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On a bifurcation governed by hysteresis nonlinearity*

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Abstract. We consider autonomous systems with a nonlinear part depending on a parameter and study Hopf bifurcations at infinity. The nonlinear part consists of the nonlinear functional term and the Prandtl–Ishlinskii hysteresis term. The linear part of the system has a special form such that the close-loop system can be considered as a hysteresis perturbation of a quasilinear Hamiltonian system. The Hamiltonian system has a continuum of arbitrarily large cycles for each value of the parameter. We present sufficient conditions for the existence of bifurcation points for the non-Hamiltonian system with hysteresis. These bifurcation points are determined by simple characteristics of the hysteresis nonlinearity.

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1 Problem statement

Consider the equation

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)f(x). \tag{1}$$

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Equations (1) with arbitrary polynomials $L(p)$, $M(p)$ satisfying condition (i) below are usual in control theory (see, e.g., [3, 11, 12]). If $M \equiv 1$, then (1) is an ordinary autonomous higher-order differential equation. The standard definition of solutions for equation (1) is as follows. Consider the system

$$z' = Az + y, \qquad y = b\,f(x), \qquad x = (c, z), \qquad z, y, b, c \in \mathbb{R}^\ell.$$

For any L, M, and f there exist a matrix A and vectors b, c (non-unique) such that this system is equivalent to equation (1).

Everywhere below it is assumed that the polynomials $L(p)$ and $M(p)$ and their degrees ℓ and m satisfy the following conditions:

(i) The polynomials $L(p)$ and $M(p)$ are coprime and $\ell > m$;

(ii) The polynomials $L(p)$ and $M(p)$ are even;

(iii) The polynomial $L(p)$ has a pair of simple imaginary roots $\pm iw_0$ ($w_0 > 0$) and $L(iw_0 n) \neq 0$ for $n = 0, 2, 3, \dots$

Condition (ii) implies that (1) is a Hamiltonian equation [6]. Suppose that the function $x^{-1}f(x)$ vanishes as $x \to \infty$ (e.g., $f(x)$ is uniformly bounded). Then due to (iii) equation (1) has a continuum of large-amplitude periodic solutions (cycles) $x_r = x_r(t)$ of all amplitudes $r = \|x_r\|_C$ with $r \geq r_0$; the period T_r of x_r goes to $2\pi/w_0$ as $r \to \infty$.

If the function $f(x, \lambda)$ depending on the parameter λ is used in place of $f(x)$ in equation (1), then this equation has a continuum of arbitrarily large cycles for each parameter value. If equation (1) with $f(x, \lambda)$ in place of $f(x)$ is perturbed by some hysteresis term, then large periodic solutions may exist for parameter values accumulating near some bifurcation points only. To be precise, the following definition from [4] is used.

Definition 1 A parameter value λ_0 is called a **Hopf bifurcation point at infinity** (shortly, a **bifurcation point** or **HBP**) **with a frequency** w_0 for some equation depending on the parameter λ if for any sufficiently large $r > 0$ there exists a λ_r such that the equation with $\lambda = \lambda_r$ has a T_r-periodic solution $x_r = x_r(t)$ and $\lambda_r \to \lambda_0$, $\|x_r\|_C \to \infty$, $T_r \to 2\pi/w_0$ as $r \to \infty$.

In this definition, r is an auxiliary parameter, in the sequel it is the amplitude of the principal harmonics of x_r. The use of an auxiliary parameter different from λ is standard in Hopf bifurcations (see, e.g., [10]).

If λ_0 is a HBP, then in arbitrary small vicinity of λ_0 there exist values λ such that the equation with these λ has arbitrarily large periodic solutions with periods arbitrarily close to $2\pi/w_0$. Let us stress that for equations with hysteresis the state of the hysteresis nonlinearity must be also periodic with the same period as the solution.

Usually Hopf bifurcations are studied for equations where the linear part (the coefficients of the polynomials L and M) depends on λ. In this situation, the bifurcation points are typically determined by the linear part only. The following result (see, [4]) is well-known for the equation

$$L\left(\frac{d}{dt};\lambda\right)x = M\left(\frac{d}{dt};\lambda\right)f(x,\lambda); \tag{2}$$

here the polynomials L and M depend continuously on λ and their degrees are supposed to be the same for all λ.

Statement 1 *Let*

$$\lim_{x\to\infty} \sup_{|\lambda-\lambda_0|\le\varepsilon} \left|\frac{f(x,\lambda)}{x}\right| = 0$$

for some $\varepsilon > 0$. *Let the polynomial* $L(p;\lambda)$ *have a pair of simple conjugate roots* $\sigma(\lambda) \pm w(\lambda)i$, *let* $\sigma(\lambda_0) = 0$, $w_0 = w(\lambda_0) > 0$. *Let the function* $\sigma(\lambda)$ *take values of both sign in every neighborhood of the point* λ_0. *Let* $L(iw_0n;\lambda_0) \neq 0$ *for* $n = 0, 2, 3, \ldots$ *Then* λ_0 *is a HBP with the frequency* w_0 *for equation* (2).

This statement can be generalized to include equations with various classes of nonlinearities, in particular, nonlinearities with delays and with hysteresis. Let us remark that the polynomial L can not be even for all λ under the conditions of Statement 1[1].

In [5], equations with linear parts independent of λ were considered, including equations with delays and with the simplest hysteresis nonlinearity called *stop*. Here we consider more complicated hysteresis nonlinearities.

Consider the perturbed equation

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)(f(x,\lambda) + H_\lambda x) \tag{3}$$

with the same linear part as in (1). Here the perturbation $H_\lambda x = (H_\lambda x)(t)$ is the output of a hysteresis nonlinearity with the input $x = x(t)$ (see, [8]); characteristics of this hysteresis nonlinearity and the function $f(x,\lambda)$ depend on the parameter $\lambda \in (a,b)$. The block-diagram of such a system is shown in Figure 1. We consider equation (3) with the Prandtl–Ishlinskii hysteresis nonlinearity H_λ (see, e.g., [2, 8, 9, 13] and the references therein), its exact definition is given in the next section.

The principal difference between theorems of this paper and results of [5] (for equations with even linear parts) is that large cycles of equation (3) with the Prandtl–Ishlinskii hysteresis term exist for different values of λ close to some HBP λ_0, while all large cycles of the equation with the stop hysteresis nonlinearity exist

[1] For every even polynomial the relation $\sigma(\lambda_0) = 0$ for its simple roots $\sigma(\lambda) \pm w(\lambda)i$ implies $\sigma(\lambda) \equiv 0$ for all λ close to λ_0; the simplicity of the roots is essential.

for a unique value $\lambda = \lambda_0$ which is exactly the point where hysteresis disappears and the equation becomes Hamiltonian. The stop can be consider as the simplest degenerate Prandtl–Ishlinskii hysteresis nonlinearity.

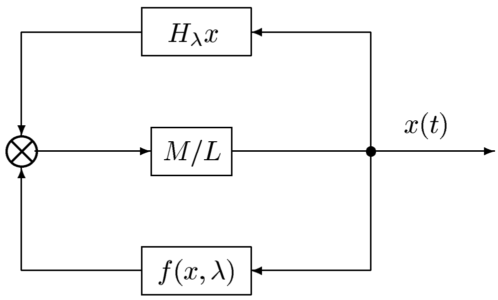

Figure 1 Block-diagram of the system considered

The main result of this paper is a sufficient condition for λ_0 to be a bifurcation point. Generically, it is close to necessary conditions. We prove that the bifurcation points of equation (3) are determined by the hysteresis nonlinearity. Rather sharp approximations of the behavior of the stop and the Prandtl–Ishlinskii nonlinearities at infinity allow to answer the question if the large cycles of equation (3) exist for $\lambda < \lambda_0$ or for $\lambda > \lambda_0$.

If a Hopf bifurcation occurs in a system without hysteresis, then the set of large cycles is one-parametric as in the definition above. For Hopf bifurcations in systems with the Prandtl–Ishlinskii hysteresis nonlinearities (such that the density of the weight function has a noncompact support) the cycles typically depend on an additional scalar parameter, i.e., the set of large cycles is two-parametric. This is discussed in Section 5. Generically, for each λ the cycles form a band in the phase space, the diameter of the band tends to infinity and its "width" tends to zero as λ goes to a bifurcation point.

The paper is organized as follows. In the next section the hysteresis nonlinearity is described. In Section 3 the results are formulated as well as simple examples. The rest of the paper contains remarks and proofs.

Hopf bifurcation points (at equilibrium or at infinity) may be studied with various methods. We do not expect that classical analytical approaches (normal forms, etc.) can be used to study bifurcations at infinity and equations with hysteresis. The method of *parameter functionalization* [7] seems to be applicable. Our methods are close to that of [1, 5]. They can be briefly described as a combination of harmonic linearization and topological methods (degree theory or vector field rotation theory).

Similar results can be formulated for equations with the Preisach hysteresis nonlinearities and some other types of hysteresis, equations with the linear part depending on the parameter[2], etc. All the results may be generalized without difficulty for equations with non-even polynomials L and M. Bifurcation points of such equations are determined by both functional and hysteresis terms of the nonlinearity; we consider even polynomials to indicate the situation where the hysteresis term only is important.

2 Prandtl–Ishlinskii nonlinearity

2.1 Stop

The Prandtl–Ishlinskii hysteresis nonlinearity is a weighted sum of the elementary hysteresis operators

$$\xi(t) = U_\rho[t_0, \xi_0]x(t), \qquad t_0 \le t \le t_1, \ \rho > 0, \tag{4}$$

called *stops*. Here the arguments are any number $\xi_0 \in [-\rho, \rho]$ and any continuous function $x = x(t)$ defined on an arbitrary segment $[t_0, t_1]$; the number is called an *initial state* or *initial value*; the function is called an *input* of the stop. The values $\xi = \xi(t)$ of operator (4) are called *outputs*. The output is also a continuous function defined on $[t_0, t_1]$, it satisfies $|\xi(t)| \le \rho$ for all t and its initial value is ξ_0, i.e., $\xi(t_0) = \xi_0$. For monotone inputs,

$$U_\rho[t_0, \xi_0]x(t) = \begin{cases} \min\{\rho, \ \xi_0 + x(t) - x(t_0)\} & \text{if } x(t) \text{ increases,} \\ \max\{-\rho, \xi_0 + x(t) - x(t_0)\} & \text{if } x(t) \text{ decreases.} \end{cases} \tag{5}$$

For each piecewise monotone continuous input, the output is defined by the semigroup identity

$$U_\rho[\tau, U_\rho[t_0, \xi_0]x(\tau)]x(t) = U_\rho[t_0, \xi_0]x(t), \qquad t \ge \tau \ge t_0.$$

To define the outputs for all continuous inputs, the operator $U_\rho[t_0, \xi_0] : x(t) \mapsto \xi(t)$ is extended by continuity in the space $C[t_0, t_1]$ from the dense set of piecewise monotone inputs to the whole space. The correctness of this procedure is proved in [8].

Figure 2 shows the trajectories of the point $\{x(t), \xi(t)\} = \{x(t), U_\rho[t_0, \xi_0]x(t)\}$ on the plane $\{x, \xi\}$. The point is always in the closed band $|\xi| \le \rho$, which is the join of the two boundary horizontal lines $\xi = \pm\rho$ and a continual number of the slanting lines $\xi = x - \theta$ with $x \in (\theta - \rho, \theta + \rho)$ (where $\theta \in \mathbb{R}$ is a parameter). If the initial value $\xi_0 = \xi(t_0)$ is not $\pm\rho$, the point $\{x(t), \xi(t)\}$ goes along the slanting line: upwards right if $x(t)$ increases and downwards left if $x(t)$ decreases. As the point reaches the horizontal line, it switches to it and goes to the right along the line $\xi = \rho$ or to the left along the line $\xi = -\rho$. The point switches again to a

[2] In such a way that a pair of conjugate roots of L lies on the imaginary axis for all λ.

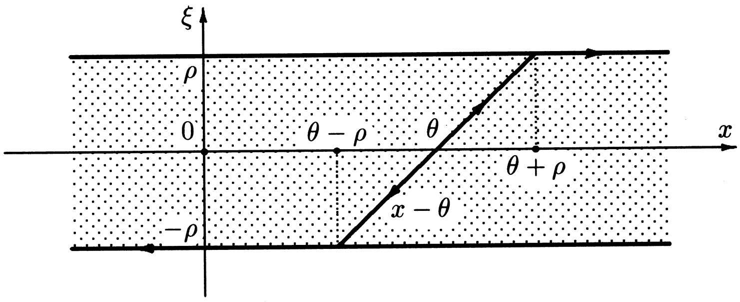

Figure 2 Stop nonlinearity

slanting line as soon as the input $x(t)$ switches from increasing to decreasing or conversely.

For any $\rho > 0$ operator (4) satisfies the Lipschitz condition

$$\|U_\rho[t_0,\xi_1^0]x_1(t) - U_\rho[t_0,\xi_2^0]x_2(t)\|_C \le |\xi_1^0 - \xi_2^0| + 2\|x_1(t) - x_2(t)\|_C \qquad (6)$$

for every $\xi_1^0, \xi_2^0 \in [-\rho,\rho],\ x_1, x_2 \in C[t_0,t_1]$.

Remark 1 If the input $x(t)$ is T-periodic and $\rho \le \Delta := (\max x(t) - \min x(t))/2$, then the output $\xi(t) = U_\rho[0,\xi_0]x(t)$ is also T-periodic for a unique initial value ξ_0. This value is defined by the formula

$$\xi_0 = U_\rho[0,\xi_1]x(t)\Big|_{t=T}$$

and is the same for all $\xi_1 \in [-\rho,\rho]$. In particular, for $\rho = \Delta$ this value is $\xi_0 = x(0) - \delta$, where $\delta := (\max x(t) + \min x(t))/2$. If $\rho > \Delta$, then the output is periodic for each initial value $\xi_0 = x(0) - \delta + \theta$ with $|\theta| \le \rho - \Delta$. In this case, periodic input and output satisfy $\xi(t) - x(t) \equiv \text{const}$.

2.2 Continual set of stops

Consider the set Ξ of all the stops $U_\rho[\cdot]$; the variable $\rho \in (0,\infty)$ is a parameter. Denote by Θ the set of all continuous functions $\xi = \xi(\rho)$, $\rho > 0$, such that $|\xi(\rho)| \le \rho$ for all ρ. Functions $\xi \in \Theta$ are called *states* of the set Ξ.

For any initial state $\xi_0 \in \Theta$ and any continuous input $x(t)$ $(t_0 \le t \le t_1)$ define the variable state of Ξ at every moment $t \in [t_0,t_1]$ by the formula

$$\xi(t;\rho) = U_\rho[t_0,\xi_0(\rho)]x(t), \qquad \rho > 0. \qquad (7)$$

The inclusion $\xi(t;\rho) \in \Theta$ for each t follows from the properties of operator (4). The function $\xi(t;\rho)$ is continuous with respect to the set of its arguments.

The output of Ξ is defined by the formula

$$\eta(t) = \int_0^\infty U_\rho[t_0, \xi_0(\rho)]x(t)\, d\mu. \tag{8}$$

Here the weight function $\mu = \mu(\rho)$ has a bounded variation $V(r;\mu)$ on each interval $[r, \infty)$ with $r > 0$ and

$$\left| \int_0^\infty \rho\, dV(\rho;\mu) \right| \le c < \infty, \tag{9}$$

which implies that $\eta(t)$ ($t_0 \le t \le t_1$) is a continuous function for any $\xi_0 \in \Theta$ and any continuous input $x(t)$ ($t_0 \le t \le t_1$). Both in (8) and (9) we use the improper Riemann–Stieltjes integral; the function $V(\rho;\mu)$ decreases and vanishes as $\rho \to \infty$, it may go to infinity as $\rho \to +0$.

The nonlinearity Ξ with input-state operator (7) and input-output operator (8) is called the Prandtl–Ishlinskii hysteresis nonlinearity; for more details and further properties, see [8].

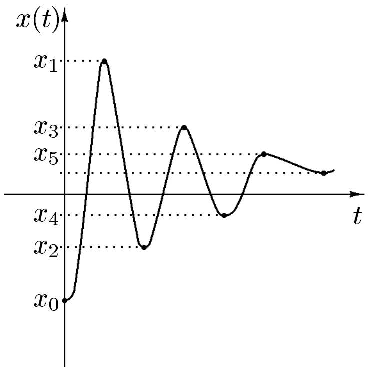

Figure 3a Input $x(t)$

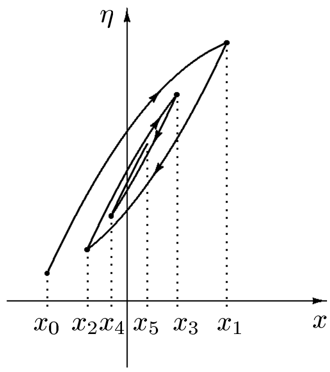

Figure 3b Input–output relation

In Figure 3b we present the input–output relations for inputs $x(t)$ of the type given in Figure 3a and the initial state $\xi_0(\rho) = -\rho$. The points x_j in both parts of Figure 3 are the same.

Consider the space of all continuous functions $\xi(\rho) : [0, \infty) \to \mathbb{R}$ such that $\|\xi\|_\Theta < \infty$, where

$$\|\xi\|_\Theta := \sup_{\rho > 0} \frac{|\xi(\rho)|}{1+\rho}.$$

It is a Banach space with the norm $\|\cdot\|_\Theta$. The state space Θ of the Prandtl–Ishlinskii hysteresis nonlinearity is a closed subset of this Banach space (we continue the states at zero by $\xi(0) = 0$).

Let us stress that generally the weight function μ is nonmonotone. Everywhere below we suppose that the function μ in (8) depends on the parameter $\lambda \in (a,b)$ (i.e., $\mu = \mu(\rho,\lambda)$), estimate (9) is valid for all λ with the same c, and

$$\int_0^\infty \rho\, dV(\rho;\mu(\cdot,\lambda)-\mu(\cdot,\lambda_1)) \to 0 \quad \text{as} \quad \lambda\to\lambda_1, \quad \lambda_1 \in (a,b) \tag{10}$$

uniformly in λ_1. Here $V(r;\mu(\cdot,\lambda)-\mu(\cdot,\lambda_1))$ denotes the variation of the function $\mu(\rho,\lambda)-\mu(\rho,\lambda_1)$ on the interval $\rho\in[r,\infty)$ for each $r>0$. Under these assumptions, (8) is a continuous operator from its domain $x\in C[t_0,t_1]$, $\xi_0\in\Theta$, $\lambda\in(a,b)$ to the space $C[t_0,t_1]$ of the outputs $\eta=\eta_\lambda(t)$.

Suppose $\mu(\rho,\lambda)\not\equiv$ const at infinity and an input $x(t)$ is T-periodic. Then variable state (7) and output (8) are T-periodic for the continuum of initial states. This follows from Remark 1. The difference between two periodic outputs for the same periodic input is always a constant.

3 Main results

Consider the equation

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)(f(x,\lambda)+\eta_\lambda(t)), \tag{11}$$

where $\eta_\lambda(t)$ is the output (8) of the Prandtl–Ishlinskii hysteresis nonlinearity with the weight function $\mu=\mu(\rho,\lambda)$ depending on λ. Here the polynomials $L(p)$, $M(p)$ satisfy conditions (i)–(iii) of Section 1. The function $f(x,\lambda)$ is supposed to be continuous with respect to the set of its arguments $x\in\mathbb{R}$, $\lambda\in(a,b)$ and uniformly bounded: $\sup|f(x,\lambda)|<\infty$. It is assumed that the function $\mu=\mu(\rho,\lambda)$ satisfies assumptions (9) and (10); the variation $V(\cdot;\mu)$ of this function depends on λ, we suppose that (9) holds for all λ with the same c.

A solution $x(t)$ of equation (11) is periodic if both the function $x(t)$ and variable state (7) of the hysteresis nonlinearity are periodic in t with the same period. Note that the initial value $\xi_0=\xi_0(\rho)$ of the periodic state is not known *a priori*.

In the following theorems we use the definition of Section 1.

Theorem 1 *Suppose that $\lambda_0\in(a,b)$ is a zero of the function*

$$\psi(\lambda)=\int_0^\infty \rho\, d\mu(\rho,\lambda) \tag{12}$$

and that $\psi(\lambda)$ takes both positive and negative values in any neighborhood of the point λ_0. Then λ_0 is a bifurcation point for equation (11) with the frequency w_0.

From (10) it follows that $\psi(\lambda)$ is a continuous function. The frequency w_0 is defined by the relation $L(\pm iw_0)=0$.

Theorem 2 *The condition* $\psi(\lambda_0) = 0$ *is necessary for* λ_0 *to be a HBP for equation* (11).

Let $M(0) \neq 0$ and let $\mu(\rho, \lambda) \not\equiv$ const at infinity. Suppose $f(x, \lambda) \equiv 0$. It follows from Remark 1 that any periodic solution $x^*(t)$ of the equation

$$L\left(\frac{d}{dt}\right) x = M\left(\frac{d}{dt}\right) \eta_\lambda(t)$$

generates the one-parameter family of solutions $x^*(t)+\theta$, $\theta \in [\theta_1, \theta_2]$. Both boundaries θ_j vanish as $\|x^*\|_C \to \infty$. The situation is similar if $f(x, \lambda) \not\equiv 0$, i.e., we consider the main equation (11). Generally, every large cycle $x_r(t)$ is included in a one-parametric set of cycles (see Section 5).

If there is more information about the weight function $\mu(\rho, \lambda)$, then we can say more precisely, for which values of λ the large-amplitude solutions of (11) exist.

In the next theorem we suppose that

$$\left| \int_0^\infty \rho^2 \, dV(\rho; \mu) \right| < \infty$$

uniformly in λ, and moreover,

$$\int_r^\infty \rho^2 \, dV(\rho; \mu) \to 0 \quad \text{as} \quad r \to \infty \quad \text{uniformly in} \quad \lambda. \tag{13}$$

Define the function

$$\varphi(\lambda) = \int_0^\infty \rho^2 \, d\mu(\rho, \lambda);$$

the integral in the right-hand side converges due to (13).

Theorem 3 *Suppose* λ_0 *is an isolated zero of the function* $\psi(\lambda)$. *Suppose the function* $\varphi(\lambda)$ *is continuous and* $\varphi(\lambda_0) \neq 0$. *Then all sufficiently large periodic solutions of equation* (11) *with* λ *sufficiently close to* λ_0 *and periods sufficiently close to* $2\pi/w_0$ *exist for the values of* λ *satisfying the inequality* $\psi(\lambda)\varphi(\lambda_0) > 0$.

It is easy to construct a function $\mu(\rho, \lambda)$ such that $\psi(\lambda_0) = 0$ for some λ_0, $\psi(\lambda) > 0$ for all $\lambda \neq \lambda_0$, and at the same time $\varphi(\lambda_0) < 0$. In this case, Theorem 3 implies that all the periodic solutions of equation (11) are uniformly bounded for all λ close to λ_0. Therefore the necessary condition $\psi(\lambda_0) = 0$ is not sufficient for λ_0 to be a bifurcation point.

Example 1 Suppose the density of the weight function $\mu(\rho, \lambda)$ is defined by

$$\frac{\partial \mu(\rho, \lambda)}{\partial \rho} = (\rho - \lambda)e^{-\rho}.$$

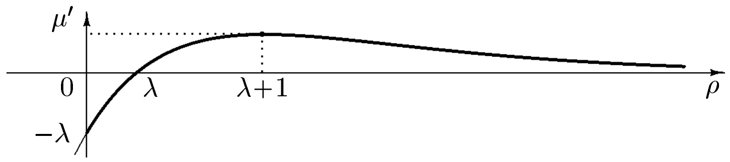

Figure 4 The density of $\mu(\rho, \lambda)$

For this function $\psi(\lambda) = 2 - \lambda$, $\varphi(\lambda) = 6 - 2\lambda$. All the assumptions of Theorems 1 and 3 are fulfilled, $\lambda_0 = 2$, large cycles exist for $\lambda < \lambda_0$.

Example 2 The simplest example of the equation that can be studied by the theorems of this section is the second order equation

$$x'' + x = U_{\rho_1}[t_0, \xi_1^0]x(t) - \lambda U_{\rho_2}[t_0, \xi_2^0]x(t) \tag{14}$$

with two stops with different signs. Theorem 1 implies that the point $\lambda_0 = \rho_1/\rho_2$ is a HBP for equation (14) with the frequency 1. Here $\psi(\lambda) = \rho_1 - \lambda\rho_2$, $\varphi(\lambda) = \rho_1^2 - \lambda\rho_2^2$. Theorem 3 implies that the large cycles exist for $\lambda > \lambda_0$ if $\rho_1 < \rho_2$ and for $\lambda < \lambda_0$ if $\rho_1 > \rho_2$.

4 Proof of Theorem 1

4.1 Change of variables

We look for periodic solutions of equation (11) with periods $2\pi/w$ close to $2\pi/w_0$. Let us change the time scaling by the formula $wt \to t$ and replace (11) by the equation

$$L\left(w\frac{d}{dt}\right)x = M\left(w\frac{d}{dt}\right)(f(x,\lambda) + \eta_\lambda(t)), \tag{15}$$

where

$$\eta_\lambda(t) = \int_0^\infty U_\rho[0, \xi_0(\rho)]x(t)\, d\mu(\rho, \lambda), \qquad t \ge 0. \tag{16}$$

One of the main common features of hysteresis is *rate-independence*, which means that the relation $x_1(t) = x_2(\tau(t))$ between inputs implies relations $\xi_1(t) = \xi_2(\tau(t))$ and $\eta_1(t) = \eta_2(\tau(t))$ between variable states and outputs of a hysteresis nonlinearity if an initial state is the same, i.e., $\xi_1(t_0) = \xi_2(\tau(t_0))$; here $\tau(t)$ is any strictly increasing continuous function. Due to rate-independence of the Prandtl–Ishlinskii operators, relation (16) implies

$$\eta_\lambda(wt) = \int_0^\infty U_\rho[0, \xi_0(\rho)]x(wt)\, d\mu(\rho, \lambda)$$

for every $w > 0$, therefore $x(wt)$ is a $2\pi/w$-periodic solution of (11) iff $x(t)$ is a 2π-periodic solution of system (15), (16). This system should be coupled with the periodicity condition for the state:

$$U_\rho[0, \xi_0(\rho)]x(t)\Big|_{t=2\pi} = \xi_0(\rho), \qquad \rho > 0. \tag{17}$$

We prove the existence of 2π-periodic solutions

$$x(t) = r \sin t + h(t), \qquad r \geq 0, \tag{18}$$

of system (15)–(17), where $h(t)$ is orthogonal to the functions $\sin t$ and $\cos t$ in $L^2 = L^2[0, 2\pi]$ (everywhere 2π-periodic functions are identified with their restrictions to the period). More precisely, we show that for every sufficiently large r there are numbers w and λ close to w_0 and λ_0, an initial state $\xi_0 = \xi_0(\rho)$ of the hysteresis nonlinearity, and a function $h = h(t)$ such that (18) is a solution of (15)–(17). Since r is arbitrarily large, so is the amplitude of (18).

Note that shifts of time in (18) generate the continuum of 2π-periodic solutions $x_\theta(t) = r \sin(t + \theta) + h(t + \theta)$, $\theta \in (0, 2\pi)$, of autonomous system (15)–(17) (the initial state of the hysteresis nonlinearity for the solution $x_\theta(t)$ is $\xi_\theta(\rho) = U_\rho[0, \xi_0(\rho)]x(\theta))$. Fixing the phase $\theta = 0$, we choose a unique solution from the continuum.

Let us remark that the original problem depends on the parameter λ, its unknown solutions are functions $x(t) = r \sin w(t + \theta) + h(w(t + \theta))$ of unknown period $2\pi/w$, the initial state $\xi_0(\cdot)$ of the hysteresis nonlinearity is an additional unknown. That is, originally we have the problem with the parameter λ and four unknowns w, r, $h(\cdot)$, $\xi_0(\cdot)$; each solution of the problem is included in the continuum of shifted solutions. Now, the amplitude r of the first harmonic of the solution is considered as a parameter, the phase $\theta = 0$ is fixed, and the unknowns are λ, w, $h(\cdot)$, and $\xi_0(\cdot)$. This choice of the parameter and the unknowns simplifies the use of topological methods.

4.2 Equivalent system

Denote $\Omega = [w_1, w_2]$, where $w_1 < w_0 < w_2$. We suppose that w_0 is the only root of the polynomial $L(iw)$ on the segment Ω, $M(iw) \neq 0$ for $w \in \Omega$, and

$$L(iwn) \neq 0, \qquad w \in \Omega,\ n = 0, 2, 3, \ldots \tag{19}$$

By assumptions (i) and (iii), this is true for any sufficiently small segment Ω. Define the orthogonal projectors

$$Pu = \frac{1}{\pi} \int_0^{2\pi} \cos(t - s) u(s)\, ds, \qquad Q = I - P$$

onto the plane $\Pi = \{\alpha \sin t + \beta \cos t\}$ and onto its orthogonal complement Π^* in L^2. Substituting (18) in equation (15) and using the projectors P, Q, we obtain

$$r\, L\left(w\frac{d}{dt}\right) \sin t = M\left(w\frac{d}{dt}\right) P(f(x(t),\lambda) + \eta_\lambda(t)),$$

$$L\left(w\frac{d}{dt}\right) h(t) = M\left(w\frac{d}{dt}\right) Q(f(x(t),\lambda) + \eta_\lambda(t)). \tag{20}$$

Since the polynomials $L(p)$, $M(p)$ are even, for all $u \in \Pi$

$$L\left(w\frac{d}{dt}\right) u = L(iw)u, \qquad M\left(w\frac{d}{dt}\right) u = M(iw)u.$$

Therefore the first equation of (20) is equivalent to the system

$$r\pi L(iw) = M(iw) \int_0^{2\pi} (f(x(t),\lambda) + \eta_\lambda(t)) \sin t \, dt,$$

$$0 = \int_0^{2\pi} (f(x(t),\lambda) + \eta_\lambda(t)) \cos t \, dt$$

of two scalar equations. We rewrite the second equation of (20) as

$$h = A(w) Q(f(x(t),\lambda) + \eta_\lambda(t)).$$

Here $A(w)$ is the linear operator that maps any function $u \in \Pi^*$ to a unique 2π-periodic solution $h = A(w)u \in \Pi^*$ of the equation

$$L\left(w\frac{d}{dt}\right) h = M\left(w\frac{d}{dt}\right) u;$$

this operator is well-defined for each $w \in \Omega$ due to (19). The operator $A(w)$ is completely continuous and self-adjoint in $\Pi^* \subset L^2$, its eigenvalues are

$$\sigma_n = \frac{M(iwn)}{L(iwn)}, \qquad n = 0, 2, 3, \ldots \tag{21}$$

The first eigenvalue σ_0 is simple, the corresponding eigenfunctions are constants; each other eigenvalue σ_n has the multiplicity 2, the corresponding eigenfunctions are $\sin nt$ and $\cos nt$. Since $\sigma_n = O(n^{-2})$, each operator $A(w)$ is completely continuous as an operator from $\Pi^* \subset L^2$ to

$$E = \{h \in C^1,\ h \in \Pi^*,\ h' \in \Pi^*\} \subset C^1 = C^1[0, 2\pi].$$

The norms of these operators are uniformly bounded:

$$\|A(w)\|_{\Pi^* \to C^1} \le c < \infty, \qquad w \in \Omega,$$

and moreover, $A(\cdot) : \Omega \times \Pi^* \to C^1$ is completely continuous with respect to w, u. Thus, 2π-periodic problem for (15)–(17) is equivalent to the system

$$\begin{cases} 0 \ = \int_0^{2\pi} (f(x(t), \lambda) + \eta_\lambda(t)) \cos t \, dt, \\ r\pi L(iw) = M(iw) \int_0^{2\pi} (f(x(t), \lambda) + \eta_\lambda(t)) \sin t \, dt, \\ h \ = A(w) Q(f(x(t), \lambda) + \eta_\lambda(t)), \\ \xi_0 \ = U_\rho[0, \xi_0] x(t)|_{t=2\pi} \end{cases} \tag{22}$$

where $\eta_\lambda(t)$ and $x(t)$ are given by (16) and (18).

Denote by Λ the segment $[\lambda_1, \lambda_2]$ such that $\psi(\lambda_1)\psi(\lambda_2) < 0$, where $\psi(\lambda)$ is function (12); such a segment with λ_1, λ_2 arbitrarily close to λ_0 exists by assumptions of Theorem 1. To prove the theorem, it suffices to show that (22) has a solution $z = \{\lambda, w, h, \xi_0\} \in \mathbf{Z} \overset{\text{def}}{=} \Lambda \times \Omega \times E \times \Theta$ for every sufficiently large r. For this purpose, we replace system (22) by the equivalent operator equation $z = B_r(z)$ and construct compact convex sets $G \subset E$ and $K \subset \Theta$ such that the continuous operator B_r maps the product $\mathbf{Z}_0 = \Lambda \times \Omega \times G \times K$ into itself. Then the existence of the solution $z = \{\lambda, w, h, \xi_0\}$ follows from the Schauder fixed point principle.

4.3 Invariant set

Consider the third equation of system (22). It follows from (9) that the outputs of the hysteresis nonlinearity are uniformly bounded. The function $f(x, \lambda)$ is also bounded by assumption. Hence, the uniform estimate

$$|f(x(t), \lambda) + \eta_\lambda(t)| \le c < \infty, \qquad t \in [0, 2\pi], \ \lambda \in (a, b), \tag{23}$$

is valid for all continuous $x(t)$ and all initial states $\xi_0 \in \Theta$. This estimate implies that the image $\mathcal{A}(\mathbf{Z})$ of the set $\mathbf{Z}$ under the completely continuous map

$$\mathcal{A} : \ \{\lambda, w, h, \xi_0\} \mapsto A(w) Q(f(r \sin t + h(t), \lambda) + \eta_\lambda(t))$$

is a compact subset of E. We define $G = \overline{\text{co}}\, \mathcal{A}(\mathbf{Z})$, i.e., G is the closure of the convex hull of $\mathcal{A}(\mathbf{Z})$.

Now consider the last equation of (22). To construct the set $K \subset \Theta$ we use *normal* states of the hysteresis nonlinearity. A state $\xi \in \Theta$ is called normal if

$$|\xi(\rho_1) - \xi(\rho_2)| \le |\rho_1 - \rho_2| \tag{24}$$

for all $\rho_1, \rho_2 > 0$. Let c_1 be so large that $2\|h\|_C \le c_1$ for all $h \in G$ and let $\tilde{\xi}(\rho)$: $[r + c_1, \infty) \to \mathbb{R}$ be any function satisfying relations (24) and

$$|\tilde{\xi}(\rho)| \le \rho - r - c_1, \qquad \rho \ge r + c_1. \tag{25}$$

By $K = K(r, \tilde{\xi})$ we denote the set of all normal states $\xi = \xi(\rho)$ such that $\xi(\rho) = \tilde{\xi}(\rho)$ for $\rho \geq r + c_1$; by definition, K is convex and compact. If $\xi_0 \in K$, $h \in G$, and $x = r \sin t + h$, then

$$\begin{aligned} |\xi_0(\rho) + x(t) - x(0)| &\leq |\xi_0(\rho)| + r + 2\|h\|_C \\ &\leq |\xi_0(\rho)| + r + c_1 \leq \rho, \qquad \rho \geq r + c_1, \end{aligned}$$

and, by the definition of the stop, $U_\rho[0, \xi_0(\rho)]x(t) = \xi_0(\rho) + x(t) - x(0)$ for all $\rho \geq r + c_1$, $0 \leq t \leq 2\pi$, therefore

$$U_\rho[0, \xi_0(\rho)]x(t)\Big|_{t=2\pi} = \xi_0(\rho) = \tilde{\xi}(\rho), \qquad \rho \geq r + c_1.$$

Since for every initial normal state ξ_0 and every continuous input $x(t)$ the variable state (7) is normal for each $t \geq t_0$ (see, e.g., [8]), it follows that

$$U_\rho[0, \xi_0]x(t)\Big|_{t=2\pi} \in K \qquad \text{for all} \qquad \xi_0 \in K,\ h \in G.$$

The first and the second equations of system (22) we rewrite in a proper form. Recall that w_0 is the only root of the even polynomial $L(iw)$ in Ω and this root is simple, i.e., $L(iw) = (w - w_0)L_1(w)$ and the real polynomial $L_1(w)$ is nonzero on the segment Ω. Therefore the second equation is equivalent to

$$w = w_0 + \frac{M(iw)}{r\pi L_1(w)} \int_0^{2\pi} (f(x(t), \lambda) + \eta_\lambda(t)) \sin t \, dt.$$

Estimates (23) and $w_1 < w_0 < w_2$ imply that the values of the right-hand side belong to the segment $\Omega = [w_1, w_2]$ if r is sufficiently large.

The first equation of system (22) is the most complicated. First note that if h satisfies the third equation of this system, then

$$\frac{1}{r} \int_0^{2\pi} \big(f(x(t), \lambda) + \eta_\lambda(t)\big) h'(t) \, dt = 0. \tag{26}$$

This equality follows from the formulas

$$u(t) = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \sin nt + \beta_n \cos nt),$$
$$h(t) = \sigma_0 \alpha_0 + \sum_{n=2}^{\infty} \sigma_n (\alpha_n \sin nt + \beta_n \cos nt)$$

for the Fourier expansions of the functions $u(t) = f(x(t), \lambda) + \eta_\lambda(t)$ and $h(t) = A(w)Qu(t)$. Adding (26) to the first equation of (22), we obtain

$$\frac{1}{r} \int_0^{2\pi} (f(x(t), \lambda) + \eta_\lambda(t))(r \cos t + h'(t)) \, dt = 0. \tag{27}$$

Since

$$\int_0^{2\pi} f(x(t),\lambda)(r\cos t + h'(t))\,dt = \int_0^{2\pi} f(x(t),\lambda)x'(t)\,dt = 0,$$

relation (27) is equivalent to

$$\frac{1}{r}\int_0^{2\pi} \eta_\lambda(t)x'(t)\,dt = 0 \tag{28}$$

and we rewrite it as

$$\lambda = \lambda + \frac{\kappa\varepsilon}{r}\int_0^{2\pi} \eta_\lambda(t)x'(t)\,dt,$$

where $\kappa = \operatorname{sign}\,\psi(\lambda_1) \neq 0$ and $\varepsilon > 0$.

Lemma 1 *Let $x(t)$ and $\eta_\lambda(t)$ be inputs and outputs* (16) *of the Prandtl–Ishlinskii hysteresis nonlinearity, let $x(t) = r\sin t + h(t)$. Then*

$$\lim_{r\to\infty} \frac{1}{r}\int_0^{2\pi} \eta_\lambda(t)x'(t)\,dt = 4\psi(\lambda), \tag{29}$$

where the convergence is uniform with respect to all h from any ball $\|h\|_{C^1} \le c$, all initial states $\xi_0 \in \Theta$ of the hysteresis nonlinearity, and all λ.

Recall that the function $\psi(\lambda)$ is continuous and $\psi(\lambda_1)\psi(\lambda_2) < 0$. By Lemma 1, there are numbers $r_0 > 0$ and $\delta > 0$ such that for any $r \ge r_0$

$$\frac{\kappa}{r}\int_0^{2\pi} \eta_\lambda(t)x'(t)\,dt > 0 \quad \text{for } \lambda \in [\lambda_1, \lambda_1 + \delta],$$

$$\frac{\kappa}{r}\int_0^{2\pi} \eta_\lambda(t)x'(t)\,dt < 0 \quad \text{for } \lambda \in [\lambda_2 - \delta, \lambda_2].$$

Therefore if $\varepsilon > 0$ is small enough the relation

$$\lambda + \frac{\kappa\varepsilon}{r}\int_0^{2\pi} \eta_\lambda(t)x'(t)\,dt \in [\lambda_1, \lambda_2] = \Lambda$$

is valid for each $r \ge r_0$ and all $\lambda \in \Lambda$, $h \in G$, $\xi_0 \in \Theta$. Thus, (22) is equivalent to the system

$$\begin{cases} \lambda &= \lambda + \frac{\kappa\varepsilon}{r}\int_0^{2\pi} \eta_\lambda(t)x'(t)\,dt, \\ w &= w_0 + \frac{M(iw)}{r\pi L_1(w)}\int_0^{2\pi} (f(x(t),\lambda) + \eta_\lambda(t))\sin t\,dt, \\ h &= A(w)Q(f(x(t),\lambda) + \eta_\lambda(t)), \\ \xi_0 &= U_\rho[0,\xi_0]x(2\pi) \end{cases}$$

which has the form $z = B_r(z)$ and as we have seen, the operator B_r maps the convex compact set $\mathbf{Z}_0$ into itself if r is sufficiently large. This completes the proof of Theorem 1. The proof of Lemma 1 is given in Section 7.

5 Another proof of Theorem 1

Here we sketch another proof of Theorem 1 and discuss a structure of the set of large cycles. The main point is that we do not use an equation in the state space Θ of the hysteresis nonlinearity to define an initial state ξ_0 for a periodic solution. Instead, initial states are defined *a priori* for all periodic inputs $x = x(t)$ by the continuous operator $x \mapsto \xi_0 = \xi_0(\rho; x, T, \alpha)$ where α is a scalar parameter.

Let $x(t)$ be a T-periodic input of the Prandtl–Ishlinskii nonlinearity, set

$$\Delta = \frac{1}{2}(\max x(t) - \min x(t)), \qquad \delta = \frac{1}{2}(\max x(t) + \min x(t)).$$

From Remark 1 it follows that the variable state (7) is T-periodic iff the initial state $\xi_0(\rho)$ satisfies

$$\xi_0(\rho) = \begin{cases} U_\rho[0, \xi_1(\rho)]x(t)|_{t=T}, & \rho \le \Delta, \\ x(0) - \delta + \theta(\rho), & \rho > \Delta. \end{cases} \tag{30}$$

Here $\xi_1(\rho) \in \Theta$ is arbitrary[3], the values $\xi_0(\rho)$ for $\rho \le \Delta$ do not depend on $\xi_1(\rho)$; the function $\theta(\rho)$ is continuous and satisfies

$$|\theta(\rho)| \le \rho - \Delta, \qquad \rho \ge \Delta. \tag{31}$$

For every such $\theta(\rho)$ the function (30) is continuous and $\xi_0(\rho) \in \Theta$.

If initial states $\xi_0(\rho)$ and $\xi_0^*(\rho)$ are defined for the same input $x(t)$ by (30) with different $\theta(\rho)$ and $\theta^*(\rho)$, then the variable states $\xi(t;\rho) = U_\rho[0, \xi_0(\rho)]x(t)$ and $\xi^*(t;\rho) = U_\rho[0, \xi_0^*(\rho)]x(t)$ coincide for $\rho \le \Delta$; for $\rho > \Delta$ the difference $\xi(t;\rho) - \xi^*(t;\rho)$ equals $\theta(\rho) - \theta^*(\rho)$ and does not depend on t. Therefore

$$\int_0^\infty U_\rho[0, \xi_0(\rho)]x(t)\, d\mu(\rho, \lambda) = \int_0^\infty U_\rho[0, \xi_0^*(\rho)]x(t)\, d\mu(\rho, \lambda), \qquad t \ge 0,$$

i.e., the outputs coincide iff

$$\int_\Delta^\infty \theta(\rho)\, d\mu(\rho, \lambda) = \int_\Delta^\infty \theta^*(\rho)\, d\mu(\rho, \lambda). \tag{32}$$

For simplicity, suppose that the weight function $\mu(\rho, \lambda)$ is strictly monotone with respect to ρ for each λ. Then due to (31), for any $\theta(\rho)$ there exists a unique[4]

[3] Below we use $\xi_1(\rho) = 0$.

[4] Existence and uniqueness of α follow from monotonicity of $\mu(\rho, \lambda)$ with respect to ρ. Since we are interested in large-amplitude solutions only, it suffices to assume that $\mu(\rho, \lambda)$ is monotone for $\rho \ge \rho_0$.

$\alpha \in [-1, 1]$ such that (32) is valid for $\theta^*(\rho) = \alpha(\rho - \Delta)$. If $x(t)$ is a T-periodic solution of equation (11) and $\xi_0(\rho)$ is the corresponding initial state of the hysteresis nonlinearity, then this $x(t)$ is also a periodic solution of (11) with the initial state

$$\xi_0(\rho; x, T, \alpha) = \begin{cases} U_\rho[0, 0]x(t)|_{t=T}, & \rho \le \Delta, \\ x(0) - \delta + \alpha(\rho - \Delta), & \rho > \Delta \end{cases} \tag{33}$$

for some unique $\alpha \in [-1, 1]$. Now we see that the system

$$\begin{cases} 0 = \int_0^{2\pi} (f(x(t), \lambda) + \eta_\lambda(t)) \cos t \, dt, \\ r\pi L(iw) = M(iw) \int_0^{2\pi} (f(x(t), \lambda) + \eta_\lambda(t)) \sin t \, dt, \\ h = A(w)Q(f(x(t), \lambda) + \eta_\lambda(t)) \end{cases} \tag{34}$$

with

$$\eta_\lambda(t) = \int_0^\infty U_\rho[0, \xi_0(\rho; x, 2\pi, \alpha)]x(t) \, d\mu(\rho, \lambda)$$

is equivalent to equation (11) for any r and any $\alpha \in [-1, 1]$. System (34) is obtained from (22), it can be studied in the same way. Under the assumptions of Theorem 1, system (34) has a solution λ, w, $x(t) = r \sin t + h(t)$ for any sufficiently large r and any $\alpha \in [-1, 1]$. Generically, the solutions are different for different α.

If $\mu(\rho, \lambda)$ is not monotone for large ρ the situation is more complicated. Generically, the main conclusion (that there exists a one-parametric set of large cycles for every r) is preserved.

Let us stress that in the previous section we have proved that periodic solutions of (11) exist for every sufficiently large r and every function $\tilde{\xi}(\rho)$ satisfying (24), (25). In fact, the solutions with different $\tilde{\xi}(\rho)$ coincide if the initial states satisfy (32) for $\rho \ge \Delta$.

The situation is different if the weight function μ satisfies $\mu \equiv \text{const}$ for $\rho \ge \rho_0$ (this is the case if the hysteresis nonlinearity can be considered as the set of stops $U_\rho[\cdot]$ with ρ ranging over a finite interval $[0, \rho_0]$). Then large cycles of equation (11) depend on one parameter r and the initial state of the hysteresis nonlinearity is determined uniquely. Namely, for each periodic solution $x(t) = r \sin t + h(t)$ with sufficiently large r this state is $\xi_0(\rho) = \rho$, $\rho \in [0, \rho_0]$.

6 Proof of Theorems 2 and 3

The proof is very simple. As we have seen in Section 4, every sufficiently large periodic solution of equation (11) has the form $x(w(t + \theta))$, where $x(t) = r \sin t + h(t)$ is a 2π-periodic solution (18) of system (15)–(17), the component $h \in E$ satisfies the uniform estimate $\|h\|_{C^1} \le c$, and relation (28) holds. Therefore the conclusion of Theorem 2 follows from Lemma 1. The conclusion of Theorem 3 follows from the more accurate lemma.

Lemma 2 *Let $x(t)$ and $\eta_\lambda(t)$ be an input and an output* (16) *of the Prandtl–Ishlinskii hysteresis nonlinearity, let $x(t) = r\sin t + h(t)$ be a 2π-periodic solution of system* (15)–(17)*. Suppose* (13) *holds. Then*

$$\left|\frac{1}{r}\int_0^{2\pi}\eta_\lambda(t)x'(t)\,dt - 4\psi(\lambda) + 4\frac{\varphi(\lambda)}{r}\right| \le C\frac{|\psi(\lambda)|}{r} + o(r^{-1}), \quad r\to\infty. \tag{35}$$

Due to (28), the integral in (35) is zero for 2π-periodic solutions of system (15)–(17), therefore

$$4\left|\psi(\lambda) - \frac{\varphi(\lambda)}{r}\right| \le C\frac{|\psi(\lambda)|}{r} + o(r^{-1}).$$

Since $\varphi(\lambda_0) \ne 0$, this implies $\psi(\lambda)\varphi(\lambda_0) > 0$ for sufficiently large r and λ close to λ_0.

7 Proof of Lemmas 1 and 2

Let $\rho \le r/3$, $\|h(t)\|_{C^1} \le c$. Consider stop (4) with the input $x(t) = r\sin t + h(t)$ and the output $\xi(t)$, $0 \le t \le 2\pi$. Set

$$t_0 = 0, \quad t_1 = \frac{\pi}{2} - \arcsin\frac{c}{r}, \quad t_2 = \frac{\pi}{2} + \arcsin\frac{c}{r},$$
$$t_3 = \pi + t_1, \quad t_4 = \pi + t_2, \quad t_5 = 2\pi.$$

Since $x'(t) = r\cos t + h'(t)$ and $\|h'(t)\|_C \le c$, the function $x(t)$ strictly increases on each of the segments $[t_0, t_1]$ and $[t_4, t_5]$ and strictly decreases on the segment $[t_2, t_3]$. The relations

$$\begin{aligned} x(t_i) &= h(t_i), && i = 0, 5;\\ x(t_i) &= h(t_i) + \sqrt{r^2 - c^2}, && i = 1, 2;\\ x(t_i) &= h(t_i) - \sqrt{r^2 - c^2}, && i = 3, 4, \end{aligned}$$

and $\|h(t)\|_C \le c$, $\|\xi(t)\|_C \le \rho \le r/3$ imply that for every sufficiently large r the equation $x(t) = x(t_0) - \xi(t_0) + \rho$ has a unique solution $t = \tau_1$ on the segment $[t_0, t_1]$; the equation $x(t) = x(t_2) - \xi(t_2) - \rho$ has a unique solution $t = \tau_3$ on the segment $[t_2, t_3]$; the equation $x(t) = x(t_4) - \xi(t_4) + \rho$ has a unique solution $t = \tau_5$ on the segment $[t_4, t_5]$. From (5) it follows that

$$\begin{aligned} \xi(t) &\equiv \rho && \text{for} \quad t \in [\tau_1, t_1] \cup [\tau_5, t_5],\\ \xi(t) &\equiv -\rho && \text{for} \quad t \in [\tau_3, t_3], \end{aligned} \tag{36}$$

and

$$\xi(t) = x(t) - x(t_i) + \xi(t_i) \quad \text{for} \quad t \in [t_i, \tau_{i+1}], \qquad i = 0, 2, 4. \tag{37}$$

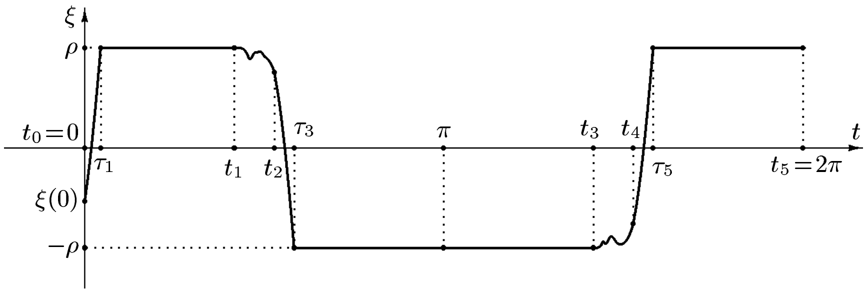

Figure 5 The output of the stop U_ρ for $x(t) = r\sin t + h(t)$

The output $\xi(t)$ is shown in Figure 5.

Set

$$I(\theta_1,\theta_2) = \frac{1}{\rho}\int_{\theta_1}^{\theta_2} x'(t)\xi(t)\,dt, \qquad 0 \le \theta_1 \le \theta_2 \le 2\pi.$$

Relations (36) imply

$$I(\tau_1,t_1) = x(t_1) - x(\tau_1), \quad I(\tau_3,t_3) = x(\tau_3) - x(t_3), \quad I(\tau_5,t_5) = x(t_5) - x(\tau_5).$$

By definition of the points τ_i,

$$x(\tau_1) = \rho + x(t_0) - \xi(t_0), \quad x(\tau_3) = -\rho + x(t_2) - \xi(t_2),$$
$$x(\tau_5) = \rho + x(t_4) - \xi(t_4).$$

Therefore

$$\begin{aligned} I(\tau_1,t_1) &= x(t_1) - x(t_0) + \xi(t_0) - \rho, \\ I(\tau_3,t_3) &= x(t_2) - x(t_3) - \xi(t_2) - \rho, \\ I(\tau_5,t_5) &= x(t_5) - x(t_4) + \xi(t_4) - \rho. \end{aligned} \tag{38}$$

From (36) and (37), we see that $|\xi(\tau_i)| = \rho$ and $x'(t) = \xi'(t)$ on the segments $[t_i,\tau_{i+1}]$, hence

$$I(t_i,\tau_{i+1}) = \frac{1}{2\rho}(\xi^2(\tau_{i+1}) - \xi^2(t_i)) = \frac{1}{2\rho}(\rho^2 - \xi^2(t_i)), \quad i = 0,2,4,$$

and due to $\|\xi(t)\|_C \le \rho$,

$$|I(t_i,\tau_{i+1})| \le |\rho - \xi(t_i)|, \quad i = 0,2; \qquad |I(t_4,\tau_5)| \le |-\rho - \xi(t_4)|. \tag{39}$$

For $t \in [t_1,t_2] \cup [t_3,t_4]$ the relations $|x'(t)| = |r\cos t + h'(t)| \le r|\cos t| + c \le 2c$ hold, therefore

$$|x(t_i) - x(t_{i+1})| \le 2c(t_{i+1} - t_i) = 4c\arcsin\frac{c}{r}, \quad i = 1,3. \tag{40}$$

The estimate

$$|\xi(\theta_1)-\xi(\theta_2)| \le \int_{\theta_1}^{\theta_2} |x'(t)|\,dt, \qquad \theta_1 \le \theta_2,$$

follows from the definition of the stop for every smooth input, hence

$$\begin{aligned} |\xi(t_1)-\xi(t_2)| &= |\rho-\xi(t_2)| \le 4c\arcsin\frac{c}{r}, \\ |\xi(t_3)-\xi(t_4)| &= |\rho+\xi(t_4)| \le 4c\arcsin\frac{c}{r}. \end{aligned} \tag{41}$$

Using (40), (41), we can rewrite equalities (38) as

$$\begin{aligned} I(\tau_1,t_1) &= x_1-x_0+\xi_0-\rho, \\ I(\tau_3,t_3) &= x_1-x_3-2\rho+2\Delta_1, \\ I(\tau_5,t_5) &= x_5-x_3-2\rho+2\Delta_2, \end{aligned}$$

where $x_i = x(t_i)$, $\xi_0=\xi(t_0)$, and $|\Delta_i| \le 4c\arcsin(c/r)$. Therefore

$$\begin{aligned} &|I(\tau_1,t_1)+I(\tau_3,t_3)+I(\tau_5,t_5)-2(x_1-x_3-2\rho)| \\ &\quad \le |x_5-x_0|+|\xi_0-\rho|+16c\arcsin\frac{c}{r}. \end{aligned}$$

Relations (39) and (41) imply

$$|I(t_0,\tau_1)| \le |\rho-\xi_0|, \quad |I(t_2,\tau_3)| \le 4c\arcsin\frac{c}{r}, \quad |I(t_4,\tau_5)| \le 4c\arcsin\frac{c}{r}.$$

Relations $\|\xi(t)\|_C \le \rho$ and $\max\{|x'(t)| : t\in[t_1,t_2], t\in[t_3,t_4]\} \le 2c$ imply

$$|I(t_1,t_2)+I(t_3,t_4)| \le 2c(t_2-t_1+t_4-t_3) = 8c\arcsin\frac{c}{r}.$$

Combining all these estimates, we obtain

$$|I(t_0,t_5)-2(x_1-x_3-2\rho)| \le |x_5-x_0|+2|\xi_0-\rho|+32c\arcsin\frac{c}{r}.$$

Multiplying by ρ and integrating over the segment $0\le\rho\le r/3$, we get

$$\begin{aligned} &\left|\int_0^{r/3} d\mu(\rho,\lambda)\int_0^{2\pi} x'(t)U_\rho[0,\xi_0(\rho)]x(t)\,dt \right.\\ &\qquad \left. -2(x_1-x_3)\int_0^{r/3}\rho\,d\mu(\rho,\lambda)+4\int_0^{r/3}\rho^2\,d\mu(\rho,\lambda)\right| \\ &\quad \le \left|\int_0^{r/3}\rho\left(|x_5-x_0|+2|\xi_0(\rho)-\rho|+32c\arcsin\frac{c}{r}\right)\,dV(\rho;\mu)\right|. \end{aligned} \tag{42}$$

Since $|x(t_0)| \le c$, $|x(t_5)| \le c$, $|\xi_0(\rho)| \le \rho$ and since the Fubini theorem is applicable to the iterated integral in (42), this estimate implies

$$\left| \int_0^{2\pi} x'(t)\,dt \int_0^{r/3} U_\rho[0,\xi_0(\rho)]x(t)\,d\mu(\rho,\lambda) - 2(x_1 - x_3)\int_0^{r/3} \rho\,d\mu(\rho,\lambda) \right| \\ \le \left| \int_0^{r/3} \rho\,(2c + 8\rho + 16\pi c)\;dV(\rho;\mu) \right|.$$

From (9) the relations

$$\frac{1}{r}\int_0^{r/3} \rho\,dV(\rho;\mu) \to 0, \qquad \frac{1}{r}\int_0^{r/3} \rho^2\,dV(\rho;\mu) \to 0 \qquad \text{as} \qquad r \to \infty$$

follow, hence

$$\frac{1}{r}\left| \int_0^{2\pi} x'(t)\,dt \int_0^{r/3} U_\rho[0,\xi_0(\rho)]x(t)\,d\mu(\rho,\lambda) \right. \\ \left. -2(x_1 - x_3)\int_0^{r/3} \rho\,d\mu(\rho,\lambda) \right| \to 0,$$

where the convergence is uniform with respect to $\xi_0 = \xi_0(\rho)$. Using the relations

$$|x_1 - x_3 - 2r| = |2\sqrt{r^2 - c^2} - 2r + h(t_1) - h(t_3)| \\ \le 2(r - \sqrt{r^2 - c^2}) + 2c \le 2(c^2/r + c), \tag{43}$$

we can rewrite this as

$$\left| \frac{1}{r}\int_0^{2\pi} x'(t)\,dt \int_0^{r/3} U_\rho[0,\xi_0(\rho)]x(t)\,d\mu(\rho,\lambda) - 4\int_0^{r/3} \rho\,d\mu(\rho,\lambda) \right| \to 0. \tag{44}$$

But the estimates $|x'(t)| \le r + c$, $|U_\rho[0,\xi_0(\rho)]x(t)| \le \rho$ imply

$$\left| \frac{1}{r}\int_0^{2\pi} x'(t)\,dt \int_{r/3}^{\infty} U_\rho[0,\xi_0(\rho)]x(t)\,d\mu(\rho,\lambda) \right| \\ \le 2\pi\left(1 + \frac{c}{r}\right)\left| \int_{r/3}^{\infty} \rho\,dV(\rho;\mu) \right|, \tag{45}$$

therefore relation (29) follows from (9) and (44). The uniformity of convergence with respect to λ in (29) follows from that in (10). This proves Lemma 1.

Suppose all the assumptions of Lemma 2 are satisfied. Then $x_0 = x_5$, $\xi_0 = \xi(t_5)$ and due to (36), $\xi_0(\rho) = \rho$ for $\rho \le r/3$. Substituting these equalities in (42) and using (43), we obtain

$$\left| \frac{1}{r} \int_0^{2\pi} x'(t)\,dt \int_0^{r/3} U_\rho[0,\xi_0(\rho)]x(t)\,d\mu(\rho,\lambda) \right.$$
$$\left. -4\int_0^{r/3} \rho\,d\mu(\rho,\lambda) + \frac{4}{r}\int_0^{r/3} \rho^2\,d\mu(\rho,\lambda) \right|$$
$$\leq 4\left(\frac{c^2}{r^2}+\frac{c}{r}\right)\left|\int_0^{r/3} \rho\,d\mu(\rho,\lambda)\right| + \frac{32c}{r}\arcsin\frac{c}{r}\left|\int_0^{r/3} \rho\,dV(\rho;\mu)\right|. \quad (46)$$

Relation (13) implies

$$\int_{r/3}^{\infty} \rho^2\,dV(\rho;\mu) \to 0, \quad r\int_{r/3}^{\infty} \rho\,dV(\rho;\mu) \to 0 \quad \text{as} \quad r\to\infty$$

with the convergence uniform with respect to λ, therefore (35) follows from (45), (46), and Lemma 2 is proved.

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