Cycle stability for Hopf bifurcation, generated by sublinear terms*

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Abstract

In this paper we consider the phenomenon of small stable cycle appearance in autonomous quasilinear systems depending on a parameter and present conditions of such cycle existence for control theory equations with scalar nonlinearities. The principal exception of the considered case from usual results about Hopf bifurcation is the degeneration of the linear part for all values of the parameter (not only at the bifurcation point). Small sublinear nonlinearities play the main role in the formulations below. Proofs of the presented results are based on the monotone operator theory.

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1 Problem statement

Consider the system $x' = \varphi(x,\lambda)$, $x \in \mathbb{R}^n$ with the scalar parameter $\lambda \in [0,1]$. Suppose that zero is an equilibrium of this system: $\varphi(0,\lambda) \equiv 0$. We study the so-called Hopf bifurcation phenomenon: the value λ_0 of the parameter is called[1] a Hopf bifurcation point if there exist arbitrary small (in some appropriate sense) nonzero periodic solutions of the system with λ arbitrary close to λ_0. In other words, small nonzero cycles arise from the equilibrium in the neighborhood of a Hopf bifurcation point.

Usual approach to study the problem is as follows.

The function $\varphi(x,\lambda)$ is supposed to be differentiable at zero, i.e., $\varphi(x,\lambda) \equiv A(\lambda)x + \psi(x,\lambda)$, where the $n \times n$ Jacobian matrix $A(\lambda)$ is continuous in λ and the continuous nonlinearity $\psi(x,\lambda) : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ is sublinear, i.e.,

$$\lim_{|x|\to 0} \psi(x,\lambda)|x|^{-1} = 0$$

uniformly in λ. If the matrix $A(\lambda)$ for $\lambda = \lambda_0$ has no imaginary eigenvalues then the value λ_0 can not be a Hopf bifurcation point.

If the matrix $A(\lambda_0)$ has a pair of conjugate imaginary eigenvalues $\pm w_0 i$ then λ_0 can be a Hopf bifurcation point. In [1] the following general conditions were presented for the value λ_0 to be a Hopf bifurcation point.

Let $\pm w_0 i$ be simple eigenvalues of the matrix $A(\lambda_0)$, let the values $\pm k w_0 i$ be regular for this matrix for any integer $k \neq \pm 1$. Suppose there exist eigenvalues $\sigma(\lambda) \pm w(\lambda)i$ of the matrix $A(\lambda)$ with λ arbitrary close to λ_0 with positive $\sigma(\lambda)$ as well as with negative $\sigma(\lambda)$ (here $\sigma(\lambda), w(\lambda)$ depend continuously on λ in some neighborhood of the point λ_0 and $\sigma(\lambda_0) = 0$, $w(\lambda_0) = w_0$). Then λ_0 is a Hopf bifurcation point for the equation $x' = A(\lambda)x + \psi(x,\lambda)$ with any sublinear $\psi(x,\lambda)$.

This result was proved in [1] with the use of special topological methods and it generalizes and continues the original paper by Hopf (see [2, 3, 4]). The existence result does not contain any additional conditions for the nonlinearity, to prove stability it is necessary to use properties of some principal nonlinear terms in the nonlinearity representation.

The stability analysis of small cycles has any sense only if all different from $\pm w_0 i$ eigenvalues of the matrix $A(\lambda_0)$ are in the left half-plane $\{z : \Re e\, z \leq 0\}$. If at least one eigenvalue has positive real part, then small cycles are unstable in any natural sense. If all eigenvalues except $\pm w_0 i$ have negative real parts, then small cycle stability or instability is defined by linear and nonlinear parts of the system considered.

In this paper we consider equations arising in control theory. These equations can be reduced to quasi-linear equations $x' = Ax + \psi(x,\lambda)$ with the matrix A independent of λ and having a pair of eigenvalues $\pm w_0 i$ on the imaginary axis. For such equations even existence results use essentially the representations of the nonlinearity and its sharp properties.

In the proofs we use methods of monotone operator theory [5, 6].

[1] The exact definition see below.

2 Main result

Consider the equation

$$L\left(\frac{d}{dt}\right) x = M\left(\frac{d}{dt}\right) f(x,\lambda). \qquad (1)$$

Here $L(p)$ and $M(p)$ are coprime polynomials with real coefficients independent of t and λ; $\ell = \deg L(p) > m = \deg M(p)$. The function $f(x,\lambda) : \mathbb{R} \times [0,1] \to \mathbb{R}$ is supposed to be continuous, let $f(0,\lambda) \equiv 0$.

Solutions of equation (1) can be defined as solutions of the system[2]

$$\frac{d\mathbf{z}}{dt} = A\mathbf{z} + \mathbf{q}\, f(x(t),\lambda), \quad x(t) = \mathbf{d}^T\mathbf{z}(t), \qquad \mathbf{z} \in \mathbb{R}^\ell, \qquad (2)$$

where the matrix A and the vectors $\mathbf{q}$ and $\mathbf{d}^T$ are defined by the polynomials $L(p)$ and $M(p)$. The polynomial $L(p)$ is the characteristic polynomial of the matrix A, its roots are the eigenvalues for the matrix. The exact formulae for system (2) construction can be found in almost any manual on control theory, see e.g. [7, 8]. The periodic solutions $x_*(t)$ of (1) and the periodic solutions $\mathbf{z}_*(t)$ of (2) satisfy the equality $x_*(t) = \mathbf{d}^T\mathbf{z}_*(t)$.

The solution $x_*(t) = \mathbf{d}^T\mathbf{z}_*(t)$ of (1) is called ***orbitally stable*** if the solution $\mathbf{z}_*(t)$ of (2) is orbitally stable. In the same way one defines orbital asymptotic stability and orbital instability of the solution $x_*(t)$. The solution $x_*(t)$ is called ***ε-small*** if $0 < \|x_*(t)\| = \max_{t\in\mathbb{R}} |x_*(t)| < \varepsilon$.

Definition 1 ([1]). *The value λ_0 of the parameter is **a Hopf bifurcation point with the frequency w_0** (shortly, **a Hopf bifurcation point**) for equation* (1) *if for any $\varepsilon > 0$ there exists a $\lambda = \lambda_\varepsilon \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ such that equation* (1) *with this λ has at least one ε-small periodic solution $x(t) = x_\lambda(t)$ of a period $T = T_\lambda \in (2\pi/w_0 - \varepsilon, 2\pi/w_0 + \varepsilon)$.*

Everywhere we suppose that the polynomial $L(p)$ has a pair of simple imaginary roots $\pm w_0 i$, in other words

$$L(p) = (p^2 + w_0^2)L_1(p), \qquad w_0 > 0,$$

where $L_1(\pm w_0 i) \neq 0$. We suppose also that the polynomial $L_1(p)$ is Hurwitzian, i.e., all its roots have strictly negative real parts. This means that all different from $\pm i w_0$ eigenvalues of the matrix A lie in the open left half-plane.

The odd and even parts

$$f_{odd}(x,\lambda) = \frac{f(x,\lambda) - f(-x,\lambda)}{2}, \qquad f_{even}(x,\lambda) = \frac{f(x,\lambda) + f(-x,\lambda)}{2}$$

of the nonlinearity $f(x,\lambda)$ play different roles in the results below. Suppose the following hypotheses are valid.

(E1) The odd part can be represented as

$$f_{odd}(x,\lambda) = a(\lambda)x|x|^{\alpha-1} + a_1(\lambda)x|x|^{\gamma-1} + \psi_0(x,\lambda), \qquad (3)$$

[2] We denote numbers by usual letters and vectors by bold ones. We use the notation $\mathbf{d}^T$ to underline that it is a row vector in contrast to the column vectors $\mathbf{z}$, $\mathbf{q}$ etc.

where $1 < \alpha < \gamma$, the function $\psi_0(x, \lambda)$ satisfies

$$\psi_0(x,\lambda)|x|^{-\gamma} \to 0, \qquad |\psi_0(x,\lambda) - \psi_0(y,\lambda)| \le c \max\{|x|^{\gamma-1}, |y|^{\gamma-1}\}\,|x-y|; \tag{4}$$

the functions $a(\lambda)$, $a_1(\lambda)$ are continuous and

$$a(\lambda_0) = 0, \qquad a_1(\lambda_0) \ne 0. \tag{5}$$

(E2) For some $\beta > 1$ the even part satisfies the estimates

$$|f_{even}(x,\lambda)| \le c|x|^{\beta}, \qquad |f_{even}(x,\lambda) - f_{even}(y,\lambda)| \le c \max\{|x|^{\beta-1}, |y|^{\beta-1}\}\,|x-y|. \tag{6}$$

(E3) The relation $\Im m\,[L_1(-iw_0)M(iw_0)] \ne 0$ is valid.

Set $\nu = \min\{\beta, \gamma\}$,

$$\kappa \overset{\text{def}}{=} a_1(\lambda_0)\Im m\,[L_1(-iw_0)M(iw_0)], \tag{7}$$

and

$$c_\alpha = \int_0^{2\pi} |\sin t|^{\alpha+1}\,dt, \qquad c_\gamma = \int_0^{2\pi} |\sin t|^{\gamma+1}\,dt, \qquad c_* = \left|\frac{c_\alpha}{c_\gamma a_1(\lambda_0)}\right|^{\frac{1}{\gamma-\alpha}}. \tag{8}$$

Theorem 1. *Let hypotheses* (E1), (E2), (E3) *be valid and*

$$1 < \alpha < \gamma < 2\beta - 1. \tag{9}$$

Let λ_0 be a limit point[3] *for the set $\Lambda_1 = \{\lambda : a(\lambda)a_1(\lambda_0) < 0\}$. Then λ_0 is a Hopf bifurcation point for equation* (1). *Moreover, there exist a vicinity $\Lambda \ni \lambda_0$ and a number $\varepsilon_0 > 0$ such that the following statements hold.*

(i) *If $\kappa < 0$, then equation* (1) *has at least one orbitally stable ε_0-small periodic solution for any $\lambda \in \Lambda \bigcap \Lambda_1$.*

(ii) *If $\kappa > 0$, then equation* (1) *has at least one orbitally unstable ε_0-small periodic solution for any $\lambda \in \Lambda \bigcap \Lambda_1$.*

(iii) *Equation* (1) *has no ε_0-small periodic solutions of any period $T \in (T_0 - \varepsilon_0, T_0 + \varepsilon_0)$ for $\lambda \in \Lambda \setminus \Lambda_1$, where $T_0 = 2\pi/w_0$.*

(iv) *Let $x_\lambda(t)$ be a periodic solution of equation* (1) *of the period $T_\lambda \in (T_0 - \varepsilon_0, T_0 + \varepsilon_0)$ with the amplitude $r_\lambda = \|x_\lambda\| \in (0, \varepsilon_0)$ for $\lambda \in \Lambda$. Then the following estimates*

$$\left|r_\lambda - c_*|a(\lambda)|^{\frac{1}{\gamma-\alpha}}\right| < \chi(\lambda)|a(\lambda)|^{\frac{1}{\gamma-\alpha}}, \qquad |T_\lambda - 2\pi/w_0| < C_1|a(\lambda)|^{\frac{\nu-1}{\gamma-\alpha}} \tag{10}$$

hold, where $\chi(\lambda) \to 0$ for $\lambda \to \lambda_0$ and there exists a $\varphi = \varphi(x_\lambda) \in [0, 2\pi)$ such that

$$\|x_\lambda(t) - r_\lambda \sin(\frac{2\pi t}{T_\lambda} + \varphi)\| < C_2|a(\lambda)|^{\frac{\nu}{\gamma-\alpha}}. \tag{11}$$

[3] In other words $a(\lambda_0) = 0$ and $\forall\, \varepsilon > 0\; \exists\, \lambda \ne \lambda_0 :\; |\lambda - \lambda_0| < \varepsilon,\; \lambda \in \Lambda_1$.

In conclusion (iv) the values $C_1, C_2 > 0$ are independent of λ and the choice of $x_\lambda(t)$; the number ν satisfies $\nu > 1$.

Note that Theorem 1 covers the case $a_1(\lambda_0) < 0$ and $a(\lambda) = (\lambda - \lambda_0)^2$. The point λ_0 is a Hopf bifurcation point in spite of the fact that $a(\lambda)$ does not intersect the zero level at $\lambda = \lambda_0$.

Condition (E3) means in particular that at least one of the polynomials $L(p)$ and $M(p)$ is not even. This implies inapplicability of Theorem 1 for the study of the equation $x'' + x = f(x, \lambda)$.

If the nonlinearity $f(x, \lambda)$ is smooth enough, then α, β and γ are integer numbers. Inequalities (9) are not valid for $\beta = 2$ and integer α and γ. This does not allow to use Theorem 1 to study equations (1) with smooth $f(x, \lambda)$ having nonzero quadratic principal terms at zero.

As an example of applications of Theorem 1 consider the equations

$$x''' + x'' + x' + x = \lambda x^3 + b(\lambda)x^4 + a_1(\lambda)x^5 + x^6 g(x, \lambda),$$

and

$$x'' + x = \frac{d}{dt}\left(\lambda x^3 + b(\lambda)x^4 + a_1(\lambda)x^5 + x^6 g(x, \lambda)\right).$$

Let $g(x, \lambda)$ be a function continuously differentiable in x and $a_1(\lambda_0) \neq 0$. Each of these equations has small nonzero periodic solutions for $\lambda a_1(0) < 0$. If $\lambda a_1(0) \geq 0$, then small periodic solutions do not exist. Theorem 1 guarantees the existence of small orbitally stable periodic solutions for the first equation if $a_1(0) > 0$, $\lambda < 0$ and for the second one if $a_1(0) < 0$, $\lambda > 0$.

3 Remarks

a. Additional information about small periodic solutions generated by Hopf bifurcation follows from the proof of Theorem 1 given below.

For example, if $\kappa < 0$ and for some $\lambda \in \Lambda$ the number of small cycles is finite, then at least one of them is orbitally asymptotically stable.

b. Theorem 1 can be continued for some less smooth nonlinearities $f(x, \lambda)$.

Theorem 2. *Let hypotheses* (E1), (E2), (E3) *be valid and in* (E2) *instead of* (6) *let*

$$|f_{even}(x, \lambda)| \leq c|x|^\beta, \qquad |f_{even}(x, \lambda) - f_{even}(y, \lambda)| \leq c \max\{|x|^\mu, |y|^\mu\}\, |x - y|, \tag{12}$$

and

$$\gamma > \alpha > 1, \quad \beta > 1, \qquad \gamma + 2\mu - 1 > 0, \quad 4\beta + 2\mu - 1 > 3\gamma. \tag{13}$$

Then conclusions (i) – (iii) *of Theorem 1 hold.*

The function $f(x, \lambda) = a(\lambda)x|x|^{\alpha-1} + a_1(\lambda)x|x|^{\gamma-1} + |x|^\beta \sin|x|^{-\delta}$ for $\delta > 0$ satisfies all the conditions of Theorem 2, here $\mu = \beta - \delta - 1$. The value μ may be negative.

The proof of Theorem 2 uses more sharp estimates of integrals of nonlinearities than in our paper, we do not give it.

c. Methods presented in the paper are applicable for the cases where the polynomials $L(p)$ and $M(p)$ depend on λ and for any λ the polynomial $L(p)$ has the same pair of simple roots $\pm w_0 i$ or the pair of imaginary roots, depending on λ.

4 Proof of Theorem 1

4.1 Scheme of the proof

At the first step of the proof we present a continuous operator $U_\lambda(\mathbf{z})$ in the phase space $\mathbb{R}^\ell$ which satisfies the following properties.

1. This operator has fixed points. Every its fixed point generates a periodic solution of system (2) (and equation (1)).
2. Fixed points of this operator can be localized in some invariant set Ω_λ that will be presented in the evident form. This invariant set is convex and closed, we construct it as an intersection of a cone with a so-called conic interval.
3. The operator is monotone with respect to the semiordering, generated by some cone. This gives us the possibility to study the stability with the use of special technics from [11, 12].
4. For λ close to λ_0 the fixed points of the operator $U_\lambda(\mathbf{z})$ are close to zero as well as the corresponding periodic cycles of system (2).

The existence of periodic cycles in the case $\kappa < 0$ follows from the Brauer fixed point principle, the stability required in (i) follows from the operator U_λ monotonicity.

If $\kappa > 0$ then the set Ω_λ is not invariant for the operator U_λ. To prove conclusion (ii) we use another operator $V_\lambda(\mathbf{z})$. Its fixed points are the fixed points of the operator $U_\lambda(\mathbf{z})$, the operator V_λ is also monotone and it transforms the set Ω_λ into itself. Its stable fixed point from the set Ω_λ is the unstable fixed point of U_λ. This fixed point generates the unstable cycle of system (2).

Statements (iii)–(iv) follow from rather simple *a priori* estimates of small periodic solutions of equation (1).

4.2 Operator U_λ

To construct the main operator U_λ we use the classical approach of parameter functionalization (see [1, 9]).

Consider the phase space $\mathbb{R}^\ell$ for system (2). The spectrum $\sigma(A)$ of the matrix A is the set of all roots of the polynomial $L(p)$; therefore the values $\pm w_0 i$ are the simple eigenvalues of this matrix. Denote the corresponding 2-dimensional invariant subspace by $E \subset \mathbb{R}^\ell$ and the complementary $(\ell - 2)$-dimensional invariant subspace by $E' \subset \mathbb{R}^\ell$. If $\ell = 2$, then E' is a point, this case is the simplest, without loss of generality we suppose in the proof that $\ell > 2$.

Suppose that the basis $\mathbf{g}$, $\mathbf{h}$ in the plane E is chosen such that

$$A\mathbf{g} = w_0\mathbf{h}, \quad A\mathbf{h} = -w_0\mathbf{g}, \qquad \mathbf{d}^T\mathbf{g} = 0, \quad \mathbf{d}^T\mathbf{h} = 1. \tag{14}$$

These equalities give simple formulae for $e^{At}\mathbf{g}$ and $e^{At}\mathbf{h}$ for real t. The relations

$$e^{At}\mathbf{g} = \sum_{n=0}^{\infty}\frac{A^n t^n}{n!}\mathbf{g} = \sum_{n=0}^{\infty}\frac{A^{2n}t^{2n}}{(2n)!}\mathbf{g} + \sum_{n=0}^{\infty}\frac{A^{2n+1}t^{2n+1}}{(2n+1)!}\mathbf{g} = \sum_{n=0}^{\infty}\frac{(-1)^n(tw_0)^{2n}}{(2n)!}\mathbf{g} + \sum_{n=0}^{\infty}\frac{(-1)^n(tw_0)^{2n+1}}{(2n+1)!}\mathbf{h}$$

and analogous ones for $e^{At}\mathbf{h}$ imply

$$e^{At}\mathbf{g} = \cos w_0 t\,\mathbf{g} + \sin w_0 t\,\mathbf{h}, \quad e^{At}\mathbf{h} = -\sin w_0 t\,\mathbf{g} + \cos w_0 t\,\mathbf{h}, \qquad t \in \mathbb{R}. \tag{15}$$

Every vector $\mathbf{z} \in \mathbb{R}^\ell$ can be represented as

$$\mathbf{z} = \xi(\mathbf{z})\mathbf{g} + \eta(\mathbf{z})\mathbf{h} + Q\mathbf{z},$$

where $\xi(\mathbf{z}),\ \eta(\mathbf{z}) \in \mathbb{R}$, $Q\mathbf{z} \in E'$.

Since the polynomial $L_1(p)$ is Hurwitzian, all different from $\pm iw_0$ eigenvalues of the matrix A have negative real parts. Therefore in the $(\ell-2)$-dimensional subspace $E' \subset \mathbb{R}^\ell$ a scalar product $(\cdot,\cdot)$ exists[4] such that the estimate

$$(\mathbf{z}', A\mathbf{z}') \le -k(\mathbf{z}',\mathbf{z}'), \qquad \mathbf{z}' \in E' \tag{16}$$

holds for some[5] $k > 0$. Below we use the inequality

$$(e^{At}\mathbf{z}', e^{At}\mathbf{z}') \le e^{-2kt}(\mathbf{z}',\mathbf{z}'), \qquad \mathbf{z}' \in E', \tag{17}$$

it follows directly from (16).[6] Let us extend this scalar product to the whole $\mathbb{R}^\ell$; put

$$(\mathbf{z}',\mathbf{g}) = (\mathbf{z}',\mathbf{h}) = 0, \quad \mathbf{z}' \in E', \qquad (\mathbf{g},\mathbf{h}) = 0, \quad (\mathbf{h},\mathbf{h}) = (\mathbf{g},\mathbf{g}) = 1.$$

We denote by $|\cdot|$ the usual Euclidean norm in $\mathbb{R}^\ell$ generated by this scalar product. We also denote by $|\cdot|$ the norms of matrices, generated by this norm.

For any

$$\mathbf{z} \in \mathbb{R}^\ell_+ \stackrel{\text{def}}{=} \{\mathbf{z}_0 \in \mathbb{R}^\ell : \xi(\mathbf{z}_0) > 0\}$$

set

$$\tau(\mathbf{z}) = \frac{1}{w_0}\left(2\pi - \arctan\frac{\eta(\mathbf{z})}{\xi(\mathbf{z})}\right). \tag{18}$$

The function $\tau(\mathbf{z})$ maps $\mathbb{R}^\ell_+$ onto the interval $\Big(3\pi/(2w_0), 5\pi/(2w_0)\Big)$, it is continuous and satisfies

$$\tau(r\,\mathbf{z}) = \tau(\mathbf{z}), \qquad r > 0. \tag{19}$$

[4] This scalar product is not unique, we choose one and fix it up to the end of the proof.
[5] One can take any $-k > \sup\{\Re e\, z,\ z \in \sigma(A),\ z \ne \pm w_0 i\}$.
[6] Denote the left-hand side of (17) as $d(t)$. Due to (16) the function $d(t)$ satisfies the differential inequality $\dot d \le -2k\,d$; $d(0) = (\mathbf{z}',\mathbf{z}')$. The inequality implies (17).

Denote by $\mathbf{z}(t; \mathbf{z}_0, \lambda)$ a unique solution of system (2) satisfying the initial condition $\mathbf{z}(0) = \mathbf{z}_0$. The required uniqueness and nonlocal continuability of solutions in a neighborhood of the origin follows from assumptions of Theorem 1.

Denote by U_λ the translation operator along the trajectories of system (2) during the time $\tau(\mathbf{z}_0)$ (this time is different for various initial points):

$$U_\lambda(\mathbf{z}_0) = \mathbf{z}(\tau(\mathbf{z}_0); \mathbf{z}_0, \lambda), \qquad \mathbf{z}_0 \in \mathbb{R}^\ell_+.$$

The idea of this construction arises to [1].

Every fixed point of the operator U_λ defines a cycle of system (2), every fixed point close to zero defines the small periodic solution $\mathbf{z}(t) = \mathbf{z}(t; \mathbf{z}_0, \lambda)$ of (2) of the period $T = \tau(\mathbf{z}_0) \in (3\pi/(2w_0), 5\pi/(2w_0))$ and the small T-periodic solution $x(t) = \mathbf{d}^T \mathbf{z}(t)$ of equation (1).

The orbital stability of the solutions $\mathbf{z}(t)$, $x(t)$ follows from the stability of the fixed point[7] $\mathbf{z}_0$ of the operator U_λ.

4.3 Cones and some necessary definitions

Put

$$K_\varepsilon = \{\mathbf{z} \in \mathbb{R}^\ell : \ \xi(\mathbf{z}) \geq 0, \ |\mathbf{z} - \xi(\mathbf{z})\mathbf{g}| \leq \varepsilon \xi(\mathbf{z})\}, \quad \varepsilon > 0.$$

The set K_ε is a cone in $\mathbb{R}^\ell$, it is convex and closed, it contains the half-line $\theta \mathbf{z}$, $\theta \geq 0$ together with every point $\mathbf{z}$ and it does not contain any strict line.

Every[8] cone K_ε generates semiordering in $\mathbb{R}^\ell$: by definition $\mathbf{z}_2 \overset{\varepsilon}{\geq} \mathbf{z}_1 \Leftrightarrow \mathbf{z}_2 - \mathbf{z}_1 \in K_\varepsilon$. We also use the notation $\mathbf{z}_2 \overset{\varepsilon}{>} \mathbf{z}_1$ if $\mathbf{z}_2 - \mathbf{z}_1 \in \operatorname{int} K_\varepsilon$. If we consider the cone K_{ε_1} with some other ε_1 we use the symbols: "$\overset{\varepsilon_1}{\geq}$", "$\overset{\varepsilon_1}{>}$", "$\overset{\varepsilon_1}{\leq}$" and "$\overset{\varepsilon_1}{<}$", they have natural sense.

This relation of partial order (we call it ***ε-semiordering***) satisfies the usual properties of ordering ([5, 6]). In particular it admits usual linear operations with inequalities, the use of limits in inequalities, any bounded (with respect either to the order or any norm) monotone ($\mathbf{z}_{n+1} \overset{\varepsilon}{\geq} \mathbf{z}_n$) sequence converges etc. The convex closed set

$$\langle \mathbf{u}, \mathbf{v} \rangle_\varepsilon = \{\mathbf{z} \in \mathbb{R}^\ell : \ \mathbf{v} \overset{\varepsilon}{\geq} \mathbf{z} \overset{\varepsilon}{\geq} \mathbf{u}\}$$

is called ***a conic interval***. All cones K_ε are normal:

$$\mathbf{z} \in \langle \mathbf{u}, \mathbf{v} \rangle_\varepsilon \Rightarrow |\mathbf{z} - \mathbf{u}|, |\mathbf{z} - \mathbf{v}| \leq N_\varepsilon |\mathbf{v} - \mathbf{u}|, \qquad \mathbf{v}, \mathbf{u} \in \mathbb{R}^\ell, \ \mathbf{v} \overset{\varepsilon}{\geq} \mathbf{u}. \tag{20}$$

The value N_ε depends on ε only, it is called ***the cone normality constant***.

The operator B is called ***ε-monotone*** if for any $\mathbf{z}_1, \mathbf{z}_2$ from its domain the relation $\mathbf{z}_2 \overset{\varepsilon}{\geq} \mathbf{z}_1$ implies $B\mathbf{z}_2 \overset{\varepsilon}{\geq} B\mathbf{z}_1$. The operator B is called ***strictly ε-monotone*** if it is monotone and moreover for any $\mathbf{z}_1, \mathbf{z}_2$ from its domain the inequalities $\mathbf{z}_2 \overset{\varepsilon}{\geq} \mathbf{z}_1$, $\mathbf{z}_1 \neq \mathbf{z}_2$ imply $B\mathbf{z}_2 \overset{\varepsilon}{>} B\mathbf{z}_1$.

[7] The fixed point $\mathbf{z}_0$ of the operator B is called stable or Lyapunov stable if for any $\delta > 0$ there exists $\delta_1 > 0$ such that $|\mathbf{z} - \mathbf{z}_0| < \delta_1$ implies $|B^n \mathbf{z} - \mathbf{z}_0| < \delta$ for any positive integer n.

[8] We consider the cones K_ε for various values of ε.

4.4 Monotonicity of the operator U_λ on the invariant set

The main goal of this subsection is the description of the invariant for U_λ set Ω_λ such that this operator is strictly ε_1-monotone on this set:

$$\mathbf{z}_2 \overset{\varepsilon_1}{\geq} \mathbf{z}_1 \;\Rightarrow\; U_\lambda(\mathbf{z}_2) \overset{\varepsilon_1}{>} U_\lambda(\mathbf{z}_1), \qquad \mathbf{z}_1, \mathbf{z}_2 \in \Omega_\lambda, \;\; \mathbf{z}_1 \neq \mathbf{z}_2, \tag{21}$$

with some ε_1, its value we describe below.

The set Ω_λ has the form

$$\Omega_\lambda = K_\varepsilon \bigcap \langle r_1\mathbf{g}, r_2\mathbf{g}\rangle_{\varepsilon_1},$$

where $0 < r_1 < r_2$. Here the parameters $\varepsilon, \varepsilon_1, r_1, r_2$ depend on λ and $\varepsilon, r_1, r_2 \to 0$, $\varepsilon_1 \to \infty$ as $\lambda \to \lambda_0$. The set Ω_λ contains its minimal and maximal elements $r_1\mathbf{g}$, $r_2\mathbf{g}$ and does not contain the origin $\mathbf{z} = 0$.

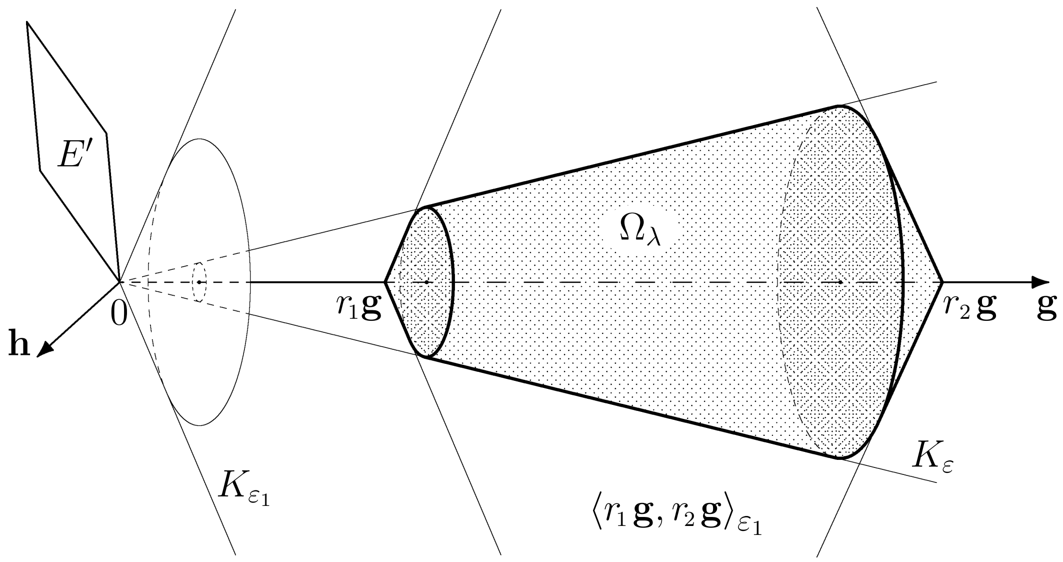

Fig. 1. The set Ω_λ

On Fig. 1 one can see the vector $\mathbf{g}$, surrounded by the cones K_ε and K_{ε_1}, and the conic interval $\langle r_1\mathbf{g}, r_2\mathbf{g}\rangle_{\varepsilon_1}$. The intersection of the conic interval with the cone K_ε is colored in grey, the ellipses (their interior colored in dark grey) inside the set Ω_λ is the intersection of the boundary of the cone K_ε with the boundary of the conic interval. For small $|\lambda - \lambda_0|$ the angle of the cone K_{ε_1} is almost π, the angle of the cone K_ε is small.

The choice of the parameters $\varepsilon, \varepsilon_1, r_1, r_2$ for every λ such that the set Ω_λ is invariant and implication (21) is valid is based on three auxiliary statements.

Set for any $\rho > 0$

$$K_\varepsilon(\rho) = \{\mathbf{z} \in K_\varepsilon : 0 < \xi(\mathbf{z}) \leq \rho\}.$$

Lemma 1. *There exist $d_0, \rho_0 > 0$ such that for any $\rho \in (0, \rho_0)$ and λ satisfying $|a(\lambda)| \leq \rho^{\gamma-\alpha}$ the operator U_λ maps the set $K_\varepsilon(\rho)$ into the interior* $\operatorname{int} K_\varepsilon$ *of the cone K_ε where $\varepsilon = d_0\rho^{\nu-1}$.*

Fig. 2 demonstrates the formulation of Lemma 1. Consider the "horn" $H = H(d_0, \nu) = \{\mathbf{z} : \ |\mathbf{z} - \xi(\mathbf{z})\mathbf{g}| \le d_0[\xi(\mathbf{z})]^\nu\}$ in the phase space. For any $\rho < \rho_0$ consider the disc $H \cap \{\xi(\mathbf{z}) = \rho\}$. Consider the cone K_ε defined by this disc. The operator U_λ maps the set $K_\varepsilon(\rho)$ into the interior of the cone K_ε. The disc is dark grey and the set $K_\varepsilon(\rho)$ is grey on Fig. 2.

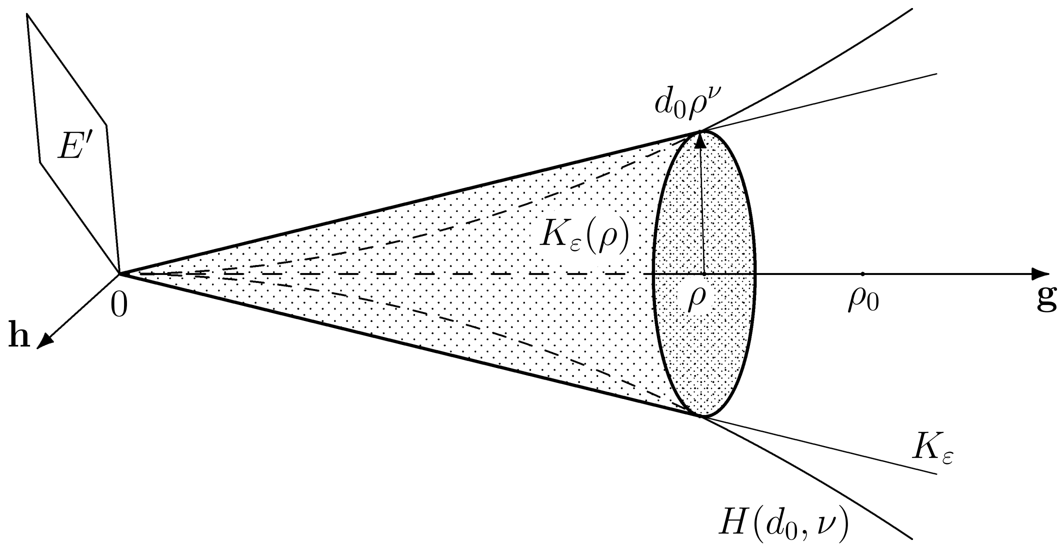

Fig. 2. Lemma 1

Lemma 2. *There exist d_1 and $\rho_1 \in (0, \rho_0)$ such that for any $\rho \in (0, \rho_1)$ and λ satisfying $|a(\lambda)| \le \rho^{\gamma-\alpha}$ the operator U_λ on the set $K_\varepsilon(\rho)$ is strictly ε_1-monotone where*[9] $\varepsilon = d_0\rho^{\nu-1}$, $\varepsilon_1 = d_1\rho^{1-\nu}$.

Define the numbers n_1, n_2 by

$$n_1^{\gamma-\alpha} = c_\alpha(2c_\gamma)^{-1}|a_1(\lambda_0)|^{-1}, \qquad n_2^{\gamma-\alpha} = \max\left\{2c_\alpha c_\gamma^{-1}|a_1(\lambda_0)|^{-1},\, 2\right\}, \tag{22}$$

where c_α, c_γ are the constants from (8). Put $r_j(\lambda) = n_j|a(\lambda)|^{\frac{1}{\gamma-\alpha}}$, $j = 1, 2$.

Lemma 3. *There exists a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $a(\lambda) \ne 0$ the inequalities*

$$\sigma_\lambda\sigma_*\Big(U_\lambda(r_1(\lambda)\mathbf{g}) - r_1(\lambda)\mathbf{g}\Big) \overset{\varepsilon_1}{>} 0, \quad \sigma_*\Big(U_\lambda(r_2(\lambda)\mathbf{g}) - r_2(\lambda)\mathbf{g}\Big) \overset{\varepsilon_1}{>} 0 \tag{23}$$

hold where $\varepsilon_1 = d_1\,[r_2(\lambda)]^{1-\nu}$ and $\sigma_\lambda = \mathrm{sign}[a(\lambda)a_1(\lambda_0)]$, $\sigma_* = \mathrm{sign}\,\kappa$.

According to Lemmas 1 – 3 there exist values $d_0, d_1 > 0$ and $\delta > 0$ such that for

$$\varepsilon = d_0 r_2^{\nu-1}(\lambda), \quad \varepsilon_1 = d_1 r_2^{1-\nu}(\lambda), \qquad a(\lambda) \ne 0, \quad |\lambda - \lambda_0| < \delta \tag{24}$$

[9]The values d_0, ρ_0 come from Lemma 1 as well as the formula for ε and the estimate for $|a(\lambda)|$.

the operator U_λ maps the set $K_\varepsilon(r_2(\lambda))$ into the interior of the cone K_ε. This operator is strictly ε_1-monotone on the set $K_\varepsilon(r_2(\lambda))$ and estimates (23) hold.

For any given λ satisfying $a(\lambda) \neq 0$, $|\lambda - \lambda_0| < \delta$ define the values ε, ε_1 by the other part of (24) and put

$$\Omega_\lambda = \{\mathbf{z} \in K_\varepsilon : r_1(\lambda)\mathbf{g} \overset{\varepsilon_1}{\leq} \mathbf{z} \overset{\varepsilon_1}{\leq} r_2(\lambda)\mathbf{g}\}.$$

The set Ω_λ (see Fig. 1) is a part of a conic interval, it is convex and closed, it contains the segment $\theta\mathbf{g}$, $r_1(\lambda) \leq \theta \leq r_2(\lambda)$ and does not contain the origin. The vectors $r_1(\lambda)\mathbf{g}$, $r_2(\lambda)\mathbf{g}$ are the minimal and the maximal elements of the set Ω_λ with respect to the ε_1-semiordering.

Since $\mathbf{z} \overset{\varepsilon_1}{\leq} r_2(\lambda)\mathbf{g}$ implies the estimate $\xi(\mathbf{z}) \leq r_2(\lambda)$ one has $\Omega_\lambda \subset K_\varepsilon(r_2(\lambda))$. Therefore the operator U_λ is strictly ε_1-monotone on Ω_λ and the relations

$$U_\lambda(\mathbf{z}) \in \operatorname{int} K_\varepsilon, \qquad U_\lambda(r_1(\lambda)\mathbf{g}) \overset{\varepsilon_1}{\leq} U_\lambda(\mathbf{z}) \overset{\varepsilon_1}{\leq} U_\lambda(r_2(\lambda)\mathbf{g})$$

hold for any $\mathbf{z} \in \Omega_\lambda$. If $\sigma_\lambda < 0$, $\sigma_* < 0$, then the estimates

$$r_1(\lambda)\mathbf{g} \overset{\varepsilon_1}{<} U_\lambda(r_1(\lambda)\mathbf{g}), \qquad U_\lambda(r_2(\lambda)\mathbf{g}) \overset{\varepsilon_1}{<} r_2(\lambda)\mathbf{g} \tag{25}$$

follow from (23) and consequently for any $\mathbf{z} \in \Omega_\lambda$

$$U_\lambda(\mathbf{z}) \in \operatorname{int} K_\varepsilon \cap \{\mathbf{z}_0 : r_1(\lambda)\mathbf{g} \overset{\varepsilon_1}{<} \mathbf{z}_0 \overset{\varepsilon_1}{<} r_2(\lambda)\mathbf{g}\} = \operatorname{int} \Omega_\lambda. \tag{26}$$

This means that for $a(\lambda)a_1(\lambda_0) < 0$, $\kappa < 0$ the set Ω_λ is invariant for the operator U_λ. Due to the Brauer principle the operator U_λ has at least one fixed point $\mathbf{z}_* \in \Omega_\lambda$. Relation (26) implies also that every fixed point $\mathbf{z}_* \in \Omega_\lambda$ is the interior point of the set Ω_λ.

The part of conclusion (i) about existence is proved. In the next subsection we prove the rest of conclusion (i): the existence of orbitally stable cycle.

4.5 Stable fixed points

Let $\kappa < 0$, $a(\lambda)a_1(\lambda_0) < 0$. To prove conclusion (i) of Theorem 1 it is sufficient to prove the existence of a stable fixed point of the operator U_λ in Ω_λ. For this we construct two sequences $\mathbf{u}_n, \mathbf{v}_n \in \Omega_\lambda$ such that

$$\lim_{n\to\infty} \mathbf{u}_n = \lim_{n\to\infty} \mathbf{v}_n = \mathbf{z}_*$$

and

$$\mathbf{u}_n \overset{\varepsilon_1}{\leq} U_\lambda(\mathbf{u}_n) \overset{\varepsilon_1}{<} \mathbf{z}_* \overset{\varepsilon_1}{<} U_\lambda(\mathbf{v}_n) \overset{\varepsilon_1}{\leq} \mathbf{v}_n. \tag{27}$$

This implies $\mathbf{z}_* = U_\lambda(\mathbf{z}_*)$. Since $\mathbf{z}_*$ is the interior point of the set Ω_λ one has $\langle \mathbf{u}_n, \mathbf{v}_n\rangle_{\varepsilon_1} \subset \Omega_\lambda$ for sufficiently large values of $n \geq n_0$. Relations (27) imply the invariance of the conic intervals $\langle \mathbf{u}_n, \mathbf{v}_n\rangle_{\varepsilon_1}$, $n \geq n_0$ for the ε_1-monotone operator U_λ and consequently

$$U_\lambda^k(\mathbf{z}) \in \langle \mathbf{u}_n, \mathbf{v}_n\rangle_{\varepsilon_1} \qquad \text{for all} \qquad \mathbf{z} \in \langle \mathbf{u}_n, \mathbf{v}_n\rangle_{\varepsilon_1}, \quad k = 1, 2, \ldots \tag{28}$$

But according to (27) one has $\mathbf{u}_n \overset{\varepsilon_1}{<} \mathbf{z}_* \overset{\varepsilon_1}{<} \mathbf{v}_n$, i.e., $\mathbf{z}_* \in \mathrm{int}\langle \mathbf{u}_n, \mathbf{v}_n\rangle_{\varepsilon_1}$ and therefore (28) implies the stability of the fixed point $\mathbf{z}_*$ of the operator U_λ.

For any $\mathbf{y} \in \Omega_\lambda$ satisfying $\mathbf{y} \overset{\varepsilon_1}{<} U_\lambda(\mathbf{y})$ put

$$\varphi(\mathbf{y}) = \sup\{|\mathbf{z} - U_\lambda(\mathbf{z})| : \ \mathbf{z} \in \Omega_\lambda, \ \mathbf{y} \overset{\varepsilon_1}{\leq} \mathbf{z} \overset{\varepsilon_1}{\leq} U_\lambda(\mathbf{z})\}.$$

From the estimates $\varphi(\mathbf{y}) \geq |\mathbf{y} - U_\lambda(\mathbf{y})| > 0$ it follows that the set

$$G(\mathbf{y}) = \{\mathbf{z} \in \Omega_\lambda : \ \mathbf{y} \overset{\varepsilon_1}{\leq} \mathbf{z} \overset{\varepsilon_1}{\leq} U_\lambda(\mathbf{z}), \ 2|\mathbf{z} - U_\lambda(\mathbf{z})| > \varphi(\mathbf{y})\}$$

is nonempty and well-defined. Let $\mathbf{z} \in G(\mathbf{y})$ and therefore $\mathbf{z} \in \Omega_\lambda$, $\mathbf{z} \neq U_\lambda(\mathbf{z})$, $\mathbf{z} \overset{\varepsilon_1}{\leq} U_\lambda(\mathbf{z})$. Then $U_\lambda(\mathbf{z}) \in \Omega_\lambda$ and since the operator U_λ is strictly ε_1-monotone on the set Ω_λ one has $U_\lambda(\mathbf{z}) \overset{\varepsilon_1}{<} U_\lambda(U_\lambda(\mathbf{z}))$. Therefore for every $\mathbf{z} \in G(\mathbf{y})$ the set $G(U_\lambda(\mathbf{z}))$ is nonempty.

The first estimate (25) implies $G(r_1(\lambda)\mathbf{g}) \neq \emptyset$. Therefore there exists a sequence $\mathbf{u}_n$ such that $\mathbf{u}_0 \in G(r_1(\lambda)\mathbf{g})$, $\mathbf{u}_n \in G(U_\lambda(\mathbf{u}_{n-1}))$, $n = 1, 2, \ldots$ By construction,

$$\mathbf{u}_0 \overset{\varepsilon_1}{\leq} U_\lambda(\mathbf{u}_0) \overset{\varepsilon_1}{\leq} \mathbf{u}_1 \overset{\varepsilon_1}{\leq} U_\lambda(\mathbf{u}_1) \overset{\varepsilon_1}{\leq} \cdots \overset{\varepsilon_1}{\leq} \mathbf{u}_n \overset{\varepsilon_1}{\leq} U_\lambda(\mathbf{u}_n) \overset{\varepsilon_1}{\leq} \cdots$$

But the relation $\mathbf{u}_n \in \Omega_\lambda$ implies the estimate $\mathbf{u}_n \overset{\varepsilon_1}{\leq} r_2(\lambda)\mathbf{g}$, $n = 0, 1, 2\ldots$ Therefore the sequences $\mathbf{u}_n$ and $U_\lambda(\mathbf{u}_n)$ converge to a common limit $\mathbf{z}_*$ and hence

$$\mathbf{z}_* = U_\lambda(\mathbf{z}_*), \qquad \mathbf{u}_n \overset{\varepsilon_1}{\leq} U_\lambda(\mathbf{u}_n) \overset{\varepsilon_1}{\leq} \mathbf{z}_* \overset{\varepsilon_1}{\leq} r_2(\lambda)\mathbf{g}.$$

This implies $U_\lambda(U_\lambda(\mathbf{u}_n)) \leq \mathbf{z}_* \overset{\varepsilon_1}{\leq} U_\lambda(r_2(\lambda)\mathbf{g})$ and since $U_\lambda(r_2(\lambda)\mathbf{g}) \overset{\varepsilon_1}{<} r_2(\lambda)\mathbf{g}$ and $U_\lambda(\mathbf{u}_n) \overset{\varepsilon_1}{<} U_\lambda(U_\lambda(\mathbf{u}_n))$, one has for all n:

$$\mathbf{u}_n \overset{\varepsilon_1}{\leq} U_\lambda(\mathbf{u}_n) \overset{\varepsilon_1}{<} \mathbf{z}_* \overset{\varepsilon_1}{<} r_2(\lambda)\mathbf{g}.$$

Let us pass to the construction of the sequence $\mathbf{v}_n$.

Denote by P the projector[10] on the convex closed set $\Pi = \{\mathbf{z} \in \Omega_\lambda : \mathbf{z}_* \overset{\varepsilon_1}{\leq} \mathbf{z} \overset{\varepsilon_1}{\leq} r_2(\lambda)\mathbf{g}\}$, where $\mathbf{z}_*$ is the limit of already given sequence $\mathbf{u}_n$. By construction of the point $\mathbf{z}_*$ the set Π is invariant for the operator U_λ and consequently $P\mathbf{z}$, $U_\lambda(P\mathbf{z}) \in \Pi$ for any $\mathbf{z} \in \mathbb{R}^\ell$. Let $0 < r < |\mathbf{z}_* - r_2(\lambda)\mathbf{g}|$, $S_r = \{\mathbf{z} \in \mathbb{R}^\ell : |\mathbf{z} - \mathbf{z}_*| = r\}$. Consider on the sphere S_r the vector fields

$$\Psi(\mathbf{z}, \theta) = \mathbf{z} - (1-\theta)U_\lambda(P\mathbf{z}) - \theta\mathbf{z}_*, \qquad 0 \leq \theta \leq 1.$$

We construct the points $\mathbf{v}_n$ in the following way. First of all, we prove that for any small r there exists the point $\hat{\mathbf{z}}_r \in S_r$ satisfying

$$\hat{\mathbf{z}}_r \overset{\varepsilon_1}{\geq} U_\lambda(\hat{\mathbf{z}}_r), \quad \hat{\mathbf{z}}_r \overset{\varepsilon_1}{\geq} \mathbf{z}_*, \quad \hat{\mathbf{z}}_r \in \Omega_\lambda. \tag{29}$$

[10] The point Px is the nearest to x point of the set Π.

Then, according to the strict ε_1-monotonicity of the operator U_λ on the set Ω_λ the relations $\mathbf{z}_* \neq \hat{\mathbf{z}}_r$ and $\mathbf{z}_* \overset{\varepsilon_1}{\leq} \hat{\mathbf{z}}_r$ imply the strict inequality $U_\lambda(\mathbf{z}_*) \overset{\varepsilon_1}{<} U_\lambda(\hat{\mathbf{z}}_r)$ and therefore the inequalities $\mathbf{z}_* \overset{\varepsilon_1}{<} U_\lambda(\hat{\mathbf{z}}_r) \overset{\varepsilon_1}{\leq} \hat{\mathbf{z}}_r$. Consequently for $\mathbf{v}_n = \hat{\mathbf{z}}_{1/n}$ estimates (27) hold and $\lim \mathbf{v}_n = \lim \mathbf{u}_n = \mathbf{z}_*$.

Thus we have to find the point $\hat{\mathbf{z}}_r \in S_r$ satisfying (29).

We prove this differently for two cases: $\Psi(\mathbf{z},\theta) = 0$ for some $\mathbf{z} \in S_r, \theta \in [0,1]$ and $\Psi(\mathbf{z},\theta) \neq 0$ for any $\mathbf{z} \in S_r$, $\theta \in [0,1]$.

Suppose that $\Psi(\hat{\mathbf{z}}_r, \hat{\theta}) = 0$ for some $\hat{\mathbf{z}}_r \in S_r$, $\hat{\theta} \in [0,1]$, i.e., $\hat{\mathbf{z}}_r = (1-\hat{\theta})U_\lambda(P\hat{\mathbf{z}}_r) + \hat{\theta}\mathbf{z}_*$. Since $U_\lambda(P\hat{\mathbf{z}}_r) \in \Pi$ and $\mathbf{z}_* \in \Pi$, one has $\hat{\mathbf{z}}_r \in \Pi$ and $U_\lambda(\hat{\mathbf{z}}_r) \in \Pi$. Therefore $P\hat{\mathbf{z}}_r = \hat{\mathbf{z}}_r$ and

$$\hat{\mathbf{z}}_r = (1-\hat{\theta})U_\lambda(\hat{\mathbf{z}}_r) + \hat{\theta}\mathbf{z}_* \overset{\varepsilon_1}{\leq} U_\lambda(\hat{\mathbf{z}}_r).$$

The definition of the function $\varphi(\mathbf{y})$ and the inequalities

$$U_\lambda(\mathbf{u}_n) \overset{\varepsilon_1}{<} \mathbf{z}_* \overset{\varepsilon_1}{\leq} \hat{\mathbf{z}}_r \overset{\varepsilon_1}{\leq} U_\lambda(\hat{\mathbf{z}}_r), \qquad n = 0,1,2,\cdots$$

imply the estimate $|\hat{\mathbf{z}}_r - U_\lambda(\hat{\mathbf{z}}_r)| \leq \varphi(U_\lambda(\mathbf{u}_n))$. But $\mathbf{u}_{n+1} \in G(U_\lambda(\mathbf{u}_n))$ hence $\varphi(U_\lambda(\mathbf{u}_n)) < 2|\mathbf{u}_{n+1} - U_\lambda(\mathbf{u}_{n+1})|$ for any n. Since $\mathbf{u}_n \to \mathbf{z}_*$ one has $\varphi(U_\lambda(\mathbf{u}_n)) \to 0$ therefore $\hat{\mathbf{z}}_r = U_\lambda(\hat{\mathbf{z}}_r)$ and (29) holds.

Let now $\Psi(\mathbf{z},\theta) \neq 0$ for all $\mathbf{z} \in S_r$, $\theta \in [0,1]$. This means that the vector fields $\Psi(\mathbf{z},0) = \mathbf{z} - U_\lambda(P\mathbf{z})$ and $\Psi(\mathbf{z},1) = \mathbf{z} - \mathbf{z}_*$ are homotopic on S_r and consequently their rotations $\gamma(\mathbf{z} - U_\lambda(P\mathbf{z}), S_r)$ and $\gamma(\mathbf{z} - \mathbf{z}_*, S_r)$ coincide, hence $\gamma(\mathbf{z} - U_\lambda(P\mathbf{z}), S_r) = 1$. Put

$$\Phi(\mathbf{z},\theta) = \mathbf{z} - (1-\theta)U_\lambda(P\mathbf{z}) - \theta r_2(\lambda)\mathbf{g}, \qquad 0 \leq \theta \leq 1.$$

From $r_2(\lambda)\mathbf{g} \notin B_r(\mathbf{z}_*) = \{\mathbf{z} \in \mathbb{R}^\ell : |\mathbf{z} - \mathbf{z}_*| \leq r\}$ it follows $\gamma(\mathbf{z} - r_2(\lambda)\mathbf{g}, S_r) = 0$. We get $\gamma(\Phi(\mathbf{z},0), S_r) \neq \gamma(\Phi(\mathbf{z},1), S_r)$ therefore there exist $\hat{\mathbf{z}}_r \in S_r$, $\hat{\theta} \in [0,1]$ such that $\Phi(\hat{\mathbf{z}}_r, \hat{\theta}) = 0$ or what is the same $\hat{\mathbf{z}}_r = (1-\hat{\theta})U_\lambda(P\hat{\mathbf{z}}_r) + \hat{\theta} r_2(\lambda)\mathbf{g}$. Consequently, $\hat{\mathbf{z}}_r \in \Pi$ and

$$\hat{\mathbf{z}}_r = (1-\hat{\theta})U_\lambda(\hat{\mathbf{z}}_r) + \hat{\theta} r_2(\lambda)\mathbf{g} \overset{\varepsilon_1}{\geq} U_\lambda(\hat{\mathbf{z}}_r).$$

Again we find $\hat{\mathbf{z}}_r$ satisfying (29). This completely proves conclusion (i) of Theorem 1.

4.6 Unstable fixed points

Put $\mathcal{P}\mathbf{z} = \xi(\mathbf{z})\mathbf{g}$. Consider the operator

$$V_\lambda(\mathbf{z}) = (2\mathcal{P} - I)(2\mathcal{P}\mathbf{z} - U_\lambda(\mathbf{z})), \qquad \mathbf{z} \in \mathbb{R}^\ell_+.$$

Let us list the properties of the operator V_λ that we use in the proof of conclusion (ii) as a separate lemma.

Lemma 4. *The following statements are valid:*

1. *Fixed points of the operators U_λ and V_λ coincide: $U_\lambda(\mathbf{z}) = \mathbf{z} \Leftrightarrow V_\lambda(\mathbf{z}) = \mathbf{z}$.*

2. *For all* $\mathbf{z} \in R^\ell_+$ *and any* $\varepsilon_1 > 0$ *the inequality* $U_\lambda(\mathbf{z}) \overset{\varepsilon_1}{>} \mathbf{z}$ *is equivalent to* $V_\lambda(\mathbf{z}) \overset{\varepsilon_1}{<} \mathbf{z}$*; the inequality* $U_\lambda(\mathbf{z}) \overset{\varepsilon_1}{<} \mathbf{z}$ *is equivalent to* $V_\lambda(\mathbf{z}) \overset{\varepsilon_1}{>} \mathbf{z}$.

3. *The analogs of Lemmas* $1-3$ *are valid obtained by replacing* U_λ *in the formulations of Lemmas* $1-3$ *with the operator* V_λ.

The first two conclusions of Lemma 4 follow from the identities

$$V_\lambda(\mathbf{z}) - \mathbf{z} = (2\mathcal{P} - I)(\mathbf{z} - U_\lambda(\mathbf{z})), \qquad U_\lambda(\mathbf{z}) - \mathbf{z} = (2\mathcal{P} - I)(\mathbf{z} - V_\lambda(\mathbf{z})), \qquad \mathbf{z} \in \mathbb{R}^\ell_+$$

and the fact that the operator $2\mathcal{P} - I$ maps the set $\operatorname{int} K_{\varepsilon_1}$ into itself. The proof of the last conclusion repeats the proof of Lemmas $1-3$ and we do not give it.

Let us construct in the case $\kappa > 0$, $a(\lambda)a_1(\lambda_0) < 0$ the solution $\mathbf{z}_*$ of the equivalent equations $\mathbf{z} = U_\lambda(\mathbf{z})$ and $\mathbf{z} = V_\lambda(\mathbf{z})$ that is unstable fixed point for the operator U_λ. This completes the proof of conclusion (ii) of Theorem 1.

For $\kappa > 0$ and $a(\lambda)a_1(\lambda_0) < 0$ relations (23) imply the inequalities

$$U_\lambda(r_1(\lambda)\mathbf{g}) \overset{\varepsilon_1}{<} r_1(\lambda)\mathbf{g}, \qquad U_\lambda(r_2(\lambda)\mathbf{g}) \overset{\varepsilon_1}{>} r_2(\lambda)\mathbf{g}.$$

According to conclusion 2 of Lemma 4 the operator V_λ satisfies the opposite inequalities

$$V_\lambda(r_1(\lambda)\mathbf{g}) \overset{\varepsilon_1}{>} r_1(\lambda)\mathbf{g}, \qquad V_\lambda(r_2(\lambda)\mathbf{g}) \overset{\varepsilon_1}{<} r_2(\lambda)\mathbf{g} \tag{30}$$

From the third conclusion of Lemma 4 it follows that the operator V_λ maps the closed domain Ω_λ in its interior. Let us define the sequence $\mathbf{z}_n$ by the equalities $\mathbf{z}_0 = r_1(\lambda)\mathbf{g}$, $\mathbf{z}_n = V_\lambda(\mathbf{z}_{n-1})$, $n = 1, 2, \ldots$ Since the operator V_λ is strictly monotone on Ω_λ relations (30) imply

$$\mathbf{z}_0 \overset{\varepsilon_1}{<} V_\lambda(\mathbf{z}_0) = \mathbf{z}_1 \overset{\varepsilon_1}{<} V_\lambda(\mathbf{z}_1) = \mathbf{z}_2 \overset{\varepsilon_1}{<} \cdots \overset{\varepsilon_1}{<} V_\lambda(\mathbf{z}_{n-1}) = \mathbf{z}_n \overset{\varepsilon_1}{<} V_\lambda(\mathbf{z}_n) = \mathbf{z}_{n+1} \overset{\varepsilon_1}{<} \cdots \overset{\varepsilon_1}{<} r_2(\lambda)\mathbf{g}.$$

Therefore the sequence $\mathbf{z}_n$ converges to some common fixed point $\mathbf{z}_* \in \operatorname{int}\Omega_\lambda$ of the operators V_λ and U_λ.

Now let us show that for every k the sequence

$$\mathbf{y}^k_0 = \mathbf{z}_k, \quad \mathbf{y}^k_n = U_\lambda(\mathbf{y}^k_{n-1}),\ n = 1, 2, \ldots$$

can not completely belong to Ω_λ. This proves the required unstability of the fixed point $\mathbf{z}_*$ of the operator U_λ. Suppose the opposite: $\mathbf{y}^k_n \in \Omega_\lambda$ for some k and all n. We have $\mathbf{z}_k \overset{\varepsilon_1}{<} V_\lambda(\mathbf{z}_k)$ and, what is the same, $\mathbf{y}^k_0 \overset{\varepsilon_1}{<} V_\lambda(\mathbf{y}^k_0)$. Therefore $\mathbf{y}^k_0 \overset{\varepsilon_1}{>} U_\lambda(\mathbf{y}^k_0)$ and since the operator U_λ is strictly monotone on Ω_λ the relations

$$\mathbf{z}_k = \mathbf{y}^k_0 \overset{\varepsilon_1}{>} U_\lambda(\mathbf{y}^k_0) = \mathbf{y}^k_1 \overset{\varepsilon_1}{>} U_\lambda(\mathbf{y}^k_1) = \mathbf{y}^k_2 \overset{\varepsilon_1}{>} \cdots \overset{\varepsilon_1}{>} U(\mathbf{y}^k_{n-1}) = \mathbf{y}^k_n \overset{\varepsilon_1}{>} U(\mathbf{y}^k_n) = \mathbf{y}^k_{n+1} \overset{\varepsilon_1}{>} \cdots$$

are valid. From $\mathbf{y}^k_n \in \Omega_\lambda$ it follows the estimate $\mathbf{y}^k_n \overset{\varepsilon_1}{\geq} r_1(\lambda)\mathbf{g} = \mathbf{z}_0$ for all n. Hence the sequence $\mathbf{y}^k_n$ converges to a fixed point $\mathbf{y}_*$ of the operator U_λ. For this fixed point the relations $\mathbf{z}_k \overset{\varepsilon_1}{>} \mathbf{y}_* \overset{\varepsilon_1}{\geq} \mathbf{z}_0$ hold. But due to monotonicity of the operator V_λ the relation $\mathbf{z}_0 \overset{\varepsilon_1}{\leq} \mathbf{y}_*$ implies

$$\mathbf{z}_0 \overset{\varepsilon_1}{<} \mathbf{z}_1 \overset{\varepsilon_1}{<} \cdots \overset{\varepsilon_1}{<} \mathbf{z}_k \overset{\varepsilon_1}{\leq} \mathbf{y}_*,$$

i.e.,, $\mathbf{z}_k \overset{\varepsilon_1}{\leq} \mathbf{y}_* \overset{\varepsilon_1}{<} \mathbf{z}_k$. This contradiction proves the unstability of the fixed point $\mathbf{z}_*$ and conclusion (ii) of Theorem 1.

4.7 Proof of conclusions (iii) – (iv)

Recall that Q projects $\mathbb{R}^\ell$ onto the eigensubspace E' of the matrix A along the plane E.

Let $x(t)$ be a periodic solution of equation (1) for λ close to λ_0 that has a small amplitude $r > 0$ and a period T close to $T_0 = 2\pi/w_0$. This means that $x(t) = \mathbf{d}^T\mathbf{z}(t)$, where $\mathbf{z}(t)$ is a T-periodic solution of system (2). Put

$$\mathbf{J}(t) = \int_0^t e^{A(t-s)}\mathbf{q}\, f(x(s), \lambda)\, ds. \tag{31}$$

Since $\|x(t)\|_C = r$[11] and $|f(x,\lambda)| \leq c_0\psi_\lambda(|x|)|x|$, where

$$\psi_\lambda(r) = |a(\lambda)|r^{\alpha-1} + r^{\nu-1}, \tag{32}$$

we have[12]

$$\|\mathbf{J}(t)\|_C \leq c_1\psi_\lambda(r)r,$$

and since $\mathbf{z}(0) = \mathbf{z}(T)$, we have $\mathbf{z}(0) - e^{AT}\mathbf{z}(0) = \mathbf{J}(T)$ and

$$(I - e^{AT})Q\mathbf{z}(0) = Q\mathbf{J}(T).$$

The spectrum of the matrix $e^{AT}Q$ does not contain the value 1 therefore the matrix $I - e^{AT}Q$ is invertable, hence

$$|Q\mathbf{z}(0)| \leq c_2|\mathbf{J}(T)| \leq c_1c_2\psi_\lambda(r)r.$$

Set $\mathbf{z}_0 = (I - Q)\mathbf{z}(0) = \mathbf{z}(0) - Q\mathbf{z}(0)$, $\rho = |\mathbf{z}_0|$. The equalities

$$\mathbf{z}(t) = e^{At}\mathbf{z}(0) + \mathbf{J}(t) = e^{At}\mathbf{z}_0 + e^{At}Q\mathbf{z}(0) + \mathbf{J}(t), \qquad x(t) = \mathbf{d}^T\mathbf{z}(t)$$

imply the estimates

$$\|x(t) - \mathbf{d}^Te^{At}\mathbf{z}_0\|_C \leq c_3(|Q\mathbf{z}(0)| + \|\mathbf{J}(t)\|_C) \leq c_4\psi_\lambda(r)r.$$

[11]Here and below we use the notation $\|u(t)\|_C = \max\{|u(t)| : 0 \leq t \leq 5\pi/(2w_0)\}$ both for scalar and vector functions $u(t)$.

[12]We denote by c_i various constants, the exact values or estimates of these constants does not play any role, we use only the existence of such constants.

Since $\mathbf{d}^T\mathbf{g} = 0$, $\mathbf{d}^T\mathbf{h} = 1$ according to the choice of $\mathbf{g}$ and $\mathbf{h}$ in (14), formulas (15) imply

$$\mathbf{d}^T e^{At}\mathbf{z}_0 = \xi(\mathbf{z}_0)\sin w_0 t + \eta(\mathbf{z}_0)\cos w_0 t$$

and therefore

$$\|x(t) - \rho\sin(w_0 t + \varphi)\|_C \le c_4\psi_\lambda(r)r, \tag{33}$$

where $\rho\cos\varphi = \xi(\mathbf{z}_0)$, $\rho\sin\varphi = \eta(\mathbf{z}_0)$. According to (33)[13] one has $|r-\rho| \le c_4\psi_\lambda(r)r$ and

$$\|x(t) - r\sin(w_0 t + \varphi)\|_C \le 2c_4\psi_\lambda(r)r. \tag{34}$$

Let us estimate the period T. Suppose r is small enough such that $|r-\rho| \le r/2$. From the equality $\mathbf{z}(0) - e^{AT}\mathbf{z}(0) = \mathbf{J}(T)$ it follows the equality $\mathbf{z}_0 - e^{AT}\mathbf{z}_0 = (I-Q)\mathbf{J}(T)$ and hence $|\mathbf{z}_0 - e^{AT}\mathbf{z}_0| \le c_1\psi_\lambda(r)r$. But $|\mathbf{z}_0 - e^{AT}\mathbf{z}_0| = 2\rho|\sin(w_0T/2)|$, this means

$$|\sin(w_0T/2)| \le c_1\psi_\lambda(r)r/(2\rho) \le c_1\psi_\lambda(r).$$

As the value $w_0T/2$ is close to π we have $|w_0T/2 - \pi| \le c_5|\sin(w_0T/2)|$ and therefore

$$|T - T_0| = 2w_0^{-1}|w_0T/2 - \pi| \le c_6\psi_\lambda(r). \tag{35}$$

Now for the proof of conclusion (iv) we have to prove the first of relations (10): this relation and estimates (34) – (35) imply the second of relations (10) and relation (11). Simultaneously we prove that the assumption of the small cycle $x(t)$ existence implies the estimate $a(\lambda)a_1(\lambda_0) < 0$; this proves conclusion (iii).

Multiply the equality $\mathbf{z}_0 - e^{AT}\mathbf{z}_0 = (I-Q)\mathbf{J}(T)$ by $\mathbf{z}_0$:

$$\rho^2(1 - \cos w_0T) = (\mathbf{z}_0, \mathbf{J}(T)).$$

Since $1 - \cos w_0T = 2\sin^2(w_0T/2) \le 2c_1^2\psi_\lambda^2(r)$ and

$$|(\mathbf{z}_0, \mathbf{J}(T) - \mathbf{J}(T_0))| \le \rho|\mathbf{J}(T) - \mathbf{J}(T_0)| \le c_7\rho|T-T_0|\|f(x(t),\lambda)\|_C \le c_7\rho\cdot c_6\psi_\lambda(r)\cdot c_0\psi_\lambda(r)r,$$

the relations

$$|(\mathbf{z}_0, \mathbf{J}(T_0))| \le 2c_1^2\psi_\lambda^2(r)\rho^2 + c_0c_6c_7\psi_\lambda^2(r)r\rho \le c_8\psi_\lambda^2(r)r\rho.$$

hold. Set

$$\mathbf{J}_* = \int_0^{T_0} e^{A(T_0-t)}\mathbf{q}\, f(r\sin(w_0t+\varphi),\lambda)\,dt.$$

The inequalities

$$|\mathbf{J}(T_0) - \mathbf{J}_*| \le c_9\|f(x(t),\lambda) - f(r\sin(w_0t+\varphi),\lambda)\|_C \le c_{10}\psi_\lambda(r)\|x(t) - r\sin(w_0t+\varphi)\|_C$$

and estimate (34) imply the relations

$$|(\mathbf{z}_0, \mathbf{J}(T_0) - \mathbf{J}_*)| \le \rho|\mathbf{J}(T_0) - \mathbf{J}_*| \le \rho\cdot c_{10}\psi_\lambda(r)\cdot 2c_4\psi_\lambda(r)r$$

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[13] $r = \|x(t)\|_C,\ \rho = \|\rho\sin(w_0t+\varphi)\|_C \ \Rightarrow\ |r-\rho| \le \|x(t) - \rho\sin(w_0t+\varphi)\|_C.$

and consequently

$$|(\mathbf{z}_0, \mathbf{J}_*)| \le 2c_4 c_{10}\psi_\lambda^2(r) r\rho + c_8\psi_\lambda^2(r) r\rho = c_{11}\psi_\lambda^2(r) r\rho.$$

But $(\mathbf{z}_0, e^{A(T_0-t)}\mathbf{q}) = \rho(\xi(\mathbf{q})\cos(w_0 t+\varphi) + \eta(\mathbf{q})\sin(w_0 s+\varphi))$, i.e.,

$$(\mathbf{z}_0, \mathbf{J}_*) = \rho\eta(\mathbf{q}) \int\limits_0^{T_0} \sin(w_0 t+\varphi) f(r\sin(w_0 t+\varphi), \lambda)\, dt = \rho\eta(\mathbf{q}) w_0^{-1} \int\limits_0^{2\pi} f_{odd}(r\sin t, \lambda)\sin t\, dt$$

and due to (4),

$$(\mathbf{z}_0, \mathbf{J}_*) = \rho\eta(\mathbf{q}) w_0^{-1}(a(\lambda)c_\alpha r^\alpha + a_1(\lambda)c_\gamma r^\gamma + \chi_1(r) r^\gamma), \qquad \chi_1(\cdot) \to 0.$$

Therefore

$$|a(\lambda)c_\alpha r^\alpha + a_1(\lambda)c_\gamma r^\gamma + \chi_1(r) r^\gamma| \le c_{11} w_0 |\eta(\mathbf{q})|^{-1}\psi_\lambda^2(r) r$$

or what is the same

$$|a(\lambda)c_\alpha r^\alpha + a_1(\lambda)c_\gamma r^\gamma + \chi_1(r) r^\gamma| \le c_{11} w_0 |\eta(\mathbf{q})|^{-1}\Big(|a(\lambda)| r^\alpha (|a(\lambda)| r^{\alpha-1} + 2r^{\nu-1}) + r^{2\nu-1}\Big).$$

Since $2\nu - 1 > \gamma$ and $a_1(\lambda) \to a_1(\lambda_0)$ for $\lambda \to \lambda_0$ the estimate

$$|a(\lambda)c_\alpha c_\gamma^{-1}/a_1(\lambda_0) + r^{\gamma-\alpha}| \le \chi_2(\lambda-\lambda_0) r^{\gamma-\alpha} + \chi_3(r)|a(\lambda)| + \chi_4(r) r^{\gamma-\alpha} \tag{36}$$

holds where $\chi_k(\cdot) \to 0$. If r and $|\lambda - \lambda_0|$ are sufficiently small, then estimate (36) implies

$$|a(\lambda)c_\alpha c_\gamma^{-1}/a_1(\lambda_0) + r^{\gamma-\alpha}| \le 2^{-1}|a(\lambda)c_\alpha c_\gamma^{-1}/a_1(\lambda_0)| + r^{\gamma-\alpha}/2.$$

Therefore

$$a(\lambda)a_1(\lambda_0) < 0, \qquad r^{\gamma-\alpha} = \theta|a(\lambda)c_\alpha c_\gamma^{-1}/a_1(\lambda_0)|, \quad 1/3 \le \theta \le 3. \tag{37}$$

Now conclusion (iii) follows from the first of relations (37). From relations (36)–(37) it follows the estimate

$$|\theta - 1||a(\lambda)c_\alpha c_\gamma^{-1}/a_1(\lambda_0)| \le C|a(\lambda)|(\chi_2(\cdot) + \chi_3(\cdot) + \chi_4(\cdot))$$

and hence $\theta \to 1$ as $\lambda \to \lambda_0$. This is the same that

$$r|a(\lambda)|^{-\frac{1}{\gamma-\alpha}} \to |c_\alpha c_\gamma^{-1}/a_1(\lambda_0)|^{\frac{1}{\gamma-\alpha}}, \qquad \lambda \to \lambda_0,$$

i.e., the first of estimates (10) and conclusion (iv) are valid. ■

5 Proofs of Lemmas

5.1 Proof of Lemma 1

The identity

$$\mathbf{z}(t;\mathbf{z}_0,\lambda) = e^{At}\mathbf{z}_0 + \int\limits_0^t e^{A(t-s)}\mathbf{q}\, f(\mathbf{d}^T\mathbf{z}(s;\mathbf{z}_0,\lambda),\lambda)\, ds \tag{38}$$

means that the operator U_λ may be defined as

$$U_\lambda(\mathbf{z}_0) = e^{A\tau(\mathbf{z}_0)}\mathbf{z}_0 + \int\limits_0^{\tau(t_0)} e^{A(\tau(\mathbf{z}_0)-s)}\mathbf{q}\, f(\mathbf{d}^T\mathbf{z}(s;\mathbf{z}_0,\lambda),\lambda)\, ds. \tag{39}$$

The last formula has two terms. Let us start from the first one

$$U_*(\mathbf{z}_0) = e^{A\tau(\mathbf{z}_0)}\mathbf{z}_0.$$

This term does not depend on λ, it has to be rather close to U_λ since $U_\lambda = U_*$ if $f \equiv 0$. According to (19) and the definition, $U_*(r\mathbf{z}_0) = rU_*(\mathbf{z}_0),\ r > 0$.

Let us linearize this operator. Since

$$U_*(\mathbf{z}_0+\mathbf{z})-U_*(\mathbf{z}_0) = e^{A\tau(\mathbf{z}_0+\mathbf{z})}(\mathbf{z}_0+\mathbf{z})-e^{A\tau(\mathbf{z}_0)}\mathbf{z}_0 = e^{A\tau(\mathbf{z}_0)}\mathbf{z}+\left(e^{A\tau(\mathbf{z}_0+\mathbf{z})} - e^{A\tau(\mathbf{z}_0)}\right)(\mathbf{z}_0+\mathbf{z}) =$$

$$= e^{A\tau(\mathbf{z}_0)}\mathbf{z} + (e^{A(\tau(\mathbf{z}_0+\mathbf{z})-\tau(\mathbf{z}_0))} - I)\, e^{A\tau(\mathbf{z}_0)}\mathbf{z}_0 + o(\mathbf{z}) =$$

$$= e^{A\tau(\mathbf{z}_0)}\mathbf{z} - \frac{1}{w_0}A\left(\arctan\frac{\eta(\mathbf{z}_0+\mathbf{z})}{\xi(\mathbf{z}_0+\mathbf{z})} - \arctan\frac{\eta(\mathbf{z}_0)}{\xi(\mathbf{z}_0)}\right) e^{A\tau(\mathbf{z}_0)}\mathbf{z}_0 + o(\mathbf{z}) =$$

$$= e^{A\tau(\mathbf{z}_0)}\mathbf{z} - \frac{1}{w_0}A\frac{\xi(\mathbf{z}_0)^2}{\xi(\mathbf{z}_0)^2+\eta(\mathbf{z}_0)^2}\left(\frac{\eta(\mathbf{z}_0)+\eta(\mathbf{z})}{\xi(\mathbf{z}_0)+\xi(\mathbf{z})} - \frac{\eta(\mathbf{z}_0)}{\xi(\mathbf{z}_0)}\right) e^{A\tau(\mathbf{z}_0)}\mathbf{z}_0 + o(\mathbf{z}) =$$

$$= e^{A\tau(\mathbf{z}_0)}\mathbf{z} - \frac{1}{w_0}A\frac{\eta(\mathbf{z})\xi(\mathbf{z}_0)-\eta(\mathbf{z}_0)\xi(\mathbf{z})}{\xi(\mathbf{z}_0)^2+\eta(\mathbf{z}_0)^2}\, e^{A\tau(\mathbf{z}_0)}\mathbf{z}_0 + o(\mathbf{z})$$

we have

$$U_*'(\mathbf{z}_0)\mathbf{z} = e^{A\tau(\mathbf{z}_0)}\mathbf{z} - \frac{1}{w_0}A\, e^{A\tau(\mathbf{z}_0)}\mathbf{z}_0\,\frac{\eta(\mathbf{z})\xi(\mathbf{z}_0)-\eta(\mathbf{z}_0)\xi(\mathbf{z})}{\xi(\mathbf{z}_0)^2+\eta(\mathbf{z}_0)^2},$$

where U_*' is the Jacobi matrix for U_*. It follows from $U_*'(r\mathbf{z}_0) \equiv U_*'(\mathbf{z}_0)$ that

$$U_*'(\mathbf{z}) = U_*'\left(\mathbf{g} + \frac{\mathbf{z}-\xi(\mathbf{z})\mathbf{g}}{\xi(\mathbf{z})}\right), \qquad \mathbf{z}\in\mathbb{R}^\ell_+.$$

Let $\varepsilon < 1$. Then $U_*'(\mathbf{z})$ satisfies the uniform[14] Lipschitz condition $|U_*'(\mathbf{z}_1) - U_*'(\mathbf{z}_2)| \le c^*|\mathbf{z}_1-\mathbf{z}_2|$ on $K_\varepsilon\setminus\{0\}$ and since $|\mathbf{z}-\xi(\mathbf{z})\mathbf{g}| \le \varepsilon\xi(\mathbf{z})$ for all $\mathbf{z}\in K_\varepsilon$ then

$$|U_*'(\mathbf{z}) - U_*'(\mathbf{g})| \le \frac{c^*}{\xi(\mathbf{z})}\,|\mathbf{z}-\xi(\mathbf{z})\mathbf{g}| \le c^*\varepsilon, \qquad \mathbf{z}\in K_\varepsilon\setminus\{0\}. \tag{40}$$

[14]The constant c^* is independent of ε.

The direct computations (since $\tau(\mathbf{g}) = T_0 \stackrel{\text{def}}{=} 2\pi/w_0$)

$$U'_*(\mathbf{g})\mathbf{z} = e^{AT_0}\mathbf{z} - \frac{1}{w_0} A\, e^{AT_0}\mathbf{g}\,\eta(\mathbf{z}) = e^{AT_0}\mathbf{z} - e^{AT_0}\mathbf{h}\,\eta(\mathbf{z}) = \xi(\mathbf{z})e^{AT_0}\mathbf{g} + e^{AT_0}Q\mathbf{z} =$$

$$= \xi(\mathbf{z})(\cos w_0T_0\mathbf{g} + \sin w_0T_0\mathbf{h}) + e^{AT_0}Q\mathbf{z} = \xi(\mathbf{z})\mathbf{g} + e^{AT_0}Q\mathbf{z}$$

imply

$$U'_*(\mathbf{g})\mathbf{z} = \xi(\mathbf{z})\mathbf{g} + e^{AT_0}Q\mathbf{z}, \qquad \mathbf{z} \in \mathbb{R}^\ell \tag{41}$$

and due to (17),

$$|U'_*(\mathbf{g})\mathbf{z} - \xi(\mathbf{z})\mathbf{g}| \le e^{-kT_0}|Q\mathbf{z}| \le e^{-kT_0}|\mathbf{z} - \xi(\mathbf{z})\mathbf{g}|. \tag{42}$$

By definition $U_*(r\,\mathbf{g}) = r\,\mathbf{g}$. Therefore for every $\mathbf{z}_0 \in K_\varepsilon \setminus \{0\}$ the relations

$$|U_*(\mathbf{z}_0) - \xi(\mathbf{z}_0)\mathbf{g}| = |U_*(\mathbf{z}_0) - U_*(\xi(\mathbf{z}_0)\mathbf{g})| = \left|\int_0^1 U'_*(s\mathbf{z}_0 + (1-s)\xi(\mathbf{z}_0)\mathbf{g})(\mathbf{z}_0 - \xi(\mathbf{z}_0)\mathbf{g})\,ds\right| \le$$

$$\le |U'_*(\mathbf{g})(\mathbf{z}_0 - \xi(\mathbf{z}_0)\mathbf{g})| + \int_0^1 |U'_*(s\mathbf{z}_0 + (1-s)\xi(\mathbf{z}_0)\mathbf{g}) - U'_*(\mathbf{g})| \cdot |\mathbf{z}_0 - \xi(\mathbf{z}_0)\mathbf{g}|\,ds \le$$

$$\le e^{-kT_0}|\mathbf{z}_0 - \xi(\mathbf{z}_0)\mathbf{g}| + c^*\varepsilon|\mathbf{z}_0 - \xi(\mathbf{z}_0)\mathbf{g}| \qquad \text{(due to convexity of the cone and (40), (42))}$$

imply

$$|U_*(\mathbf{z}_0) - \xi(\mathbf{z}_0)\mathbf{g}| \le (e^{-kT_0} + c^*\varepsilon)\varepsilon\xi(\mathbf{z}_0). \tag{43}$$

Now let us consider the second term in the right-hand side of (39). Since $|f(x,\lambda)| \le |a(\lambda)||x|^\alpha + c_1|x|^\nu$ and $\|\mathbf{z}(t;\mathbf{z}_0,\lambda)\|_C \le c_2|\mathbf{z}_0|$ relation (39) implies

$$|U_\lambda(\mathbf{z}_0) - U_*(\mathbf{z}_0)| \le c_3|a(\lambda)|\,|\mathbf{z}_0|^\alpha + c_3|\mathbf{z}_0|^\nu.$$

Let us choose some small parameter $\rho > 0$ (taking part in the definition of the set $K_\varepsilon(\rho)$) such that $|a(\lambda)| \le \rho^{\gamma-\alpha}$. If $\mathbf{z}_0 \in K_\varepsilon(\rho)$, $\varepsilon \le 1$, then $|\mathbf{z}_0|/\sqrt{2} \le \xi(\mathbf{z}_0) \le \rho$. Therefore

$$|U_\lambda(\mathbf{z}_0) - U_*(\mathbf{z}_0)| \le c_4|a(\lambda)|\,[\xi(\mathbf{z}_0)]^\alpha + c_4[\xi(\mathbf{z}_0)]^\nu \le c_4\rho^{\gamma-1}\xi(\mathbf{z}_0) + c_4\rho^{\nu-1}\xi(\mathbf{z}_0) \le c_5\rho^{\nu-1}\xi(\mathbf{z}_0).$$

The last relation and (43) imply

$$|U_\lambda(\mathbf{z}_0) - \xi(\mathbf{z}_0)\mathbf{g}| \le \Big((e^{-kT_0} + c^*\varepsilon)\varepsilon + c_5\rho^{\nu-1}\Big)\xi(\mathbf{z}_0). \tag{44}$$

Since the ball $\{\mathbf{z} : |\mathbf{z} - \xi(\mathbf{z}_0)\mathbf{g}| \le r\xi(\mathbf{z}_0)\}$ belongs to the interior of $K_\varepsilon \setminus \{0\}$ for $r < \varepsilon/\sqrt{1+\varepsilon^2}$ one has $U_\lambda(\mathbf{z}_0) \in K_\varepsilon \setminus \{0\}$ if

$$(e^{-kT_0} + c^*\varepsilon)\varepsilon + c_5\rho^{\nu-1} < \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}. \tag{45}$$

If $\varepsilon = d_0\rho^{\nu-1}$, then for $d_0 > c_5(1 - e^{-kT_0})^{-1}$ and $\rho \le \rho_0$ with sufficiently small positive $\rho_0 = \rho_0(c_5)$ inequality (45) holds and therefore the operator U_λ maps the set $K_\varepsilon(\rho)$ into the interior of the cone K_ε. ∎

5.2 Proof of Lemma 2

During the proof we use some relations obtained in the proof of Lemma 1 and the function (32), i.e., $\psi_\lambda(r) = |a(\lambda)|r^{\alpha-1} + r^{\nu-1}$; this function is used along the whole paper. Due to assumptions of Theorem 1 for any r small enough

$$|f(x,\lambda)| \le c\psi_\lambda(r)|x|, \qquad |f(x,\lambda) - f(y,\lambda)| \le c\psi_\lambda(r)|x-y|, \qquad |x|,|y| \le r,$$

where the constant c is independent of r and λ. Since for all $\mathbf{z}_1, \mathbf{z}_2$ from the sufficiently small ball $B(r_0) = \{|\mathbf{z}_0| \le r_0\}$ the estimates

$$\|\mathbf{z}(t;\mathbf{z}_1,\lambda)\|_C \le c_1|\mathbf{z}_1|, \quad \|\mathbf{z}(t;\mathbf{z}_1,\lambda) - \mathbf{z}(t;\mathbf{z}_2,\lambda)\|_C \le c_2|\mathbf{z}_1 - \mathbf{z}_2|$$

hold, the relations

$$\|f(\mathbf{d}^T\mathbf{z}(t;\mathbf{z}_1,\lambda),\lambda)\|_C \le c_3\psi_\lambda(r)|\mathbf{z}_1|$$

and

$$\|f(\mathbf{d}^T\mathbf{z}(t;\mathbf{z}_1,\lambda),\lambda) - f(\mathbf{d}^T\mathbf{z}(t;\mathbf{z}_2,\lambda),\lambda)\|_C \le c_4\psi_\lambda(r)|\mathbf{z}_1 - \mathbf{z}_2|$$

are valid for any $\mathbf{z}_1, \mathbf{z}_2 \in B(r)$, $r \le r_0$. Let $\xi(\mathbf{z}_2) \ge \xi(\mathbf{z}_1) > 0$. From (39) it follows that

$$\begin{aligned} |U_\lambda(\mathbf{z}_2) - U_\lambda(\mathbf{z}_1) - U_*(\mathbf{z}_2) + U_*(\mathbf{z}_1)| \;&\le\; C_1|\tau(\mathbf{z}_2) - \tau(\mathbf{z}_1)| \cdot \|f(\mathbf{d}^T\mathbf{z}(t;\mathbf{z}_1,\lambda),\lambda)\|_C + \\ &+\; C_2\|f(\mathbf{d}^T\mathbf{z}(t;\mathbf{z}_1,\lambda),\lambda) - f(\mathbf{d}^T\mathbf{z}(t;\mathbf{z}_2,\lambda),\lambda)\|_C. \end{aligned}$$

Since $(\arctan x)' = (1+x^2)^{-1} \le 1$ the relations

$$|\tau(\mathbf{z}_1) - \tau(\mathbf{z}_2)| = \frac{1}{w_0}\left|\arctan\frac{\eta(\mathbf{z}_1)}{\xi(\mathbf{z}_1)} - \arctan\frac{\eta(\mathbf{z}_2)}{\xi(\mathbf{z}_2)}\right| \le \frac{1}{w_0}\left|\frac{\eta(\mathbf{z}_1)}{\xi(\mathbf{z}_1)} - \frac{\eta(\mathbf{z}_2)}{\xi(\mathbf{z}_2)}\right| =$$

$$= \frac{1}{w_0\xi(\mathbf{z}_1)\xi(\mathbf{z}_2)}|(\eta(\mathbf{z}_1) - \eta(\mathbf{z}_2))\xi(\mathbf{z}_2) + (\xi(\mathbf{z}_2) - \xi(\mathbf{z}_1))\eta(\mathbf{z}_2)| \le \frac{|\mathbf{z}_1 - \mathbf{z}_2||\mathbf{z}_2|}{w_0\xi(\mathbf{z}_1)\xi(\mathbf{z}_2)}$$

imply the estimate

$$|\tau(\mathbf{z}_2) - \tau(\mathbf{z}_1)| \le \frac{|\mathbf{z}_1 - \mathbf{z}_2||\mathbf{z}_2|}{w_0\,\xi(\mathbf{z}_1)\xi(\mathbf{z}_2)}, \qquad \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^\ell_+.$$

Therefore

$$|U_\lambda(\mathbf{z}_2) - U_\lambda(\mathbf{z}_1) - U_*(\mathbf{z}_2) + U_*(\mathbf{z}_1)| \le c_5\,\psi_\lambda(r)|\mathbf{z}_2 - \mathbf{z}_1|\Big(\frac{|\mathbf{z}_1||\mathbf{z}_2|}{\xi(\mathbf{z}_1)\xi(\mathbf{z}_2)} + 1\Big),$$

where $r = \max\{|\mathbf{z}_1|, |\mathbf{z}_2|\}$. Suppose $\mathbf{z}_1, \mathbf{z}_2 \in K_\varepsilon(\rho)$. We consider $\varepsilon < 1$ hence $r \le \sqrt{2}\rho$ and $|\mathbf{z}_1|/\xi(\mathbf{z}_1) \le \sqrt{2}$, $|\mathbf{z}_2|/\xi(\mathbf{z}_2) \le \sqrt{2}$. If $|a(\lambda)| \le \rho^{\gamma-\alpha}$, then

$$\psi_\lambda(r) \le \rho^{\gamma-\alpha}(\sqrt{2}\rho)^{\alpha-1} + (\sqrt{2}\rho)^{\nu-1}$$

and therefore

$$|U_\lambda(\mathbf{z}_2) - U_\lambda(\mathbf{z}_1) - U_*(\mathbf{z}_2) + U_*(\mathbf{z}_1)| \le C\rho^{\nu-1}|\mathbf{z}_2 - \mathbf{z}_1|,$$

where the value C is independent of small $\rho > 0$. But from (40) the estimate

$$|U_*(\mathbf{z}_2) - U_*(\mathbf{z}_1) - U_*'(\mathbf{g})(\mathbf{z}_2 - \mathbf{z}_1)| \le c^*\varepsilon|\mathbf{z}_2 - \mathbf{z}_1|$$

follows. Here $\varepsilon = d_0\rho^{\nu-1}$ and hence the estimate

$$|U_\lambda(\mathbf{z}_2) - U_\lambda(\mathbf{z}_1) - U_*'(\mathbf{g})(\mathbf{z}_2 - \mathbf{z}_1)| \le C_0\rho^{\nu-1}|\mathbf{z}_1 - \mathbf{z}_2| \tag{46}$$

is valid for some C_0. Put

$$\xi_0 = \xi(\mathbf{z}_2 - \mathbf{z}_1), \quad \mathbf{z}_0' = Q(\mathbf{z}_2 - \mathbf{z}_1), \quad \tilde{\xi} = \xi(U_\lambda(\mathbf{z}_2) - U_\lambda(\mathbf{z}_1)), \quad \mathbf{y} = U_\lambda(\mathbf{z}_2) - U_\lambda(\mathbf{z}_1) - \tilde{\xi}\mathbf{g}.$$

According to (41), $U_*'(\mathbf{g})(\mathbf{z}_2 - \mathbf{z}_1) = \xi_0\mathbf{g} + e^{AT_0}\mathbf{z}_0'$. So estimate (46) can be rewritten as

$$\left((\tilde{\xi} - \xi_0)^2 + |\mathbf{y} - e^{AT_0}\mathbf{z}_0'|^2\right)^{1/2} \le C_0\rho^{\nu-1}\sqrt{\xi_0^2 + |\mathbf{z}_2 - \mathbf{z}_1 - \xi_0\mathbf{g}|^2}.$$

Now

$$|\tilde{\xi} - \xi_0| \le C_0\rho^{\nu-1}\sqrt{\xi_0^2 + |\mathbf{z}_2 - \mathbf{z}_1 - \xi_0\mathbf{g}|^2}$$

and since $|e^{AT_0}\mathbf{z}_0'| \le e^{-kT_0}|\mathbf{z}_0'| \le e^{-kT_0}|\mathbf{z}_2 - \mathbf{z}_1 - \xi_0\mathbf{g}|$ one has

$$|\mathbf{y}| \le e^{-kT_0}|\mathbf{z}_2 - \mathbf{z}_1 - \xi_0\mathbf{g}| + C_0\rho^{\nu-1}\sqrt{\xi_0^2 + |\mathbf{z}_2 - \mathbf{z}_1 - \xi_0\mathbf{g}|^2}.$$

Let $\mathbf{z}_2 \ne \mathbf{z}_1$, $\mathbf{z}_2 \overset{\varepsilon_1}{\ge} \mathbf{z}_1$ for $\varepsilon_1 = d_1\rho^{1-\nu}$, i.e., $\xi_0 > 0$, $|\mathbf{z}_2 - \mathbf{z}_1 - \xi_0\mathbf{g}| \le d_1\rho^{1-\nu}\xi_0$. This implies

$$\tilde{\xi} \ge \left(1 - C_0\sqrt{\rho^{2\nu-2} + d_1^2}\right)\xi_0, \qquad |\mathbf{y}| \le \left(e^{-kT_0}d_1\rho^{1-\nu} + C_0\sqrt{\rho^{2\nu-2} + d_1^2}\right)\xi_0.$$

If $\rho \le \rho_1$ and d_1, ρ_1 are small enough, then these estimates imply the inequality $|\mathbf{y}| < \varepsilon_1\tilde{\xi} = d_1\rho^{1-\nu}\tilde{\xi}$ and, what is the same, the required inequality $U_\lambda(\mathbf{z}_2) \overset{\varepsilon_1}{>} U_\lambda(\mathbf{z}_1)$. ■

5.3 Proof of Lemma 3

Let $r > 0$. The equalities $U_*(r\mathbf{g}) = r\mathbf{g}$, $\tau(r\mathbf{g}) = T_0$ and (39) imply the relation

$$U_\lambda(r\mathbf{g}) = r\mathbf{g} + \int_0^{T_0} e^{A(T_0-s)}\mathbf{q}\, f(\mathbf{d}^T\mathbf{z}(s; r\mathbf{g}, \lambda), \lambda)\, ds \tag{47}$$

and since $\|f(\mathbf{d}^T\mathbf{z}(t; r\mathbf{g}, \lambda), \lambda)\|_C \le C_0\psi_\lambda(r)r$ for $\psi_\lambda(r) = |a(\lambda)|r^{\alpha-1} + r^{\nu-1}$, the estimate

$$|U_\lambda(r\mathbf{g}) - r\mathbf{g}| \le C_1\psi_\lambda(r)r \tag{48}$$

holds. Let us estimate the component $\xi(U_\lambda(r\mathbf{g}) - r\mathbf{g})$ of the vector $U_\lambda(r\mathbf{g}) - r\mathbf{g}$. Put

$$\mathbf{w}_\lambda(r) = \int_0^{T_0} e^{A(T_0-s)}\mathbf{q}\, f(r\mathbf{d}^T e^{As}\mathbf{g}, \lambda)\, ds.$$

From (47) it follows that

$$|U_\lambda(r\mathbf{g}) - r\mathbf{g} - \mathbf{w}_\lambda(r)| \le C_2\|f(\mathbf{d}^T\mathbf{z}(t;r\mathbf{g},\lambda),\lambda) - f(r\mathbf{d}^Te^{At}\mathbf{g},\lambda)\|_C,$$

hence

$$|U_\lambda(r\mathbf{g}) - r\mathbf{g} - \mathbf{w}_\lambda(r)| \le C_3\psi_\lambda(r)\|\mathbf{z}(t;r\mathbf{g},\lambda) - re^{At}\mathbf{g}\|_C.$$

But due to (38)

$$\|\mathbf{z}(t;r\mathbf{g},\lambda) - re^{At}\mathbf{g}\|_C \le C_4\|f(\mathbf{d}^T\mathbf{z}(t;r\mathbf{g},\lambda),\lambda)\|_C$$

and therefore

$$|U_\lambda(r\mathbf{g}) - r\mathbf{g} - \mathbf{w}_\lambda(r)| \le C_3C_4\psi_\lambda(r)\|f(\mathbf{d}^T\mathbf{z}(t;r\mathbf{g},\lambda),\lambda)\|_C \le C\psi_\lambda^2(r)r.$$

Consequently

$$|\xi(U_\lambda(r\mathbf{g}) - r\mathbf{g} - \mathbf{w}_\lambda(r))| \le C\psi_\lambda^2(r)r. \tag{49}$$

Since

$$e^{At}(\xi\mathbf{g} + \eta\mathbf{h}) = (\xi\cos w_0t - \eta\sin w_0t)\mathbf{g} + (\xi\sin w_0t + \eta\cos w_0t)\mathbf{h}$$

and[15] $\mathbf{d}^T\mathbf{g} = 0$, $\mathbf{d}^T\mathbf{h} = 1$, one has

$$\xi(\mathbf{w}_\lambda(r)) = \int_0^{T_0}(\xi(\mathbf{q})\cos w_0t + \eta(\mathbf{q})\sin w_0t)f(r\sin w_0t,\lambda)\,dt = \frac{\eta(\mathbf{q})}{w_0}\int_0^{2\pi} f_{odd}(r\sin t,\lambda)\sin t\,dt$$

and according to (3),

$$\xi(\mathbf{w}_\lambda(r)) = \frac{\eta(\mathbf{q})}{w_0}[c_\alpha a(\lambda)r^\alpha + c_\gamma a_1(\lambda)r^\gamma] + r^\gamma\varphi(r),$$

where $\varphi(r) \to 0$ as $r \to 0$. Therefore

$$\xi(\mathbf{w}_\lambda(r_j(\lambda))) = \frac{\eta(\mathbf{q})}{w_0}\Big(c_\alpha n_j^\alpha a(\lambda)|a(\lambda)|^{\frac{\alpha}{\gamma-\alpha}} + c_\gamma n_j^\gamma a_1(\lambda)|a(\lambda)|^{\frac{\gamma}{\gamma-\alpha}}\Big) + n_j^\gamma|a(\lambda)|^{\frac{\gamma}{\gamma-\alpha}}\varphi(n_j|a(\lambda)|^{\frac{1}{\gamma-\alpha}}),$$

this implies

$$\xi(\mathbf{w}_\lambda(r_j(\lambda))) = \Big(c_\alpha n_j^\alpha\frac{\eta(\mathbf{q})}{w_0}\Big(\operatorname{sign} a(\lambda) + c_\alpha^{-1}c_\gamma n_j^{\gamma-\alpha}a_1(\lambda)\Big) + n_j^\gamma\varphi(n_j|a(\lambda)|^{\frac{1}{\gamma-\alpha}})\Big)|a(\lambda)|^{\frac{\gamma}{\gamma-\alpha}}$$

for $j = 1,2$. Due to (22)

$$c_\alpha^{-1}c_\gamma n_1^{\gamma-\alpha}|a_1(\lambda_0)| = 1/2, \qquad c_\alpha^{-1}c_\gamma n_2^{\gamma-\alpha}|a_1(\lambda_0)| \ge 2$$

and since $a_1(\lambda) \to a_1(\lambda_0) \ne 0$, $a(\lambda) \to 0$ for $\lambda \to \lambda_0$, for all λ from a sufficiently small interval $|\lambda - \lambda_0| < \delta$ the estimates

$$\operatorname{sign}[\eta(\mathbf{q})a(\lambda)]\xi(\mathbf{w}_\lambda(r_1(\lambda))) \ge \frac{1}{3}c_\alpha n_1^\alpha\frac{|\eta(\mathbf{q})|}{w_0}|a(\lambda)|^{\frac{\gamma}{\gamma-\alpha}},$$

[15] See the choice (14) of the vectors $\mathbf{g}$ and $\mathbf{h}$.

$$\text{sign}[\eta(\mathbf{q})a_1(\lambda_0)]\xi(\mathbf{w}_\lambda(r_2(\lambda))) \geq \frac{1}{3}c_\alpha n_2^\alpha \frac{|\eta(\mathbf{q})|}{w_0} a(\lambda)|^{\frac{\gamma}{\gamma-\alpha}}$$

are valid. Lemma 5 (see the next subsection) gives us

$$\eta(\mathbf{q}) = \frac{\Im m\, M(w_0 i)L_1(-w_0 i)}{w_0\,|L_1(w_0 i)|^2} \tag{50}$$

and consequently $\text{sign}[\eta(\mathbf{q})a_1(\lambda_0)] = \sigma_*$, $\text{sign}[\eta(\mathbf{q})a(\lambda)] = \sigma_\lambda\sigma_*$. Therefore

$$\sigma_\lambda\sigma_*\xi(\mathbf{w}_\lambda(r_1(\lambda))) \geq \frac{c_\alpha}{3}n_1^\alpha\frac{|\eta(\mathbf{q})|}{w_0}|a(\lambda)|^{\frac{\gamma}{\gamma-\alpha}},\ \ \sigma_*\xi(\mathbf{w}_\lambda(r_2(\lambda))) \geq \frac{c_\alpha}{3}n_2^\alpha\frac{|\eta(\mathbf{q})|}{w_0}|a(\lambda)|^{\frac{\gamma}{\gamma-\alpha}}. \tag{51}$$

Now we have

$$\psi_\lambda(r_j(\lambda)) \leq n_j^{\alpha-1}|a(\lambda)|^{\frac{\gamma-1}{\gamma-\alpha}} + n_j^{\nu-1}|a(\lambda)|^{\frac{\nu-1}{\gamma-\alpha}} \leq c_0|a(\lambda)|^{\frac{\nu-1}{\gamma-\alpha}}$$

for $c_0 \geq n_j^{\alpha-1} + n_j^{\nu-1}$, $|a(\lambda)| \leq 1$, $j = 1, 2$ and according to (48) and (49)

$$|U_\lambda(r_j(\lambda)\mathbf{g}) - r_j(\lambda)\mathbf{g} - \xi(U_\lambda(r_j(\lambda)\mathbf{g}) - r_j(\lambda)\mathbf{g})\mathbf{g}| \leq |U_\lambda(r_j(\lambda)\mathbf{g}) - r_j(\lambda)\mathbf{g}| \leq C_1c_0n_j|a(\lambda)|^{\frac{\nu}{\gamma-\alpha}} \tag{52}$$

and

$$|\xi(U_\lambda(r_j(\lambda)\mathbf{g}) - r_j(\lambda)\mathbf{g}) - \xi(\mathbf{w}_\lambda(r_j(\lambda)))| \leq Cc_0^2n_j|a(\lambda)|^{\frac{2\nu-1}{\gamma-\alpha}}. \tag{53}$$

Since $2\nu - 1 > \gamma$ the estimates (51) – (53) imply for any $a(\lambda) \neq 0$ (if $|a(\lambda)|$ is sufficiently small) the inequalities

$$|U_\lambda(r_1(\lambda)\mathbf{g}) - r_1(\lambda)\mathbf{g} - \xi(U_\lambda(r_1(\lambda)\mathbf{g}) - r_1(\lambda)\mathbf{g})\mathbf{g}| < \sigma_\lambda\sigma_* d_1 n_2^{1-\nu}|a(\lambda)|^{\frac{1-\nu}{\gamma-\alpha}}\xi(U_\lambda(r_1(\lambda)\mathbf{g}) - r_1(\lambda)\mathbf{g})$$

and

$$|U_\lambda(r_2(\lambda)\mathbf{g}) - r_2(\lambda)\mathbf{g} - \xi(U_\lambda(r_2(\lambda)\mathbf{g}) - r_2(\lambda)\mathbf{g})\mathbf{g}| < \sigma_* d_1 n_2^{1-\nu}|a(\lambda)|^{\frac{1-\nu}{\gamma-\alpha}}\xi(U_\lambda(r_2(\lambda)\mathbf{g}) - r_2(\lambda)\mathbf{g}).$$

This coincide with (23) with $\varepsilon_1 = d_1[r_2(\lambda)]^{1-\nu}$. ■

5.4 Lemma 5

Lemma 5. *Equality* (50) *holds.*

Proof. From the definition of the vectors **g** and **h** it follows that

$$\begin{aligned} L_1(A)\mathbf{h} &= \Re e\, L_1(w_0 i)\mathbf{h} - \Im m\, L_1(w_0 i)\mathbf{g}, \\ L_1(A)\mathbf{g} &= \Re e\, L_1(w_0 i)\mathbf{g} + \Im m\, L_1(w_0 i)\mathbf{h}. \end{aligned}$$

This together with (14) implies

$$\mathbf{d}^T L_1(A)\mathbf{h} = \Re e\, L_1(w_0 i), \qquad \mathbf{d}^T L_1(A)\mathbf{g} = \Im m\, L_1(w_0 i).$$

It follows from the standard formulas

$$\mathbf{d}^T L_1(A)(A - w_0 i)\mathbf{q} = M(-w_0 i), \qquad \mathbf{d}^T L_1(A)(A + w_0 i)\mathbf{q} = M(w_0 i)$$

of general linear system theory that

$$w_0\mathbf{d}^T L_1(A)\mathbf{q} = \Im m\, M(w_0 i), \qquad \mathbf{d}^T L_1(A)A\mathbf{q} = \Re e\, M(w_0 i).$$

Since $L_1(A)Q\mathbf{q} = 0$ (see [13]) one has

$$\begin{aligned} w_0\xi(\mathbf{q})\mathbf{d}^T L_1(A)\mathbf{g} \,+\, w_0\eta(\mathbf{q})\mathbf{d}^T L_1(A)\mathbf{h} &= \Im m\, M(w_0 i),\\ w_0\xi(\mathbf{q})\mathbf{d}^T L_1(A)\mathbf{h} \,-\, w_0\eta(\mathbf{q})\mathbf{d}^T L_1(A)\mathbf{g} &= \Re e\, M(w_0 i) \end{aligned}$$

and consequently,

$$\begin{aligned} w_0\xi(\mathbf{q})\Im m\, L_1(w_0 i) \,+\, w_0\eta(\mathbf{q})\Re e\, L_1(w_0 i) &= \Im m\, M(w_0 i),\\ w_0\xi(\mathbf{q})\Re e\, L_1(w_0 i) \,-\, w_0\eta(\mathbf{q})\Im m\, L_1(w_0 i) &= \Re e\, M(w_0 i). \end{aligned}$$

The last two relations prove the lemma. ■

Remark. Theorem 1 does not contain statements about the uniqueness of small cycles for system (2). However, the uniqueness and the asymptotic stability are clear under some additional assumptions on smoothness of the function $f(x, \lambda)$.

For example, let all the conditions of Theorem 1 be valid. Let additionally the smallest term in representation (3) of the odd part $f_{odd}(x, \lambda)$ satisfy

$$|\psi_0(x, \lambda) - \psi_0(y, \lambda)| \le o(r^{\gamma-1})|x - y|, \qquad r = \max\{|x|, |y|\}, \tag{54}$$

and let

$$|f'_{even}(x, \lambda) - f'_{even}(y, \lambda)| \le c\, \max\{|x|^{\beta-2}, |y|^{\beta-2}\}|x - y|. \tag{55}$$

To be definite, suppose $\kappa < 0$. Then the operator U_λ is contracting in some appropriate norm in the vicinity of every fixed point $\mathbf{z}_* \in \Omega_\lambda$. In particular, this is true for the cone norm $\|\mathbf{z}\|_{\varepsilon_1} = \min\{\theta : -\theta\mathbf{g} \overset{\varepsilon_1}{\le} \mathbf{z} \overset{\varepsilon_1}{\le} \theta\mathbf{g}\}$:

$$\|U_\lambda(\mathbf{z}) - \mathbf{z}_*\|_{\varepsilon_1} \le c_0\|\mathbf{z} - \mathbf{z}_*\|_{\varepsilon_1} \qquad \text{for} \qquad \|\mathbf{z} - \mathbf{z}_*\|_{\varepsilon_1} \le r(\lambda), \tag{56}$$

where $c_0 = c_0(\lambda) = 1 - c|a(\lambda)|^{\frac{\gamma-1}{\gamma-\alpha}}$ (it is possible to use some other norms). Therefore the topological index of any fixed point $\mathbf{z}_* \in \Omega_\lambda$ of the operator U_λ equals 1. On the other hand, the operator U_λ maps closed domain Ω_λ in its interior, this implies that the sum of indices of all the fixed points contained in Ω_λ is also to 1. This means that the operator U_λ has a unique fixed point $\mathbf{z}_*$ in Ω_λ. From (56) it follows the exponential orbital stability of the cycle $\mathbf{z}^*(t)$ which contains the point $\mathbf{z}_*$ and Lyapunov stability of every periodic solution $\mathbf{z}^*(t + \varphi)$, $\varphi \in R$.

It is easily shown that any cycle of system (2) contains some fixed point of the operator U_λ lying in Ω_λ. Therefore $\mathbf{z}^*(t)$ is a unique small cycle.

If estimates (54), (55) are valid and $\kappa > 0$, then the operator V_λ is contracting in a vicinity of its fixed point $\mathbf{z}_* \in \Omega_\lambda$ and again the uniqueness of a small cycle takes place.

The authors do not know if it is possible to guarantee the uniqueness without additional assumptions on nonlinearity smoothness.

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