Remark on Rotation of Bilinear Vector Fields

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In the paper we present a formula for rotation of vector fields defined in the space with a spacial algebraic structure. Partially, this formula is applicable on the complex plane and in 4-dimensional space of quaternions.

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1. Introduction

Consider a planar continuous vector field Φx. Let it be non-zero on the simple (without selfintersections) close curve Γ. Then the rotation $\gamma = \gamma(\Phi, \Gamma)$ (see [1, 2]) is well-defined of this field along this curve. The plane can be considered as the set of complex numbers with the operation of multiplying. If $\Phi(z) = \varphi_1(z)\varphi_2(z)$, then the rotations $\gamma(\varphi_1, \Gamma)$ and $\gamma(\varphi_2, \Gamma)$ are also well-defined and

$$\gamma(\Phi, \Gamma) = \gamma(\varphi_1, \Gamma) + \gamma(\varphi_2, \Gamma). \tag{1}$$

This formula follows from the properties of the multiplication of the complex numbers:

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$$

and from the definition of the planar rotation as the winding number. Formula (1) is valid if Γ is the boundary of any domain (not necessary for simple curves).

In this paper we formulate a simple theorem that generalizes formula (1) for some other even dimensional spaces that have an algebraic structure, partially such generalizations are valid for the quaternions. From this theorem the "fundamental theorem of algebra" for the quaternions [3] and for Cayley numbers [4] follow.

The notion of rotation of vector fields is equivalent to the notion of degree of a mapping, we use the rotation due to family traditions.

2. The main result

Consider the space $\mathbb{R}^n$ where $n = 2k$ is an even positive integer.

Let B denotes a bilinear mapping $R^n \times R^n \mapsto R^n$. By the definition of the bilinearity there exist the matrices $M(x)$ and $M^*(x)$, $x \in \mathbb{R}^n$ satisfying the following conditions:

$$B(x, y) = M(x)y = M^*(y)x, \quad x, y \in \mathbb{R}^n. \tag{2}$$

We will say that the mapping B is non-degenerate, if $B(x, y) = 0$ implies $x = y = 0$. This is equivalent to the property that the matrices $M(x)$ and $M^*(x)$ are non-degenerate for all non-zero $x \in \mathbb{R}^n$. In particular for a non-degenerated bilinear mapping B all determinants $\det M(x)$ and $\det M^*(x)$ for $x \in \mathbb{R}^n, x \neq 0$ have the same signature. Thus we can define the number

$$\sigma(B) = \mathrm{sign}\det M(x), \quad \sigma^*(B) = \mathrm{sign}\det M^*(x).$$

The pair $(\sigma(B), \sigma^*(B))$ will be called *the signature* of the (non-degenerated) bilinear mapping B.

Non-degenerated bilinear mappings exist ([5]) only for the dimensions $2, 4, 8$, with examples given by the multiplications of complex numbers, quaternions, and the Cayley octaves. Note, that there exist bilinear mappings which are not equivalent to the aforementioned standard products. Let us consider, for example, the case $n = 2$. Then for $x = (u, v),\ y = (u, v)$ the mappings

$$\begin{aligned} B_1(x, y) &= (au - bv, av + bu), & B_2(x, y) &= (au + bv, -av + bu), \\ B_3(x, y) &= (-au + bv, av + bu), & B_4(x, y) &= (-au + bv, av + bu) \end{aligned}$$

have the signatures $(+1, +1)$, $(-1, +1)$, $(-1, -1)$, $(+1, -1)$. Note also that

$$\sigma(B) = \sigma^*(B) = +1 \tag{3}$$

if there exist an element $w \in \mathbb{R}^n$ such that $B(w, x) \equiv B(x, w) \equiv x$ (that is if there exist the unit with respect to the form B).

Let us consider in $\mathbb{R}^n$ a domain D with the boundary ∂D and consider two non-degenerate continuous vector fields $\Phi(x)$ and $\Psi(x)$ on ∂D.

Theorem 1. *If $\Phi, \Psi \neq 0$ on ∂D, then the equality*

$$\gamma\big(B(\Phi(x), \Psi(x)), \partial D\big) = \sigma(B)^* \gamma\big(\Phi(x), \partial D\big) + \sigma(B) \gamma\big(\Psi(x), \partial D\big) \tag{4}$$

is valid.

We do not consider odd dimensional spaces: non-degenerate multiplications do not exist there. But in 1 dimensional space the usual multiplication is nondegenerate and formula (4) is not valid. An unquisitive reader will easily find out the source of such apparent contradiction.

Let us stress here that formula (4) is not only the formula for rotation of products but also a formula for rotation of fractions.

3. Proof

To prove the theorem let us continue both vector-fields Φ and Ψ into the interior of D in such a way that the following conditions hold:

1. Both fields have only finite numbers of zeros and all these zeroes are simple: the corresponding Jacobi matrices are non-degenerate.

2. The zeroes $x_1, \ldots, x_m$ of the field Φ are different with the zeroes $y_1, \ldots, y_k$ of the field Ψ:
$$x_i \neq y_j, \qquad i = 1, \ldots, m; \quad j = 1, \ldots, k.$$

This is possible according to Sard Lemma (see, e.g., [6], Ch. 1, Theorem 1.2). Now formula (4) follows from the relations

$$\begin{aligned}\gamma\Big(B\big(\Phi(x),\Psi(x)\big),\partial D\Big) &= \sum_{i=1}^{m} \operatorname{Ind} B\big(\Phi(x),\Psi(x)\big)\Big|_{x=x_i} + \sum_{j=1}^{k} \operatorname{Ind} B\big(\Phi(x),\Psi(x)\big)\Big|_{x=y_j}\\ &= \sum_{i=1}^{m} \sigma^*(B) \operatorname{Ind} \Phi(x)\Big|_{x=x_i} + \sum_{j=1}^{k} \sigma(B) \operatorname{Ind} \Psi(x)\Big|_{x=y_j}\\ &= \sigma^*(B)\gamma(\Phi,\partial D) + \sigma(B)\gamma(\Psi,\partial D).\end{aligned}$$

The equalities

$$\sum_{i=1}^{m} \operatorname{Ind} \Phi(x)\big|_{x=x_i} = \gamma(\Phi,\partial D) \quad \text{and} \quad \sum_{j=1}^{k} \operatorname{Ind} \Psi(x)\big|_{x=y_j} = \gamma(\Psi,\partial D)$$

are obvious. The formulae

$$\begin{aligned}\operatorname{Ind} B\big(\Phi(x),\Psi(x)\big)\Big|_{x=x_i} &= \operatorname{Ind} B\big(\Phi(x),\Psi(x_i)\big)\Big|_{x=x_i} = \operatorname{Ind} M^*\big(\Psi(x_i)\big)\Phi(x)\big|_{x=x_i}\\ &= \operatorname{sign}\det M^*\big(\Psi(x_i)\big)\ \operatorname{Ind} \Phi(x)\big|_{x=x_i} = \sigma^*(B) \operatorname{Ind} \Phi(x)\big|_{x=x_i}\end{aligned}$$

for any i and the formulae

$$\begin{aligned}\operatorname{Ind} B\big(\Phi(x),\Psi(x)\big)\Big|_{x=y_j} &= \operatorname{Ind} B\big(\Phi(y_j),\Psi(x)\big)\Big|_{x=y_j} = \operatorname{Ind} M\big(\Phi(y_i)\big)\Psi(x)\big|_{x=y_i}\\ &= \operatorname{sign}\det M\big(\Phi(y_j)\big)\ \operatorname{Ind} \Psi(x)\big|_{x=y_j} = \sigma(B) \operatorname{Ind} \Psi(x)\big|_{x=y_j}\end{aligned}$$

for any j follow from the continuity and from the formula

$$\operatorname{Ind} AF(x)\big|_{x=x^*} = \operatorname{sign}\det A \operatorname{Ind} F(x)\big|_{x=x^*}$$

that is valid for any non-degenerated matrix A and for any operator F; the last formula follows directly from the Rotation Product Formula ([6], Ch. 1, Theorem 7.2).

4. Discussion

We consider an equation with the leading 'quadratic'term:

$$B(x,x) + f(x) = 0.$$

Here $|f(x)|/(|x|^2) \to 0$ as $|x| \to \infty$. By the theorem this has at least one root, if the signature of B is either $(+1,+1)$ or $(-1,-1)$. The rotations of the field $B(x,x)+f(x)$ on the spheres of large radii r equal either 2 or -2.

Let us generalize this principle. Let $\mathbb{W}$ be the set of all finite word of letters l and r. Then we can correspond to each such word $w \in \mathbb{W}$ a mapping $P_w(x) : R^n \to R^n$ by the following rules

- The quadratic mapping $B(x,x)$ is corresponded to the empty word.
- If P_w is defined, then to the concatenation lw the mapping $B(x, P_w(x))$ is corresponded.
- If P_w is defined, then to the concatenation rw the mapping $B(P_w(x), x)$ is corresponded.

Remark. If the bilinear mapping B creates the associative multiplication $x \circledcirc y = B(x,y)$, then

$$P_w(x) = x^{m+2} = \overbrace{x \circledcirc \ldots \circledcirc x}^{m+2}$$

for any word w of the length m; the equation has more 'usual' form $x^{m+2}+f(x)=0$. If this multiplication is not associative, then the notion x^n loses any sense, there exist various such terms: $x \circledcirc (x \circledcirc x)$ may be different from $(x \circledcirc x) \circledcirc x$. Here to describe various possible principal terms we need to use P_w.

Theorem 2. *Suppose either m is odd or the signature of B is either $(+1,+1)$ or $(-1,-1)$. For each word w of the length m the equation*

$$P_w(x) + f(x) = 0$$

with the leading monomial term has at leas one root if

$$|f(x)|/(|x|^{m+2}) \to 0 \qquad \textit{as} \qquad |x| \to \infty.$$

Consider the set of complex numbers, or quaternions, or Cayley numbers, and let the bilinear mapping B be given by the corresponding product structure. This mapping is non-degenerated because there is no divisors of zero in the set of complex numbers, or quaternions, or Cayley numbers; also in this case

$$\sigma(B) = \sigma^*(B) = 1$$

by (3).

Thus Theorem 2 implies immediately that each polynomial with the single monomial term of the highest degree in the corresponding structures has roots. Theorem 2 may be applied for some non-standard multiplications, for instance, given by the form B_3 above.

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