NONLOCAL THEOREMS ON HOPF BIFURCATIONS

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Dedicated to our friend Nikolai Bobylev

Consider the systems $\dot{z} = g(z,\lambda)$, $z \in \mathbb{R}^\ell$, such that the point $z = 0$ is an equilibrium for any value of the scalar parameter $\lambda \in [\lambda_1, \lambda_2]$; the function $g(z,\lambda)$ is continuous. The value $\lambda_0 \in (\lambda_1, \lambda_2)$ is called a *Hopf bifurcation point* for this system if for any sufficiently small $\varepsilon > 0$ there is a $\lambda_\varepsilon \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ such that for $\lambda = \lambda_\varepsilon$ the system has a cycle $\Gamma_\varepsilon \subset B_\varepsilon = \{z \in \mathbb{R}^\ell : |z| \le \varepsilon\}$. If the system is linearizable at the equilibrium point $z = 0$ and has the form

$$\frac{dz}{dt} = A(\lambda)z + f(z,\lambda), \qquad z \in \mathbb{R}^\ell, \tag{1}$$

where $A(\lambda)$ is a continuous matrix-valued function and $\max_{\lambda\in[\lambda_1,\lambda_2]} |f(z,\lambda)| = o(z)$ as $z \to 0$, then according to the Hopf Bifurcation Theorem the main linear term is responsible for the existence of cycles in a small neighborhood of the zero. More presicely, if the matrix $A(\lambda)$ has a pair of simple conjugate eigenvalues $\sigma(\lambda) \pm w(\lambda)i$, which cross transversally the imaginary axis at some points $\pm w_0 i$ for $\lambda = \lambda_0$, and if the so-called *non-resonance* condition is valid for the rest of the spectrum of $A(\lambda)$ (which means that the values $nw_0 i$ are not the eigenvalues of $A(\lambda_0)$ for $n = 0, 2, 3, \ldots$), then λ_0 is a Hopf bifurcation point for system (1).

Here we present sufficient conditions for the existence of a branch of cycles that originates from the zero equilibrium and goes to infinity. The existence of this branch is defined by the linear term of system (1) like in the local Hopf Bifurcation Theorem. The main point is that we use the information about the matrix $A(\lambda)$ on the whole segment $[\lambda_1, \lambda_2]$ as well as a global estimate of the nonlinearity $f(z,\lambda)$. To be precise, we introduce the following definition.

Definition. *A set of cycles is called a* continuous branch connecting the balls B_{ρ_1} and B_{ρ_2} *with* $0 < \rho_1 < \rho_2$ *if for any open set* G *satisfying* $B_{\rho_1} \subset G \subset B_{\rho_2}$ *there is at least one cycle* Γ *of equation* (1) *for at least one* $\lambda \in [\lambda_1, \lambda_2]$ *such that* $\Gamma \subset \overline{G}$, $\Gamma \cap \partial G \ne \varnothing$ *where* $\overline{G}$ *and* ∂G *are the closure and the boundary of* G. *A set of cycles is called a* global continuous branch *if it is a continuous branch connecting the balls* B_{ρ_1} *and* B_{ρ_2} *for any* $0 < \rho_1 < \rho_2$.

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Assume that the matrix $A(\lambda)$ has the simple conjugate eigenvalues $\sigma(\lambda) \pm w(\lambda)i$ for all $\lambda \in [\lambda_1, \lambda_2]$ (real functions $\sigma(\lambda)$, $w(\lambda)$ are continuous) and define the minimal segment $[w_1, w_2]$ such that $0 < w_1 \le w(\lambda) \le w_2$ for all λ. Denote by $\mathfrak{F}(q)$ the class of all continuous functions $f(z, \lambda)$ satisfying the *sector estimate*

$$|f(z,\lambda)| \le q|z|, \qquad z \in \mathbb{R}^\ell, \quad \lambda \in [\lambda_1, \lambda_2]. \tag{2}$$

Theorem 1. *Let $\sigma(\lambda_1)\sigma(\lambda_2) < 0$. Let $A(\lambda)$ do not have the imaginary eigenvalues of the form nwi for all $\lambda \in [\lambda_1, \lambda_2]$, $w \in [w_1, w_2]$ and $n = 0, 2, 3, \dots$ (this is a counterpart of the non-resonance condition of the Hopf Bifurcation Theorem). Then there exists a $q > 0$ such that system (1) with any nonlinearity $f \in \mathfrak{F}(q)$ has a global continuous branch of cycles.*

Simple modifications of this theorem are valid if instead of the global estimate (2) the estimate $|f(z,\lambda)| \le q|z|$ holds in some domain $r_1 \le |z| \le r_2$ or if the nonlinearity satisfies $|f(z,\lambda)| \le q|z| + c$ instead of (2). These estimates with appropriate q, r_1, r_2, c imply the existence of a continuous branch of cycles connecting some balls B_{ρ_1} and B_{ρ_2}, their radii are defined by the coefficients and the domain of the estimate. It is easy to extend Theorem 1 to other classes of equations, in particular equations with delays and partial differential equations. Let us stress that the conditions of the theorem are satisfied for some systems (1) that can not be linearized at the point of the zero equilibrium.

The authors do not know a good algorithm to obtain estimates of the coefficient q such that condition (2) guarantees the existence of a continuous branch of cycles for systems of the general form (1). The following theorem is a basis for simple algorithms to estimate q for higher order scalar equations

$$L\Big(\frac{d}{dt}, \lambda\Big)x = \varphi(x, x', \dots, x^{(\ell-1)}, \lambda), \tag{3}$$

where $L(p,\lambda) = p^\ell + a_1(\lambda)p^{\ell-1} + \dots + a_\ell(\lambda)$. In this particular case, we use the sector estimate of the form

$$|\varphi(x_0, x_1, \dots, x_{\ell-1}, \lambda)| \le q\big(\mu_0 x_0^2 + \mu_1 x_1^2 + \dots + \mu_{\ell-1}x_{\ell-1}^2\big)^{1/2} \tag{4}$$

with some fixed $\mu_k \ge 0$ such that $\mu_0 + \dots + \mu_{\ell-1} > 0$.

Consider on the plane Π of the variables (w, λ) the level curves of the nonnegative function $\Psi(w,\lambda) = |L(wi,\lambda)|\big(\mu_0 + \mu_1 w^2 + \dots + \mu_{\ell-1}w^{2\ell-2}\big)^{-1/2}$. Let the closed contour $C = C(q_0)$ belong to the level set $\Psi(w,\lambda) = q_0$ of this function for some $q_0 > 0$ and let C be the boundary of a bounded open domain $D = D(q_0)$. Let the set $D \cup C$ lie in the half-plane $w > 0$ of Π.

Theorem 2. *Let $\Psi(nw,\lambda) \ge q_0$ for all $(w,\lambda) \in D \cup C$ and $n = 0, 2, 3, \dots$ Let the winding number of the real planar vector field $\big(\operatorname{Re} L(wi,\lambda), \operatorname{Im} L(wi,\lambda)\big)$ on the boundary C of the domain D be non-zero. Let estimate (4) be valid with any $q < q_0$. Then the set of cycles of equation (3) in its phase space $\mathbb{R}^\ell$ is a global continuous branch.*

In particular examples considered by the authors the coefficient q determined by Theorem 2 is of the same order as the coefficients of the polynomial $L(p,\lambda)$.