On resonant differential equations with unbounded nonlinearities*

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Abstract

We present a method to study asymptotically linear degenerate problems with sublinear unbounded nonlinearities. The method is based on the uniform convergence to zero of projections of nonlinearity increments onto some finite-dimensional spaces. Such convergence was used for the analysis of resonant equations with bounded nonlinearities by many authors [1, 2, 4, 5, 7, 11–13, 17]. The unboundedness of nonlinear terms complicates essentially the analysis of most problems: existence results, approximate methods, systems with parameters, stability, dissipativity, etc. In this paper we present statements on projection convergence for unbounded nonlinearities and apply them to various resonant asymptotically linear problems: existence of forced periodic oscillations and unbounded sequences of such oscillations; existence of unbounded solutions; the sharp analysis of integral equations with the simple degeneration of the linear part (a scalar two-point boundary value problem is considered as an example); existence of nontrivial cycles for higher order autonomous ODEs; Hopf bifurcations at infinity.

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1. Projections of nonlinearity increments

1.1. Introduction. Consider the equation $x = Tx$ in a Banach space E with an asymptotically linear[1] completely continuous operator T. If its main linear part $x = Ax$ is nondegenerate (has no nontrivial solutions), then various problems related to this equation are rather simple. If 1 is an eigenvalue of the linear operator A (degenerate case) the analysis is much more complicated, it is necessary to use some properties of sublinear nonlinearities $F = T - A$. An important property that can be used is the uniform convergence of the values $\Delta F = \phi\big(F(\xi e + h) - F(\xi e)\big)$ to zero as $\xi \to \infty$. Here $\phi(\cdot)$ is some linear functional, the convergence must be uniform with respect to all normed eigenvectors e of the operator A, $Ae = e$, and with respect to all vectors $h \in E$ from some special sets depending on ξ. The same convergence can be used also in some problems that can not be reduced directly to equations of the type $x = Tx$.

Here we present sufficient conditions for the convergence $\Delta F \to 0$ for unbounded nonlinearities F and consider some applications to ordinary differential equations. Most applications are concerned with periodic and Dirichlet problems for second order equations and for equations arising in control theory.

Theorems 1 and 2 on the convergence $\Delta F \to 0$ are formulated in the space L^2 for nonlinearities $Fx(t) = f\big(t, x(t)\big)$; the uniformity of the convergence is proved for rather special classes of increments $h(\cdot)$. The choice of formulations is determined by the applications.

The applications presented are new for the case of unbounded nonlinearities. If the nonlinearity is bounded, the corresponding analogs of our theorems are known.

The paper is organized as follows.

In the next subsection we present three theorems. The theorems are used in all other results of the paper, in some sense all other results are applications of Theorems 1 and 2. Theorem 3 is slightly more general, we do not present its applications. The last subsection of this section contains some remarks.

In Section 2 there are results about forced periodic oscillations. In Section 3 we give a generalization of results by R.Ortega and J.Alonso concerning the existence of unbounded solutions. In Section 4 we consider integral equations and two-point BVPs. Sections 5 and 6 are concerned with self-induced oscillations: in Section 5 we present existence theorems for cycles of higher order quasilinear ODEs; in Section 6 Hopf bifurcations at infinity are considered. In Section 7 we give a lemma to compute some values that we use throughout the paper. The other sections contain proofs.

1.2. Convergence conditions. Let $\Theta(\xi) : [0, +\infty) \to [1, \infty)$ be a monotone increasing sublinear at infinity function:

$$\lim_{\xi \to \infty} \frac{\Theta(\xi)}{\xi} = 0.$$

We suppose that for any $M > 0$ there exists a number $C = C(M)$ such that $\Theta(M\xi) \leq C\Theta(\xi)$ for all $\xi \geq C$.

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[1]$Tx = Ax + Fx$ where A is linear and F is sublinear, i.e., the nonlinear operator F satisfies $\|Fx\| \cdot \|x\|^{-1} \to 0$ at infinity.

Let a function $e(\cdot)$ be defined on the interval $[a, b]$, let it be $k > 1$ times continuously differentiable. Let its derivative $e'(\cdot)$ take the zero value in a finite set of critical points. Let the highest order of tangency of the graph of $e(\cdot)$ with a horizontal line equal k. The last assumption means that if $e'(t_0) = 0$, then at least one of the numbers $e''(t_0), \ldots, e^{(k)}(t_0)$ is nonzero.

Let a continuous scalar-valued function $f(t, x) : [a, b] \times \mathbb{R} \to \mathbb{R}$ satisfy the estimate

$$|f(t, x)| \leq c\Theta(|x|), \qquad t \in [a, b], \ x \in \mathbb{R} \tag{1}$$

and the Lipschitz condition with respect to the first variable:

$$|f(t, x) - f(s, x)| \leq c\Theta(|x|)|t - s|, \qquad t, s \in [a, b], \ x \in \mathbb{R}. \tag{2}$$

Let $g(t) : [a, b] \to \mathbb{R}$ satisfy the Lipschitz condition $|g(t) - g(s)| \leq c|t - s|$.

Theorem 1. *Let*

$$\lim_{\xi \to \infty} \frac{\Theta^{2k}(\xi)}{\xi} = 0. \tag{3}$$

Then for any $R > 0$ the following relation is valid:

$$\lim_{\xi \to \infty} \sup_{\|h(t)\|_{C^1} \leq R\Theta(\xi)} \left| \int_a^b g(t)\Big(f\big(t, \xi e(t) + h(t)\big) - f\big(t, \xi e(t)\big) \Big) \, dt \right| = 0. \tag{4}$$

Relation (4) is the convergence $\Delta F \to 0$ for the nonlinearity $Fx(t) = f\big(t, x(t)\big)$.

With the use of some additional information about the function f, it is possible to weaken condition (3). Let the function f satisfy the Lipschitz condition

$$|f(t, x) - f(t, y)| \leq d(\xi)|x - y|, \qquad |x|, |y| \geq \xi, \tag{5}$$

where $d(\xi)$ decreases and $d(\xi) \to 0$ as $\xi \to \infty$. We suppose that for any $\mu > 0$ there exists a number $C_* = C_*(\mu)$ such that the estimate $d(\mu\xi) \leq C_*(\mu)d(\xi)$ holds for all $\xi \geq C_*$. We use this estimate and the relation $\Theta(M\xi) \leq C(M)\Theta(\xi)$ without special references.

Theorem 2. *Let $e(\tau) \neq 0$ whenever $e'(\tau) = 0$. Let (1), (2), and (5) be valid and*

$$\lim_{\xi \to \infty} \frac{\Theta^{2k}(\xi)d^{2k-2}(\xi)}{\xi} = \lim_{\xi \to \infty} \frac{\Theta^2(\xi)}{\xi} = 0. \tag{6}$$

Then for any $R > 0$ relation (4) holds.

We use the function $\Theta(\cdot)$ in both Theorems 1 and 2 twice: in the estimate for $\|h\|$ in supremum in (4) and in the estimate for the function f. In the next theorem we use two different functions in these places.

If relation (4) is valid for some functions $f_j = f_j(t, x)$ with the same $\Theta = \Theta(\xi)$, then it is valid for their sum $f^0 = \sum_j f_j$. The functions f_j may satisfy (1) and (5) with different Θ_j and d_j; naturally, $f^0 = \sum_j f_j$ satisfies (1) with $\Theta(\xi) = \max_j \Theta_j(\xi)$.

Theorem 3. *Let $|f(t,x)| \leq c\Theta_*(|x|)$. Let either*

$$\lim_{\xi \to \infty} \frac{\Theta_*^{2k-1}(\xi)\Theta(\xi)}{\xi} = 0, \tag{7}$$

or $e(\tau) \neq 0$ if $e'(\tau) = 0$, (5) holds, and

$$\lim_{\xi \to \infty} \frac{\Theta_*(\xi)\Theta^{2k-1}(\xi)d^{2k-2}(\xi)}{\xi} = \lim_{\xi \to \infty} \frac{\Theta_*(\xi)\Theta(\xi)}{\xi} = 0. \tag{8}$$

Then for any $R > 0$ relation (4) holds.

Condition (8) of this theorem can be more restrictive than (7).

1.3. Remarks. **A.** The Lipschitz condition (5) with $d(\xi) \to 0$ is rather restrictive. Under this condition it is possible to prove statements on relation (4), which do not use the derivatives of $e(t)$ and $h(t)$, statements about vector-functions ([12]), etc. Final formulations are weaker than Theorem 2 (see [7]).

The conclusion of Theorems 2 is valid for functions $e(t)$ satisfying $e(\tau) = e'(\tau) = 0$ for some τ, the corresponding analog of (6) is more restrictive. In natural applications the assumptions of Theorem 2 are valid.

The conclusion of Theorems 2 is also valid for monotone differentiable functions $e(t)$ satisfying $e'(t) \geq \eta > 0$ in $[a, b]$. In this case, relation (4) follows from $\Theta^2(\xi)/\xi \to 0$ as $\xi \to \infty$.

Theorems $1 - 3$ can be easily extended for piecewise differentiable functions $e(t)$.

B. The natural case is $k = 2$. Theorems $1 - 3$ are formulated for arbitrary k due to rare but possible applications. Consider an example. Let $e(t)$ be an eigenfunction of the differential operator $Lx = x^{(\text{IV})}$ with the boundary conditions $x'(0) = x''(0) = x'(1) = x''(1) = 0$. The leading eigenvalue of this operator is 0, the corresponding eigenfunction is a constant function. Its derivative is identically zero and Theorems $1-3$ are inapplicable. Other eigenvalues (they are equal to μ^4, where μ denotes any nonzero root of the transcendental equation $\cos\mu \cdot \cosh\mu = 1$) correspond to eigenfunctions $e(t)$ with $k = 3$, both Theorems 1 and 2 are applicable.

In general, conditions (6) are different for different k. For the function $f(t, x) = x^\alpha + b(t)$ conditions (6) do not depend on k and have the form $2\alpha < 1$.

C. Our theorems can be easily modified if the nonlinearity is a sum of functions with different asymptotic properties at infinity. Suppose $f = f_1 + f_2$ and mes $\{t : e(t) = 0\} = 0$.

Let f_1 satisfy all the conditions of Theorem 1 or 2 and let f_2 be uniformly bounded. Then for any $R > 0$ relation (4) holds.

In the following example Theorem 3 can be used. Let $|f_1(t, x)| = (1 + |x|)^\alpha$ and $|f_2(t, x)| \leq (1 + |x|)^\beta$ for some $\beta < \alpha$. Let $k = 2$. Suppose the function f_2 does not satisfy condition (5). Then Theorem 1 with $\Theta(\xi) = (1 + \xi)^\alpha$ is applicable if $\alpha < 1/4$, Theorem 2 is inapplicable. Theorem 3 with $\Theta(\xi) = (1 + \xi)^\alpha$ is applicable if $\alpha < 1/2$ and $\beta < (1 - \alpha)/3$, since f_1 satisfies (1), (5) with $\Theta_1(\xi) = (1 + \xi)^\alpha$, $d_1(\xi) = \alpha(1 + \xi)^{\alpha-1}$ and f_2 satisfies (1) with $\Theta_2(\xi) = (1 + \xi)^\beta$.

D. Theorems 1 and 2 give sufficient conditions for (4). The necessity of these conditions can be illustrated with the following example.

Fix an $\alpha \in (0, 1)$ and set

$$[a, b] = [-1, 1], \ f(x) = (1 + |x|)^\alpha, \ \Theta(\xi) = (1 + \xi)^\alpha, \ e(t) = 1 + t^2, \ h(t) \equiv \xi^\alpha, \ g(t) \equiv 1.$$

Here $k = 2$; $f'(x) = \alpha(1 + |x|)^{\alpha - 1} \operatorname{sign} x$ and therefore $d(\xi) = \alpha(1 + \xi)^{\alpha - 1}$. The inequality $\alpha < 1/2$ is necessary and sufficient for (4) as well as for conditions (6) of Theorem 2.

E. Let $[a, b] = [0, 2\pi]$ and $e(t) = \sin t$. It would be interesting to obtain sufficient conditions for the relation

$$\lim_{\xi \to \infty} \sup_{\|h\|_{C^1} \le R\,\Theta(\xi)} \left| \int_0^{2\pi} g(t) \Big(f\big(t, \xi e(t) + h(t), \xi e(t_w) + h(t_w)\big) - f\big(t, \xi e(t), \xi e(t_w)\big) \Big) dt \right| = 0,$$

where $t_w = t - w$ if $t \ge w$ and $t_w = 2\pi + t - w$ if $w > t$. This relation can be applied to study periodic oscillations in systems with nonlinearities $Fx(t) = f\big(t, x(t), x(t - w)\big)$. Some simple conditions are considered in [13].

2. Existence of forced periodic oscillations

2.1. Second order equation. Everywhere we use the following definition.

Definition 1. *We say that the nonlinearity f has **a proper growth at infinity** if it satisfies (1), (2), and either relation (3) is valid for $k = 2$ or estimate (5) holds and relations (6) are valid for $k = 2$.*

Consider the equation

$$x'' + n^2 x = f(x) + b(t), \tag{9}$$

where n is a positive integer, $b(\cdot)$ is continuous and 2π-periodic, and $f(\cdot)$ is continuous and unbounded in general. The theorems below generalize some results from [11] where bounded nonlinearities f are considered.

Set

$$\Psi(\xi) = \int_0^{2\pi} \sin nt \, f(\xi \sin nt) \, dt, \tag{10}$$

$$\psi_+ = \liminf_{\xi \to +\infty} |\Psi(\xi)|, \qquad \psi^+ = \limsup_{\xi \to +\infty} |\Psi(\xi)|, \tag{11}$$

$$\bar{b} = \int_0^{2\pi} e^{int} b(t) \, dt.$$

Limits (11) may be finite or infinite, computation of such limits see in Section 7. The function $\Psi(\cdot)$ is always odd.

Theorem 4. *Let f have a proper growth at infinity. If relations*

$$\psi_+ < |\bar{b}| < \psi^+ \tag{12}$$

hold, then equation (9) has an unbounded sequence of 2π-periodic solutions. If

$$|\bar{b}| < \psi_+, \tag{13}$$

then equation (9) has at least one 2π-periodic solution and the set of all such solutions is bounded.

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If $\psi^+ < |\bar{b}|$, then the set of 2π-periodic solutions of (9) is bounded but may be empty. If $|\bar{b}| = \psi_+$ or $|\bar{b}| = \psi^+$, then the knowledge of the values ψ_+, ψ^+, and $|\bar{b}|$ is not sufficient to determine if the set of 2π-periodic solutions is bounded or not. A special case $\bar{b} = \psi_+ = 0$ is essentially different from the case $|\bar{b}| > 0$, the corresponding analysis for bounded nonlinearities $f(\cdot)$ can be found in [11]. The reasons why these cases are different do not depend on boundedness or unboundedness of $f(\cdot)$.

2.2. Control theory equations with delay. Analogs of Theorem 4 are valid for forced oscillations in more complex systems. Let us formulate one statement for control theory equations with delay.

Consider the equation

$$L\left(\frac{d}{dt}\right)x(t) = M\left(\frac{d}{dt}\right)\left(f\left(x(t)\right) + f_1\left(x(t-w)\right) + b(t)\right). \qquad (14)$$

Here $L(p)$ and $M(p)$ are coprime real polynomials, the degree of the polynomial $L(p)$ is greater than the degree of $M(p)$. If $M(p) \equiv 1$ and $w = 0$, then (14) is a usual higher order ODE. Let $L(\pm ni) = 0$ for some positive integer n and let $L(\pm mi) \neq 0$ for any integer $m \neq n$. Let limits (11) be determined by the function

$$\Psi(\xi) = \int_0^{2\pi} \sin nt\, f(\xi \sin nt)\, dt + \cos nw \int_0^{2\pi} \sin nt\, f_1(\xi \sin nt)\, dt.$$

Theorem 5. *Let both functions f and f_1 have a proper growth at infinity. If relations (12) are valid, then equation (14) has an unbounded sequence of 2π-periodic solutions. If relation (13) holds, then equation (14) has at least one 2π-periodic solution and the set of all such solutions is bounded.*

Theorem 5 generalizes Theorem 4 in two directions: its linear part is more general and there is a delay in the nonlinearity. The proofs of both theorems are almost the same, we omit the proof of Theorem 5. The analogs of this theorem for bounded f and f_1 can be found in [8].

3. Existence of unbounded solutions

Consider again equation (9) with the 2π-periodic function b. Now we are interested in nonperiodic solutions of (9), namely, we study the existence of unbounded (at $+\infty$ or at $-\infty$) solutions. This problem was analyzed by J.M. Alonso and R. Ortega for equations with bounded nonlinearities ([1]). Theorems 1 and 2 allow to obtain similar results for equation (9) with unbounded nonlinearities f.

Let limits (11) be finite and let
$$|\bar{b}| > \psi^+. \qquad (15)$$

For example, limits (11) are finite if the odd part of f is bounded.

Let any initial values $x(0), x'(0)$ define a unique solution of (9).

Theorem 6. *Let f have a proper growth at infinity. Then any solution of* (9) *with sufficiently large* $|x(0)| + |x'(0)|$ *is unbounded, i.e., at least one of the following relations holds:*

$$\lim_{t \to +\infty} |x(t)| + |x'(t)| = +\infty, \qquad \lim_{t \to -\infty} |x(t)| + |x'(t)| = +\infty. \tag{16}$$

This theorem is similar to Proposition 3.4 from [1]. The main condition (15) is less restrictive than its analog from Proposition 3.4 even for bounded f. From the proof below it follows that under the assumptions of Theorem 6 there exist solutions satisfying both relations (16).

4. Integral equations, two-point BVP

4.1. Integral equation.

Consider the integral equation of Hammerstein type

$$x(t) = \int_a^b K(t, s) f_1\big(s, x(s)\big) \, ds \tag{17}$$

with the nonlinearity $f_1(t, x) = \mu x + f(t, x)$, where $f(t, x)$ is sublinear in x and $\mu \neq 0$. Let the linear operator

$$Ax(t) = \int_a^b K(t, s) x(s) \, ds$$

be completely continuous in $L^2 = L^2(a, b)$. If μ is not a characteristic value of A, then equation (17) has at least one solution and the set of all solutions of (17) is bounded.

Let μ be a simple characteristic value of A (i.e., μ^{-1} is a simple eigenvalue), let $e(\cdot)$ be a corresponding normed eigenfunction of A. The adjoint operator A^* has the eigenfunction $g(\cdot)$ that corresponds to the characteristic value μ and satisfies $(g, e)_{L^2} = 1$, let $g(\cdot)$ be Lipschitz continuous. Let A be a continuous operator from L^2 to C^1 and from C^1 to C^2. Suppose that $e'(t) = 0$ for a finite number of critical points and that $e(t) \neq 0$, $e''(t) \neq 0$ at these points.

Set

$$\Psi(\xi) = \int_0^{2\pi} g(t) f\big(t, \xi e(t)\big) \, dt,$$

$$\varphi_{\pm} = \liminf_{\xi \to \pm\infty} \Psi(\xi), \quad \varphi^{\pm} = \limsup_{\xi \to \pm\infty} \Psi(\xi). \tag{18}$$

The following theorem gives the sharp conditions for the existence of solutions to equation (17) and for the boundedness of the solution set. Moreover, the index $\mathrm{ind}_\infty \Phi$ at infinity ([3, 16]) of the vector field $\Phi(x) = x - \mu Ax - Af(\cdot, x)$ in L^2 is calculated.

Theorem 7. *Let the nonlinearity f have a proper growth at infinity. Let A satisfy all the conditions above. Then*

i. If either $\varphi^+ < 0 < \varphi_-$ *or* $\varphi^- < 0 < \varphi_+$, *then the index at infinity of the vector field* $\Phi(x)$ *is well defined,* $|\mathrm{ind}_\infty \Phi| = 1$; *equation* (17) *has at least one solution;*

ii. If either $\varphi_- > 0$ *and* $\varphi_+ > 0$ *or* $\varphi^- < 0$ *and* $\varphi^+ < 0$, *then the index at infinity of the vector field* $\Phi(x)$ *is well defined,* $\mathrm{ind}_\infty \Phi = 0$;

iii. If either $\varphi_- < 0 < \varphi^-$ *or* $\varphi_+ < 0 < \varphi^+$, *or both these relations are valid, then the index at infinity of the vector field* $\Phi(x)$ *is not defined and there exists an unbounded sequence* $x_n \in L^2$, $\|x_n\|_{L^2} \to \infty$, *of solutions of equation* (17).

The only situation which is not considered in Theorem 7 is that one of the limits (18) is zero and neither $\varphi_- < 0 < \varphi^-$ nor $\varphi_+ < 0 < \varphi^+$.

If $\varphi_+ < 0 < \varphi^+$, then the unbounded sequence of solutions of equation (17) has the form $x_n = \xi_n e + h_n$ with $\xi_n \to +\infty$ and $\|h_n\|_{L^2}/\xi_n \to 0$; here $(g, h_n)_{L^2} = 0$. If $\varphi_- < 0 < \varphi^-$, then $\xi_n \to -\infty$. If $0 \in (\varphi_-, \varphi^-) \cap (\varphi_+, \varphi^+)$, then there exist two sequences of solutions $x_n^\pm = \xi_n^\pm e + h_n^\pm$ with $\xi_n^\pm \to \pm\infty$.

If $\varphi_- = \varphi^-$ and $\varphi_+ = \varphi^+$, then the nonlinearity is asymptotically homogeneous in the sense of [6] and the case 3 of Theorem 7 is impossible.

Some methods to compute limits (18) are discussed below (see also [12]) in Section 7.

4.2. Application to two-point BVP. Consider the problem

$$x'' + n^2 x + f(t, x) = 0, \quad x(0) = x(\pi) = 0. \tag{19}$$

Here n is a positive integer. Suppose that $f(t, x) : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is continuous in x and uniformly Lipschitz continuous in t. Set

$$e(t) = \sqrt{\frac{2}{\pi}} \sin nt, \qquad \Psi(\xi) = \int_0^\pi e(t) f(t, \xi e(t)) \, dt \tag{20}$$

and define the values (18) for this function.

Theorem 8. *Let the nonlinearity f have a proper growth at infinity. If either $\varphi^+ < 0 < \varphi_-$ or $\varphi^- < 0 < \varphi_+$, then problem (19) has at least one solution and the set $K \subset C^2$ of all the solutions of (19) is bounded. If either $\varphi_- < 0 < \varphi^-$ or $\varphi_+ < 0 < \varphi^+$ or both these relations hold, then problem (19) has an infinite number of solutions, the set K is unbounded.*
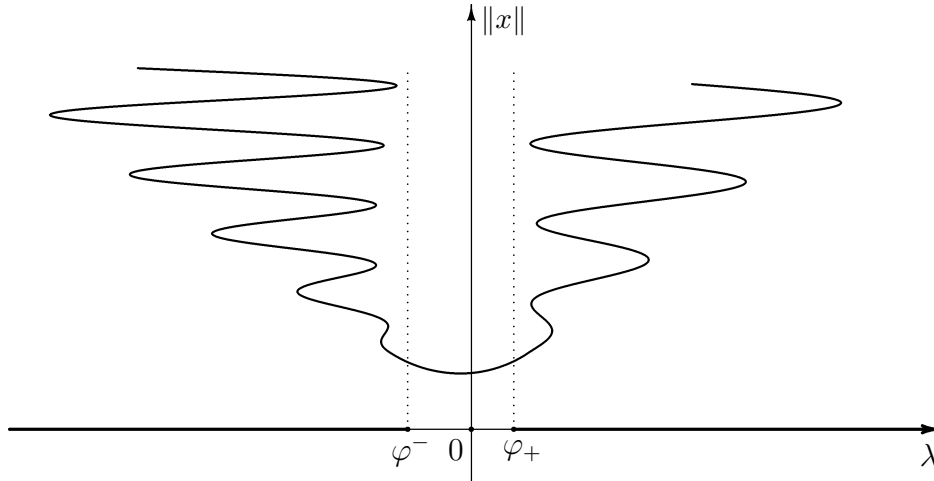

Fig. 1: The set of asymptotic bifurcation points for problem (21).

This theorem follows directly from Theorem 7. As an illustration, consider the set of solutions of the problem

$$x'' + n^2 x + f(t, x) - \lambda e(t) = 0, \quad x(0) = x(\pi) = 0 \tag{21}$$

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with a scalar parameter λ. Define the function Ψ by (20), let the values (18) satisfy

$$-\infty = \varphi_- < \varphi^- < 0 < \varphi_+ < \varphi^+ = \infty.$$

Then for any $\xi \in \mathbb{R}$ there exists at least one λ and a function h, $(e, h)_{L^2} = 0$, such that $x = \xi e + h$ is a solution of problem (21). If f'_x is sufficiently small, then these λ and h are unique for each ξ.

The set $(-\infty, \varphi^-] \cup [\varphi_+, \infty)$ is the set of asymptotic bifurcation points ([16]) for problem (21). On Fig. 1 this set is drawn by thick horizontal lines.

5. Cycles in autonomous higher oder ODE

Consider the equation

$$L\left(\frac{d}{dt}\right)x = f(x). \tag{22}$$

Here L is a real polynomial, f is continuous and sublinear, $f(0) = 0$.

Theorem 9. *Let the polynomials $L(wi)$ and $\Im m\, L(wi)$ have a pair of common real roots $\pm w_0$ $(w_0 > 0)$ of the same odd multiplicity. Let there exist an interval $\Omega = [w_1, w_2]$, $w_0/2 < w_1 < w_0 < w_2$, such that*

$$L(nwi) \neq 0, \quad w \in \Omega, \ n = 0, \pm 2, \pm 3, \dots; \qquad \Im m\, L(wi) \neq 0, \quad w \in \Omega, \ w \neq w_0.$$

Let

$$q < \inf_{w \in \Omega} \inf_{n=0,2,3,4,\dots} |L(nwi)|$$

and let the inequality

$$|\Im m\, L(wi)|\left(|L(nwi)|^2 - q^2\right) + n\left(q^2 - |\Im m\, L(wi)|^2\right) \text{sign}\left(\Im m\, L(wi)\right) \Im m\, L(nwi) > 0 \tag{23}$$

be valid for $w = w_1$, $w = w_2$ and $n = 2, 3, 4, \dots$. Suppose that the nonlinearity has a proper growth at infinity and $|f(x)| \leq q|x|$, $x \in \mathbb{R}$. Suppose that the derivative $f'(0)$ exists, $f'(0) \neq 0$, and the function

$$\Psi(\xi) \stackrel{\text{def}}{=} \int_0^{2\pi} \sin t\, f(\xi \sin t)\, dt \tag{24}$$

for all sufficiently large $\xi > 0$ satisfies the estimate

$$\Psi(\xi) f'(0) \leq -c_0 < 0. \tag{25}$$

Then there exists a nontrivial cycle of equation (9) *with a period $T \in [2\pi/w_2, 2\pi/w_1]$.*

Equations $L(\frac{d}{dt})x = M(\frac{d}{dt})f(x)$ with bounded nonlinearities $f(x)$ were considered in [2]. As an example, consider the equation

$$x''' + x'' + x' + x = f(x). \tag{26}$$

Here $L(p) = p^3 + p^2 + p + 1$, $\Im m\, L(wi) = w(1 - w^2)$, and $w_0 = 1$. Suppose that the nonlinearity f has a proper growth at infinity and (25) holds for all sufficiently large ξ. Let $|f(x)| \leq 0.745|x|$, $x \in \mathbb{R}$. Then equation (26) has at least one cycle with a period $T \in [6.283, 7.652]$.

6. Hopf bifurcation at infinity

Consider the differential equation

$$L\left(\frac{d}{dt}, \lambda\right)x = M\left(\frac{d}{dt}, \lambda\right)f(x, \lambda), \qquad (27)$$

where real polynomials $L(p, \lambda)$ and $M(p, \lambda)$ of degrees ℓ and m, $\ell > m$, are coprime for each value of the scalar parameter $\lambda \in \Lambda = (a, b)$.

Definition 2. *A parameter value λ_0 is called a **Hopf bifurcation point at infinity** (shortly, a **bifurcation point** or **HBP**) for equation (27) with a frequency w_0 if for any sufficiently large $r > 0$ there exists a λ_r such that equation (27) with $\lambda = \lambda_r$ has a T_r-periodic solution $x_r = x_r(t)$ and $\lambda_r \to \lambda_0$, $\|x_r\|_C \to \infty$, $T_r \to 2\pi/w_0$ as $r \to \infty$.*

The following result is formulated in [9]. Suppose the continuous nonlinearity $f(x, \lambda)$ is sublinear in x. Suppose the polynomial L has a pair of simple conjugate roots $\sigma(\lambda) \pm w(\lambda)i$ ($w(\lambda) > 0$) depending continuously on λ, where $\sigma(\lambda_0) = 0$ and the function $\sigma(\lambda)$ takes values of both sign in every neighborhood of the point λ_0. Suppose $L(nw(\lambda_0)i, \lambda_0) \neq 0$ for $n = 0, 2, 3, \ldots$ Then λ_0 is a Hopf bifurcation point at infinity for equation (27) with the frequency $w(\lambda_0)$.

Here we consider the differential equation

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)f(x, \lambda), \qquad (28)$$

where the linear part is independent of the parameter. We suppose that the real polynomials L and M are coprime[2], their degrees satisfy $\ell > m$, and the polynomial L has a pair of imaginary roots $\pm i w_0$, $w_0 > 0$. The function $f(x, \lambda) : \mathbb{R} \times \Lambda \to \mathbb{R}$ is continuous with respect to its arguments and sublinear uniformly in λ, i.e., $|f(x, \lambda)| \leq \Theta(|x|)$, where $\Theta(\cdot)$ is independent of λ.

Theorem 10. *Let the following conditions hold:*
1. The number w_0 is a root of the polynomials $\Im m[L(wi)M(-wi)]$ and $L(wi)$ of the same odd multiplicity K and $L(nw_0i) \neq 0$ for every $n = 0, 2, 3, 4, \ldots$;
2. The function f has a proper growth at infinity;
3. For every λ there exists the limit

$$\psi(\lambda) \overset{\text{def}}{=} \lim_{\xi \to +\infty} \int_0^{2\pi} \sin t \, f(\xi \sin t, \lambda) \, dt \qquad (29)$$

and the function $\psi(\lambda)$ takes values of both sign in every neighborhood of the point λ_0.
Then λ_0 is a Hopf bifurcation point at infinity for equation (28) with the frequency w_0.

Lemma 1 below explains when limit (29) exists.
Assumption 1 of Theorem 10 implies that the polynomial $\Im m[L(wi)M(-wi)]$ is nonzero. One can show that the identity $\Im m[L(wi)M(-wi)] \equiv 0$ holds iff both polynomials L and M are even. In this case, the equation

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)f(x)$$

---

[2]Above we supposed this for the polynomial, depending on the parameter.

with any continuous nonlinearity f is Hamiltonian ([14, 15]). Moreover, if assumption 1 holds and f is bounded, then this equation has a continuum of periodic cycles $x(\cdot; \xi)$ such that $\|x(\cdot; \xi)\|_C \to \infty$ and $T(\xi) \to 2\pi/w_0$ as $\xi \to \infty$, where $T(\xi)$ is the period of the cycle $x(t; \xi)$ and $\xi \geq \xi_0$ is a parameter. Therefore the identity $\Im m[L(wi)M(-wi)] \equiv 0$ implies that all the values $\lambda \in \Lambda$ are Hopf bifurcation points at infinity for equation (28) with the frequency w_0.

The simplest example of equation that can be studied with Theorem 10 is

$$x''' + x'' + x' + x = f(x, \lambda).$$

We do not give the proof of Theorem 10. A close result is proved in [13] for equations (28) with bounded nonlinearities. To obtain the proof for unbounded nonlinearities, it suffices to combine the method of [13] with Theorems 1 and 2.

7. Computation of φ_+, φ^+

In this section, we present algorithms to compute limits (18) for function (10).

Lemma 1. *Let* $f = f_1 + f_2 + f_3 + f_4 + f_5$, *where* $f_1(x) = \frac{1}{2}[f(x) + f(-x)]$ *is the unbounded even part of* $f(x)$, *other functions* $f_i(x)$ *are odd and bounded. Let* $f_2(x) \to 0$ *as* $x \to \infty$, $f_3(x) \to \pm\overline{f} \neq 0$ *as* $x \to \pm\infty$, *let the function* f_4 *have sublinear primitive* F_4:

$$\lim_{x \to \infty} \frac{F_4(x)}{x} = 0, \qquad F_4(x) = \int_0^x f_4(u)\, du.$$

Then

$$\varphi_+ = 4\overline{f} + \liminf_{\xi \to +\infty} \int_0^{2\pi} \sin nt\, f_5(\xi \sin nt)\, dt, \quad \varphi^+ = 4\overline{f} + \limsup_{\xi \to +\infty} \int_0^{2\pi} \sin nt\, f_5(\xi \sin nt)\, dt.$$

The proof of this simple statement coincides with its analog from [11] for bounded nonlinearities.

Since f_1 is even and $f_2(x) \to 0$ as $x \to \infty$, it follows that

$$\int_0^{2\pi} \sin nt\, f_1(\xi \sin nt)\, dt = \lim_{\xi \to \infty} \int_0^{2\pi} \sin nt\, f_2(\xi \sin nt)\, dt = 0.$$

The relations $f_3(x) \to \pm\overline{f}$ as $x \to \pm\infty$ imply

$$\lim_{\xi \to +\infty} \int_0^{2\pi} \sin nt\, f_3(\xi \sin nt)\, dt = 4\overline{f}.$$

Finally,

$$\int_0^{2\pi} \sin nt\, f_4(\xi \sin nt)\, dt = 4 \int_0^{\pi/2} \sin t\, f_4(\xi \sin t)\, dt$$

$$= 4 \int_\varepsilon^{\pi/2-\varepsilon} \sin t\, f_4(\xi \sin t)\, dt + 4 \int_{[0,\varepsilon] \cup [\pi/2-\varepsilon, \pi/2]} \sin t\, f_4(\xi \sin t)\, dt$$

$$\leq 4\xi^{-1} \int_\varepsilon^{\pi/2-\varepsilon} \tan t\, dF_4(\xi \sin t) + 8\varepsilon \sup |f_4(x)|$$

$$= c\varepsilon + 4\xi^{-1} \tan t\, F_4(\xi \sin t) \Big|_{t=\varepsilon}^{t=\pi/2-\varepsilon} - 4 \int_\varepsilon^{\pi/2-\varepsilon} \frac{F_4(\xi \sin t)}{\xi \cos^2 t}\, dt.$$

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Since ε is arbitrarily small and $F_4(x)/x \to 0$ as $x \to \infty$, this implies

$$\lim_{\xi \to +\infty} \int_0^{2\pi} \sin nt \, f_4(\xi \sin nt) \, dt = 0$$

and the lemma is proved.

For example, f_4 has a sublinear primitive if $f_4(x)$ is periodic or almost periodic with zero average, the functions $\sin x^3$ and $\sin \sqrt{|x|} \operatorname{sign} x$ also have sublinear primitives. The primitive of the functions $\sin \ln(1 + |x|) \operatorname{sign} x$ and $arctanx$ are not sublinear. Nonlinearities with sublinear derivatives were considered in [1] for problems on the existence of unbounded solutions.

8. Proof of Theorems 1 and 2

8.1. Change of variables. The first part of the proof is common for both theorems. Under their conditions, $\Theta^2(\xi)/\xi \to 0$. Consider a decreasing function $\delta = \delta(\xi) : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\lim_{\xi \to \infty} \delta(\xi) = 0, \qquad \lim_{\xi \to \infty} \frac{\Theta^2(\xi)}{\xi \delta^{2k-2}(\xi)} = 0. \tag{30}$$

Since $\Theta(\xi) \geq 1$, $k \geq 2$, this implies also

$$\lim_{\xi \to \infty} \delta_*(\xi) = 0, \qquad \delta_*(\xi) \stackrel{\mathrm{def}}{=} \frac{\Theta(\xi)}{\xi \delta^k(\xi)}. \tag{31}$$

Under the assumptions of Theorems 1 and 2 the choice of δ is different, but it is not essential now.

Let $t_1 < t_2 < \cdots < t_{n-1}$ be the set of all critical points of the function $e(\cdot)$ in the interval (a, b), set $t_0 = a$, $t_n = b$. Consider the intervals $o_0 = [t_0, t_0 + \delta_0)$, $o_n = (t_n - \delta_n, t_n]$, and $o_j = (t_j - \delta_j, t_j + \delta_j)$ for $1 \leq j \leq n - 1$, where $\delta_j = \delta(\xi)$ if $e'(t_j) = 0$ and $\delta_j = \delta_*(\xi)$ otherwise (the relation $e'(t_j) \neq 0$ may hold for $j = 0, n$ only). The estimate $\operatorname{mes}\{\bigcup o_j\} \leq 2n\delta + 2\delta_*$ implies $\operatorname{mes}\{\bigcup o_j\} \to 0$ as $\xi \to \infty$. Suppose that ξ is so large that the closures of the intervals o_j are disjoint. Then the set $[a, b] \setminus \bigcup o_j$ is the unification of n disjoint segments $[a_j, b_j] \subset (t_j, t_{j+1})$ (the points a_j, b_j depend on ξ, the segments $[a_j, b_j]$ become larger as ξ increases). By construction, $[a_j - \delta_j/2, b_j + \delta_j/2] \subset (t_j, t_{j+1})$ for each j and the function $e(\cdot)$ is strictly monotone on every interval (t_j, t_{j+1}). Moreover, the relations $e^{(\ell)}(t_j) \neq 0$ with $\ell = \ell_j \leq k$ imply the estimate

$$|e'(t)| \geq c_1 \delta^{k-1}, \qquad t \in [a_j - \delta_j/2, b_j + \delta_j/2], \ j = 0, \ldots, n-1, \tag{32}$$

where $c_1 > 0$ is independent of ξ.

To prove relation (4) we estimate separately the integrals over the sets $\bigcup[a_j, b_j]$ and $\bigcup o_j$ and show that for some appropriate choice of δ both integrals tend to zero as $\xi \to \infty$.

First, fix some j and consider one segment $[a_j, b_j]$ and the corresponding integral

$$I(a_j, b_j) = \int_{a_j}^{b_j} g(t) \Big(f\big(t, \xi e(t) + h(t)\big) - f\big(t, \xi e(t)\big) \Big) \, dt. \tag{33}$$

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Since the function $e(\cdot)$ is strictly monotone on the segment $[a_j - \delta_j/2, b_j + \delta_j/2]$, the inverse function $e^{-1}(\cdot)$ is well defined on the segment $e([a_j - \delta_j/2, b_j + \delta_j/2])$ and (32) implies

$$\left| \frac{d}{d\tau} e^{-1}(\tau) \right| \le c_1^{-1} \delta^{1-k}, \qquad \tau \in e([a_j - \delta_j/2, b_j + \delta_j/2]). \tag{34}$$

From (31) and $\|h\|_{C^1} \le R\,\Theta(\xi)$ the relations $\xi^{-1}\|h\|_{C^1} = o(\delta^k)$ and $\xi^{-1}\|h\|_{C^1} = o(\delta^{k-1}\delta_*)$, $\xi \to \infty$ follow, therefore estimate (32) implies that the function $e(\cdot) + \xi^{-1}h(\cdot)$ is strictly monotone on the segment $[a_j, b_j]$, the values of this function belong to the segment $e([a_j - \delta_j/2, b_j + \delta_j/2])$, and

$$|e'(t) + \xi^{-1}h'(t)| \ge c_1 \delta^{k-1}/2, \qquad t \in [a_j, b_j] \tag{35}$$

if ξ is sufficiently large. Let us change the variable in integral (33) by the formula $e(\tau) = e(t) + \xi^{-1}h(t)$; the change $t \to \tau$ is correct due to the monotonicity of both sides of this formula. Let us prove two preliminary estimates.

The relations

$$\tau = e^{-1}\big(e(t) + \xi^{-1}h(t)\big) = t + e^{-1}\big(e(t) + \xi^{-1}h(t)\big) - e^{-1}\big(e(t)\big)$$

and (34) imply the estimate $|\tau - t| \le c_1^{-1}\xi^{-1}\delta^{1-k}|h(t)|$, therefore

$$|\tau - t| \le r_1 \frac{\Theta(\xi)}{\xi \delta^{k-1}(\xi)}$$

(here and everywhere r_m are positive constants, their exact values do not play any role). The next preliminary estimate follows from the formula $e'(\tau) = \big(e'(t) + \xi^{-1}h'(t)\big)t'(\tau)$ which implies

$$|t'(\tau) - 1| = \left| \frac{e'(\tau)}{e'(t) + \xi^{-1}h'(t)} - 1 \right| = \left| \frac{e'(\tau) - e'(t) - \xi^{-1}h'(t)}{e'(t) + \xi^{-1}h'(t)} \right|$$

$$\le r_2 \frac{|\tau - t| + \Theta(\xi)/\xi}{\delta^{k-1}(\xi)} \le r_3 \frac{\Theta(\xi)}{\xi \delta^{2k-2}(\xi)}.$$

From (30) we see that $t'(\tau)$ is close to 1 for large ξ.

Now come back to integral (33). Since

$$\int_{a_j}^{b_j} g(t)f\big(t, \xi e(t) + h(t)\big)\,dt = \int_{\tau(a_j)}^{\tau(b_j)} g\big(t(\tau)\big)\, f\big(t(\tau), \xi e(\tau)\big)\, t'(\tau)\,d\tau,$$

it follows that $I(a_j, b_j) \le \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$, where

$$\varepsilon_1 = \left| \int_{\tau(a_j)}^{\tau(b_j)} g\big(t(\tau)\big)f\big(t(\tau), \xi e(\tau)\big)\, t'(\tau)\,d\tau - \int_{a_j}^{b_j} g\big(t(\tau)\big)f\big(t(\tau), \xi e(\tau)\big)\, t'(\tau)\,d\tau \right|,$$

$$\varepsilon_2 = \left| \int_{a_j}^{b_j} g\big(t(\tau)\big)f\big(t(\tau), \xi e(\tau)\big)\, t'(\tau)\,d\tau - \int_{a_j}^{b_j} g(\tau)f\big(t(\tau), \xi e(\tau)\big)\, t'(\tau)\,d\tau \right|,$$

$$\varepsilon_3 = \left| \int_{a_j}^{b_j} g(\tau)f\big(t(\tau), \xi e(\tau)\big)\, t'(\tau)\,d\tau - \int_{a_j}^{b_j} g(\tau)f\big(\tau, \xi e(\tau)\big)\, t'(\tau)\,d\tau \right|,$$

$$\varepsilon_4 = \left| \int_{a_j}^{b_j} g(\tau)\,f\big(\tau, \xi e(\tau)\big)\, t'(\tau)\,d\tau - \int_{a_j}^{b_j} g(\tau)f\big(\tau, \xi e(\tau)\big)\,d\tau \right|.$$

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Estimates (1), (2), and $|g(t) - g(s)| \le |t - s|$ imply

$$\varepsilon_1, \varepsilon_2, \varepsilon_3 \le r_4 \Theta(\xi) \max |\tau - t(\tau)|, \qquad \varepsilon_4 \le r_5 \Theta(\xi) \max |t'(\tau) - 1|,$$

and from the above estimates for $|\tau - t(\tau)|$ and $|t'(\tau) - 1|$, we conclude

$$\varepsilon_1, \varepsilon_2, \varepsilon_3 \le r_6 \frac{\Theta^2(\xi)}{\xi \delta^{k-1}(\xi)}, \qquad \varepsilon_4 \le r_7 \frac{\Theta^2(\xi)}{\xi \delta^{2k-2}(\xi)}, \qquad I(a_j, b_j) \le r_8 \frac{\Theta^2(\xi)}{\xi \delta^{2k-2}(\xi)}.$$

Therefore relation (30) implies that all integrals (33) vanish as $\xi \to \infty$.

8.2. Completion of the proof. It remains to show that the integral

$$J(\xi) = \int_{\bigcup o_j} g(t) \Big(f\big(t, \xi e(t) + h(t)\big) - f\big(t, \xi e(t)\big) \Big) \, dt$$

vanishes as $\xi \to \infty$ uniformly w.r.t. h, $\|h\|_{C^1} \le R\Theta(\xi)$, for some appropriate $\delta(\xi)$ satisfying (30).

Under the conditions of Theorem 1, set

$$\delta(\xi) = \frac{1}{\Theta(\xi)} \Big(\frac{\Theta^{2k}(\xi)}{\xi} \Big)^{\frac{1}{3k-3}},$$

then

$$\frac{\Theta^2(\xi)}{\xi \delta^{2k-2}(\xi)} = \Big(\frac{\Theta^{2k}(\xi)}{\xi} \Big)^{\frac{1}{3}}.$$

Due to (3), these relations and the estimates

$$|J(\xi)| \le r_9 \Theta(\xi) \mathrm{mes}\Big\{ \bigcup o_j \Big\} \le r_9 \Theta(\xi) \big(2n\delta(\xi) + 2\delta_*(\xi) \big) \le r_{10} \Big(\Theta(\xi)\delta(\xi) + \frac{\Theta^2(\xi)}{\xi \delta^{2k-2}(\xi)} \Big)$$

imply (30) and $J(\xi) \to 0$. Theorem 1 is proved.

Suppose that the conditions of Theorem 2 are satisfied. Consider an interval o_j. If $e'(t_j) = 0$, then by assumption of this theorem $e(t_j) \ne 0$ and relation (5) implies for $t \in o_j$

$$|f\big(t, \xi e(t) + h(t)\big) - f\big(t, \xi e(t)\big)| \le d\Big(\min_{t \in \bigcup o_j} \{ |\xi e(t)|, |\xi e(t) + h(t)| \} \Big) |h(t)| \le r_{11} d(\xi)\Theta(\xi),$$

hence

$$\Big| \int_{o_j} g(t) \Big(f\big(t, \xi e(t) + h(t)\big) - f\big(t, \xi e(t)\big) \Big) \, dt \Big| \le r_{12} d(\xi)\Theta(\xi)\delta_j(\xi) = r_{12} d(\xi)\Theta(\xi)\delta(\xi).$$

The relation $e'(t_j) \ne 0$ may be valid for $j = 0, n$; in this case $\delta_j = \delta_*(\xi)$ and

$$\Big| \int_{o_j} g(t) \Big(f\big(t, \xi e(t) + h(t)\big) - f\big(t, \xi e(t)\big) \Big) \, dt \Big| \le r_{13} \Theta(\xi)\delta_*(\xi) \le r_{13} \frac{\Theta^2(\xi)}{\xi \delta^{2k-2}(\xi)}.$$

Therefore

$$|J(\xi)| \le r_{14} \Big(d(\xi)\Theta(\xi)\delta(\xi) + \frac{\Theta^2(\xi)}{\xi \delta^{2k-2}(\xi)} \Big).$$

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Under the conditions of Theorem 2, we set

$$\delta(\xi) = \Big(\frac{\Theta^2(\xi)}{\xi}\Big)^{\frac{1}{3k-3}} \min\big\{\big(d(\xi)\Theta(\xi)\big)^{-\frac{1}{3}}, 1\big\},$$

then

$$\frac{\Theta^2(\xi)}{\xi\delta^{2k-2}(\xi)} = \max\Big\{\Big(\frac{\Theta^{2k}(\xi)d^{2k-2}(\xi)}{\xi}\Big)^{\frac{1}{3}}, \Big(\frac{\Theta^2(\xi)}{\xi}\Big)^{\frac{1}{3}}\Big\}$$

and

$$d(\xi)\Theta(\xi)\delta(\xi) = \begin{cases} \Big(\dfrac{\Theta^{2k}(\xi)d^{2k-2}(\xi)}{\xi}\Big)^{\frac{1}{3k-3}} & \text{if} \ \ d(\xi)\Theta(\xi) \geq 1, \\ d(\xi)\Theta(\xi)\Big(\dfrac{\Theta^2(\xi)}{\xi}\Big)^{\frac{1}{3k-3}} & \text{if} \ \ d(\xi)\Theta(\xi) < 1, \end{cases}$$

i.e.,

$$d(\xi)\Theta(\xi)\delta(\xi) \leq \Big(\frac{\Theta^2(\xi)}{\xi\delta^{2k-2}(\xi)}\Big)^{\frac{1}{k-1}}.$$

Therefore relations (6) imply (30) and $J(\xi) \to 0$. Theorem 2 is also proved.

9. Proof of Theorem 4

9.1. General scheme. The proof consists of several steps. First we formulate a simple auxiliary statement on the explicit form of the superposition operator in some 2-dimensional subspace of L^2. Then the periodic problem is replaced by the equivalent operator equation

$$x = A\big(\mu x - f(x) - b\big) \tag{36}$$

of the Hammerstein type with a proper linear operator A and a proper μ.

In the next step, the index at infinity of the vector field $x - A(\mu x - f(x) - b)$ is calculated for the case $\psi_+ > |\bar{b}|$. It turns out that the index is not zero, this proves the second part of Theorem 4. The index is calculated by the usual homotopic methods: all the zeroes of some deformation $\Phi = \Phi(\lambda, x)$ satisfy an *a priori* estimate, $\Phi(1, x) = x - A\big(\mu x - f(x) - b\big)$, and $\Phi(0, \cdot)$ is a vector field of the Landesman–Lazer type. The index of such vector fields was calculated, e.g. in [5, 17, 18].

The case $\psi_+ < |\bar{b}| < \psi^+$ is more cumbersome. We prove that for any ξ_0 there exist $\xi_* > \xi_0$ and $\xi^* > \xi_*$ such that equation (36) has at least one solution $x(t) = \xi\sin(nt + \lambda) + h(t)$ with $\xi \in (\xi_*, \xi^*)$. This proves the first part of Theorem 4.

9.2. Planar mapping. If $x = x(t)$ is a 2π-periodic solution of equation (9), then

$$\int_0^{2\pi}\Big(f\big(x(t)\big) + b(t)\Big)\cos nt\, dt = \int_0^{2\pi}\Big(f\big(x(t)\big) + b(t)\Big)\sin nt\, dt = 0. \tag{37}$$

Consider the orthogonal projector

$$Pu(t) = \frac{1}{\pi}\int_0^{2\pi}\cos n(t - \tau)\, u(\tau)\, d\tau$$

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in $L^2 = L^2(0, 2\pi)$. Equalities (37) are equivalent to $Pf(x) + Pb = 0$. Denote by Π_n the 2-dimensional subspace of L^2 spanned on the functions $\cos nt$ and $\sin nt$. By definition, $PL^2 = \Pi_n$ and any $e \in \Pi_n$ has the form $e = \xi \sin(nt + \lambda)$, where $\xi \geq 0$ and $\lambda \in [0, 2\pi)$. The proof of the following lemma is by simple computations and we omit it.

Lemma 2. *For every $\xi \geq 0$ and $\theta \in [0, 2\pi)$*

$$Pf\big(\xi \sin(nt + \theta)\big) = \frac{1}{\pi} \Psi(\xi) \sin(nt + \theta), \tag{38}$$

where Ψ is defined by (10).

9.3. Equivalent integral equation.
Consider the linear operator $x = Au$ that maps any function $u = u(t) \in L^2$ to a unique solution x of the problem

$$-x'' + x = u, \qquad x(0) = x(2\pi), \ x'(0) = x'(2\pi).$$

This operator acts in L^2, it is completely continuous in L^2, self-adjoint, and positive definite. Moreover, A is an integral operator which can be considered in various functional spaces: it is continuous as an operator from $L^2(0, 2\pi)$ to $C^1(0, 2\pi)$ and from $C(0, 2\pi)$ to $C^2(0, 2\pi)$. Its spectrum is the sequence of eigenvalues $1, 2, 5, \ldots, n^2+1, \ldots$. The eigenvalue 1 is simple, it corresponds to constant eigenfunctions; the other eigenvalues have the multiplicity 2. The eigenvalue $n^2 + 1$ corresponds to the eigenfunctions $\xi \sin(nt + \theta)$ with arbitrary $\xi > 0$ and $\theta \in [0, 2\pi)$, these eigenfunctions and the zero function form the plane Π_n in L^2.

Now we can rewrite the 2π-periodic problem for equation (9) equivalently in the form (36) with $\mu = n^2 + 1$. A function $x = x(t) \in L^2$ is a solution of operator equation (36) iff it is a classical 2π-periodic solution of equation (9).

The principal linear part $x = \mu Ax$ of equation (36) is degenerate at infinity: the linear operator $I - \mu A$ has the nontrivial kernel Π_n. Recall that P is the orthogonal projector onto the plane Π_n in L^2. Set $Q = I - P$.

9.4. The case $\psi_+ > |\bar{b}|$.
This estimate implies that $\Psi(\xi) \neq 0$ and has the same sign for all sufficiently large ξ. Fix an $a \in \mathbb{R}$ such that $\psi_+ > |a| > |\bar{b}|$ and $a\Psi(\xi) > 0$ for large ξ. Define the function

$$s(\xi) = \begin{cases} 1 & \text{if} \ \ \xi \ \geq \ 1, \\ \xi & \text{if} \ \ |\xi| < \ 1, \\ -1 & \text{if} \ \ \xi \ \leq -1 \end{cases} \tag{39}$$

and consider the deformation

$$\Phi(\lambda, x) = x - A\big(\mu x - \lambda f(x) - a(1 - \lambda)s(x)/4 - b\big).$$

To prove the existence of at least one solution of equation (36), it suffices to establish the two facts: to prove that all the zeroes of the deformation $\Phi(\lambda, x)$ satisfy an *a priori* estimate and to show that the index at infinity of the vector field $\Phi(0, x)$ is nonzero.

We begin with the *a priori* estimate. Let $x(t) = \xi \sin(nt + \theta) + h(t)$ where $h = Qx$, and let $\Phi(\lambda, x) = 0$. Then $Q\Phi(\lambda, x) = 0$ and $P\Phi(\lambda, x) = 0$. Since the linear operator $I - \mu A$ is continuously invertible in the subspace $QL^2 \subset L^2$, the equality $Q\Phi(\lambda, x) = 0$ implies the estimate $\|h\|_{C^1} \leq c\Theta(\xi)$. The equality $P\Phi(\lambda, x) = 0$ can be rewritten as

$$\lambda Pf(x) + (1 - \lambda)aPs(x)/4 = -Pb. \tag{40}$$

Theorems 1 and 2 imply $Pf(x) - Pf\big(\xi \sin(nt+\theta)\big) \to 0$ and similarly $Ps(x) - Ps\big(\xi \sin(nt+\theta)\big) \to 0$ as $\xi \to +\infty$. From (38) it follows that

$$Pf\big(\xi \sin(nt + \theta)\big) = \frac{1}{\pi}\Psi(\xi)\sin(nt + \theta), \quad Ps\big(\xi \sin(nt + \theta)\big) = \left(\frac{4}{\pi} + O(\xi^{-1})\right)\sin(nt + \theta).$$

Therefore

$$\liminf_{\xi \to +\infty} \|\lambda Pf(x) + (1 - \lambda)aPs(x)/4\|_{L^2} = \liminf_{\xi \to +\infty} \frac{1}{\sqrt{\pi}}|\lambda\Psi(\xi) + (1 - \lambda)a| \geq \frac{|a|}{\sqrt{\pi}}.$$

Since $\sqrt{\pi}\,\|Pb\|_{L^2} = |\bar{b}| < |a|$, relation (40) is not true for large ξ. This proves the required *a priori* estimate.

Now consider the vector field $\Phi(0, x) = x - A(\mu x - as(x)/4 - b)$ where the nonlinearity satisfies $s(\xi) \to \pm 1$ as $\xi \to \pm\infty$. The computation of the index at infinity of such vector fields can be found, for example, in [5, 17, 18]. It is proved there that $|\operatorname{ind}\Phi(0, \cdot)| = 1$ if $|a| > |\bar{b}|$.

The *a priori* estimate guarantees that $\operatorname{ind}\Phi(\lambda, \cdot)$ does not depend on λ. Therefore,

$$\operatorname{ind}\Phi(1, \cdot) = \operatorname{ind}\Phi(0, \cdot) \neq 0$$

and the conclusion of Theorem 4 follows from the general degree theory.

9.5. Unbounded sequence of solutions.
Suppose that estimates (12) hold. Then there exist unbounded sequences ξ_k and ξ^k such that $\xi_k < \xi^k < \xi_{k+1}$ and

$$|\Psi(\xi_k)| + \varepsilon < |\bar{b}| < |\Psi(\xi^k)| - \varepsilon \tag{41}$$

for some $\varepsilon > 0$. Without loss of generality, suppose that all ξ_k are sufficiently large and therefore the supremum in (4) for $[a, b] = [0, 2\pi]$ and trigonometric functions e and g is sufficiently small for all $\xi \geq \xi_k$.

Let us show that the field $x - A\big(\mu x - f(x) - b\big)$ is nonzero on the boundary $\partial\Omega_k$ of the set

$$\Omega_k = \{x = \xi \sin(nt + \theta) + h(t): \ h = Qx; \ \|h\|_{L^2} \leq R_1\Theta(\xi^k) + 1; \ \xi_k \leq \xi \leq \xi^k\} \subset L^2$$

with a proper $R_1 > 0$ and that the rotation γ_k of this vector field on $\partial\Omega_k$ is either 1 or -1. The relation $\gamma_k \neq 0$ implies that each set Ω_k contains at least one solution of equation (36); since the sets Ω_k are disjoint for different k and $\xi_k \to \infty$, this implies the conclusion of Theorem 4.

Let us calculate the value $|\gamma_k|$ for some fixed k. Consider the deformation

$$\Xi(\lambda, x) = x - \mu Ax + Af(x) + Ab + \lambda A\big(Pf(Px) - f(x)\big), \qquad \lambda \in [0, 1].$$

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For $\lambda = 0$ and $\lambda = 1$

$$\Xi(0, x) = x - A(\mu x - f(x) - b), \qquad \Xi(1, x) = x - A(\mu x - Pf(Px) - b).$$

First we prove that the deformation Ξ is nonzero on $\partial\Omega_k$. Let $\Xi(\lambda, x) = 0$ for some $\lambda \in [0, 1]$ and $x = \xi \sin(nt + \theta) + h(t)$, $h = Qx$. Then $Q\Xi(\lambda, x) = 0$ and $P\Xi(\lambda, x) = 0$. The first equality implies the estimates

$$\|h\|_{L^2} \le R_1\Theta(\xi), \qquad \|h\|_{C^1} \le c\Theta(\xi), \tag{42}$$

where the constants c and R_1 are independent of λ, ξ, k. We use this R_1 in the definition of Ω_k. Therefore the relations $\|h\|_{L^2} = R_1\Theta(\xi^k) + 1$, $\xi_k \le \xi \le \xi^k$ imply $Q\Xi(\lambda, x) \ne 0$.

The rest of the boundary $\partial\Omega_k$ consists of the functions $x = \xi \sin(nt + \theta) + h(t)$ with $\xi = \xi_k$ and $\xi = \xi^k$. The equality $P\Xi(\lambda, x) = 0$ can be rewritten as $P[f(Px) + b] = (1 - \lambda)P[f(Px) - f(x)]$. By Lemma 2, the left-hand side of this equality is

$$P[f(Px) + b] = \frac{1}{\pi}\Psi(\xi)\sin(nt + \theta) + Pb$$

and due to (41) its norm satisfies $\sqrt{\pi}\|P[f(Px) + b]\|_{L^2} > \varepsilon$ for $\xi = \xi_k$ and $\xi = \xi^k$. The right-hand side $(1 - \lambda)P[f(Px) - f(x)]$ vanishes as $\xi \to +\infty$ according to Theorems 1, 2, therefore the relation $\Xi(\lambda, x) = 0$ is impossible for sufficiently large $\xi = \xi_k$, $\xi = \xi^k$.

Thus, $\Xi(\lambda, x) \ne 0$ on $\partial\Omega_k$. Therefore the rotation $\gamma(\Xi(\lambda, \cdot), \partial\Omega_k)$ of the vector field $\Xi(\lambda, \cdot)$ on $\partial\Omega_k$ is the same for each λ. For $\lambda = 1$ this rotation can be computed by the rotation product formula ([16], Theorem 22.4, p. 117; [3], Theorem 3.15, p. 72), which in our case has the form

$$\gamma(\Xi(1, \cdot), \partial\Omega_k) = (-1)^\alpha \gamma\big(Pf(\cdot) + Pb, \partial Z_k\big).$$

Here α is an integer and $\gamma(Pf(\cdot) + Pb, \partial Z_k)$ is the rotation of the vector field $Pf(x) + Pb$ on the boundary ∂Z_k of the annulus $Z_k = \{x = \xi \sin(nt + \theta) : \xi_k\sqrt{\pi} \le \|x\|_{L^2} = \xi\sqrt{\pi} \le \xi^k\sqrt{\pi}\} \subset \Pi_n$; this rotation is defined by the formula $\gamma(Pf(\cdot) + Pb, \partial Z_k) = \gamma_2 - \gamma_1$, where γ_1 and γ_2 are the rotations of the field $Pf(x) + Pb$ on the circles $\|x\|_{L^2} = \xi_k\sqrt{\pi}$ and $\|x\|_{L^2} = \xi^k\sqrt{\pi}$, $x \in \Pi_n$.

By Lemma 2,

$$Pf(x) = \frac{\Psi(\xi)}{\pi\xi}x, \qquad \|x\|_{L^2} = \xi\sqrt{\pi}, \quad x \in \Pi_n, \tag{43}$$

hence relations $\sqrt{\pi}\|Pb\|_{L^2} = |\bar{b}|$ and (41) imply

$$\|Pf(x)\|_{L^2} < \|Pb\|_{L^2}, \quad \|x\|_{L^2} = \xi_k\sqrt{\pi}; \qquad \|Pf(x)\|_{L^2} > \|Pb\|_{L^2}, \quad \|x\|_{L^2} = \xi^k\sqrt{\pi}.$$

By the Rouché theorem ([16]), it follows from these estimates that γ_1 is equal to the rotation of the constant vector field Pb on the circle $\|x\|_{L^2} = \xi_k\sqrt{\pi}$, i.e., $\gamma_1 = 0$; the rotation γ_2 is equal to the rotation of field (43) on the circle $\|x\|_{L^2} = \xi^k\sqrt{\pi}$, i.e., $\gamma_2 = 1$. Therefore,

$$\gamma_k = \gamma(\Xi(1, \cdot), \partial\Omega_k) = (-1)^\alpha(\gamma_2 - \gamma_1) = (-1)^\alpha.$$

Theorem 4 is completely proved.

10. Proof of Theorem 6

Denote by $x(t; \zeta, \eta)$ the solution of equation (9), satisfying $x(0; \zeta, \eta) = \zeta$, $x'(0; \zeta, \eta) = \eta$. Denote by $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ the operator of translation along the trajectories of equation (9) from the moment 0 to the moment 2π. This operator maps any initial point $\{\zeta, \eta\}$ to the point $\{x(2\pi), x'(2\pi)\}$ of the trajectory of the solution $x(t) = x(t; \zeta, \eta)$ in the phase plane $\mathbb{R}^2$.

Consider the function

$$h(t) = h(t; \zeta, \eta) \overset{\text{def}}{=} x(t; \zeta, \eta) - \zeta \cos nt - \frac{\eta}{n} \sin nt. \tag{44}$$

This function is the solution of the Cauchy problem

$$h'' + n^2 h = f\big(x(t; \zeta, \eta)\big) + b(t), \qquad h(0) = h'(0) = 0,$$

hence it satisfies

$$h(t) = \frac{1}{n} \int_0^t \sin n(t - \tau)\Big(f\big(x(\tau; \zeta, \eta)\big) + b(\tau)\Big)\, d\tau,$$

$$h'(t) = \int_0^t \cos n(t - \tau)\Big(f\big(x(\tau; \zeta, \eta)\big) + b(\tau)\Big)\, d\tau$$

and therefore

$$\|h\|_{C^1} \le R_0 \Theta(|\zeta| + |\eta|), \tag{45}$$

where $R_0 > 0$ is independent of ζ and η.

By Theorems 1 and 2,

$$\lim_{\xi \to \infty} \sup_{\|h\|_{C^1} \le R\Theta(\xi)} \left| \int_0^{2\pi} \sin(nt+\theta) f\big(\xi \sin(nt+\theta_1) + h(t)\big)\, dt - \Psi(\xi)\cos(\theta_1 - \theta) \right| = 0 \tag{46}$$

and the convergence in (46) is uniform w.r.t. all θ, θ_1 for each $R > 0$. Relations (44)–(46) imply that for each $\varepsilon > 0$ there exists a $\xi_0 = \xi_0(\varepsilon)$ such that the inequality

$$\int_0^{2\pi} \sin(nt + \theta) f\big(x(t; \zeta, \eta)\big)\, dt \le \psi^+ + \varepsilon$$

is valid for all $\theta \in [0, 2\pi)$ whenever $|\zeta| + |\eta| \ge \xi_0$.

Set $\varepsilon = (|\bar{b}| - \psi^+)/3$; estimate (15) implies $\varepsilon > 0$. Let θ be any number such that

$$-\int_0^{2\pi} b(t) \sin(nt + \theta)\, dt \ge |\bar{b}| - \varepsilon = \psi^+ + 2\varepsilon. \tag{47}$$

Consider the linear guiding function

$$V(\zeta, \eta; \theta) = -\eta \sin \theta + n\zeta \cos \theta$$

on the plane $\{\zeta, \eta\} \in \mathbb{R}^2$. If $V(\zeta, \eta; \theta) > 0$ is sufficiently large, then $|\zeta| + |\eta| \ge \xi_0(\varepsilon)$ and therefore

$$V\big(\mathcal{P}(\zeta, \eta); \theta\big) - V(\zeta, \eta; \theta) = V\big(x(2\pi; \zeta, \eta), x'(2\pi; \zeta, \eta); \theta\big) - V(\zeta, \eta; \theta)$$
$$= nh(2\pi; \zeta, \eta) \cos \theta - h'(2\pi; \zeta, \eta) \sin \theta$$
$$= -\int_0^{2\pi} \sin(nt + \theta) b(t)\, dt - \int_0^{2\pi} \sin(nt + \theta) f\big(x(t; \zeta, \eta)\big)\, dt \ge \varepsilon.$$

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Similarly, if $-V(\zeta, \eta; \theta) > 0$ is sufficiently large, then $V\big(\mathcal{P}^{-1}(\zeta, \eta); \theta\big) - V(\zeta, \eta; \theta) \leq -\varepsilon$.

Thus, there are disjoint half-planes $V(\zeta, \eta; \theta) \geq c$ and $V(\zeta, \eta; \theta) \leq -c$ such that any sequence $\{\zeta, \eta\}, \mathcal{P}(\zeta, \eta), \ldots, \mathcal{P}^k(\zeta, \eta), \ldots$ starting in the first half-plane and any sequence

$$\{\zeta, \eta\}, \mathcal{P}^{-1}(\zeta, \eta), \ldots, \mathcal{P}^{-k}(\zeta, \eta), \ldots$$

starting in the second half-plane belong to the same half-planes as their initial values and go to infinity. Therefore the solution $x(t; \zeta, \eta)$ of equation (9) satisfies the first of estimates (16) whenever $\{\zeta, \eta\}$ belongs to the half-plane $V(\zeta, \eta; \theta) \geq c$ and the second of estimates (16) whenever $\{\zeta, \eta\}$ lies in the half-plane $V(\zeta, \eta; \theta) \leq -c$.

Let us choose another θ such that (47) holds. For this θ we can repeat all the arguments exactly in the same way as above. Consequently there is another pair of half-planes consisting of the initial values of unbounded solutions. The unity of these four half-planes is the whole plane $\{\zeta, \eta\}$ without a bounded set (a parallelogram). Any solution with the initial value out of this parallelogram satisfies at least one of relations (16).

Theorem 6 is proved. The main idea of the proof for bounded f was published in [1].

11. Proof of Theorem 7

The proof of this theorem is close to the proof of the main result from [12], some ideas are similar to the proof of Theorem 4 from that paper. The computations of the index at infinity of the vector field $\Phi x = x - \mu A x - A F(x)$ are different for different conclusions of Theorem 7. Here and below by $F(x)$ we denote the superposition operator $F\big(x(\cdot)\big) = f\big(\cdot, x(\cdot)\big)$.

11.1. Conclusion (i). Let $\varphi^+ < 0 < \varphi_-$ (the case $\varphi^- < 0 < \varphi_+$ can be studied in a similar way). Set $p(x) = (g, x)_{L^2}$ and $Qx = x - p(x)e$ for all $x \in L^2 = L^2(a, b)$ and consider the deformation

$$\Xi_1(\lambda, x) = Qx - \mu A Q x - \lambda A F(x) + (1 - \lambda)\mu^{-1}s\big(p(x)\big)e, \qquad \lambda \in [0, 1],$$

where $s(\cdot)$ is function (39). Since $e = \mu A e$, the equalities $\Xi_1(0, x) = Qx - \mu A Q x + \mu^{-1}s\big(p(x)\big)e$ and $\Xi_1(1, x) = \Phi(x)$ are valid. To prove conclusion (i) of Theorem 7, it suffices to establish the relation $|\operatorname{ind}_\infty \Xi_1(0, \cdot)| = 1$ and to prove an *a priori* estimate $\|x\|_{L^2} \leq const < \infty$ for all the zeroes of the deformation Ξ_1.

The first assertion is simple, since the equation $\Xi_1(0, x) = 0$ has a unique solution $x = 0$ and the index of this solution equals either 1 or -1. This follows from the rotation product formula cited above. Indeed, $\Xi_1(0, \cdot)$ is the direct sum of the scalar field $\chi(\xi e) = \mu^{-1}s(\xi)e$, $\xi \in \mathbb{R}$ and the vector field $h - \mu A h \in E$, where $E = QL^2$ is a subspace of L^2 with co-dimension 1. The index of the zero $\xi = 0$ of the scalar field $\chi(\cdot)$ equals $\operatorname{sign}\mu$, while the index of the zero $h = 0$ of the vector field $h - \mu A h \in E$ equals $(-1)^\beta$, where β is the sum of the multiplicities of all the real eigenvalues of the operator μA that are greater than 1, hence $\operatorname{ind}_\infty \Xi_1(0, \cdot) = (-1)^\beta \operatorname{sign}\mu$.

Now let us prove an *a priori* estimate for the zeroes of Ξ_1. If $\Xi_1(\lambda, x) = 0$ for some $x = \xi e + h$, $h \in E$, and $\lambda \in [0, 1]$, then

$$0 = Q\Xi_1(\lambda, x) = h - \mu A h - \lambda A Q F(\xi e + h), \tag{48}$$

$$0 = \mu p\big(\Xi_1(\lambda, x)\big) = -\lambda p\big(F(\xi e + h)\big) + (1 - \lambda)s(\xi). \tag{49}$$

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The first equality rewritten as $h = \lambda(I - \mu AQ)^{-1}AQF(x)$ implies the estimate $\|h\|_{C^1} \leq c\Theta(|\xi|)$. Consider the second equality. If $|\xi| \geq 1$, then $s(\xi) = \operatorname{sign}\xi$. Theorems 1 and 2 imply

$$p\big(F(\xi e + h)\big) - p\big(F(\xi e)\big) = p\big(F(\xi e + h)\big) - \Psi(\xi) \to 0, \qquad \xi \to \infty,$$

hence

$$\mu p\big(\Xi_1(\lambda, x)\big) + \lambda \Psi(\xi) - (1 - \lambda)\operatorname{sign}\xi \to 0, \qquad \xi \to \pm\infty$$

and therefore

$$\mu p\big(\Xi_1(\lambda, x)\big) \geq 1 - \lambda - \lambda\varphi^+ + o(1), \qquad \xi \to +\infty,$$
$$\mu p\big(\Xi_1(\lambda, x)\big) \leq \lambda - 1 - \lambda\varphi_- + o(1), \qquad \xi \to -\infty.$$

Since $1 - \lambda - \lambda\varphi^+ \geq \min\{1, -\varphi^+\} > 0$ and $\lambda - 1 - \lambda\varphi_- \leq -\min\{1, \varphi_-\} < 0$, we see that system (48), (49) has no solutions if $|\xi|$ is sufficiently large. This proves the required *a priori* estimate.

11.2. Conclusion (ii). To be definite, consider the case $\varphi_- > 0$, $\varphi_+ > 0$. The other case ($\varphi^- < 0$, $\varphi^+ < 0$) is similar. Define the deformation

$$\Xi_2(\lambda, x) = Qx - \mu AQx - \lambda AF(x) - (1 - \lambda)\mu^{-1}\varphi_+ e, \qquad \lambda \in [0, 1].$$

Now $\Xi_2(1, x) = \Phi(x)$ and $\Xi_2(0, x) = Qx - \mu AQx - \mu^{-1}\varphi_+ e$; the equation $\Xi_2(0, x) = 0$ has no solutions and therefore $\operatorname{ind}_\infty \Xi_2(0, \cdot) = 0$. To prove the conclusion (ii) of Theorem 7, it suffices to establish an *a priori* estimate $\|x\|_{L^2} \leq const < \infty$ for all the zeroes $x = \xi e + h$ of the deformation Ξ_2. This *a priori* estimate follows from the estimate $\|h\|_{C^1} \leq c\Theta(|\xi|)$ and the relations

$$\mu p\big(\Xi_2(\lambda, x)\big) + \lambda\Psi(\xi) + (1 - \lambda)\varphi_+ = -\lambda p\big(F(x)\big) + \lambda p\big(F(\xi e)\big) \to 0, \qquad \xi \to \pm\infty$$

and

$$\liminf_{\xi \to \pm\infty}\big(\lambda\Psi(\xi) + (1 - \lambda)\varphi_+\big) \geq \min\{\varphi_+, \varphi_-\} > 0.$$

11.3. Conclusion (iii). Let $\varphi_+ < 0 < \varphi^+$ (we omit the similar proof for the case $\varphi_- < 0 < \varphi^-$). We need to prove the existence of the zeroes $x_n = \xi_n e + h_n$ of the vector field Φ with the norms $\|x_n\|_{L^2} \to \infty$. As everywhere above, for each zero $x = \xi e + h$ the equality $Q\Phi(x) = 0$ implies estimates (42). Set $\varepsilon = \frac{1}{3}\min\{\varphi^+, -\varphi_+\}$ and fix a ξ_0 such that

$$|p\big(F(\xi e + h)\big) - \Psi(\xi)| < \varepsilon \qquad (50)$$

for all $\xi \geq \xi_0$, $\|h\|_{C^1} \leq c\Theta(\xi)$, where c comes from (42); the existence of such ξ_0 follows from (4). By definition of φ_+, φ^+, there exist numbers $\xi_* > \xi_0$ and $\xi^* > \xi_*$ such that

$$\Psi(\xi_*) < -2\varepsilon, \quad \Psi(\xi^*) > 2\varepsilon. \qquad (51)$$

Consider the set $\Omega = \{x \in L^2 : \|Qx\|_{L^2} \leq R_1\Theta(\xi^*) + 1, \ p(x) \in [\xi_*, \xi^*]\}$, where R_1 is defined in (42). The boundary $\partial\Omega$ of Ω consists of three parts:

$$G_* = \{\|Qx\|_{L^2} \leq R_1\Theta(\xi^*) + 1, \ p(x) = \xi_*\}, \quad G^* = \{\|Qx\|_{L^2} \leq R_1\Theta(\xi^*) + 1, \ p(x) = \xi^*\},$$

and
$$G = \{\|Qx\|_{L^2} = R_1\Theta(\xi^*) + 1, \ p(x) \in [\xi_*, \xi^*]\}.$$
The vector field Φ is nonzero on G due to the estimates $\|h\|_{L^2} \leq R_1\Theta(\xi) \leq R_1\Theta(\xi^*)$, which are valid for all the solutions $x = \xi e + h$ of $Q\Phi(x) = 0$ in Ω. Relations (50), (51) imply

$$\mu p\big(\Phi(x)\big) = -p\big(F(x)\big) + \Psi\big(p(x)\big) - \Psi\big(p(x)\big) > -\varepsilon + 2\varepsilon = \varepsilon > 0, \quad x \in G_*, \ \|h\|_{C^1} \leq c\Theta(\xi),$$
$$\mu p\big(\Phi(x)\big) = -p\big(F(x)\big) + \Psi\big(p(x)\big) - \Psi\big(p(x)\big) < \varepsilon - 2\varepsilon = -\varepsilon < 0, \quad x \in G^*, \ \|h\|_{C^1} \leq c\Theta(\xi),$$

hence $\Phi(x) \neq 0$ on $G_* \bigcup G^*$. Thus, $\Phi \neq 0$ on $\partial\Omega$. We also see that $\mu p\big(\Phi(x)\big) > 0$ if $Q\Phi(x) = 0$, $p(x) = \xi_*$ and $\mu p\big(\Phi(x)\big) < 0$ if $Q\Phi(x) = 0$, $p(x) = \xi^*$.

Now we can calculate the rotation $\gamma(\Phi, \partial\Omega)$ of the field Φ on $\partial\Omega$ using an appropriate deformation and Theorem 3 from [10]. Let us reformulate this theorem for our problem.

Let X be a Banach space. Let some completely continuous operator $\mathcal{A} = \{A_1, A_2\}$ be defined on $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 is the ball $\|x\|_X \leq r$ and $\Omega_2 = [\xi_*, \xi^*]$. Let for any $\xi \in \Omega_2$ the vector field $x - A_1(x, \xi)$ be nondegenerate on $\partial\Omega_1 = \{\|x\|_X = r\}$ and let the rotation $\gamma_1 = \gamma(I - A_1(\cdot, \xi), \partial\Omega_1)$ be nonzero. Then γ_1 is the same for all $\xi \in \Omega_2$. Denote by $K(\xi)$ the nonempty set of solutions x of the equation $x = A_1(x, \xi)$ for each $\xi \in \Omega_2$.

Statement 1. *If $\xi_* - A_2(x, \xi_*) < 0$ for all $x \in K(\xi_*)$ and $\xi^* - A_2(x, \xi^*) > 0$ for all $x \in K(\xi^*)$, then $\gamma(\mathcal{A}, \partial\Omega) = \gamma_1$.*

In our case, this statement implies that either $\gamma(\Phi, \partial\Omega) = 1$ or $\gamma(\Phi, \partial\Omega) = -1$. Therefore there exists at least one zero $x_* \subset \Omega$ of the vector field Φ; relation $x_* \in \Omega$ implies $(g, x_*)_{L^2} \geq \xi_*$. Since the numbers ξ_* and ξ^* satisfying (51) may be chosen arbitrarily large, there is a sequence $\{x_n\}$ of zeroes of the vector field Φ with unbounded norms: $\|x_n\|_{L^2} \to \infty$. Theorem 7 is proved.

12. Proof of Theorem 9

12.1. The choice of unknowns. First, let us rescale the time in (22). For any $w \in \Omega = [w_1, w_2]$ every 2π-periodic solution $x(t)$ of the equation

$$L\left(w\frac{d}{dt}\right)x = f(x) \tag{52}$$

determines the $(2\pi/w)$-periodic solution $x(wt)$ of equation (22). We consider (52) instead of (22) and prove that for some $w \in \Omega$ equation (52) has at least one nontrivial 2π-periodic solution $x(t) = r\sin t + h(t)$. Here $r > 0$, $h(t)$ is 2π-periodic, and $Ph = 0$;

$$Px(t) \stackrel{\text{def}}{=} \frac{1}{\pi}\int_0^{2\pi}\cos(t-\tau)\,x(\tau)\,d\tau.$$

Let us stress that any nonstationary periodic solution $x(t)$ of autonomous equation (52) generates the continuum of periodic solutions $x(t + \alpha)$. We delete the nonuniqueness by fixing the solution that has zero projection onto $\cos t$ and a positive projection onto $\sin t$. At the same time, the frequency w is considered as an additional unknown.

12.2. Linear subspaces and operators. The first linear operator is the projector P. Set $Q = I - P$ and define the subspaces $\Pi = PL^2$ and $\Pi^* = QL^2$ in $L^2 = L^2(0, 2\pi)$. These subspaces are orthogonal, Π is two-dimensional, Π^* has co-dimension 2.

Denote by $A(w)$ ($w \in \Omega$) the linear operator that maps any function $u \in \Pi^*$ to a unique solution $x \in \Pi^*$ of the linear equation

$$L\left(w\frac{d}{dt}\right)x = u(t). \tag{53}$$

Consider the operators $A(w)Q$. Their norms in L^2 are uniformly bounded:

$$\left\|A(w)Q\right\|_{L^2 \to L^2} = q_*(w) \le c_* = \stackrel{\text{def}}{=} \sup_{w \in \Omega} q_*(w) < \infty, \qquad q_*(w) = \sup_{n \in \mathbb{Z}, n \neq \pm 1} \left|L(nwi)\right|^{-1}.$$

Each operator $A(w)Q$ is completely continuous from L^2 to C and continuous from C to C^1. Moreover, $A(\cdot)Q : \Omega \times L^2 \to L^2$ is completely continuous w.r.t. both arguments w and $u = u(t)$.

We use the notation P_n for the following orthogonal projectors in L^2. By P_0 we denote the orthogonal projector onto the one-dimensional subspace Π_0 of constant functions; P_n with $n = 2, 3, 4, \dots$ is the projector onto the two-dimensional linear span Π_n of the functions $\sin nt$ and $\cos nt$. Each subspace Π_n is invariant for the operator $A(w)Q$. The one-dimensional subspace Π_0 corresponds to the eigenvalue $\left(L(0)\right)^{-1}$; each plane Π_n corresponds to the pair $\left(L(\pm nwi)\right)^{-1}$.

Lemma 3. *Let $w \in \Omega$. The functions $x(t) = r\sin t + h(t)$, $h \in \Pi^*$, and $u \in L^2$ satisfy equation (53) iff $h = A(w)Qu$ and*

$$\pi r \, \Re e \, L(wi) = \int_0^{2\pi} \sin t \; u(t) \, dt, \qquad \pi r \, \Im m \, L(wi) = \int_0^{2\pi} \cos t \; u(t) \, dt.$$

Relations $h = A(w)Qu$, $u \in C$ imply

$$(P_n h', P_n u)_{L^2} = n \, \Im m \, L(nwi)\|P_n h\|_{L^2}^2, \qquad n = 2, 3, 4, \dots$$

The proof is by simple computations and we omit it.

12.3. Main deformation. By Lemma 3, the problem on 2π-periodic solutions of equation (52) is equivalent to the system

$$\begin{cases} \pi \, \Re e \, L(wi) = \dfrac{1}{r} \displaystyle\int_0^{2\pi} \sin t \; f\left(r\sin t + h(t)\right) dt, \\[2mm] \pi \, \Im m \, L(wi) = \dfrac{1}{r} \displaystyle\int_0^{2\pi} \cos t \; f\left(r\sin t + h(t)\right) dt, \\[2mm] h = A(w)Qf\left(r\sin t + h(t)\right). \end{cases} \tag{54}$$

Consider in the space $\mathbb{E} = \{\mathbb{R} \times \mathbb{R} \times \Pi^*\}$ the deformation

$$\Phi_\lambda(r, w, h) = \begin{cases} \lambda\pi \, \Re e \, L(wi) - \dfrac{1}{r} \displaystyle\int_0^{2\pi} \sin t \; f\left(r\sin t + \lambda h(t)\right) dt, \\[2mm] \pi \, \Im m \, L(wi) - \dfrac{\lambda}{r} \displaystyle\int_0^{2\pi} \cos t \; f\left(r\sin t + h(t)\right) dt, \\[2mm] h - \lambda A(w)Qf\left(r\sin t + h(t)\right) \end{cases} \tag{55}$$

of the vector field Φ_1 to the field

$$\Phi_0(r, w, h) = \begin{cases} -\dfrac{1}{r} \displaystyle\int_0^{2\pi} \sin t\ f(r \sin t)\, dt, \\ \pi \Im m\, L(wi), \\ h. \end{cases}$$

Here $\lambda \in [0,1]$ is the parameter of the deformation. For $\lambda = 1$ any zero of the vector field Φ_1 is a solution of (54); we should prove the existence of such zeroes. For $\lambda = 0$ the vector field Φ_0 has a simple form and it is easy to compute its topological characteristics on proper sets.

12.4. Rotation of the vector field Φ_0. Consider the set

$$G = G(\rho, R, w_1, w_2, c_1) \stackrel{\text{def}}{=} \{r \in [\rho, R],\ w \in [w_1, w_2],\ \|h\|_{L^2} \leq c_1 \Theta(R)\} \subset \mathbb{E}.$$

The positive constants ρ, R, and c_1 are determined by the values w_1, w_2, q, the polynomial L, and the function f. The number ρ is sufficiently small, the numbers R and c_1 are sufficiently large.

If we choose the parameters ρ, R, and c_1 such that deformation (55) is nondegenerate on the boundary ∂G of the domain G and if the rotation $\gamma(\Phi_0, \partial G)$ of the vector field Φ_0 on ∂G is nonzero, then Theorem 9 will be proved.

By the rotation product formula, $\gamma(\Phi_0, \partial G)$ is the product of three rotations. The first is the rotation γ_r of the scalar vector field

$$-\frac{1}{r} \int_0^{2\pi} \sin t\ f(r \sin t)\, dt = -\frac{\Psi(r)}{r} \tag{56}$$

on the boundary of the interval $[\rho, R]$; the second is the rotation γ_w of the scalar field $\pi \Im m\, L(wi)$ on the boundary of the interval Ω; the third is the rotation γ_h of the vector field h on the sphere $\{h \in \Pi^* : \|h\|_{L^2} = c_1 \Theta(R)\}$.

The rotation γ_r is either 1 or -1. This follows from condition (25): for small $r = \rho$ the component (56) of Φ_0 has the same sign as $-f'(0)$; for large $r = R$ this component has the opposite sign coinciding with the sign of $-\Psi(R)$. Of course, ρ should be sufficiently small and R should be sufficiently large.

The rotation γ_w is also either 1 or -1. The component $\pi \Im m\, L(wi)$ has opposite signs for $w = w_1$ and $w = w_2$, since the polynomial $\Im m\, L(wi)$ has a unique root w_0 inside $[w_1, w_2]$ and this root is of odd multiplicity.

The rotation γ_h equals 1. We see that $\gamma(\Phi_0, \partial G) \neq 0$.

12.5. Auxiliary estimates. Below we use the notation $\|\cdot\| = \|\cdot\|_{L^2}$.

Lemma 4. *Let $w \in \Omega$, denote*

$$q_1(w) \stackrel{\text{def}}{=} \left|\Im m\, L(wi)\right|, \qquad q_2(w) \stackrel{\text{def}}{=} \inf_{n=0,2,3,4,\dots} \left|L(nwi)\right|.$$

If the second and the third components of deformation (55) *are zero, then* $q_1(w) \leq q$ *and*

$$\|h\| \leq r \sqrt{\pi \, \frac{q^2 - q_1^2(w)}{q_2^2(w) - q^2}}.$$

Proof. The equality $h = \lambda A(w) \, Q f \big(r \sin t + h \big)$ implies $\big| L(nwi) \big| \|P_n h\| = \lambda \|P_n f(x)\|$ for $n = 0, 2, 3, \ldots$ The equality

$$\pi r \, \Im m \, L(wi) = \lambda \int_0^{2\pi} \cos t f\big(x(t) \big) \, dt$$

implies $\sqrt{\pi} \, r |\Im m \, L(wi)| \leqslant \lambda \|P f(x)\|$. Therefore

$$\pi r^2 \big| \Im m \, L(wi) \big|^2 + \sum_{n=0,2,3,\ldots} \big| L(nwi) \big|^2 \|P_n h\|^2 \leq \lambda^2 \|f(x)\|^2$$

and from $\lambda \leq 1$ it follows that

$$\pi r^2 \big| \Im m \, L(wi) \big|^2 + \sum_{n=0,2,3,\ldots} \big| L(nwi) \big|^2 \|P_n h\|^2 \leq q^2 \|x\|^2 = q^2 \Big(\pi r^2 + \sum_{n=0,2,3,\ldots} \|P_n h\|^2 \Big), \quad (57)$$

consequently $\pi r^2 q_1^2(w) + q_2^2(w) \|h\|^2 \leq q^2 \big(\pi r^2 + \|h\|^2 \big)$. Here $q < q_2(w)$ by assumption, hence the conclusion of Lemma 4 is true.

Lemma 5. *There exist numbers c and c_2 such that the estimates $\|h\|_{L^2}, \|h\|_{C^1} \leq cr$ and $\|h\|_{L^2}, \|h\|_{C^1} \leq c_2 \Theta(r)$ hold for any zero $\{r, w, h\}$ of deformation* (55).

This lemma follows from Lemma 4 and from the properties of the operators $A(w)Q$.

12.6. Nondegeneracy of Φ_λ. To complete the proof of Theorem 9, it remains to show that deformation (55) is nonzero on the boundary ∂G of the domain G for some sufficiently small ρ and sufficiently large R and c_1. The boundary ∂G is the join of the sets

$$G_\rho = \{ r = \rho, \ w \in [w_1, w_2], \ \|h\| \leq c_1 \Theta(R) \},$$

$$G_R = \{ r = R, \ w \in [w_1, w_2], \ \|h\| \leq c_1 \Theta(R) \},$$

$$G_w = \{ r \in [\rho, R], \ w = w_j, \ j = 1, 2, \ \|h\| \leq c_1 \Theta(R) \},$$

$$G_h = \{ r \in [\rho, R], \ w \in [w_1, w_2], \ \|h\| = c_1 \Theta(R) \}.$$

Let $\Phi_\lambda(r, w, h) = 0$ for some $\{r, w, h\} \in G$ and $\lambda \in [0, 1]$. Then

$$\lambda^2 \frac{\Re e \, L(wi)}{\Im m \, L(wi)} \int_0^{2\pi} \cos t \ f\big(r \sin t + h(t) \big) \, dt = \int_0^{2\pi} \sin t \ f\big(r \sin t + \lambda h(t) \big) \, dt. \quad (58)$$

First consider this equality for small r. By assumption, w_0 is a root of the same multiplicity K for the polynomials $L(iw)$ and $\Im m \, L(iw)$. This means that the multiplicity of this root for the polynomial $\Re e \, L(iw)$ is greater or equal than K, therefore the function

$\Re e\,L(wi)/\Im m\,L(wi)$ is continuous on Ω. Since $f(x) = f'(0)x + o(x)$, $x \to 0$, the term of order r in the left-hand side of (58) equals zero. In the right-hand side, the term of order r equals

$$\int_0^{2\pi} \sin t \; f'(0)\big(r\sin t + \lambda h(t)\big)\,dt = \pi f'(0)\,r,$$

i.e., this term is nonzero. The other terms have smaller order w.r.t. r due to Lemma 5. Therefore equality (58) cannot be valid for sufficiently small r, hence $\{r, w, h\} \notin G_\rho$ if ρ is sufficiently small.

The relation $\{r, w, h\} \notin G_R$ for sufficiently large R follows from (58) due to Theorems 1 and 2. These theorems imply

$$\int_0^{2\pi} \cos t \; f\big(r\sin t + h(t)\big)\,dt \to 0,$$

$$\int_0^{2\pi} \sin t \; f\big(r\sin t + \lambda h(t)\big)\,dt - \int_0^{2\pi} \sin t \; f(r\sin t)\,dt \to 0$$

as $r \to \infty$ and from (25) it follows that

$$\left| \int_0^{2\pi} \sin t \; f\big(r\sin t + \lambda h(t)\big)\,dt \right| \geq \frac{c_0}{2|f'(0)|} > 0$$

for all $r \geq r_0 > 0$. Therefore relation (58) cannot be valid for large r.

The proof of relation $\{r, w, h\} \notin G_w$ is the most cumbersome. Since

$$\int_0^{2\pi} f\big(r\sin t + h(t)\big)\big(r\cos t + h'(t)\big)\,dt = 0$$

and the second component of (55) is zero, we have

$$\pi\,\Im m\,L(wi)r^2 + \lambda \int_0^{2\pi} h'(t)\,f\big(r\sin t + h(t)\big)\,dt = 0.$$

Equivalently,

$$\pi\,\Im m\,L(wi)r^2 + \lambda \sum_{n=2,3,4,\dots} \int_0^{2\pi} P_n h'(t)\,P_n f\big(r\sin t + h(t)\big)\,dt = 0.$$

But $h(t) = \lambda A(w)Qf\big(r\sin t + h(t)\big)$ and by Lemma 3,

$$\lambda \int_0^{2\pi} P_n h'(t)\,P_n f\big(r\sin t + h(t)\big)\,dt = n\,\Im m\,L(wni)\|P_n h\|^2, \qquad n = 2, 3, 4, \dots,$$

therefore

$$\pi\,\Im m\,L(wi)r^2 + \sum_{n=2,3,4,\dots} n\,\Im m\,L(wni)\|P_n h\|^2 = 0,$$

i.e.,

$$\pi\big|\Im m\,L(wi)\big|r^2 + \mathrm{sign}\big(\Im m\,L(wi)\big) \sum_{n=2,3,4,\dots} n\,\Im m\,L(wni)\|P_n h\|^2 = 0. \tag{59}$$

From relation (57) rewritten as

$$\sum_{n=0,2,3,\ldots} \left(\left|L(wni)\right|^2 - q^2\right)\|P_n h\|^2 \leq \pi r^2 \left(q^2 - \left|\Im m\, L(wi)\right|^2\right),$$

it follows the estimate

$$\sum_{n=0,2,3,\ldots} \Bigl(\left|\Im m\, L(wi)\right|\left(\left|L(nwi)\right|^2 - q^2\right)$$

$$+ n\left(q^2 - \left|\Im m\, L(wi)\right|^2\right)\operatorname{sign}\left(\Im m\, L(wi)\right)\Im m\, L(nwi)\Bigr)\|P_n h\|^2 \leq 0.$$

Due to (23), this estimate is impossible for $w = w_j$, $j = 1, 2$ if $h \neq 0$. If $h = 0$, then the second component of (55) equals $\pi \Im m\, L(wi)$ and differs from zero for $w = w_j$, $j = 1, 2$. Therefore in both cases $\{r, w, h\} \notin G_w$.

Finally, Lemma 5 implies $\{r, w, h\} \notin G_h$ if we choose $c_1 > c_2$. Theorem 9 is completely proved.

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