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ORDINARY
DIFFERENTIAL EQUATIONS

Continua of Cycles of Higher-Order Equations

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1. INTRODUCTION

In the present paper, we suggest theorems on the existence of continua of cycles for autonomous higher-order ordinary differential equations

$$L(d/dt, \lambda)x = f\left(x, x', \dots, x^{(\ell-1)}, \lambda\right) \qquad (1)$$

with a scalar parameter λ ranging in a bounded interval Λ. The existence of continua of cycles is determined by the linear part of the equation, i.e., by the polynomial

$$L(p, \lambda) = p^{\ell} + a_1(\lambda)p^{\ell-1} + \dots + a_{\ell}(\lambda) \qquad (2)$$

of degree $\ell \geq 2$ with coefficients continuously depending on λ. The nonlinearity is subjected to two conditions, namely, continuity and the validity of the estimate

$$|f\left(x_0, x_1, \dots, x_{\ell-1}, \lambda\right)| \leq q\left(\mu_0 x_0^2 + \mu_1 x_1^2 + \dots + \mu_{\ell-1} x_{\ell-1}^2\right)^{1/2}, \qquad (3)$$

where $\mu_j \geq 0$ and $\mu_0 + \dots + \mu_{\ell-1} > 0$. This estimate is a natural analog of the ordinary two-sided sector estimate $|f(x)| \leq q|x|$. By virtue of (3), zero is an equilibrium of Eq. (1) for each λ.

Our main theorems guarantee the existence of a continuum of cycles issuing from zero and going to infinity. (A rigorous definition is given in the following section.) The theorems contain assumptions about the polynomial (2) that guarantee the existence of such a continuum of cycles for Eq. (1) with an arbitrary continuous nonlinearity satisfying the estimate (3) for $q < q_0$, where q_0 is determined by the polynomial (2) and the numbers μ_j. We suggest a constructive method for the computation of q_0, which can readily be implemented on a computer. In the examples given in the paper, MAPLE has been used to perform the computations.

We point out that the theorems on global continua of cycles apply simultaneously to the entire class of equations (1) with common linear part and common coefficients q and μ_j in the estimate (3) of the continuous nonlinearity. No information on the differentiability of the nonlinearity at any point is required.

Our method can also be used in the analysis of local continua of cycles. The main classical example of the existence of such continua is the theorems on Andronov–Hopf bifurcations (e.g., see [1] and the bibliography therein). The known theorems are based on the linearization of the equation around the equilibrium; information on the linearization permits one to prove the existence of a continuum of cycles in a small neighborhood of this point. Theorems based on the differentiability of the nonlinearity only at the equilibrium were stated for the first time in [2]. Similar theorems are valid for bifurcations at infinity [3]. In the present paper, we neither assume the differentiability nor use linearizations. The main results of [2, 3] for higher-order equations readily follow from the theorems given here.

In conclusion, we present results that can be treated as conditions for the existence of cycles in situations where the nonlinearity is known only approximately. Here we prove the existence of a continuum of cycles in the domain lying between two concentric spheres; the equilibrium, whose exact position is not known, is localized in the ball bounded by the inner sphere.

Although the stability of cycles is important, related issues are not discussed in the present paper. Their analysis requires additional information about the nonlinearities.

The main theorems can be generalized to the case of systems $z' + A(\lambda)z = f(z,\lambda)$ in the space $z \in \mathbf{R}^\ell$. However, the authors do not know any effective method for estimating the admissible coefficient q in bounds similar to (3) for the nonlinearities in such systems. Our results can be further developed for delay equations, equations with hysteresis nonlinearities, partial differential equations, and equations in Banach spaces.

The definitions of continua of cycles used in the paper are close to the notion of continua of solutions of operator equations introduced and studied by topological methods in [4, 5]. Later, theorems on continua (usually local) were proved by numerous authors by analytic and geometric methods. Apparently, problems on continua of cycles cannot be directly reduced to the classical analysis scheme for branches of solutions of operator equations.

2. GLOBAL CONTINUA OF CYCLES

2.1. Main Theorem

Consider Eq. (1) for $\lambda \in \Lambda = [\lambda_1, \lambda_2]$. Let the nonlinearity $f(x_0, \dots, x_{\ell-1}, \lambda)$ and the polynomial $L(p,\lambda)$ be jointly continuous. In general, all considered functions are assumed to be continuous, but their differentiability is not assumed.

The cycles of Eq. (1) are considered in its phase space $\mathbf{R}^\ell$. We say that cycles form a *complete continuum* for $\lambda \in \Lambda$ if the boundary of each bounded open set $G \subset \mathbf{R}^\ell$ containing the origin has a nonempty intersection with at least one cycle of Eq. (1) for at least one $\lambda \in \Lambda$.

Consider the vector field $\Phi(w,\lambda) = (\operatorname{Re} L(wi,\lambda), \operatorname{Im} L(wi,\lambda))$ on the plane (w,λ). Let D be a bounded open domain on this plane, and let Γ be boundary of D. If $L(wi,\lambda) \neq 0$ for $(w,\lambda) \in \Gamma$, then the field $\Phi = \Phi(w,\lambda)$ has no zeros, or, which is the same, no singular points on Γ; therefore, the *rotation* $\gamma(\Phi, D)$ of Φ on Γ is well defined (e.g., see [6]).

We set

$$\Psi(w,\lambda) = |L(wi,\lambda)| \left(\mu_0 + \mu_1 w^2 + \cdots + \mu_{\ell-1} w^{2\ell-2}\right)^{-1/2}. \tag{4}$$

Theorem 1. *Let D be an open domain lying in the rectangle $w \in [w_1, w_2]$, $\lambda \in \Lambda = [\lambda_1, \lambda_2]$, where $w_1 > 0$. Let the relations*

$$\Psi(w,\lambda) \geq q_0 \quad \textit{for all} \quad (w,\lambda) \in \Gamma, \tag{5}$$

$$\Psi(nw,\lambda) \geq q_0 \quad \textit{for all} \quad (w,\lambda) \in \Gamma \cup D, \quad n = 0, 2, 3, \dots, \tag{6}$$

be valid for some $q_0 > 0$, so that the rotation $\gamma(\Phi, D)$ is well defined. Suppose that $\gamma(\Phi, D) \neq 0$ and the estimate (3) *is valid for all $(x_0, \dots, x_{\ell-1}) \in \mathbf{R}^\ell$ and $\lambda \in \Lambda$ with some $q < q_0$. Then Eq.* (1) *has a complete continuum of cycles with periods $T \in [2\pi/w_2, 2\pi/w_1]$ for $\lambda \in \Lambda$.*

If all assumptions of Theorem 1 are valid, then the boundary of each bounded open set $G \subset \mathbf{R}^\ell$ containing the origin has a nonempty intersection with at least one cycle of Eq. (1) entirely lying in the closure $\bar{G}$ of G and with at least one cycle entirely lying in the complement $\mathbf{R}^\ell \backslash G$ of G.

The relation $\gamma(\Phi, D) \neq 0$ implies that the vector field Φ has at least one singular point (w_0, λ_0) in D. If such a point is unique, then its topological index, or, which is the same, the Poincaré index, is equal to $\gamma(\Phi, D)$. For example, if the coefficients of the polynomial (2) are continuously differentiable in a neighborhood of λ_0 and the partial derivatives Φ_w^j and Φ_λ^j of the components of Φ at (w_0, λ_0) satisfy the relation $\Phi_w^1 \Phi_\lambda^2 - \Phi_\lambda^1 \Phi_w^2 \neq 0$, then the topological index of the singular point (w_0, λ_0) of Φ is nonzero and is equal to $\operatorname{sgn}(\Phi_w^1 \Phi_\lambda^2 - \Phi_\lambda^1 \Phi_w^2)$. It follows from the estimate (6) that

$$L(nw_0 i, \lambda_0) \neq 0 \quad \text{for} \quad n = 0, 2, 3, \dots, \tag{7}$$

which implies that the points (nw_0, λ_0) are not singular points of the vector field Φ for integer $n \neq \pm 1$.

Theorem 1 justifies a simple algorithm for the construction of estimates (3) providing the existence of complete continua of cycles for specific equations (1). The nonnegative coefficients μ_j in such estimates must satisfy the only constraint $\mu_0 + \cdots + \mu_{\ell-1} > 0$.

Consider the main situation, in which Φ has an isolated singular point (w_0, λ_0) with nonzero topological index and with $w_0 > 0$ and relation (7) is valid. Here as the domain D one can use the connected component $D(q_0)$ of the open set $\{(w,\lambda): \Psi(w,\lambda) < q_0\}$ containing (w_0,λ_0). Since (w_0,λ_0) is an isolated zero of the nonnegative function $\Psi(w,\lambda)$, it follows that $D(q_0)$ is nonempty for $q_0 > 0$. If $D(q_0)$ is a bounded domain lying together with its boundary $\Gamma = \Gamma(q_0)$ in the half-plane $w > 0$, containing the only zero (w_0,λ_0) of the field Φ, and satisfying condition (6), then it satisfies all assumptions of Theorem 1; therefore, for $q < q_0$, the estimate (3) implies that there exists a complete continuum of cycles of Eq. (1). The maximum value q_0 for which $D(q_0)$ satisfies all these conditions can readily be found with the use of computers. (Below we present related examples.) For sufficiently small $q_0 > 0$, all assumptions of Theorem 1 are necessarily valid for $D(q_0)$; this follows from the fact that $D(q_0)$ shrinks to the point (w_0,λ_0) as $q_0 \to +0$.

Continua of cycles can be studied in function spaces. Let us present one related assertion. Consider the space $C_0 = C_0([0,2\pi],\mathbf{R})$ of scalar continuous 2π-periodic functions $x(t)$ with the uniform norm; we identify these functions with their restrictions to the period. By V we denote the subspace of codimension 1 in C_0 consisting of all functions $x(t)$ orthogonal to the function $\cos t$ in the space $L^2 = L^2([0,2\pi],\mathbf{R})$. We assume that all assumptions of Theorem 1 are valid. Then on the boundary of any bounded open set in C_0 containing the origin, there is at least one point $x(t) \in V$ such that the function $x(wt)$ is a nonstationary periodic solution of Eq. (1) for at least one λ and for some $w > 0$, where $(w,\lambda) \in D$.

The space C_0 can be replaced by another space, say, C, C^k, or L^p. Instead of V, one can use other hyperplanes.

2.2. Nonlinearities Independent of the Derivatives

If the nonlinearity in Eq. (1) depends only on the variable x and is independent of the derivatives, then the estimate of the admissible coefficient q occurring in condition (3) can be improved. Here we apply the considerations used by V.M. Popov and E. Garber in absolute stability problems and frequency criteria for the absence of cycles (e.g., see [7]). Consider the equation

$$L(d/dt,\lambda)x = f(x,\lambda). \tag{8}$$

Theorem 2. *Let the domain D lie in the rectangle $w \in [w_1,w_2]$, $\lambda \in \Lambda = [\lambda_1,\lambda_2]$, where $w_1 > 0$. Suppose that, at each point $(w,\lambda) \in \Gamma$ of the boundary Γ of D, either $|L(wi,\lambda)| \geq q_0 > 0$ or the numbers $\operatorname{Im} L(nwi,\lambda)$ are nonzero and have the same sign for all positive integer n.*[1] *Suppose that $\gamma(\Phi,D) \neq 0$, $|L(nwi,\lambda)| \geq q_0$ for all $(w,\lambda) \in \Gamma\cup D$, $n = 0,2,3,\ldots$, and the estimate $|f(x,\lambda)| \leq q|x|$ $(x \in \mathbf{R},\ \lambda \in \Lambda)$ is valid for $q < q_0$. Then Eq. (8) has a complete continuum of cycles with periods $T \in [2\pi/w_2, 2\pi/w_1]$ for $\lambda \in \Lambda$.*

2.3. Control Systems

Let us present an analog of Theorem 1 for a class of systems occurring in control theory. Consider the equation

$$L(d/dt,\lambda)x = M(d/dt,\lambda)f\left(x,x',\ldots,x^{(k)},\lambda\right) \tag{9}$$

for $\lambda \in \Lambda = [\lambda_1,\lambda_2]$. Here the polynomials $L(p,\lambda) = p^\ell + a_1(\lambda)p^{\ell-1} + \cdots + a_\ell(\lambda)$ and $M(p,\lambda) = b_0(\lambda)p^m + \cdots + b_m(\lambda)$ are coprime for each λ, and their degrees admit the estimate $\ell > m + k$. Equations of the form (9) are used in the description of single-circuit systems consisting of a linear integrating unit with linear-fractional transfer function $M(p,\lambda)/L(p,\lambda)$ and a nonlinear feedback $f\left(x,x',\ldots,x^{(k)},\lambda\right)$ involving derivatives. The solutions of Eq. (9) are treated, as is customary in control theory, as solutions of the equivalent first-order system in the space $\mathbf{R}^\ell$, referred to as the state space (e.g., see [8]). If $M(p,\lambda) \equiv 1$, then Eq. (9) acquires the form (1).

Let $\mu_0 \geq 0,\ldots,\mu_k \geq 0$ $(\mu_0 + \mu_1 + \cdots + \mu_k > 0)$ be given numbers, and let

$$|f(x_0,\ldots,x_k,\lambda)| \leq q\left(\mu_0 x_0^2 + \cdots + \mu_k x_k^2\right)^{1/2}, \quad x_j \in \mathbf{R}, \quad \lambda \in \Lambda. \tag{10}$$

Just as above, we assume that the vector field $\Phi(w,\lambda) = (\operatorname{Re} L(wi,\lambda), \operatorname{Im} L(wi,\lambda))$ has no singular points on the boundary Γ of a domain $D \subset [w_1,w_2]\times[\lambda_1,\lambda_2]$, where $w_1 > 0$.

[1] Here and throughout the following, this assumption does not exclude the case in which both conditions are simultaneously valid at some point (w,λ).

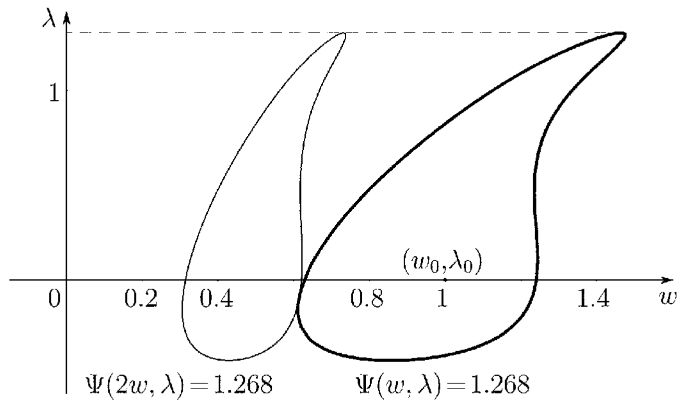

Fig. 1. The domain $D = D(q_0)$ for Eq. (11).

Theorem 3. *Let* $\gamma(\Phi, D) \neq 0$ *and* $M(wi, \lambda) \neq 0$ *for* $(w, \lambda) \in \Gamma \cup D$, *let the function*

$$\Psi(w, \lambda) = \left(\mu_0 + \mu_1 w^2 + \cdots + \mu_k w^{2k}\right)^{-1/2} |L(wi, \lambda)|/|M(wi, \lambda)|$$

satisfy the estimates (5) *and* (6) *for* $q_0 > 0$, *and let the estimate* (10) *be valid for* $q < q_0$. *Then Eq.* (9) *has a complete continuum of cycles for* $\lambda \in \Lambda$.

The proof of Theorems 1 and 2 is given in Section 5. The proof of Theorem 3 is similar to that of Theorem 1, and we omit it. Theorem 2 can also be generalized to the case of Eq. (9).

3. EXAMPLES

In all examples, we assume that the estimate (3) has the form $|f(\cdots)| \leq q|x|$. The simplest equation $x'' + \lambda x' + x = f(x, \lambda)$, where $|f(x, \lambda)| \leq q|x|$ $(q < 1)$, has a complete continuum of cycles in the phase plane for $\lambda = 0$; for $\lambda \neq 0$, there are no cycles. If $q = 1$, then cycles may be absent; for example, this is the case for $f(x, \lambda) = x$. Theorem 1 gives only $q < 0.6$ instead of the sharp estimate $q < 1$.

Consider the equation

$$x''' + (2 - \lambda)x'' + \left(1 - 2\lambda + 2\lambda^2\right) x' + (2 + \lambda/2)x = f\left(x, x', x'', \lambda\right). \quad (11)$$

Let $|f(x_0, x_1, x_2, \lambda)| \leq 1.268\,|x_0|$. Then Eq. (11) has a complete continuum of cycles with periods $1.355\pi < T < 3.271\pi$ in the phase space $\mathbf{R}^3$ for $-0.426 < \lambda < 1.304$. This follows from Theorem 1. Here we have used the domain $D = D(q_0)$ bounded by the level line $\Psi(w, \lambda) = q_0$ of the function $\Psi(w, \lambda) = |L(wi, \lambda)|$ for $q_0 \cong 1.475$. This value q_0 is chosen so as to ensure that the contour $\Psi(w, \lambda) = q_0$ is tangent to the contour $\Psi(2w, \lambda) = q_0$ (see Fig. 1). For larger values of q_0, condition (6) fails for $n = 2$.

The vector field $\Phi(w, \lambda)$ has the unique singular point $(w_0, \lambda_0) = (1, 0)$, and its topological index is equal to 1. Since this singular point [i.e., a zero of the function $\Psi(w, \lambda)$] lies in $D(q_0)$, we have $\gamma(\Phi, D(q_0)) = 1$.

If the nonlinearity in Eq. (11) depends only on x and λ, then Theorem 2 permits one to refine the parameter range $(-0.426 < \lambda < 0.347)$ and the estimate of the cycle periods but does not improve the estimate of q_0.

As another example, we consider the equation

$$x''' + (2 + \lambda)x'' + \left(1 - 2\lambda + 2\lambda^2\right) x' + (2 + \lambda/2)x = f\left(x, x', x'', \lambda\right). \quad (12)$$

Here the vector field $\Phi(w, \lambda)$ has two singular points $(w_0, \lambda_0) = (1, 0)$ and $(w_1, \lambda_1) \cong (0.918, 0.914)$ with topological indices 1 and -1. If $0 < q < q_0 \cong 0.348$, then the relation $\Psi(w, \lambda) = q$ defines two

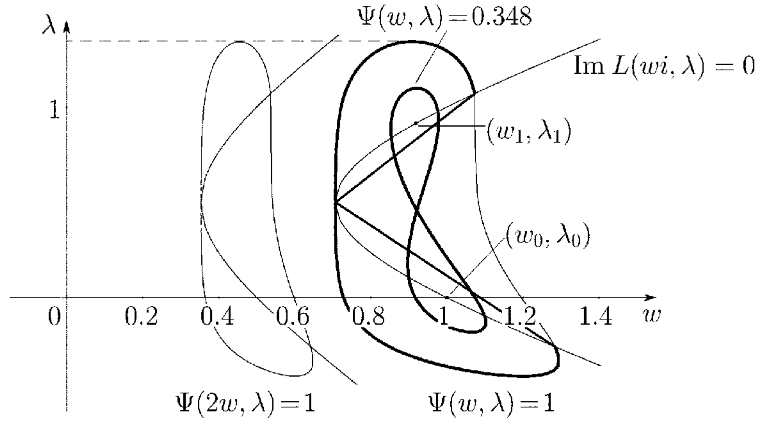

Fig. 2. The domains D for Eqs. (12) and (13).

disjoint simple closed contours; one of them surrounds the point (w_0, λ_0), and the other surrounds the point (w_1, λ_1). [These points are zeros of $\Psi(w, \lambda)$.] For $q_0 \cong 0.348$, the curve $\Psi(w, \lambda) = q_0$ is figure-eight shaped (see Fig. 2). All assumptions of Theorem 1 are valid for each of the two connected domains $D = D(q_0)$ and $D' = D'(q_0)$ surrounded by the loops of this figure-eight curve. Therefore, for

$$|f(x_0, x_1, x_2, \lambda)| \le 0.348\,|x_0|,$$

Eq. (12) has two complete continua of cycles. The cycles of one continuum exist for $-0.181 < \lambda < 0.457$, and their periods satisfy the estimate $1.814\pi < T < 2.230\pi$; for the other continuum of cycles, we have

$$0.457 < \lambda < 1.099, \qquad 2.044\pi < T < 2.345\pi.$$

Since each of the domains $D = D(q_0)$ and $D' = D'(q_0)$ contains one singular point of Φ, it follows that the rotations $\gamma(\Phi, D(q_0))$ and $\gamma(\Phi, D'(q_0))$ are equal to the topological indices 1 and -1 of these points. For $q > q_0$, the curve $\Psi(w, \lambda) = q$ is a simple closed contour bounding a domain $D(q)$ that contains both singular points of Φ. Therefore, $\gamma(\Phi, D(q)) = 0$, and the assumptions of Theorem 1 fail for $q > q_0$.

If the nonlinearity depends only on x and λ and Eq. (12) has the form

$$x''' + (2+\lambda)x'' + \left(1 - 2\lambda + 2\lambda^2\right)x' + (2 + \lambda/2)x = f(x, \lambda), \tag{13}$$

then Theorem 2 can be used. It follows from this theorem that the estimate $|f(x, \lambda)| \le q|x|$ with any $q < 1$ guarantees the existence of two complete continua of cycles of Eq. (13) for $-0.413 < \lambda < 0.5$ and for $0.5 < \lambda < 1.344$. Here $q_0 = 1$ is chosen so as to ensure that $\Psi(w, \lambda) = 1$ is tangent to the hyperbola $\operatorname{Im} L(wi, \lambda) = 0$ $(w > 0)$. The point of tangency is the vertex $(w_*, \lambda_*) = \left(\sqrt{0.5}, 0.5\right)$ of the hyperbola, and the tangent is vertical (Fig. 2). The hyperbola and the closed contour $\Psi(w, \lambda) = 1$ have another two points of intersection, $(w', \lambda') \cong (1.284, -0.258)$ and $(w'', \lambda'') \cong (1.073, 1.071)$. All assumptions of Theorem 2 are valid for the domain D_1 whose boundary consists of the straight line segment with endpoints (w_*, λ_*) and (w', λ') and the arc of the curve $\Psi(w, \lambda) = 1$ lying above this segment and joining its endpoints as well as for the domain D_2 bounded by the straight line segment with endpoints (w_*, λ_*) and (w'', λ'') and the arc of the curve $\Psi(w, \lambda) = 1$ lying above this segment. The estimate $\operatorname{Im} L(wi, \lambda) < 0$ is valid for all points (except for the endpoints) of both segments.

Figure 2 represents also the curves $\Psi(2w, \lambda) = 1$ and $\operatorname{Im} L(2wi, \lambda) = 0$; their arrangement justifies the estimates $|L(nwi, \lambda)| \ge 1$ and $\operatorname{Im} L(nwi, \lambda) < 0$ for the domains D_1 and D_2 and their boundaries for all positive integers $n \ge 2$.

4. LOCAL CONTINUA OF CYCLES

4.1. Continua of Cycles in a Neighborhood of the Equilibrium

We again consider Eq. (1). In its phase space $\mathbf{R}^\ell$, we introduce the Euclidean norm

$$|\xi| = \left(x_0^2 + \cdots + x_{\ell-1}^2\right)^{1/2},$$

where $\xi = (x_0, x_1, \ldots, x_{\ell-1})$. We assume that the estimate (3) is valid not for all $\xi \in \mathbf{R}^\ell$ but only in some ball $B_\varrho = \left\{\xi \in \mathbf{R}^\ell : \ |\xi| < \varrho\right\}$ and consider the problem on the existence of cycles in this case.

We say that the cycles of Eq. (1) form a local continuum in a neighborhood of the zero equilibrium for $\lambda \in \Lambda$ if there exists a ball B_R $(R > 0)$ such that the boundary of each neighborhood G of zero contained in this ball has a nonempty intersection with at least one cycle of Eq. (1) for at least some $\lambda \in \Lambda$. Under the assumptions of the following theorem, the ball B_R is determined by the polynomial (2), the coefficients μ_j and q in the estimate (3), and the radius ϱ of the ball B_ϱ in which this estimate holds. One can write out explicit estimates for R. In the case of Eq. (8), such estimates follow from the formulas in Subsection 4.3.

Theorem 4. *Let $D \subset [w_1, w_2] \times \Lambda$ $(w_1 > 0)$. Suppose that the function* (4) *satisfies the estimates* (5) *and* (6) *with some $q_0 > 0$ in the domain D and on its boundary Γ, $\gamma(\Phi, D) \neq 0$, and the estimate* (3) *is valid for all $\xi = (x_0, \ldots, x_{\ell-1}) \in B_\varrho$ $(\varrho > 0)$ and all $\lambda \in \Lambda$ with some $q < q_0$. Then Eq.* (1) *has a local continuum of cycles in a neighborhood of the zero equilibrium for $\lambda \in \Lambda = [\lambda_1, \lambda_2]$.*

4.2. Continua of Cycles at Infinity

Suppose that, instead of the estimate (3), the global estimate

$$|f(x_0, x_1, \ldots, x_{\ell-1}, \lambda)| \le \left(q^2\left(\mu_0 x_0^2 + \mu_1 x_1^2 + \cdots + \mu_{\ell-1} x_{\ell-1}^2\right) + c^2\right)^{1/2} \tag{14}$$

is valid for some $c > 0$. We say that the cycles of Eq. (1) form a continuum at infinity for $\lambda \in \Lambda$ if there exists a ball B_r such that the boundary of each open bounded set G containing this ball has a nonempty intersection with at least one cycle of Eq. (1) for at least one $\lambda \in \Lambda$.

Theorem 5. *Suppose that $D \subset [w_1, w_2] \times \Lambda$ $(w_1 > 0)$, the function* (4) *satisfies the estimates* (5) *and* (6) *for some $q_0 > 0$ in the domain D and on its boundary Γ, $\gamma(\Phi, D) \neq 0$, and the estimate* (14) *is valid for all $(x_0, \ldots, x_{\ell-1}) \in \mathbf{R}^\ell$ and $\lambda \in \Lambda$ with some $q < q_0$. Then Eq.* (1) *has a continuum of cycles at infinity for $\lambda \in \Lambda$.*

Theorem 4 and 5 are close to Andronov–Hopf bifurcation theorems. The main difference from the known results is that we do not assume that Eq. (1) can be linearized at zero (under the assumptions of Theorem 4) or at infinity (under the assumptions of Theorem 5).

4.3. Systems with Approximately Known Nonlinearities

In this subsection, we present conditions for the existence of a continuum of cycles in a spherical layer around the origin. We consider Eq. (8) and assume that the nonlinearity satisfies the estimate

$$|f(x, \lambda)| \le \left(q^2 x^2 + \varepsilon^2\right)^{1/2} \quad \text{for} \quad |x| \le \varrho, \qquad \lambda \in \Lambda = [\lambda_1, \lambda_2], \tag{15}$$

which is close to (14) but is assumed to be valid only in a bounded domain. Here we assume that $\varepsilon > 0$ is much less than ϱ; precise assumptions will be given below. A natural example is as follows: ε is of the order of the computer roundoff error, and all other quantities are of the order of unity.

We say that the cycles of Eq. (8) form a continuum joining the balls B_r and B_R $(r < R)$ in the phase space $\mathbf{R}^\ell$ if the boundary of each open set G satisfying the inclusions $B_r \subset G \subset B_R$ has a nonempty intersection with at least one cycle for at least one $\lambda \in \Lambda$.

Let the relations

$$|L(nwi,\lambda)| \geq q_* > 0 \quad \text{for} \quad (w,\lambda) \in \Gamma \cup D, \quad n = 0, 2, 3, \ldots, \tag{16}$$

similar to (6), be valid in the domain D and on its boundary Γ. For $1 \leq k \leq \ell$ and $q < q_*$, we set

$$\sigma_k(w) = \left[\frac{1-w^{2k}}{1-w^2}\right]^{1/2}, \qquad \nu_k(q) = \sup_{(w,\lambda)\in D}\left[\frac{1}{2\left(|L(0,\lambda)|^2 - q^2\right)} + \sum_{n=2}^{\infty}\frac{\sigma_k^2(nw)}{|L(nwi,\lambda)|^2 - q^2}\right]^{1/2};$$

here and throughout the following, we adopt the convention that $\sigma_k(1) = k$.

Theorem 6. *Suppose that $D \subset [w_1, w_2] \times \Lambda$ for some $w_1 > 0$, the estimate* (16) *is valid, $\gamma(\Phi, D) \neq 0$, and one of the following assertions holds at each point $(w,\lambda) \in \Gamma$: either $|L(wi,\lambda)| \geq q_0 > 0$, or the numbers $\operatorname{Im} L(nwi,\lambda)$ are nonzero and have the same sign for all positive integers n. Let $0 < r_* < R_*$, and let the nonlinearity $f(x,\lambda)$ satisfy the estimate* (15) *for some $q < \min\{q_0, q_*\}$ and for some*

$$\varepsilon < \frac{r_*\left(q_0^2 - q^2\right)^{1/2}}{\sqrt{2}\left(\sigma_\ell(w_2) + q_0\nu_\ell(q)\right)}, \qquad \varrho > \frac{\sqrt{2}}{\sigma_\ell(w_1)}\left(R_* + q_*\nu_1(0)\left(\frac{q^2R_*^2 + \varepsilon^2\sigma_\ell^2(w_1)}{q_*^2 - q^2}\right)^{1/2}\right). \tag{17}$$

Then Eq. (8) *has a continuum of cycles joining the balls B_{r_*} and B_{R_*} in the phase space $\mathbf{R}^\ell$.*

Theorem 6 supplements Theorems 4 and 5 for the case of nonlinearities independent of derivatives. It provides estimates for the radii of the balls B_R (for $\varepsilon = 0$ and $\mu_0 = 1$) and B_r (for $\varrho = \infty$, $\mu_0 = 1$, and $c = \varepsilon$) used in the definitions of continua of cycles in a neighborhood of the equilibrium and at infinity.

The proof of Theorem 6 is given in Section 6. The proof of Theorems 4 and 5 is not represented, since it is based on the scheme used in the proof of Theorems 1 and 2 and does not contain any new ideas.

5. PROOF OF THEOREMS 1 AND 2

5.1. Linear Spaces and Operators

In the proof, we use the standard spaces C, C^k, L^2, and $W^{k,2}$ of scalar functions $x(t)$ defined for $0 \leq t \leq 2\pi$. Functions defined on the entire line and periodic with period 2π are identified with their restrictions to the interval $0 \leq t \leq 2\pi$.

Consider the linear problem

$$L(wd/dt,\lambda)x = y(t), \qquad x(0) = x(2\pi), \qquad \ldots, \qquad x^{(\ell-1)}(0) = x^{(\ell-1)}(2\pi). \tag{18}$$

If $L(nwi,\lambda) \neq 0$ for all integer n, then, for each function $y(t) \in L^2$, problem (18) has a unique solution $x(t) \in W^{\ell,2}$ satisfying the equation almost everywhere. Let $(w,\lambda) \in \Gamma \cup D$. Then the estimate (6) implies the relation $L(nwi,\lambda) \neq 0$ for all integer $n \neq \pm 1$, whence it follows that there exists a unique solution $x(t) \in W^{\ell,2} \cap E^\perp$ of problem (18) for each $y(t) \in E^\perp$, where $E^\perp$ is the orthogonal complement of the plane $E = \{a \sin t + b \cos t : \ a, b \in \mathbf{R}\}$ in L^2. In other words, the operator $H = H(w,\lambda)$ acting from the subspace $E^\perp$ of L^2 to the subspace $W^{\ell,2} \cap E^\perp$ of $W^{\ell,2}$ and taking each function $y(t)$ to the unique solution $x(t) = Hy(t)$ of problem (18) in $E^\perp$ is well defined and bounded. If $y(t) \in C$, then this solution is a classical solution, i.e., $x(t) = Hy(t) \in C^\ell \cap E^\perp$ for $y(t) \in C \cap E^\perp$.

For an arbitrary function $y(t) \in L^2$, all solutions of problem (18) are given by the formulas $x(t) = r\sin(t+\phi) + H(w,\lambda)Qy(t)$ and $r \operatorname{Re}\left(L(wi,\lambda)e^{i\phi}\right) = \beta(y(t))$, $r \operatorname{Im}\left(L(wi,\lambda)e^{i\phi}\right) = \alpha(y(t))$, where Q is the orthogonal projection onto the subspace $E^\perp$ of L^2 and

$$\alpha(y(t)) = \frac{1}{\pi}\int_0^{2\pi} y(t)\cos t\,dt, \qquad \beta(y(t)) = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin t\,dt.$$

If $L(wi,\lambda)\neq 0$, then the solution is unique; if $L(wi,\lambda)=0$, then solutions exist only for $y(t)\in E^{\perp}$; in this case, r and ϕ are arbitrary. The assumption $\gamma(\Phi,D)\neq 0$ in Theorems 1 and 2 implies that there exists at least one zero $(w,\lambda)\in D$ of the function $L(wi,\lambda)$.

In what follows, we use the relations

$$\frac{1}{\|y\|_{L^2}^2}\sum_{k=0}^{\ell-1}\mu_k w^{2k}\left\|H^{(k)}y\right\|_{L^2}^2 \le \max_{n=0,2,3,\dots}\sum_{k=0}^{\ell-1}\frac{\mu_k w^{2k}n^{2k}}{|L(nwi,\lambda)|^2}=\max_{n=0,2,3,\dots}\Psi^{-2}(nw,\lambda), \tag{19}$$

where $H^{(0)}=H=H(w,\lambda)$ and $H^{(k)}y(t)=d^k(Hy(t))/dt^k$ for $k\ge 1$ and $y=y(t)\in E^{\perp}$.

5.2. An Equivalent Problem

Instead of Eq. (1), we consider the equation

$$L(wd/dt,\lambda)x=f\left(x,wx',\dots,w^{\ell-1}x^{(\ell-1)},\lambda\right) \tag{20}$$

with an additional unknown $w>0$. Each 2π-periodic solution $x(t)$ of Eq. (20) determines the $(2\pi/w)$-periodic solution $x(wt)$ of Eq. (1).

We seek 2π-periodic solutions of Eq. (20) satisfying the condition $\alpha(x(t))=0$. It follows from the relations in Subsection 5.1 that a function $x(t)$ is such a solution if and only if it has the form $x(t)=r\sin t+H(w,\lambda)h(t)$ for $h(t)\in E^{\perp}$ and the relations

$$r\operatorname{Re}L(wi,\lambda)=\beta(Y(t)),\qquad r\operatorname{Im}L(wi,\lambda)=\alpha(Y(t)),\quad h(t)=QY(t) \tag{21}$$

hold, where

$$Y(t)=f\left(r\sin t+Hh(t),w(r\sin t+Hh(t))',\dots,w^{\ell-1}(r\sin t+Hh(t))^{(\ell-1)},\lambda\right);$$

here $H=H(w,\lambda)$. Since each solution $x(t)$ of Eq. (20) is embedded in the continuum of solutions $x(t+\phi)$ ($\phi\in\mathbf{R}$), it follows that the condition $\alpha(x(t))=0$ is not an additional constraint in the problem on the existence of periodic solutions.

Consider an open bounded set $G\subset\mathbf{R}^{\ell}$. Let ∂G be the boundary of this set, and let $\bar G$ be its closure. On the space of all continuous vector functions $\xi(t):[0,2\pi]\to\mathbf{R}^{\ell}$ with the uniform norm, we define a continuous functional $\varphi_G(\xi(t))=\varphi_G(\xi_1(t),\dots,\xi_\ell(t))$ by the relations

$$\varphi_G(\xi(t))=\begin{cases}-\min_{z\in\partial G,\ t\in[0,2\pi]}|z-\xi(t)| & \text{if}\quad \xi(t)\in\bar G \text{ for all } t\in[0,2\pi]\\ \max_{z\in\partial G,\ \xi(t)\notin\bar G,\ t\in[0,2\pi]}|z-\xi(t)| & \text{if}\quad \xi(\tau)\notin\bar G \text{ for at least one } \tau\end{cases}$$

and supplement system (21) by the equation

$$\varphi_G\left(r\sin t+Hh(t),w(r\sin t+Hh(t))',\dots,w^{\ell-1}(r\sin t+Hh(t))^{(\ell-1)}\right)=0, \tag{22}$$

where $H=H(w,\lambda)$. The resulting system (21), (22) consists of three scalar equations and one equation in the subspace $E^{\perp}$ of L^2 and contains three scalar unknowns w, λ, and r and one unknown vector function $h=h(t)\in E^{\perp}$. By definition, $\varphi_G(\xi(t))=0$ if and only if the curve $\xi(t)$ has at least one common point with ∂G and entirely lies in $\bar G$. Therefore, each solution $(w,\lambda,r,h)\in\mathbf{R}^3\times E^{\perp}$ of system (21), (22) with component $w>0$ determines a 2π-periodic solution $x(t)=r\sin t+Hh(t)$ of Eq. (20) such that the cycle $\left(x(wt),wx'(wt),\dots,w^{\ell-1}x^{(\ell-1)}(wt)\right)$ of Eq. (1) has a nonempty intersection with the boundary of G. Consequently, to prove the existence of a complete continuum of cycles, it suffices to prove the solvability of system (21), (22) for an arbitrary choice of a bounded open set G containing the origin. The proof is based on a topological method.

Note that the inclusion $h(t)\in L^2$ implies the relations $Hh(t)\in W^{\ell,2}\subset C^{\ell-1}$ and, further, $Y(t)\in C$. If relations (21) are valid, then $h(t)=QY(t)\in C$, and consequently, $Hh(t)\in C^{\ell}$. Therefore, the formula $x(t)=r\sin t+H(w,\lambda)h(t)$ defines classical periodic solutions $x(t)$ and $x(wt)$ of Eqs. (20) and (1) for each solution $(w,\lambda,r,h)\in\mathbf{R}^3\times E^{\perp}$ $(w>0)$ of system (21), (22).

5.3. Homotopy

We set $\Theta_w^s = r\operatorname{Re} L(wi,\lambda) - s\beta(Y(t))$,

$$\Theta_\lambda^s = r\operatorname{Im} L(wi,\lambda) - s\alpha(Y(t)), \qquad \Theta_r^s = \varphi_G\Big(r\sin t + sHh(t), swx'(t),\ \dots, sw^{\ell-1}x^{(\ell-1)}(t)\Big),$$

and $\Theta_h^s = h(t) - sQY(t)$, where $x(t) = r\sin t + H(w,\lambda)h(t)$, $Y(t) = f\Big(x(t),\ \dots,\ w^{\ell-1}x^{(\ell-1)}(t),\ \lambda\Big)$, and $0 \le s \le 1$. Consider a compact deformation $\Theta^s = (\Theta_w^s, \Theta_\lambda^s, \Theta_r^s, \Theta_h^s)$ joining the completely continuous vector fields $\Theta^0 = \Theta^0(w,\lambda,r,h)$ and $\Theta^1 = \Theta^1(w,\lambda,r,h)$ in the space $\mathbf{R}^3 \times E^\perp$. By definition, the singular points (or, which is the same, zeros) of the vector field Θ^1 are the solutions of system (21), (22).

Lemma 1. *Let the assumptions of Theorem* 1 *or Theorem* 2 *be valid. Then the deformation* $\Theta^s(w,\lambda,r,h)$ $(0 \le s \le 1)$ *has no zeros on the boundary of the set*

$$\Omega = \Big\{(w,\lambda,r,h) \in \mathbf{R}^3 \times E^\perp :\ (w,\lambda) \in D,\ r_1 < r < r_2,\ \|h\|_{L^2} < c_0 r\Big\}$$

for sufficiently small $r_1 > 0$ *and sufficiently large* r_2 *and* c_0.

The proof of the lemma is given in the next subsection. The lemma implies the equality $\gamma(\Theta^0,\Omega) = \gamma(\Theta^1,\Omega)$ of rotations of the vector fields Θ^0 and Θ^1 on the boundary of the domain Ω for sufficiently large $1/r_1$, r_2, and c_0. The components $\Theta_w^0 = r\operatorname{Re} L(wi,\lambda)$, $\Theta_\lambda^0 = r\operatorname{Im} L(wi,\lambda)$, $\Theta_r^0 = \varphi_G(r\sin t, 0,\dots,0)$, and $\Theta_h^0 = h$ of the field Θ^0 have a simple form; by virtue of standard theorems on the product of rotations [6, 9], this field satisfies the relation $\gamma(\Theta^0,\Omega) = \gamma(\Phi,D)\gamma(\Theta_r^0,J)$, where $\gamma(\Theta_r^0,J)$ is the rotation of the scalar field $\Theta_r^0(r) = \varphi_G(r\sin t,0,\dots,0)$ on the boundary of the interval $J = (r_1,r_2)$. Since the open set G contains the origin and is bounded, we have $\Theta_r^0(r_1) < 0$ for sufficiently small $r_1 > 0$ and $\Theta_r^0(r_2) > 0$ for sufficiently large r_2. These estimate imply that $\gamma(\Theta_r^0,J) = 1$ and hence $\gamma(\Theta^1,\Omega) = \gamma(\Theta^0,\Omega) = \gamma(\Phi,D) \ne 0$. By the nonzero rotation principle, the relation $\gamma(\Theta^1,\Omega) \ne 0$ guarantees that there exists at least one singular point (w,λ,r,h) of the field Θ^1, or, which is the same, a solution of system (21), (22), in the domain Ω. The proof of Theorems 1 and 2 is complete.

5.4. Proof of Lemma 1

The boundary of Ω is the union of the sets $\Omega_h = \{(w,\lambda) \in D\cup\Gamma,\ r \in [r_1,r_2],\ \|h\|_{L^2} = c_0 r\}$, $\Omega_{w,\lambda} = \{(w,\lambda) \in \Gamma,\ r \in [r_1,r_2],\ \|h\|_{L^2} \le c_0 r\}$, and $\Omega_r^j = \{(w,\lambda) \in D\cup\Gamma,\ r = r_j,\ \|h\|_{L^2} \le c_0 r_j\}$ for $j = 1,2$; we must prove the relation $\Theta^s(w,\lambda,r,h) \ne 0$ on each of these sets for sufficiently large c_0 and r_2 and for sufficiently small $r_1 > 0$.

Let $\Theta^s(w,\lambda,r,h) = 0$ and $(w,\lambda) \in D\cup\Gamma$. The relations $\Theta_w^s = \Theta_\lambda^s = \Theta_h^s = 0$ imply that $\pi r^2|L(wi,\lambda)|^2 + \|h\|_{L^2}^2 = \pi s^2(\alpha^2+\beta^2) + s^2\|QY(t)\|_{L^2}^2 = s^2\|Y(t)\|_{L^2}^2$, where $\alpha = \alpha(Y(t))$ and $\beta = \beta(Y(t))$. It follows from the estimate (3) that

$$\|Y(t)\|_{L^2}^2 \le q^2 \sum_{k=0}^{\ell-1} \mu_k w^{2k} \left\|x^{(k)}(t)\right\|_{L^2}^2 = q^2 \sum_{k=0}^{\ell-1} \mu_k w^{2k} \left(\pi r^2 + \left\|H^{(k)}h(t)\right\|_{L^2}^2\right),$$

where $x(t) = r\sin t + H(w,\lambda)h(t)$. By virtue of the estimates (19) and (6), we have

$$\|Y(t)\|_{L^2}^2 - q^2\pi r^2 \sum_{k=0}^{\ell-1} \mu_k w^{2k} \le q^2\|h\|_{L^2}^2 \max_{n=0,2,3,\dots} \Psi^{-2}(nw,\lambda) \le \left(\frac{q}{q_0}\right)^2 \|h\|_{L^2}^2.$$

Therefore,

$$\pi r^2|L(wi,\lambda)|^2 + \|h\|_{L^2}^2 \le s^2\|Y(t)\|_{L^2}^2 \le q^2\pi r^2 m(w) + (q/q_0)^2\|h\|_{L^2}^2$$

for $m(w) = \mu_0 + \mu_1 w^2 + \cdots + \mu_{\ell-1} w^{2\ell-2}$; since $|L(wi,\lambda)|^2 = m(w)\Psi^2(w,\lambda)$, we have

$$\left(1 - q^2/q_0^2\right) \|h\|_{L^2}^2 \le \pi r^2 m(w) \left(q^2 - \Psi^2(w,\lambda)\right). \tag{23}$$

The estimate $q < q_0$ and inequality (23) imply that $\|h\|_{L^2} \le c_* r$ for $c_*^2 = \pi m_* / \left(q^{-2} - q_0^{-2}\right)$, where $m_* = \max\{m(w)\colon (w,\lambda) \in D \cup \Gamma\}$. Since $\|h\|_{L^2} = c_0 r$ on Ω_h, it follows that the deformation Θ^s has no zeros $(w,\lambda,r,h) \in \Omega_h$ for $c_0 > c_*$.

If $\Psi(w,\lambda) \ge q_0 > q$ and $r > 0$, then relation (23) fails. Therefore, assumption (5) of Theorem 1 implies that the deformation Θ^s has no zeros $(w,\lambda,r,h) \in \Omega_{w,\lambda}$.

Let the nonlinearity have the form $f(x,\lambda)$, and let the assumptions of Theorem 2 be valid. By construction, an arbitrary zero of the deformation Θ^s specifies a solution

$$x(t) = r \sin t + H(w,\lambda) h(t) \in W^{\ell,2}$$

of the equation $L(wd/dt,\lambda)x = sf(x,\lambda)$. By multiplying this equation by $x'(t)$ and by integrating the resulting relation over the interval $0 \le t \le 2\pi$, we obtain

$$\pi \sum_{n=1}^{\infty} n \operatorname{Im} L(nwi,\lambda) \left(\alpha_n^2 + \beta_n^2\right) = 0, \tag{24}$$

where $\alpha_n = \alpha\left(x\left(n^{-1}t\right)\right)$ and $\beta_n = \beta\left(x\left(n^{-1}t\right)\right)$. By the assumptions of Theorem 2, at each point $(w,\lambda) \in \Gamma$, either we have $\Psi(w,\lambda) \ge q_0$ or all numbers $\operatorname{Im} L(nwi,\lambda)$ $(n = 1, 2, \ldots)$ are nonzero and have the same sign. Therefore, at least one of relations (23) and (24) is valid at each point of the set $\Omega_{w,\lambda}$, and hence the deformation Θ^s also has no zeros $(w,\lambda,r,h) \in \Omega_{w,\lambda}$ under the assumptions of Theorem 2.

It remains to consider the sets Ω_r^1 and Ω_r^2. Since $0 \in G$ and G is bounded, it follows that there exists a $\delta = \delta(G) > 0$ such that

$$\varphi_G(\xi(t)) < 0 \quad \text{for} \quad \|\xi_j\|_C \le \delta, \quad j = 1, \ldots, l, \qquad \varphi_G(\xi(t)) > 0 \quad \text{for} \quad \|\xi_1\|_C \ge 1/\delta, \tag{25}$$

where $\xi(t) = (\xi_1(t), \ldots, \xi_\ell(t))$. By the first estimate in (25) and the uniform boundedness of the norms $\|H(w,\lambda)\|_{L^2 \to C^{\ell-1}} \le c < \infty$ of the operators $H(w,\lambda) : E^\perp \to W^{\ell,2} \cap E^\perp$ for all $(w,\lambda) \in D \cup \Gamma$ on the set Ω_r^1 for any sufficiently small $r_1 > 0$, we have the estimate $\Theta_r^s(w,\lambda,r,h) < 0$. The relations $2\pi \|r \sin t + sH(w,\lambda)h(t)\|_C^2 \ge \|r \sin t\|_{L^2}^2 + \|sH(w,\lambda)h(t)\|_{L^2}^2 \ge \pi r^2$, together with the second estimate in (25), imply that $\Theta_r^s(w,\lambda,r,h) > 0$ on Ω_r^2 for sufficiently large r_2. Therefore, the deformation Θ^s has no zeros (w,λ,r,h) on the boundary $\Omega_h \cup \Omega_{w,\lambda} \cup \Omega_r^1 \cup \Omega_r^2$ of Ω for $c_0 > c_*$ provided that $r_1 > 0$ is sufficiently small and r_2 is sufficiently large. The proof of Lemma 1 is complete.

6. PROOF OF THEOREM 6

6.1. Outline of the Proof

The proof mainly reproduces that of Theorems 1 and 2. However, in this case, we use another deformation $\Theta = (\Theta_w^s, \Theta_\lambda^s, \Theta_r^s, \Theta_h^s)$, which differs from that introduced in Section 5 in the definition of the third scalar component Θ_r^s. Namely,

$$\Theta_w^s = r \operatorname{Re} L(wi,\lambda) - s\beta(Y), \qquad \Theta_\lambda^s = r \operatorname{Im} L(wi,\lambda) - s\alpha(Y), \qquad \Theta_h^s = h - sQY,$$
$$\Theta_r^s = \varphi_G\left(sx + (1-s)v_r,\ swx' + (1-s)w_1 v_r', \ldots, sw^{\ell-1} x^{(\ell-1)} + (1-s) w_1^{\ell-1} v_r^{(\ell-1)}\right),$$

where $x = r \sin t + H(w,\lambda)h(t)$, $v_r = r \sin t$, and $Y = f(x(t),\lambda)$. The use of the deformation defined in Section 5 is also possible; the new deformation has been introduced for refining estimates for the radii B_{r_*} and B_{R_*} of the balls joined by a continuum of cycles. The following analog of Lemma 1 plays a key role.

Lemma 2. *Let all assumptions of Theorem* 6 *be valid. Let*

$$r_1\left(q_0^2-q^2\right)^{1/2}>\varepsilon\sqrt{2},\qquad r_1\sigma_\ell\left(w_2\right)+\nu_\ell(q)\left(q^2r_1^2+2\varepsilon^2\right)^{1/2}<r_*, \tag{26}$$

$$r_2\sigma_\ell\left(w_1\right)>R_*\sqrt{2},\qquad r_2+q_*\nu_1(0)\left(\left(q^2r_2^2+2\varepsilon^2\right)/\left(q_*^2-q^2\right)\right)^{1/2}<\varrho, \tag{27}$$

and $B_{r_*}\subset G\subset B_{R_*}$. *If* $c_1=(\varrho-r_2)\sqrt{\pi}/\nu_1(0)$, *then there are no zeros of the deformation* $\Theta^s=\Theta^s(w,\lambda,r,h)$ $(0\le s\le 1)$ *on the boundary of the set*

$$\Omega=\left\{(w,\lambda,r,h)\in\mathbf{R}^3\times E^\perp:\ (w,\lambda)\in D,\ r_1<r<r_2,\ \|h\|_{L^2}<c_1\right\}.$$

Since $r_*<R_*$, it follows from (26) and (27) that $r_1<r_2$. The existence of numbers r_1 and r_2 satisfying relations (26) and (27) follows from the estimate (17).

It follows from Lemma 2 that $\gamma\left(\Theta^1,\Omega\right)=\gamma\left(\Theta^0,\Omega\right)$. Therefore, to prove the theorem, it suffices to prove the relation $\gamma\left(\Theta^0,\Omega\right)\ne 0$. Just as in Section 5, in the computation of the rotation of the vector field Θ^0 on the boundary of Ω, one can use the theorem on the product of rotations of the components of this field, which implies that $\gamma\left(\Theta^0,\Omega\right)=\gamma(\Phi,D)\gamma\left(\Theta_r^0,J\right)$, where $J=(r_1,r_2)$. Since $r_1\sigma_\ell\left(w_1\right)<r_*$ and $r_2\sigma_\ell\left(w_1\right)/\sqrt{2}>R_*$, we have

$$\left(v_{r_1}(t),w_1v_{r_1}'(t),\dots,w_1^{\ell-1}v_{r_1}^{(\ell-1)}(t)\right)\in B_{r_*}$$

for all t and the relation

$$\left(v_{r_2}(\tau),w_1v_{r_2}'(\tau),\dots,w_1^{\ell-1}v_{r_2}^{(\ell-1)}(\tau)\right)\notin\bar{B}_{R_*}$$

holds for at least one τ, where $v_r(t)=r\sin t$. (Here $\bar{B}_{R_*}$ is the closure of the open ball B_{R_*}.) Since

$$\Theta_r^0(r)=\varphi_G\left(v_r(t),w_1v_r'(t),\dots,w_1^{\ell-1}v_r^{(\ell-1)}(t)\right),\qquad B_{r_*}\subset G\subset B_{R_*},$$

it follows that $\Theta_r^0\left(r_1\right)<0<\Theta_r^0\left(r_2\right)$. Therefore, $\gamma\left(\Theta_r^0,J\right)=1$ and, further,

$$\gamma\left(\Theta^0,\Omega\right)=\gamma(\Phi,D)\ne 0.$$

It remains to prove Lemma 2.

6.2. Proof of Lemma 2

Let $\Theta^s(w,\lambda,r,h)=0$ at some point of the closure $\bar{\Omega}$ of Ω. The inclusion $(w,\lambda,r,h)\in\bar{\Omega}$ is equivalent to the relations $(w,\lambda)\in D\cup\Gamma$, $r\in[r_1,r_2]$, and $\|h\|_{L^2}\le c_1$. It suffices to show that the zero (w,λ,r,h) of the deformation Θ^s does not lie on the boundary of Ω, i.e., $(w,\lambda)\in D$, $r_1<r<r_2$ and $\|h\|_{L^2}<c_1$.

It follows from the definition of the operators $H(w,\lambda)$ that $\|H(w,\lambda)\|_{L^2\to C}\le\nu_1(0)/\sqrt{\pi}$; therefore, the function $x(t)=r\sin t+H(w,\lambda)h(t)$ admits the estimate $\|x(t)\|_C\le r_2+c_1\nu_1(0)/\sqrt{\pi}=\varrho$. This, together with the estimate (15), implies the estimate $\|Y(t)\|_{L^2}^2\le q^2\|x\|_{L^2}^2+2\pi\varepsilon^2$ for the norm of the function $Y(t)=f(x(t),\lambda)$, or, which is the same, $\|Y(t)\|_{L^2}^2\le q^2\left(\pi r^2+\|H(w,\lambda)h\|_{L^2}^2\right)+2\pi\varepsilon^2$. Just as in Section 5, from the relations $\Theta_w^s=\Theta_\lambda^s=\Theta_h^s=0$, we obtain

$$\pi r^2|L(wi,\lambda)|^2+\|h\|_{L^2}^2=\pi s^2\left(\alpha^2(Y(t))+\beta^2(Y(t))\right)+s^2\|QY(t)\|_{L^2}^2=s^2\|Y(t)\|_{L^2}^2$$

(here and in Section 5, the components Θ_w^s, Θ_λ^s, and Θ_h^s of the deformation are defined in the same way), and therefore,

$$\pi r^2|L(wi,\lambda)|^2+\|h\|_{L^2}^2\le q^2\left(\pi r^2+\|H(w,\lambda)h\|_{L^2}^2\right)+2\pi\varepsilon^2. \tag{28}$$

By (16) and (19), we have $\|H(w,\lambda)\|_{L^2\to L^2}\le q_*^{-1}$, and consequently,

$$\left(1-q^2/q_*^2\right)\|h\|_{L^2}^2\le\pi r^2\left(q^2-|L(wi,\lambda)|^2\right)+2\pi\varepsilon^2,$$

which is an analog of the estimate (23). Therefore,

$$\left(1-q^2/q_*^2\right)\|h\|_{L^2}^2 \le \pi r^2 q^2 + 2\pi\varepsilon^2, \qquad 0 \le \pi r^2\left(q^2 - |L(wi,\lambda)|^2\right) + 2\pi\varepsilon^2;$$

since $r \in [r_1, r_2]$, it follows that

$$\|h\|_{L^2}^2 \le \pi q_*^2\left(q^2 r_2^2 + 2\varepsilon^2\right)/\left(q_*^2 - q^2\right), \qquad |L(wi,\lambda)|^2 \le q^2 + 2\varepsilon^2/r_1^2.$$

By the second estimate in (27), we have

$$q_*^2\left(q^2 r_2^2 + 2\varepsilon^2\right)/\left(q_*^2 - q^2\right) < (\varrho - r_2)^2/\nu_1^2(0),$$

and consequently, $\|h\|_{L^2}^2 < \pi(\varrho - r_2)^2/\nu_1^2(0) = c_1^2$. This completes the proof of the estimate $\|h\|_{L^2} < c_1$.

By virtue of the first estimate in (26), $q^2 + 2\varepsilon^2/r_1^2 < q_0^2$, and therefore, $|L(wi,\lambda)| < q_0$. Moreover, it follows from (24) that the numbers $\operatorname{Im} L(nwi,\lambda)$ cannot have the same sign for all positive integers n. [The validity of (24) for the zeros of the deformation Θ^s can be proved in the same way as in Section 5.] This, together with the assumptions of the theorem, implies that $(w,\lambda) \notin \Gamma$, i.e., $(w,\lambda) \in D$.

It remains to prove the estimates $r_1 < r < r_2$. The upper bound follows from the relation $\Theta_r^s(w,\lambda,r,h) = 0$. This relation, together with the definition of the functional $\varphi_G(\cdots)$ and the inclusion $G \subset B_{R_*}$, implies the estimate $\|\xi_1^2(t) + \cdots + \xi_\ell^2(t)\|_C \le R_*^2$, where

$$\xi_k(t) = s w^{k-1} x^{(k-1)}(t) + (1-s) w_1^{k-1} v_r^{(k-1)}(t).$$

Since

$$\|\xi_k(t)\|_{L^2}^2 \ge \pi\alpha^2(\xi_k(t)) + \pi\beta^2(\xi_k(t)) = \pi r^2\left(s w^{k-1} + (1-s) w_1^{k-1}\right)^2 \ge \pi r^2 w_1^{2k-2}$$

and $\|\xi_1\|_{L^2}^2 + \cdots + \|\xi_\ell\|_{L^2}^2 \le 2\pi\|\xi_1^2 + \cdots + \xi_\ell^2\|_C$, we have $\pi r^2\left(1 + w_1^2 + \cdots + w_1^{2\ell-2}\right) \le 2\pi R_*^2$, or, which is the same, $r\sigma_\ell(w_1) \le R_*\sqrt{2}$; by virtue of the first estimate in (27), we see that $r < r_2$.

The proof of the estimate $r > r_1$ is more complicated. It is based on the relation

$$\left\|x^2 + w^2(x')^2 + \cdots + w^{2\ell-2}\left(x^{(\ell-1)}\right)^2\right\|_C \le \left(r\sigma_\ell(w) + \nu_\ell(q)\left(\|h\|_{L^2}^2 - q\|Hh\|_{L^2}^2/\pi\right)^{1/2}\right)^2,$$

which follows from the definition of the operator $H = H(w,\lambda)$ and the estimates $q < q_*$ and (16) (we omit the proof). This relation, together with (28), implies the estimate

$$\left\|x^2 + w^2(x')^2 + \cdots + w^{2\ell-2}\left(x^{(\ell-1)}\right)^2\right\|_C \le \left(r\sigma_\ell(w) + \nu_\ell(q)\left(q^2 r^2 + 2\varepsilon^2\right)^{1/2}\right)^2.$$

Since $\left\|v_r^2 + w_1^2(v_r')^2 + \cdots + w_1^{2\ell-2}\left(v_r^{(\ell-1)}\right)^2\right\|_C \le r^2\sigma_\ell^2(w_1)$ and $w_1 \le w \le w_2$, we have the estimate

$$\left\|\xi_1^2 + \cdots + \xi_\ell^2\right\|_C \le \left(r\sigma_\ell(w_2) + \nu_\ell(q)\left(q^2 r^2 + 2\varepsilon^2\right)^{1/2}\right)^2,$$

where $\xi_k = s w^{k-1} x^{(k-1)}(t) + (1-s) w_1^{k-1} v_r^{(k-1)}(t)$. Since $\Theta_r^s(w,\lambda,r,h) = 0$ and $B_{r_*} \subset G$, we find that $r_*^2 \le \|\xi_1^2 + \cdots + \xi_\ell^2\|_C$. Therefore, $r_* \le r\sigma_\ell(w_2) + \nu_\ell(q)\left(q^2 r^2 + 2\varepsilon^2\right)^{1/2}$, which, together with the second estimate in (26), implies that $r_1 < r$. The proof of Lemma 2 and Theorem 6 is complete.

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