1-parametric. Stationary solutions $u(x, y) \equiv const$ are generically isolated.

The linear problem (1) – (2) with $f \equiv 0$ has the continuum of twice 2π-periodic solutions $u_{r,\phi}(x, y) = r \sin(x - 2y + \phi)$ and no other twice periodic solutions. These periodic solutions satisfy $\|u_{r,\phi}(\cdot, \cdot)\|_C \to 0$ as $r \to 0$ and $\|u_{r,\phi}(\cdot, \cdot)\|_C \to \infty$ as $r \to \infty$. Under appropriate conditions, the nonlinear problem (1) – (2) has a similar continual set of twice periodic solutions.

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PROPOSITION 1. *Let $|f(u, u_1, u_2)| \leq q|u|$ for all $u, u_1, u_2 \in \mathbb{R}$ with $q < 3$. Then for any $r > 0$ equation* (1) *has classical twice periodic non-stationary solutions of the form $v_r(w_1(r)x + w_2(r)y + \phi)$, where $\phi \in \mathbb{R}$ and the 2π-periodic functions $v_r(\cdot)$ satisfy*

$$\Big|\frac{1}{\pi}\int_0^{2\pi} v_r(t)\, e^{it}\, dt\Big| = r. \tag{3}$$

In this paper we present similar results for scalar higher order PDEs.

The results are close to theorems on the existence of the so-called continuous branches of cycles for various ordinary differential equations with a scalar parameter. Equation (1) does not depend on a parameter, but it includes two independent variables, which leads to similar results. If an autonomous equation includes $n > 2$ independent variables, than the structure of the set of multi-periodic solutions may be even reacher.

The functions $w_j(r)$ and the mapping $r \mapsto v_r(\cdot)$ in the following analogs of Proposition 1 may be non-unique. Under appropriate additional assumptions on the smoothness of the nonlinearity, it is possible to state uniqueness, regularity and some further properties of $w_j(r)$ and $v_r(\cdot)$. Here we restrict ourselves to the existence theorems only and state sufficient conditions for the existence of continuous branches of twice periodic solutions.

2. Main theorem.

2.1. Branches of solutions. Let $L = L(p_1, p_2)$ be a real polynomial with constant coefficients. It generates the differential operator

$$\mathcal{L}u = L(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})u.$$

Let the complex-valued function $L(w_1 i, w_2 i)$ of the real variables w_1, w_2 have on the plane (w_1, w_2) a zero (w_1^*, w_2^*). Then the operator $\mathcal{L}$ maps twice periodic functions $r\sin(w_1^* x + w_2^* y + \phi)$ to zero.

Consider the equation

$$\mathcal{L}u = f(u, u_x, u_y, \ldots). \tag{4}$$

Let $|L(nw_1 i, nw_2 i)| \to \infty$ as $n \to \infty$ uniformly with respect to (w_1, w_2) from the closure $\mathcal{D}$ of some vicinity of the point (w_1^*, w_2^*) and let for some integer $K = K(L) > 0$ (generically, K is the degree $\ell \geq 2$ of the polynomial L) the relation

$$\lim_{n\to\infty} \sup_{(w_1,w_2)\in\mathcal{D}} n^{1-K}|L(nw_1 i, nw_2 i)| = \infty \tag{5}$$

be valid. The polynomial $L(p_1,p_2) = p_1^2 + p_2^2 + 2p_1 - p_2 + 5$ that defines the left-hand part of equation (1) satisfies these assumptions for $(w_1^*, w_2^*) = (1,2)$ and $K = 2$.

Let us fix numbers $\mu_{(j-k,k)} \ge 0$ with $\mu_{(0,0)} > 0$. Let $\mathfrak{F}(q)$ denote the class of all continuous nonlinearities f satisfying the estimate

$$(6) \qquad |f(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, \ldots, u_{(0,m)})| \le q \sum_{j=0}^{m} \sum_{k=0}^{j} \mu_{(j-k,k)} |u_{(j-k,k)}|$$

and such that $m < K = K(L)$. In (4) the arguments $u_{(j-k,k)}$ of the nonlinearity are replaced by the partial derivatives $\partial^j u/\partial x^{j-k}\partial y^k$.

An important role in our constructions plays the continuous planar vector field

$$\Phi(w_1,w_2) = \big(\Re e\, L(w_1 i, w_2 i), \Im m\, L(w_1 i, w_2 i)\big)$$

on the plane (w_1,w_2). Suppose that the zero (w_1^*,w_2^*) of this vector field is isolated; therefore a topological index of the zero (w_1^*,w_2^*) of Φ is defined ([1]). Denote this index by $\gamma = \gamma\big(\Phi; (w_1^*,w_2^*)\big)$.

THEOREM 1. *Let $\gamma \ne 0$. Then there exists a $q_0 > 0$ such that for any $q < q_0$ equation* (4) *with any $f \in \mathfrak{F}(q)$ has classical twice periodic non-stationary solutions of the form $v_r\big(w_1(r)x + w_2(r)y + \phi\big)$ for all $r > 0$, where the 2π-periodic functions v_r satisfy* (3).

Remark that (3) implies $\|v_r\|_{C^\ell} \to 0$ as $r \to 0$ and $\|v_r\|_C \to \infty$ as $r \to \infty$.

From the next Theorem 2 it follows that condition (3) may be replaced by its analog $\psi(v) = r$ with any continuous functional $\psi : C \to \mathbb{R}$ such that $\psi(v) \to 0$ as $\|v\|_C \to 0$ and $\psi(v) \to \infty$ as $\|v\|_C \to \infty$, where $C = C[0,2\pi]$. More precisely, it means that under the assumptions of Theorem 2 equation (4) has classical twice periodic non-stationary solutions of the form $v_r\big(w_1(r)x + w_2(r)y + \phi\big)$ with $\psi(v_r) = r$ for all $r > 0$. The same is true for the functionals defined on the spaces L^p, C^k, etc. For example, Theorem 2 guarantees the existence of periodic solutions with all positive amplitudes $\|u(\cdot,\cdot)\|_C = \|v(\cdot)\|_C = r$. Also, this remark is valid for the theorems of the next subsection.

Let us consider a Banach space $\mathbb{B}$ and a set-valued mapping $\xi : \mathbb{R}^+ \to 2^{\mathbb{B}}$. Let $\xi(r) \ne \varnothing$ for any $r \in \mathbb{R}^+ = (0,\infty)$. Following Mark Krasnosel'skii ([2, 3]), we say that $\xi = \xi(r)$ is a continuous branch in $\mathbb{B}$ for $r \in [r_1,r_2]$ if the boundary ∂G of any open set $G \subset \mathbb{R}^+ \times \mathbb{B}$ such that $\{(r_1,z) : z \in \xi(r_1)\} \subset G$ and $\{(r_2,z) : z \in \xi(r_2)\} \bigcap G = \varnothing$ contains at least one point (r,z) with $r \in [r_1,r_2]$ and $z \in \xi(r)$.

The usual way to prove that $\xi = \xi(r)$ is a continuous branch is as follows. Suppose that we can construct a bounded open set $\Omega \subset \mathbb{B}$ and completely continuous vector fields $\Xi_r = \Xi_r(z)$ in $\mathbb{B}$ so that $\xi(r) = \{z \in \overline{\Omega} : \Xi_r(z) = 0\}$ for each $r \in [r_1, r_2]$ where $\overline{\Omega}$ is the closure of Ω. Let for all $r \in [r_1, r_2]$ the relation $\partial\Omega \bigcap \xi(r) = \varnothing$ hold, the rotation $\gamma(\Xi_r, \partial\Omega)$ of the vector field Ξ_r on the boundary $\partial\Omega$ of Ω be non-zero (see, e.g. [2]) and $\Xi_r = \Xi_r(z)$ depend continuously on r uniformly with respect to $z \in \overline{\Omega}$. Then $\xi = \xi(r)$ is a continuous branch in $\mathbb{B}$ for $r \in [r_1, r_2]$. One can also use domains $\Omega = \Omega_r$ depending on r.

Denote by $\mathfrak{x} = \mathfrak{x}(r)$ the set-valued mapping that sends each $r \in \mathbb{R}^+$ to the set of all triples $\big(w_1(r), w_2(r), v_r\big)$ such that $v_r\big(w_1(r)x + w_2(r)y\big)$ is a twice periodic non-stationary solution of equation (4) and the 2π-periodic function $v_r = v_r(\cdot)$ satisfies (3).

Theorem 2. *If the conditions of Theorem 1 are satisfied and* $f \in \mathfrak{F}(q)$ *with* $q < q_0$, *then the set-valued mapping* $\mathfrak{x} = \mathfrak{x}(r)$ *is a continuous branch in the space* $\mathbb{R} \times \mathbb{R} \times C$ *for every segment* $r \in [r_1, r_2] \subset \mathbb{R}^+$.

2.2. Estimates for q_0. In what follows, it is more convenient to use instead of (6) the similar estimate

$$|f(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, \dots, u_{(0,m)})| \le q\sqrt{\sum_{j=0}^{m}\sum_{k=0}^{j}\mu_{(j-k,k)}u^2_{(j-k,k)}}. \tag{7}$$

Consider on the plane (w_1, w_2) the real-valued non-negative function

$$\Psi(w_1, w_2) = \frac{|L(w_1 i, w_2 i)|}{\sqrt{\sum_{j=0}^{m}\sum_{k=0}^{j}\mu_{(j-k,k)}w_1^{2j-2k}w_2^{2k}}}. \tag{8}$$

Let us surround the zero (w_1^*, w_2^*) of the function (8) by a simple contour Γ such that

$$\Psi(w_1, w_2) \ge q_0, \qquad (w_1, w_2) \in \Gamma; \tag{9}$$

$$\Psi(nw_1, nw_2) \ge q_0, \qquad (w_1, w_2) \in \mathcal{D}, \quad n = 0, 2, 3, 4, \dots, \tag{10}$$

where $\mathcal{D}$ is the closed bounded domain with the boundary Γ and $q_0 > 0$. Denote by $\gamma(\Phi, \Gamma)$ the winding number of the vector field Φ along the contour Γ. Suppose that relation (5) is valid for the domain $\mathcal{D}$ and denote by $\mathfrak{F}_1(q)$ the class of all continuous nonlinearities f satisfying (7) and such that $m < K$.

Theorem 3. *Let relations* (9), (10) *hold. Let* $\gamma(\Phi, \Gamma) \ne 0$. *Let* $q < q_0$. *Then equation* (4) *with any* $f \in \mathfrak{F}_1(q)$ *has classical twice periodic non-stationary solutions of the form* $v_r(w_1(r)x + w_2(r)y + \phi)$ *for all* $r > 0$, *where the* 2π*-periodic functions* v_r *satisfy* (3).

If the nonlinearity f does not depend on derivatives ($m = 0$), i.e., the equation has the form

$$\mathcal{L}u = f(u), \tag{11}$$

then condition (9) may be weakened in the following way. We assume that for any $(w_1, w_2) \in \Gamma$ either (9) is valid with $\Psi(w_1, w_2) = |L(w_1 i, w_2 i)|$, or all the numbers $\Im m\, L(nw_1 i, nw_2 i)$ for $n = 1, 2, \ldots$ have the same signature. Recall that $|L(nw_1 i, nw_2 i)| \to \infty$ as $n \to \infty$ uniformly with respect to $(w_1, w_2) \in \mathcal{D}$.

THEOREM 4. *Let the function $\Psi(w_1, w_2) = |L(w_1 i, w_2 i)|$ satisfy* (10) *and let for any $(w_1, w_2) \in \Gamma$ either* (9) *be valid, or all numbers $\Im m\, L(nw_1 i, nw_2 i)$ for $n = 1, 2, \ldots$ have the same signature. Let $\gamma(\Phi, \Gamma) \neq 0$. Then equation* (4) *with any continuous f satisfying $|f(u)| \le q|u|$ with any $q < q_0$ has classical twice periodic non-stationary solutions of the form $v_r(w_1(r)x + w_2(r)y + \phi)$ for all $r > 0$, where the 2π-periodic functions v_r satisfy* (3).

As a particular case, consider equation (11) with a linear part such that the factorization $\Im m\, L(w_1 i, w_2 i) = (\alpha_1 w_1 + \alpha_2 w_2) L_1(w_1, w_2)$ holds with some polynomial L_1. Set $u(x, y) = v(-\alpha_2 x + \alpha_1 y)$. Then the equation for $v(t)$ has the form $L_2(\frac{d}{dt})v = f(v)$ where $L_2 = L_2(p)$ is an even polynomial. The existence of a continuum of cycles of all positive amplitudes for such equations with the nonlinearities satisfying sector estimates $|f(v)| \le q|v|$ was proved in [4].

2.3. Example. As an example, consider problem (1)–(2). Proposition 1 is a consequence of Theorem 3 (see Fig. 1). For $q_0 = 3$ the contour Γ defined by $\Psi(w_1, w_2) = q_0$ (the large contour in Fig. 1) is tangent to the contour $\Psi(2w_1, 2w_2) = q_0$ (the small one). Here $q_0 = 3$ is the largest q_0 that may be obtained by Theorem 3.

2.4. Equations with shifts. Similar results are valid for equations with shifts:

$$\mathcal{L}u = f(u(x, y), u(x - h_1, y - h_2), u_x(x, y), u_x(x - h_1, y - h_2), \ldots). \tag{12}$$

Analogs of Theorems 1 and 3 for equation (12) can be obtained by straightforward changes of formulations. Also, it is easy to generalize these theorems to equations with shifts in the linear part.

2.5. Solutions of general form. An interesting problem is if there exist twice periodic solutions of general form $u(x, y)$ that can not be represented as $v(w_1 x + w_2 y)$. It would be also interesting to obtain some conditions for the uniqueness of a twice periodic solution for a fixed r or for a fixed ratio

w_1/w_2 (here we identify all the shifts $u(x+\phi_x, y+\phi_y)$ as a unique solution). Authors do not know any results of this type.

3. Proof of Theorem 3. We give the proof of Theorem 3 only. Theorem 1 follows from this theorem. The proofs of Theorems 2 and 4 may be obtained by rather standard modifications of the following scheme and we omit them.

To simplify the notation, we suppose that the polynomial $L(pw_1i, pw_2i)$ of one variable p has the same degree ℓ for all $(w_1, w_2) \in \mathcal{D}$. This means that $K(L) = \ell$ and all derivatives included in f are of order less than ℓ.

3.1. Ordinary differential equation. We look for twice periodic solutions to equation (4) of the form $u(x, y) = v(w_1x + w_2y)$. Substituting this formula in (4), we obtain the following ordinary differential equation for the unknown function $v = v(t)$:

$$\mathcal{L}_t v = \mathcal{L}_t(w_1, w_2)v \overset{\text{def}}{=} L(w_1\frac{d}{dt}, w_2\frac{d}{dt})v = f(v, w_1v', w_2v', \ldots). \tag{13}$$

We couple this equation with the 2π-periodic boundary conditions and look for solutions

$$v(t) = r\sin t + h(t) \tag{14}$$

such that the Fourier series of the 2π-periodic function $h(t)$ does not contain the harmonics $\sin t$ and $\cos t$. We are going to prove that for any $r > 0$ there exist a point $(w_1, w_2) = (w_1(r), w_2(r)) \in \mathcal{D}$ and a 2π-periodic function $h = h_r = h_r(\cdot)$ such that formula (14) defines a classical solution $v = v_r = v_r(t)$ of equation (13). Since $(0, 0) \notin \mathcal{D}$ (due to (5)) and the functions v_r satisfy (3), the corresponding twice periodic solutions $u_{r,\phi} = v_r(w_1(r)x+w_2(r)y+\phi)$ of equation (4) are non-stationary for all $r > 0$, $\phi \in \mathbb{R}$ and therefore for this set of solutions the conclusion of Theorem 3 holds.

Everywhere below $r > 0$ is considered as a parameter; the unknowns are the numbers $w_1 = w_1(r)$ and $w_2 = w_2(r)$ as well as the component $h = h_r = h_r(t)$ of the 2π-periodic solution (14) of equation (13). Remark that every non-stationary 2π-periodic solution $v(t)$ of this autonomous equation is included in the continuum $v(t+\phi)$ of such solutions (which define the same cycle in the phase space of equation (13) for all $\phi \in \mathbb{R}$), but at most one of them has the form (14) with $r > 0$.

3.2. Auxiliary constructions. We use the spaces C, C^k, L^2 and $W^{k,2}$ of functions $x(t) : [0, 2\pi] \to \mathbb{R}$ with the usual norms. Consider the projector

$$\mathcal{P}x(t) = \frac{1}{\pi}\int_0^{2\pi} \cos(t-s)\, x(s)\, ds$$

onto the plane E spanned on the functions $\sin t$ and $\cos t$ and the projector $\mathcal{Q} = I - \mathcal{P}$ onto the subspace $E^\perp$ that consists of all the functions $x = x(t)$ such that $\mathcal{P}x = 0$. The projectors $\mathcal{P}$ and $\mathcal{Q}$ act in all the above spaces. In L^2 they are orthogonal.

Consider the differential operator $\mathcal{L}_t = \mathcal{L}_t(w_1, w_2)$ with the 2π-periodic boundary conditions. Since (10) implies $L(nw_1 i, nw_2 i) \neq 0$ for all $(w_1, w_2) \in \mathcal{D}$, $n = 0, 1, 2, \ldots$, the kernel of this differential operator is the plane E if $L(w_1 i, w_2 i) = 0$ (in particular, this is the case for $(w_1, w_2) = (w_1^*, w_2^*)$) and the kernel is zero if $L(w_1 i, w_2 i) \neq 0$.

For each $(w_1, w_2) \in \mathcal{D}$ denote by $H = H(w_1, w_2)$ the linear operator that maps any function $y = y(t) \in E^\perp \bigcap L^2$ to a unique solution $x = Hy \in E^\perp \bigcap W^{\ell,2}$ of the equation $\mathcal{L}_t x = y(t)$. The existence of the solution $x = x(t)$ follows from $y \in E^\perp$ and $L(nw_1 i, nw_2 i) \neq 0$ for $n = 0, 1, 2, \ldots$, the uniqueness follows from $x \in E^\perp$. Operators $H : E^\perp \bigcap L^2 \to E^\perp \bigcap W^{\ell,2}$ are continuous; the norms of the operators H and the operators

$$H^{(k)} = H^{(k)}(w_1, w_2) = \frac{d^k}{dt^k} H$$

acting in the subspace $E^\perp \bigcap L^2$ of L^2 satisfy the estimates

$$\begin{aligned} \|H\|_{L^2 \to L^2} &\leq \max\nolimits_{n=0,2,3,\ldots} |L(w_1 ni, w_2 ni)|^{-1}, \\ \|H^{(k)}\|_{L^2 \to L^2} &\leq \max\nolimits_{n=0,2,3,\ldots} n^k |L(w_1 ni, w_2 ni)|^{-1} \end{aligned} \tag{15}$$

for all $(w_1, w_2) \in \mathcal{D}$, $k = 1, \ldots, \ell$.

Also, the map $(w_1, w_2, y) \mapsto H(w_1, w_2)h$ of the product $\mathcal{D} \times (E^\perp \bigcap L^2)$ to $W^{\ell-1,2}$ is completely continuous and each operator $H(w_1, w_2)$ maps continuously $E^\perp \bigcap C$ to C^ℓ.

3.3. Equivalent system. We look for solutions of equation (13) of the form $v(t) = r \sin t + H(w_1, w_2) y(t)$ with $y = y(t) \in E^\perp \bigcap L^2$. For each $r > 0$ set

$$\mathcal{Z}y = \mathcal{Z}_r(w_1, w_2)y = f\big(r \sin t + Hy, w_1(r \sin t + Hy)', w_2(r \sin t + Hy)', \ldots\big).$$

Since the operators $H = H(w_1, w_2)$ act from $E^\perp \bigcap L^2$ to $W^{\ell,2}$, the function f is continuous, and the order of the highest derivative included in f is by assumption less than ℓ, it follows that each operator $\mathcal{Z} = \mathcal{Z}_r(w_1, w_2)$ acts from $E^\perp \bigcap L^2$ to C. Moreover, the map $(w_1, w_2, y) \mapsto \mathcal{Z}_r(w_1, w_2)y$ from $\mathcal{D} \times (E^\perp \bigcap L^2)$ to C is completely continuous.

For each $r>0$ consider the system

$$\pi r \Re e\, L(w_1 i, w_2 i) = \int_0^{2\pi} \mathcal{Z} y(t)\sin t\, dt, \quad \pi r \Im m\, L(w_1 i, w_2 i) = \int_0^{2\pi} \mathcal{Z} y(t)\cos t\, dt,$$
$$y(t) = \mathcal{Q}\mathcal{Z} y(t). \tag{16}$$

By construction, every its solution $(w_1, w_2, y) \in \mathcal{D} \times (E^\perp \bigcap L^2)$ defines the 2π-periodic solution $v(t) = r\sin t + H(w_1, w_2)y(t) \in W^{\ell,2}$ of equation (13). Due to $\mathcal{Z}y(t) \in C$, the last equation of (16) implies $y \in C$ and therefore $Hy \in C^\ell$, i.e., $v = v(t)$ is a classical 2π-periodic solution of (13).

3.4. Deformation. For each $r > 0$ consider in the space $\mathbb{E} = \mathbb{R}^2 \times (E^\perp \bigcap L^2)$ a completely continuous deformation $\Theta = \big(\Theta_w, \Theta_y\big)$ with the components

$$\begin{aligned} \Theta_w(w_1, w_2, y, \xi) =& \Big(\Re e\, L(w_1 i, w_2 i) - \frac{\xi}{\pi r}\int_0^{2\pi} \mathcal{Z}y(t)\sin t\, dt, \\ & \qquad \Im m\, L(w_1 i, w_2 i) - \frac{\xi}{\pi r}\int_0^{2\pi} \mathcal{Z}y(t)\cos t\, dt\Big) \in \mathbb{R}^2, \\ \Theta_y(w_1, w_2, y, \xi) =& y - \xi \mathcal{Q}\,\mathcal{Z}y \in E^\perp \cap L^2, \end{aligned}$$

where $(w_1, w_2, y) \in \mathcal{D} \times E^\perp \bigcap L^2 \subset \mathbb{E}$ and $\xi \in [0,1]$ is the parameter of the deformation. By definition, for $\xi = 1$ every zero (w_1, w_2, y) of the vector field $\Theta(w_1, w_2, y, 1)$ is a solution of system (16).

Set

$$\eta = \eta(w_1, w_2) = \sum_{j=0}^{m}\sum_{k=0}^{j} \mu_{(j-k,k)} w_1^{2j-2k} w_2^{2k}, \quad c^2 = \frac{\pi q^2}{(1-q^2 q_0^{-2})} \max_{(w_1,w_2)\in\mathcal{D}} \eta(w_1, w_2)$$

(here $q < q_0$ by assumption) and define the set

$$\mathbb{D} = \mathbb{D}_r = \{(w_1, w_2, y) \in \mathbb{E} :\ (w_1, w_2) \in \mathcal{D},\ \|y\|_{L^2} \le (c+1)\, r\}.$$

To prove the existence of at least one zero of the vector field $\Theta(w_1, w_2, y, 1)$ inside $\mathbb{D}$, it is sufficient to show that the deformation $\Theta(w_1, w_2, y, \xi)$ is non-degenerate on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$ for all ξ (i.e., $\Theta(w_1, w_2, y, \xi) \ne 0$ for all $(w_1, w_2, y) \in \partial\mathbb{D}$, $\xi \in [0,1]$) and that for $\xi = 0$ the rotation $\gamma(\Theta^0, \partial\mathbb{D})$ of the vector field $\Theta^0 = \Theta(w_1, w_2, y, 0)$ on $\partial\mathbb{D}$ is non-zero.

3.5. Non-degeneracy of $\Theta(w_1, w_2, y, \xi)$. Everywhere in the following $\|\cdot\| = \|\cdot\|_{L^2}$. First, let us prove an *a priori* estimate for $\|\mathcal{Z}y\|$. The main estimate (7) implies

$$\|\mathcal{Z}y\|^2 \le q^2 \sum_{j=0}^{m} \sum_{k=0}^{j} \mu_{(j-k,k)} w_1^{2j-2k} w_2^{2k} \big(\pi r^2 + \|H^{(j)}y\|^2\big),$$

where $H^{(0)} = H$. From the definition of the operators $H^{(j)} = H^{(j)}(w_1, w_2)$, it follows that

$$\sum_{j=0}^{m}\sum_{k=0}^{j} \mu_{(j-k,k)} w_1^{2j-2k} w_2^{2k} \|H^{(j)}y\|^2 \le \|y\|^2 \max_{n=0,2,3,\dots} \sum_{j=0}^{m}\sum_{k=0}^{j} \frac{\mu_{(j-k,k)} w_1^{2j-2k} w_2^{2k} n^{2j}}{|L(w_1 ni, w_2 ni)|^2}$$

(this estimate is close to (15)). By definition of the function (8), the double sum in the right-hand part of this estimate equals $\Psi^{-2}(nw_1, nw_2)$, therefore (10) implies

$$\sum_{j=0}^{m}\sum_{k=0}^{j} \mu_{(j-k,k)} w_1^{2j-2k} w_2^{2k} \|H^{(j)}y\|^2 \le \|y\|^2 \max_{n=0,2,3,\dots} \Psi^{-2}(nw_1, nw_2) \le q_0^{-2}\|y\|^2$$

and consequently,

(17)

$$\|\mathcal{Z}y\|^2 \le \pi r^2 q^2 \sum_{j=0}^{m}\sum_{k=0}^{j} \mu_{(j-k,k)} w_1^{2j-2k} w_2^{2k} + q^2 q_0^{-2}\|y\|^2 = \pi r^2 q^2 \eta + q^2 q_0^{-2}\|y\|^2$$

for all $(w_1, w_2) \in \mathcal{D}$, $y \in E^{\perp} \bigcap L^2$.

Now note that for every zero of the deformation $\Theta = \Theta(w_1, w_2, y, \xi)$ the equality

$$\pi r^2 |L(w_1 i, w_2 i)|^2 + \|y\|^2 = \xi^2 \|\mathcal{Z}y\|^2$$

holds. Since $0 \leqslant \xi \leqslant 1$ and $|L(w_1 i, w_2 i)|^2 = \eta(w_1, w_2)\Psi^2(w_1, w_2)$, from this equality and the estimate (17) it follows that

$$\pi r^2 \eta(w_1, w_2)\big(\Psi^2(w_1, w_2) - q^2\big) + \big(1 - q^2 q_0^{-2}\big)\|y\|^2 \le 0$$

and due to $r > 0$, $\eta = \eta(w_1, w_2) > 0$, $q_0 > q > 0$,

$$\|y\|^2 \le \frac{\pi q^2 \eta(w_1, w_2)}{1 - q^2 q_0^{-2}} r^2, \qquad \Psi(w_1, w_2) < q.$$

The first of these two estimates implies $\|y\| \le cr$. The second estimate and condition (9) imply $(w_1, w_2) \notin \Gamma$. Since for every point of the boundary $\partial\mathbb{D}$ of the domain $\mathbb{D}$ at least one of the relations $\|y\| = (c+1)r$ and $(w_1, w_2) \in \Gamma$ holds, it follows that the deformation Θ has no zeros on $\partial\mathbb{D}$ (i.e., Θ is non-degenerate on $\partial\mathbb{D}$ for all ξ).

3.6. Rotation of the vector field $\Theta^0 = \Theta(w_1, w_2, y, 0)$. By definition, the components of this vector field have the simple form

$$\Theta_w^0 = \big(\Re e\, L(w_1 i, w_2 i),\ \Im m\, L(w_1 i, w_2 i)\big) = \Phi(w_1, w_2), \qquad \Theta_y^0 = y.$$

Directly from the Rotation Product Formula (see [2] or any other book on the degree theory) it follows that $\gamma\big(\Theta^0, \partial\mathbb{D}\big) = \gamma(\Phi, \Gamma)$. This rotation is non-zero by assumption, which completes the proof.

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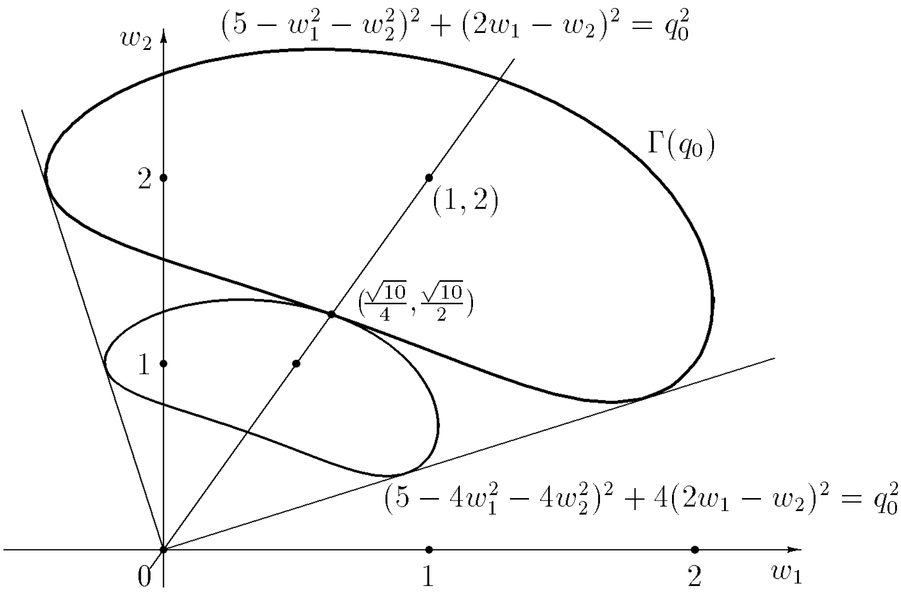

Fig. 1. Example; $q_0 = 3$.

FIG. 1. *Example;* $q_0 = 3$.