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# Small Periodic Solutions Generated by Sublinear Terms

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The paper is concerned with Hopf bifurcations in systems of autonomous ordinary differential equations with a parameter. The principal distinction between usual theorems on Hopf bifurcations and our results is that here the linearized equation is degenerate and independent of the parameter. We present sufficient conditions for a parameter value to be a bifurcation point and analyze properties of small cycles arising in the vicinity of the equilibrium. Sublinear nonlinearities play the main role in the results obtained. 

*Key Words:* Hopf bifurcation; autonomous system; periodic oscillations.

## 1. INTRODUCTION

In this paper, a system

$$x' = \mathscr{A}(\lambda)\, x + f(x, \lambda), \qquad x \in \mathbb{R}^m \tag{1}$$

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is considered with a matrix $\mathscr{A}(\lambda)$ depending continuously on the scalar parameter $\lambda \in (0, 1)$ and a continuous nonlinearity $f(x, \lambda)$. The origin is supposed to be an equilibrium of the system for every $\lambda$, i.e., $f(0, \lambda) \equiv 0$. It is assumed that $f(x, \lambda)$ is smaller than any linear function in a vicinity of the origin[3]:

$$\lim_{|x| \to 0} \sup_{\lambda \in (0,1)} \frac{|f(x, \lambda)|}{|x|} = 0; \tag{2}$$

such nonlinearities are called *sublinear at zero*, or simply *sublinear*.

We study the *Hopf bifurcation*: the existence of arbitrarily small periodic cycles of system (1) for parameter values[4] close to some point $\lambda_0$. More precisely, the following definition is used.

DEFINITION 1.1. A value $\lambda_0$ of the parameter is called[5] a *Hopf bifurcation point with the frequency* $w_0$ *for system* (1) if for every sufficiently small $r > 0$ there exists a $\lambda = \lambda(r)$ such that system (1) with this $\lambda$ has a nonstationary periodic solution $x(t; r)$ with a period $T(r)$ and $\lambda(r) \to \lambda_0$, $T(r) \to 2\pi/w_0$, $\|x(\cdot\,; r)\|_C \to 0$ as $r \to 0$.

In other words, $\lambda_0$ is a Hopf bifurcation point with the frequency $w_0$ if for values of $\lambda$ arbitrarily close to $\lambda_0$ there is a one-parameter set of periodic cycles of system (1) with arbitrarily small amplitudes and with periods arbitrarily close to $2\pi/w_0$. The use of an additional parameter $r$ different from $\lambda$ is usual in Hopf bifurcations, starting from the original works of Poincare, Andronov, Hopf [1, 2, 3]. A natural parameter is the amplitude of the cycle or close quantities.

If the matrix $\mathscr{A}(\lambda)$ has a pair of simple conjugate eigenvalues $\mu(\lambda) \neq \bar{\mu}(\lambda)$ that cross the imaginary axis for $\lambda = \lambda_0$ and the other spectrum of $\mathscr{A}(\lambda)$ satisfies simple additional conditions, then the parameter value $\lambda_0$ is a Hopf bifurcation point with the frequency $w_0 = |\Im m\, \mu(\lambda_0)| = |\mu(\lambda_0)|$. For example, a sufficient additional condition is that the spectrum $\sigma(\mathscr{A}(\lambda_0))$ of $\mathscr{A}(\lambda_0)$ does not contain the points $ikw_0$, $k = 0, 2, 3, \ldots$ (it means the absence of *resonance*).

[3] The notation $|\cdot|$ is used for norms in finite-dimensional spaces as well as for the modula of real and complex numbers.

[4] Sometimes values of the parameter are called *points*.

[5] We say simply *a bifurcation point* if it is clear, which equation and frequency are meant or if we do not want to specify the frequency value.

This is a sharp statement, since the condition $iw_0 \in \sigma(\mathscr{A}(\lambda_0))$ is necessary for $\lambda_0$ to be a HBP with the frequency $w_0$. Various modifications of this statement and further important facts about Hopf bifurcations (also referred to as Andronov–Hopf bifurcations, Poincare continuation of periodic solutions etc.) as well as other related problems are studied in a very great number of monographs and papers. We refer to books [4, 5, 4, 7, 8, 9, 10] and the bibliography observed therein.

An important point is the smoothness of the right-hand side of system (1). If the matrix-valued function $\mathscr{A}(\lambda)$ and the nonlinearity $f(x, \lambda)$ are analytic or sufficiently smooth, then the functions $\lambda(r)$, $T(r)$ in the definition above and the set of small periodic solutions $x(\cdot\,; r)$ are also analytic or smooth and there are algorithms to construct their asymptotic expansions. Generically, in the phase space of a smooth system (1) there is the two-dimensional locally invariant integral manifold containing the cycle $x(\cdot\,; r)$ for $\lambda = \lambda(r)$; this manifold is tangent to the invariant plane of the matrix $\mathscr{A}(\lambda)$ corresponding to the eigenvalues $\mu(\lambda)$, $\bar{\mu}(\lambda)$, the behavior of the trajectories on and outside the manifold is well-studied. In the space $\{x, \lambda\}$ the cycles $x(\cdot\,; r)$ form the two-dimensional smooth surface, a "cup," passing through the origin (generically, system (1) does not have other small cycles).

Another important problem is the stability of small cycles $x(\cdot\,; r)$. For planar and analytic systems it was studied already in [2, 3], for some further results we refer to [4] again.

Classical methods to study Hopf bifurcations for smooth systems (with various smoothness) are normal forms, central manifold theorems, implicit function theorems. In fact, the sufficient condition formulated above for $\lambda_0$ to be a HBP of a continuous system (1) without any additional requirements on smoothness was first obtained in [10] by topological methods and a special technique of *parameter functionalization*.

The main condition that the eigenvalues $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ of the matrix $\mathscr{A}(\lambda)$ cross the imaginary axis at the bifurcation point can have different forms. Usually it is supposed that $\Re e\, \mu(\lambda_0) = 0$, $\Re e\, \mu'(\lambda_0) \neq 0$ (the *transversality* condition). A more complicated situation $\Re e\, \mu(\lambda_0) = \Re e\, \mu'(\lambda_0) = \cdots = \Re e\, \mu^{(k)}(\lambda_0) = 0$, $\Re e\, \mu^{(k+1)}(\lambda_0) \neq 0$ and some close cases are considered for example in [4, 11]. In [10], where no smoothness is assumed, the main condition is that the function $\Re e\, \mu(\lambda)$ takes both positive and negative values in every neighborhood of its zero $\lambda_0$.

In all the situations described above, the Hopf bifurcation occurs due to a special behaviour of the linear part $\mathscr{A}(\lambda)$ of the system, the only requirements on the nonlinearity are condition (2) and the continuity.

In this paper we suggest sufficient conditions for the Hopf bifurcation for systems (1) such that the eigenvalues $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ of the matrix $\mathscr{A}(\lambda)$

are imaginary for all parameter values. The conditions use essentially asymptotics of the nonlinear term $f(x, \lambda)$ at the origin.

All the functions in the theorems below are supposed to be continuous; sometimes, we do not mention this in the formulations. Any additional smoothness is not assumed. Everywhere the principal term of the nonlinearity is Lipschitz continuous on small balls centered at the origin, the Lipschitz coefficient vanishes as the radius of the ball goes to zero.

Typically, small cycles exist either for $\lambda > \lambda_0$ or for $\lambda < \lambda_0$ only; some systems have continua of cycles in a vicinity of the origin for $\lambda = \lambda_0$ and have no cycles for $\lambda \neq \lambda_0$ (e.g., linear systems $x' = A(\lambda)\, x$, some Hamiltonian and reversible systems etc.). The paper suggests a method to answer the question for which $\lambda$ the small cycles exist.

We do not study neither the stability of the cycles nor the behavior of the trajectories in the phase space. All constructions are made in functional spaces. The continuity of the functions $\lambda(r)$, $T(r)$ and of the set of cycles $x(\cdot\,; r)$ is not studied.

The paper is organized as follows.

In the next section we present a rather simple theorem on the Hopf bifurcation and its applications. This Theorem 2.1 cannot be applied, e.g., to systems with nonlinearities having nonzero quadratic principal terms. Section 3 contains essentially more general Theorem 3.1 and its applications to systems with quadratic nonlinearities. Let us stress that Theorem 2.1 follows directly from Theorem 3.1. To formulate Theorem 3.1, we need an auxiliary statement, Lemma 3.1. In Section 4 we discuss properties of small cycles, generated by the Hopf bifurcation. Theorem 4.1 gives the information if periods of the small cycles are less or greater than $2\pi$. In Theorem 4.2 we analyze if the cycles exist for $\lambda < \lambda_0$ or for $\lambda > \lambda_0$.

Section 5 contains some miscellaneous remarks on the subject. In particular, there are multiplicity results (Theorem 5.1) and results about continuous branches of cycles (Theorem 5.2). Throughout the paper we consider non-Hamiltonian systems. The last subsection of Section 5 contains remarks on some other situations.

The last part of the paper (Sections 6–8) contains the proofs.

We use the original method that can be briefly described as a combination of the harmonic linearization and topological methods: the degree theory or the vector field rotation theory. The method (or its simple modifications) is applicable to related problems with nonsmooth nonlinearities of various types: delays, hysteresis, etc. (see, e.g., [12]). It can be applied to study weak resonances in Hopf bifurcations [13]. Also, our method can be used to study the usual situation (the eigenvalues $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ cross the imaginary axis at the bifurcation point), in particular, the sufficient condition above for $\lambda_0$ to be a HBP can be easily proved.

## 2. EXISTENCE OF BIFURCATION POINTS

Throughout the paper we study the system

$$\begin{cases} x' = y, \\ y' = -x + f(x, y, z, \lambda), \\ z' = A(\lambda)\, z + g(x, y, z, \lambda). \end{cases} \tag{3}$$

Here $x$ and $y$ are scalar variables, $z$ is a $m_1$-dimensional vector, $m_1 = m-2$. The functions

$$f(x, y, z, \lambda)\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{m_1}\times(0, 1) \to \mathbb{R},$$

$$g(x, y, z, \lambda)\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{m_1}\times(0, 1) \to \mathbb{R}^{m_1}$$

are continuous w.r.t. the set of their arguments; $f(0, 0, 0, \lambda) \equiv g(0, 0, 0, \lambda) \equiv 0$. The $m_1 \times m_1$ matrix $A(\lambda)$ depends continuously on $\lambda$. The relation between systems (3) and general system (1) is discussed in Section 5, it turns out that almost any system (1) locally has form (3).

THEOREM 2.1. *Let there exist a point $\lambda_0 \in (0, 1)$ and its vicinity $\Lambda \ni \lambda_0$ such that the following conditions hold:*

1. *The numbers ki do not belong to the spectrum of the matrix $A(\lambda_0)$ for all $k \in \mathbb{Z}$;*

2. *The function $g(x, y, z, \lambda)$ can be represented as $g(x, y, z, \lambda) = G(x, y, \lambda) + \Gamma(x, y, z, \lambda)$, where $G(x, y, \lambda)$ for some $\gamma > 1$ satisfies the estimate*

$$|G(x, y, \lambda)| \leqslant c_1 v_1^{\gamma}, \qquad \lambda \in \Lambda \tag{4}$$

*for all sufficiently small $v_1 = |x| + |y|$ and where $|\Gamma(x, y, z, \lambda)|/|z| \to 0$ as $|x| + |y| + |z| \to 0$ uniformly w.r.t. $\lambda \in \Lambda$;*

3. *The function $f(x, y, z, \lambda)$ can be represented as $f(x, y, z, \lambda) = F(x, y, \lambda) + \Phi(x, y, z, \lambda)$, where $\Phi(x, y, z, \lambda)$ for some $\beta > 1$ satisfies the estimate*

$$|\Phi(x, y, z, \lambda)| \leqslant c_1\, v_2^{\beta}, \qquad \lambda \in \Lambda \tag{5}$$

*for all sufficiently small $v_2 = |x| + |y| + |z|^{1/\gamma}$;*

4. *The function* $F(x, y, \lambda)$ *for some* $\alpha \in (1, \beta)$ *satisfies the estimate*

$$|F(x, y, \lambda)| \leqslant c_1 \, v_1^{\alpha}, \qquad \lambda \in \Lambda \tag{6}$$

*for all sufficiently small* $v_1 = |x| + |y|$;

5. *The function* $F(x, y, \lambda)$ *satisfies for some* $v \in (0, \alpha - 1]$ *the Lipschitz condition*

$$|F(x_1, y_1, \lambda) - F(x_2, y_2, \lambda)| \leqslant c_1 \max\{|x_i|^v, |y_i|^v\} \, (|x_1 - x_2| + |y_1 - y_2|) \tag{7}$$

*for all sufficiently small* $|x_i|$, $|y_i|$, $i = 1, 2$ *and for* $\lambda \in \Lambda$;

6. *In any vicinity of the point* $\lambda_0$ *there exist* $\lambda^-$, $\lambda^+$ *such that the function*

$$d_0(\lambda, r) \stackrel{\text{def}}{=} \int_0^{2\pi} \cos t \, F(r \sin t, r \cos t, \lambda) \, dt, \qquad r > 0 \tag{8}$$

*satisfies the relations*

$$\lim_{r \to +0} \frac{d_0(\lambda^-, r)}{r^\beta} = -\infty, \qquad \lim_{r \to +0} \frac{d_0(\lambda^+, r)}{r^\beta} = +\infty; \tag{9}$$

7. *The exponents* $\alpha$, $\beta$, *and* $v$ *satisfy*

$$\alpha + v \geqslant \beta. \tag{10}$$

*Then* $\lambda_0$ *is a Hopf bifurcation point with the frequency* 1 *for system* (3).

Due to condition (9), the function $d_0(\lambda, r)$ with $\lambda = \lambda^\pm$ is greater than any terms[6] of order $\beta$ for small $r$, therefore this function determines the main terms in some equations below. The functions $F(x, y, \lambda)$ and $d_0(\lambda, r)$ may have different orders at the origin, this is the case in the following Corollary 2, where the principal even terms of the function $F(x, y, \lambda)$ vanish as we pass to $d_0(\lambda, r)$ by formula (8). In fact, the terms even in $x$ and odd in $y$ only contribute to integral (8). The function $d_0(\lambda, r)$ is of the same order as the greatest of such terms, this order should be less than $\beta$. Some additional uncontrollable terms arising in the calculations are[7] $O(r^\beta)$ due to condition (10).

[6] We say that the function $\psi(\xi)\colon \mathbb{R}^{k_1} \to \mathbb{R}^{k_2}$ is of order $\delta$ at the point $\xi = 0$ if for some $c_2 \geqslant c_1 > 0$ the inequalities $c_1 \, |\xi|^\delta \leqslant |\psi(\xi)| \leqslant c_2 \, |\xi|^\delta$ hold for any sufficiently small $|\xi|$.

[7] We write $\psi(r, \lambda) = o(r^\delta)$ if $r^{-\delta} \sup\{|\psi(r, \lambda)| : \lambda \in \Lambda\} \to 0$ as $r \to +0$, we write $\psi(r, \lambda) = O(r^\delta)$ if $\sup\{|\psi(r, \lambda)| : \lambda \in \Lambda\} \leqslant C r^\delta$ for all sufficiently small $r$.

Theorem 2.1 may be simplified if the function $F(x, y, \lambda)$ is smooth. In this case it is natural to consider integer $\alpha$, $\beta$, and $\nu$. Suppose conditions 1 and 2 of Theorem 2.1 are satisfied.

COROLLARY 2.1. *Let the function $f(x, y, z, \lambda)$ have the form*

$$f(x, y, z, \lambda) = \sum_{k=0}^{3} a_k(\lambda)\, x^k y^{3-k} + \Phi(x, y, z, \lambda),$$

*where $\Phi(x, y, z, \lambda)$ satisfies* (5) *with $\beta = 4$. Let $3a_0(\lambda_0) + a_2(\lambda_0) = 0$ and let the function $3a_0(\lambda) + a_2(\lambda)$ take the values of both sign in any vicinity of the point $\lambda_0$. Then $\lambda_0$ is a bifurcation point with the frequency* 1 *for system* (3).

COROLLARY 2.2. *Let the function $f(x, y, z, \lambda)$ have the form*

$$f(x, y, z, \lambda) = \sum_{k=0}^{4} b_k(\lambda)\, x^k y^{4-k} + \sum_{k=0}^{5} a_k(\lambda)\, x^k y^{5-k} + \Phi(x, y, z, \lambda), \tag{11}$$

*where $\Phi(x, y, z, \lambda)$ satisfies* (5) *with $\beta = 6$. Let $5a_0(\lambda_0) + a_2(\lambda_0) + a_4(\lambda_0) = 0$ and let the function $5a_0(\lambda) + a_2(\lambda) + a_4(\lambda)$ take the values of both sign in any vicinity of the point $\lambda_0$. Then $\lambda_0$ is a bifurcation point with the frequency* 1 *for system* (3).

Under the hypotheses of Corollary 1, one can take

$$F(x, y, \lambda) \stackrel{\text{def}}{=} \sum_{k=0}^{3} a_k(\lambda)\, x^k y^{3-k}.$$

In this case

$$d_0(\lambda, r) = \frac{\pi}{4} r^3 (3a_0(\lambda) + a_2(\lambda)). \tag{12}$$

Therefore all the conditions of Theorem 2.1 are satisfied for $\alpha = 3$, $\nu = 2$. To calculate integral (8) we use the equalities

$$\int_0^{2\pi} \sin^{k_1} t \cos^{k_2} t \, dt = 0,$$

they are valid iff at least one of the integers $k_1, k_2 \geqslant 0$ is odd.

Under the hypotheses of Corollary 2, put

$$F(x, y, \lambda) \stackrel{\text{def}}{=} \sum_{k=0}^{4} b_k(\lambda)\, x^k y^{4-k} + \sum_{k=0}^{5} a_k(\lambda)\, x^k y^{5-k}. \tag{13}$$

In this case,

$$d_0(\lambda, r) = \frac{\pi}{8} r^5 (5a_0(\lambda) + a_2(\lambda) + a_4(\lambda))$$

and all the conditions of Theorem 2.1 are satisfied for $\alpha = 4$, $v = 3$. Note that the function $d_0(\lambda, r)$ does not depend on the forth order terms of expansion (13).

Theorem 2.1 is inapplicable if

$$f(x, y, z, \lambda) = \sum_{k=0}^{2} b_k(\lambda)\, x^k y^{2-k} + \sum_{k=0}^{3} a_k(\lambda)\, x^k y^{3-k} + \Phi(x, y, z, \lambda) \tag{14}$$

with nonzero quadratic terms and $\Phi(x, y, z) = O(v_2^\beta)$. Here $\alpha = 2$, $v = 1$ and (10) implies $\beta \leqslant 3$. At the same time, $d_0(\lambda, r)$ is given by formula (12), therefore condition (9) is valid iff $\beta > 3$. Systems with nonlinearities (14) are considered in the next section.

Function (12) determines bifurcation points for systems (3) with the nonlinearities

$$f(x, y, z, \lambda) = \sum_{i=0}^{N} b_i(\lambda)\, |x|^{p_i}\, |y|^{q_i} + \sum_{k=0}^{3} a_k(\lambda)\, x^k y^{3-k} + \Phi(x, y, z, \lambda),$$

where $2 < p_i + q_i < 3$ and $\Phi(x, y, z, \lambda)$ satisfies (5) with $\beta > 3$. Evidently, these nonlinearities are at most twice differentiable at the origin.

## 3. MORE ACCURATE RESULT

In this section, we suppose that system (3) satisfies conditions 1–5 of Theorem 2.1, but relation (10) is not true. For such systems, function (8) does not determine bifurcation points any more. Actually, some other function plays the role of $d_0(\lambda, r)$ in the following generalization of Theorem 2.1. To introduce the necessary notation, we start with the study of small periodic solutions of the auxiliary equation[8]

$$x'' + x = F(x, x', \lambda). \tag{15}$$

This problem is equivalent to the $2\pi$-periodic problem

$$w^2 x'' + x = F(x, wx', \lambda), \qquad x(0) = x(2\pi), \quad x'(0) = x'(2\pi) \tag{16}$$

[8] If $\Phi(x, y, z, \lambda) \equiv 0$ equation (15) is equivalent to the system of the first two equations of (3).

with the unknowns $x(t)$ and $w>0$. Any solution[9] $x_*(t)$ of problem (16) with some $w=w_*$ determines the $2\pi/w_*$-periodic solution $x_*(tw_*)$ of equation (15).

Note that each nonstationary solution $x(t)$ of autonomous problem (16) generates a continuum of the solutions $x_\tau = x(t+\tau)$, $0 \leqslant \tau < 2\pi$ with the same cyclic trajectory on the phase plane. To avoid this *a priori* lack of uniqueness, we couple problem (16) with some additional restriction that extracts exactly one solution from the continuum. More precisely, we look for solutions of the form

$$x(t) = r \sin t + h(t), \tag{17}$$

where $r>0$ and the Fourier expansion of $h(t)$ does not contain the first harmonics, i.e.,

$$\int_0^{2\pi} h(t) \cos t \, dt = \int_0^{2\pi} h(t) \sin t \, dt = 0. \tag{18}$$

Denote by $E$ the space of continuous $2\pi$-periodic functions $h(t)$ satisfying (18). Set

$$Px(t) \overset{\text{def}}{=} \frac{1}{\pi} \int_0^{2\pi} \cos(t-s)\, x(s)\, ds, \qquad Q \overset{\text{def}}{=} I - P. \tag{19}$$

Then $Q$ projects the space of all continuous $2\pi$-periodic functions onto $E$ along the plane spanned on the functions $\cos t$ and $\sin t$. Substituting (17) in (16), one obtains the equation

$$r(1-w^2) \sin t + w^2 h'' + h = F(r \sin t + h, wr \cos t + wh', \lambda),$$

which is equivalent to the system

$$0 = \int_0^{2\pi} \cos t \, F(r \sin t + h(t), wr \cos t + wh'(t), \lambda)\, dt, \tag{20}$$

$$\pi[1-w^2]\, r = \int_0^{2\pi} \sin t \, F(r \sin t + h(t), wr \cos t + wh'(t), \lambda)\, dt, \tag{21}$$

$$w^2 h'' + h = QF(r \sin t + h(t), wr \cos t + wh'(t), \lambda), \tag{22}$$

[9] Everywhere we consider the classical solutions only; $2\pi$-periodic functions are identified with their restrictions to the segment $[0, 2\pi]$.

coupled with the periodicity conditions

$$h(0) = h(2\pi), \qquad h'(0) = h'(2\pi). \tag{23}$$

Here the unknowns are $w$, $r > 0$, and $h = h(t) \in E$.

First consider the system of equation (21) and problem (22)–(23), disregarding equation (20). For the moment, $w$ and $h = h(t)$ are unknowns and $\lambda$, $r$ are parameters. For every $w > 1/2$ we denote by $B(w)$ the linear operator that maps any function $u(t) \in E$ to a unique solution $h = B(w)\, u \in E \cap C^2$ of the equation $w^2 h'' + h = u(t)$ satisfying conditions (18) and (23). The existence follows from $u \in E$ and the uniqueness follows from $h \in E$.

LEMMA 3.1. *Suppose the function* $F(x, y, \lambda)$ *satisfies relations* (6) *and* (7) *with some* $\alpha > 1$, $v > 0$. *Then there is a* $K > 0$ *and an* $\varepsilon > 0$ *such that the following statements are valid:*

(i) *For every* $r \in (0, \varepsilon)$ *and every* $\lambda \in \Lambda$ *system* (21)–(23) *has a unique solution*[10]

$$w_* = w_*(\lambda, r), \qquad h_*(t) = h_*(t, \lambda, r) \in E \cap C^2 \tag{24}$$

*such that*

$$|w_* - 1| < K r^{\alpha - 1}, \qquad \|h_*(t)\|_{C^1} < K r^{\alpha}; \tag{25}$$

(ii) *Functions* (24) *and* $h'_*(t) = \frac{\partial}{\partial t} h_*(t, \lambda, r)$ *are continuous w.r.t. the set of all their arguments* $\lambda$, $r$, *and* $t$;

(iii) *For every* $r \in (0, \varepsilon)$, $\lambda \in \Lambda$ *the iterations*

$$w_0 = 1, \qquad h_0 \equiv 0,$$

$$h_{n+1}(t) = B(w_n)\, QF(r \sin t + h_n(t), w_n r \cos t + w_n h'_n(t), \lambda), \tag{26}$$

$$w_{n+1} = \left(1 - \frac{1}{\pi r} \int_0^{2\pi} \sin t\, F(r \sin t + h_n(t), w_n r \cos t + w_n h'_n(t), \lambda)\, dt\right)^{\frac{1}{2}}$$

*converge to solution* (24) *of system* (21)–(23), *moreover, the estimates*

$$\|h_n(t) - h_*(t)\|_{C^1} \leqslant K^{n+1} r^{vn+\alpha}, \qquad |w_n - w_*| \leqslant K^{n+1} r^{vn+\alpha-1} \tag{27}$$

*hold for all* $n = 0, 1, 2, \ldots$

[10] In other words, system (21)–(23) determines a unique implicit function $\{\lambda, r\} \to \{w, h\}$ which maps $\Lambda \times (0, \varepsilon)$ into $\mathbb{R} \times (E \cap C^2)$.

Set

$$d_*(\lambda, r) \overset{\text{def}}{=} \int_0^{2\pi} \cos t\, F(r \sin t + h_*(t), w_* r \cos t + w_* h_*'(t), \lambda)\, dt. \quad (28)$$

By Lemma 3.1, the system of all three equations (20)–(22) coupled with conditions (18) and (23) is equivalent to the equation $d_*(\lambda, r) = 0$ which is obtained by substituting functions (24) into equation (20). Therefore every solution $(\lambda^*, r^*)$, $r^* > 0$ of the equation $d_*(\lambda, r) = 0$ determines the solution $x(t) = r^* \sin t + h_*(t, \lambda^*, r^*)$ of problem (16) with $w = w_*(\lambda^*, r^*)$, $\lambda = \lambda^*$. Now we can formulate the main result for system (3).

THEOREM 3.1. *Let conditions* 1–5 *of Theorem* 2.1 *be satisfied. Suppose in every vicinity of the point* $\lambda_0$ *there are points* $\lambda^-$, $\lambda^+$ *such that function* (28) *satisfies the relations*

$$\lim_{r \to +0} \frac{d_*(\lambda^-, r)}{r^\beta} = -\infty, \qquad \lim_{r \to +0} \frac{d_*(\lambda^+, r)}{r^\beta} = +\infty. \quad (29)$$

*Then* $\lambda_0$ *is a bifurcation point with the frequency* 1 *for system* (3).

In particular, (29) holds if

$$d_*(\lambda, r) = a(\lambda)\, r^{\beta_1} + o(r^{\beta_1}), \qquad \beta_1 < \beta, \quad (30)$$

where $a(\lambda)$ takes the values of both sign in any vicinity of the point $\lambda_0$.

Evidently, under condition (29) of Theorem 3.1 the equation $d_*(\lambda, r) = 0$ has at least one solution $\lambda(r) \in (\lambda^-, \lambda^+)$ for any $r$ small enough.

To verify relations (29), one can use the approximations

$$d_n(\lambda, r) \overset{\text{def}}{=} \int_0^{2\pi} \cos t\, F(r \sin t + h_n(t), w_n r \cos t + w_n h_n'(t), \lambda)\, dt$$

of function (28), where $w_n = w_n(\lambda, r)$, $h_n(t) = h_n(t, \lambda, r)$ are given by (26). The Lipschitz condition (7) and estimates (25), (27) imply

$$\begin{gathered} |d_n(\lambda, r) - d_*(\lambda, r)| \leqslant C(n)\, r^{\nu(n+1)+\alpha}, \\ \lambda \in \Lambda, \quad r \in (0, \varepsilon), \quad n = 0, 1, 2, \ldots, \end{gathered} \quad (31)$$

therefore relations (29) are equivalent to the relations

$$\lim_{r \to +0} \frac{d_N(\lambda^-, r)}{r^\beta} = -\infty, \qquad \lim_{r \to +0} \frac{d_N(\lambda^+, r)}{r^\beta} = +\infty, \quad (32)$$

where the integer $N \geqslant 0$ is defined by

$$\nu N + \alpha < \beta \leqslant \nu(N+1) + \alpha. \tag{33}$$

In particular, if (10) is valid, then $N = 0$ and relations (29) are equivalent to (9); therefore Theorem 2.1 follows from Theorem 3.1. If $N > 0$ Theorem 2.1 is inapplicable. To apply Theorem 3.1, one should construct the functions $h_N(t, \lambda, r)$ and $w_N(\lambda, r)$ by iteration procedure (26) (which requires to solve $N$ linear periodic problems for equations of the form $w^2 h'' + h = u(t)$), then calculate the function $d_N(\lambda, r)$ and check (32).

COROLLARY 3.1. *Suppose conditions* 1, 2 *of Theorem* 2.1 *hold and the function* $f(x, y, z, \lambda)$ *has form* (14), *where* $\Phi(x, y, z, \lambda)$ *satisfies* (5) *with* $\beta > 3$. *Suppose the function*

$$b_1(\lambda)(b_0(\lambda) + b_2(\lambda)) + 3a_0(\lambda) + a_2(\lambda)$$

*takes the values of both sign in any vicinity of the point* $\lambda_0$. *Then* $\lambda_0$ *is a bifurcation point with the frequency* 1 *for system* (3).

Under the assumptions of Corollary 3.1 we take

$$F(x, y, \lambda) = \sum_{k=0}^{2} b_k(\lambda)\, x^k y^{2-k} + \sum_{k=0}^{3} a_k(\lambda)\, x^k y^{3-k}. \tag{34}$$

Now conditions 1–5 of Theorem 2.1 are satisfied for $\alpha = 2$, $\nu = 1$. We can assume without loss of generality that $\beta \leqslant 4$ (if conditions 1–5 are valid for $\beta > 4$, they are also valid for $\beta = 4$), therefore relations (29) are equivalent to (32) for $N = 1$. Substituting $x = r \sin t$, $y = r \cos t$ in (34), one obtains

$$F(r \sin t, r \cos t, \lambda) = r^2(\varphi_0 + \varphi_2(t)) + r^3(\varphi_1 + \varphi_3(t)),$$

where

$$\varphi_0 = \frac{b_0(\lambda) + b_2(\lambda)}{2}, \qquad \varphi_2(t) = \frac{b_1(\lambda)}{2} \sin 2t + \frac{b_0(\lambda) - b_2(\lambda)}{2} \cos 2t,$$

$$\varphi_1(t) = C_1(\lambda) \sin t + C_2(\lambda) \cos t, \qquad \varphi_3(t) = C_3(\lambda) \sin 3t + C_4(\lambda) \cos 3t$$

(we do not use the exact values of the coefficients $C_i(\lambda)$). Therefore

$$\begin{aligned} w_1 &= \left(1 - \frac{1}{\pi r} \int_0^{2\pi} \sin t\, F(r \sin t, r \cos t, \lambda)\, dt\right)^{\frac{1}{2}} \\ &= \sqrt{1 - C_1(\lambda)\, r^2} = 1 + O(r^2). \end{aligned} \tag{35}$$

The function $h_1 = h_1(t, \lambda, r) \in E$ is a periodic solution of the equation $h'' + h = QF(r \sin t, r \cos t, \lambda)$, i.e., $h_1'' + h_1 = r^2(\varphi_0 + \varphi_2(t)) + r^3\varphi_3(t)$, hence

$$h_1 = r^2\varphi_0 - \frac{r^2}{3}\varphi_2(t) - \frac{r^3}{8}\varphi_3(t). \tag{36}$$

Using expressions (35), (36) for $w_1$ and $h_1$ in

$$d_1(\lambda, r) = \int_0^{2\pi} \cos t\, F(r \sin t + h_1, w_1 r \cos t + w_1 h_1', \lambda)\, dt,$$

after simple calculations, one obtains

$$d_1(\lambda, r) = [b_1(\lambda)(b_0(\lambda) + b_2(\lambda)) + 3a_0(\lambda) + a_2(\lambda)]\, r^3 + O(r^4),$$

the second order terms disappear. By assumption, the cubic term takes the values of both sign in any vicinity of the point $\lambda_0$, therefore relations (32) and (29) hold and all the conditions of Theorem 3.1 are valid.

## 4. PROPERTIES OF SMALL CYCLES

Here we discuss, which of the estimates $T > 2\pi$ or $T < 2\pi$ holds for the periods of small cycles generated by the Hopf bifurcation and determine if cycles arise for $\lambda > \lambda_0$ or for $\lambda < \lambda_0$. A cycle (a periodic solution) is said to be small if both its amplitude is small and its period $T$ is close to $2\pi$. Everywhere it is assumed that system (3) satisfies all the conditions of Theorem 3.1, therefore $\lambda_0$ is a bifurcation point with the frequency 1. Small cycles are considered for the values of $\lambda$ from a sufficiently small vicinity of the point $\lambda_0$.

By Lemma 3.1, functions (24) are well defined and $w_*(\lambda, r) = 1 + O(r^{\alpha-1})$. In the following theorem we suppose that $w_*(\lambda, r)$ can be represented as

$$w_*(\lambda, r) = 1 + D_0(\lambda)\, r^{\alpha 1} + o(r^{\alpha_1}) \tag{37}$$

with some $\alpha_1 < \beta - 1$. By the second estimate of (27), this relation is equivalent to

$$w_{N+1}(\lambda, r) = 1 + D_0(\lambda)\, r^{\alpha_1} + o(r^{\alpha_1}), \tag{38}$$

where $N$ is defined by (33).

THEOREM 4.1. *If $D_0(\lambda_0) < 0$, then the period $T$ of any sufficiently small periodic solution of system* (3) *with $\lambda$ sufficiently close to $\lambda_0$ is greater than $2\pi$. If $D_0(\lambda_0) > 0$, then the periods of all these solutions are less than $2\pi$.*

If $D_0(\lambda_0) = 0$, then the answer is more cumbersome; we do not consider this case.

From the proof of Theorem 4.1 (see Section 8) it follows that the periods of all sufficiently small periodic solutions $\{x, y, z\}$ of system (3) satisfy the asymptotic relation

$$T = 2\pi(1 - D_0(\lambda)\,\rho^{\alpha_1}) + o(\rho^{\alpha_1}),$$

where $\rho = \|x\|_C$.

If the function $F(x, y, \lambda)$ has form (13) and $\beta = 6$, then $N = 0$ and

$$w_{N+1} = w_1 = \left(1 - \frac{r^4}{8}[a_1(\lambda) + a_3(\lambda) + 5a_5(\lambda)]\right)^{\frac{1}{2}}$$

$$= 1 - \frac{r^4}{16}[a_1(\lambda) + a_3(\lambda) + 5a_5(\lambda)] + O(r^5).$$

If $F(x, y, \lambda)$ has form (34) and $\beta > 3$, then[11] $N = 1$, the functions $w_1$ and $h_1$ are given by (35), (36), therefore substituting the expressions for $F$, $w_1$, and $h_1$ in formula (26) and calculating, we obtain

$$w_{N+1} = w_2$$

$$= 1 - \frac{r^2}{24}(b_1^2(\lambda) + 4b_0^2(\lambda) + 10b_0(\lambda)\,b_2(\lambda) + 10b_2^2(\lambda) + 3a_1(\lambda) + 9a_3(\lambda)) + O(r^3).$$

Thus, Corollaries 1–3 can be supplemented with the following statement.

COROLLARY 4.1. *Under the conditions of Corollary* 2, *the estimate $a_1(\lambda_0) + a_3(\lambda_0) + 5a_5(\lambda_0) > 0$ implies that the period $T$ of any sufficiently small cycle of system* (3) *with $\lambda$ close enough to $\lambda_0$ is greater than $2\pi$, the opposite estimate $a_1(\lambda_0) + a_3(\lambda_0) + 5a_5(\lambda_0) < 0$ implies $T < 2\pi$. Under the conditions of Corollary* 3, *the sign of $T - 2\pi$ for any sufficiently small cycle is the same as the sign of the value*

$$b_1^2(\lambda_0) + 4b_0^2(\lambda_0) + 10b_0(\lambda_0)\,b_2(\lambda_0) + 10b_2^2(\lambda_0) + 3a_1(\lambda_0) + 9a_3(\lambda_0)$$

*if this value is nonzero.*

[11] We again replace $\beta$ with $\min\{\beta, 4\}$ as in Corollary 3.

In particular, under the conditions of Corollary 1, the relation $a_1(\lambda_0)+3a_3(\lambda_0)\neq 0$ implies $\operatorname{sign}(T-2\pi)=\operatorname{sign}[a_1(\lambda_0)+3a_3(\lambda_0)]$.

Now let us pass to the question if small cycles exist for $\lambda<\lambda_0$ or for $\lambda>\lambda_0$.

Suppose the function $d_*(\lambda, r)$ has form (30); then its principal term vanishes at the bifurcation point $\lambda_0$. To determine for which $\lambda$ small cycles exist, we need to know the next term (following the principal one) in expansion (30).

Let

$$d_*(\lambda, r)=a(\lambda)\, r^{\beta_1}+b(\lambda)\, r^{\beta_2}+o(r^{\beta_2}), \qquad \beta_1<\beta_2<\beta, \tag{39}$$

where $a(\lambda_0)=0$. Suppose for simplicity that either

$$a(\lambda)(\lambda_0-\lambda)<0 \tag{40}$$

or

$$a(\lambda)(\lambda_0-\lambda)>0 \tag{41}$$

for $\lambda\neq\lambda_0$ from a small vicinity of the point $\lambda_0$. These assumptions imply relations (29).

THEOREM 4.2. *Suppose $b(\lambda_0)\neq 0$. Then sufficiently small periodic cycles of system* (3) *exist for the values of $\lambda$ satisfying $a(\lambda)\, b(\lambda_0)<0$ only. In other words, if $b(\lambda_0)>0$, then relation* (40) (*resp., relation* (41)) *implies that the small cycles exist for $\lambda<\lambda_0$ (resp., for $\lambda>\lambda_0$); if $b(\lambda_0)<0$, then relation* (40) (*resp.,* (41)) *implies that the small cycles exist for $\lambda>\lambda_0$ (resp., $\lambda<\lambda_0$).*

From the proof of Theorem 4.2 given below it follows that the amplitude $\rho=\|x\|_C$ of the first component of any sufficiently small periodic solution is related with $\lambda$ by the asymptotic equality

$$a(\lambda)=-b(\lambda)\, \rho^{\beta_2-\beta_1}+o(\rho^{\beta_2-\beta_1}).$$

By (31), formula (39) for $d_*(\lambda, r)$ is valid iff the function $d_N(\lambda, r)$ has the same form

$$d_N(\lambda, r)=a(\lambda)\, r^{\beta_1}+b(\lambda)\, r^{\beta_2}+o(r^{\beta_2}), \qquad \beta_1<\beta_2<\beta, \tag{42}$$

for $N$ defined by (33). It is easily seen that for smooth functions $F(x, y, \lambda)$ the expansion of $d_N(\lambda, r)$ contains only odd powers of $r$. Therefore 3 and 5 are the smallest exponents that fit into (42), hence one should know the terms of all orders up to at least 5 in the expansion of $F(x, y, \lambda)$ to use Theorem 4.2. If some quadratic terms of $F(x, y, \lambda)$ are nonzero, then $\alpha=2$,

$\nu = 1$ and the estimates $\beta > \beta_2 \geqslant 5$ imply $N \geqslant 3$, therefore at least three iterations (26) are required to find the coefficients in (42). For the polynomial $F(x, y, \lambda)$ of degree 5 written in the general form, the related calculations are too cumbersome to be presented here (even the resulting expressions[12] for $a(\lambda)$ and $b(\lambda)$ are rather large). We restrict ourselves with the two particular examples.

COROLLARY 4.2. *Let $f(x, y, z, \lambda)$ have the form*

$$\sum_{k=0}^{3} a_k(\lambda)\, x^k y^{3-k} + \sum_{k=0}^{4} c_k(\lambda)\, x^k y^{4-k} + \sum_{k=0}^{5} b_k(\lambda)\, x^k y^{5-k} + \Phi(x, y, z, \lambda),$$

*where $\Phi(x, y, z)$ satisfies* (5) *with $\beta = 6$. Let $3a_0(\lambda_0) + a_2(\lambda_0) = 0$, let the relation*

$$0 \neq b_* \stackrel{\text{def}}{=} 5b_0(\lambda_0) + b_2(\lambda_0) + b_4(\lambda_0) - a_0(\lambda_0)\, a_1(\lambda_0) + a_2(\lambda_0)\, a_3(\lambda_0)$$

*be valid, and let an $\varepsilon > 0$ exist such that either*

$$b_* \,[3a_0(\lambda) + a_2(\lambda)]\,(\lambda_0 - \lambda) < 0, \qquad \lambda \neq \lambda_0,\ |\lambda - \lambda_0| < \varepsilon, \tag{43}$$

*or*

$$b_* \,[3a_0(\lambda) + a_2(\lambda)]\,(\lambda_0 - \lambda) > 0, \qquad \lambda \neq \lambda_0,\ |\lambda - \lambda_0| < \varepsilon. \tag{44}$$

*Then $\lambda_0$ is a bifurcation point for system* (3) *with the frequency* 1. *Moreover, if estimate* (43) *is valid then sufficiently small cycles exist only for $\lambda < \lambda_0$; if estimate* (44) *holds then small cycles exist for $\lambda > \lambda_0$.*

Under the conditions of Corollary 5, one has $\alpha = 3$, $\nu = 2$, $N = 1$, and

$$w_1 = 1 - \frac{r^2}{8}\,(a_1(\lambda) + 3a_3(\lambda)) + O(r^3),$$

$$h_1 = \frac{r^3}{32}\,((a_3(\lambda) - a_1(\lambda)) \sin 3t + (a_2(\lambda) - a_0(\lambda)) \cos 3t) + O(r^4).$$

Formula (42) reads as

$$d_1(\lambda, r) = \frac{\pi r^3}{4}\,(3a_0(\lambda) + a_2(\lambda)) + \frac{\pi r^5}{16}\, b(\lambda) + O(r^6),$$

[12] The coefficients $a(\lambda)$, $b(\lambda)$ of $d_N(\lambda, r)$ are the polynomials over the coefficients of the polynomial $F(x, y, \lambda)$.

where

$$b(\lambda)=10b_0(\lambda)+2b_2(\lambda)+2b_4(\lambda)-5a_0(\lambda)\,a_1(\lambda)$$
$$-12a_0(\lambda)\,a_3(\lambda)-a_1(\lambda)\,a_2(\lambda)-2a_2(\lambda)\,a_3(\lambda).$$

The relation $3a_0(\lambda_0)+a_2(\lambda_0)=0$ implies that $b(\lambda_0)=2b_*$.

In the following example the function $F(x, y, \lambda)=F(y, \lambda)$ contains the terms of all orders from 2 to 5.

COROLLARY 4.3. *Suppose*

$$f(x, y, z, \lambda)=c_0(\lambda)\,y^2+c_1(\lambda)\,y^3+c_2(\lambda)\,y^4+c_3(\lambda)\,y^5+\Phi(x, y, z, \lambda),$$

*where* $\Phi(x, y, z, \lambda)$ *satisfies* (5) *with* $\beta=6$, *the relations*

$$c_1(\lambda_0)=0, \qquad c_3(\lambda_0)\neq 0$$

*are valid, and either*

$$c_3(\lambda_0)\,c_1(\lambda)\,(\lambda_0-\lambda)<0, \qquad \lambda\neq\lambda_0,\ |\lambda-\lambda_0|<\varepsilon, \tag{45}$$

*or*

$$c_3(\lambda_0)\,c_1(\lambda)\,(\lambda_0-\lambda)>0, \qquad \lambda\neq\lambda_0,\ |\lambda-\lambda_0|<\varepsilon. \tag{46}$$

*Then* $\lambda_0$ *is a bifurcation point for system* (3) *with the frequency* 1. *Estimate* (45) *implies that small cycles exist only for* $\lambda<\lambda_0$; *estimate* (46) *implies that small cycles exist for* $\lambda>\lambda_0$.

Here $N=3$, i.e., three iterations (26) are required; calculations give

$$d_3(\lambda, r)=\frac{3\pi r^3}{4}\,c_1(\lambda)+\frac{\pi r^5}{24}\,[15c_3(\lambda)-16c_0^2(\lambda)\,c_1(\lambda)]+O(r^6).$$

## 5. REMARKS

### 5.1. *On Reduction to System* (3)

Consider generic system (1), where the nonlinearity satisfies (2) and let the matrix $\mathscr{A}(\lambda)$ have the pair of simple imaginary eigenvalues $\pm iw(\lambda)$, $w(\lambda)>0$ for all $\lambda\in(0,1)$. If the function $f(x, \lambda)$ satisfies some minimal smoothness assumptions, such a system can be reduced to form (3) in a vicinity of the origin. First note that in some appropriate basis the matrix $\mathscr{A}(\lambda)$ has the form

$$\mathscr{A}(\lambda) = \begin{pmatrix} 0 & w(\lambda) & 0 & \dots & 0 \\ -w(\lambda) & 0 & 0 & \dots & 0 \\ 0 & 0 & c_{11}(\lambda) & \dots & c_{1m_1}(\lambda) \\ & \dots & & \dots & \\ 0 & 0 & c_{m_1 1}(\lambda) & \dots & c_{m_1 m_1}(\lambda) \end{pmatrix}$$

and system (1) reads as

$$\begin{cases} x_1' = \quad w(\lambda)\, x_2 & + f_1(x_1, x_2, \ldots, x_m, \lambda), \\ x_2' = -w(\lambda)\, x_1 & + f_2(x_1, x_2, \ldots, x_m, \lambda), \\ x_3' = \quad \sum_{j=1}^{m_1} c_{1j}(\lambda)\, x_{j+2} & + f_3(x_1, x_2, \ldots, x_m, \lambda), \\ \dots \quad \dots & \dots \\ x_m' = \quad \sum_{j=1}^{m_1} c_{m_1 j}(\lambda)\, x_{j+2} & + f_m(x_1, x_2, \ldots, x_m, \lambda). \end{cases}$$

Rescaling the time by the linear (for each $\lambda$) transformation $\tau = w(\lambda)\, t$, one obtains

$$\begin{cases} x_1' = \quad x_2 + \tilde{f}_1(x_1, x_2, \ldots, x_m, \lambda), \\ x_2' = -x_1 + \tilde{f}_2(x_1, x_2, \ldots, x_m, \lambda), \\ \dots \end{cases}$$

and finally, the nonlinear coordinate transform

$$x = x_1, \quad y = x_2 + \tilde{f}_1(x_1, x_2, \ldots, x_m, \lambda), \quad z_1 = x_3, \quad \ldots, \quad z_{m_1} = x_m \tag{47}$$

brings the system to form (3). If the function $\tilde{f}_1(x_1, x_2, \ldots, x_m, \lambda)$ is continuously differentiable, then transformation (47) is a diffeomorphism[13] (for each $\lambda$) in a vicinity of the origin, hence systems (1) and (3) are locally equivalent.

## 5.2. *Existence of Cycles with Different Periods*

If the matrix $\mathscr{A}(\lambda)$ has two pairs of simple imaginary eigenvalues, then one and the same parameter value $\lambda_0$ can be both a bifurcation point with

[13] This follows from (2), the reader can easily check the details.

the frequency $w_0$ and a bifurcation point with some other frequency $w_1 \neq w_0$. In particular, under appropriate assumptions on the nonlinearities $f_k(x_1, x_2, x_3, x_4, \lambda)$, this is the case for the system

$$\begin{cases} x_1' = x_2, \\ x_2' = -x_1 + f_2(x_1, x_2, x_3, x_4, \lambda), \\ x_3' = \sqrt{2}\, x_4, \\ x_4' = -\sqrt{2}\, x_3 + f_4(x_1, x_2, x_3, x_4, \lambda), \end{cases} \tag{48}$$

here $w_0 = 1$, $w_1 = \sqrt{2}$. The linear system with $f_2(\dots) \equiv f_4(\dots) \equiv 0$ is a trivial example. More interesting results can be derived from Theorems 2.1 and 3.1 for system (48) with the nonlinearities $f_k(\dots)$ of the form

$$f_2(x_1, x_2, x_3, x_4, \lambda) = F_2(x_1, x_2, \lambda) + \Phi_2(x_1, x_2, x_3, x_4, \lambda),$$
$$f_4(x_1, x_2, x_3, x_4, \lambda) = F_4(x_3, x_4, \lambda) + \Phi_4(x_1, x_2, x_3, x_4, \lambda).$$

## 5.3. *On the Iteration Procedure*

To check condition (29) or equivalent condition (32) of Theorem 3.1, $N$ steps of iteration procedure (26) are required, the number $N$ is determined by (33). It follows from the proof of Lemma 3.1 given in the next section that the functions $h_n = h_n(t, \lambda, r)$, $w_n = w_n(\lambda, r)$ can be replaced with the functions[14] $\tilde{h}_n = h_n + O(r^{\nu n+\alpha})$, $\tilde{w}_n = w_n + O(r^{\nu n+\alpha-1})$ at each step of the iteration procedure, i.e., one can ignore the terms of order $\nu n+\alpha$ and higher order terms in the expansions of the functions $h_n$ and $rw_n$. This will lead to the error $O(r^\beta)$ in the expressions for $d_N(\lambda, r)$ and $d_*(\lambda, r)$, which is small enough to check (29).

## 5.4. *Multiplicity of Solutions*

In Theorem 3.1 we establish that for every sufficiently small $r > 0$ system (3) with some $\lambda = \lambda_r$ has at least one $2\pi/w_r$-periodic solution $\{x(t; r), y(t; r), z(t; r)\}$ such that $x(t; r) = r \sin w_r t + h(w_r t; r)$, where $h(t; r)$ satisfies (18) and

$$\|x(t; r)\|_C + \|y(t; r)\|_C + \|z(t; r)\|_C \to 0, \qquad \lambda_r \to \lambda_0, \quad w_r \to 1 \quad \text{as} \quad r \to +0.$$

[14] Here $\tilde{h}_n = h_n + O(r^{\nu n+\alpha})$ means that $\|\tilde{h}_n - h_n\|_{C^1} = O(r^{\nu n+\alpha})$.

Just a minor modification of the formulations and proofs leads to sufficient conditions for existence of $k>1$ families of small cycles parameterized by $r$.

THEOREM 5.1. *Let conditions* 1–5 *of Theorem* 2.1 *be satisfied. Suppose there exist functions* $\ell_0(r), \ell_1(r), ..., \ell_k(r)$, *where* $k \geqslant 1$, $0<r \leqslant r_0$ *such that*

$$\ell_0(r)<\ell_1(r)< \cdots <\ell_k(r), \qquad \ell_j(r) \to \lambda_0 \quad \textit{as} \quad r \to +0, j=0, 1, ..., k$$

*and function* (28) *satisfies either the relations*

$$(-1)^j \lim_{r \to +0} r^{-\beta}\, d_*(\ell_j(r), r)=-\infty, \qquad j=0, 1, ..., k, \tag{49}$$

*or the relations*

$$(-1)^j \lim_{r \to +0} r^{-\beta} d_*(\ell_j(r), r)=+\infty, \qquad j=0, 1, ..., k. \tag{50}$$

*Then* $\lambda_0$ *is a bifurcation point with the frequency* 1 *for system* (3). *Moreover, there are functions* $\lambda_j(r)$, $w_j(r)$, $j=1, ..., k$ *satisfying*

$$\ell_0(r)<\lambda_1(r)<\ell_1(r)< \cdots <\lambda_k(r)<\ell_k(r),$$

$$\lambda_j(r) \to \lambda_0, w_j(r) \to 1 \quad \textit{as} \quad r \to +0$$

*such that for every small* $r>0$ *and every* $j=1, ..., k$ *system* (3) *with* $\lambda=\lambda_j(r)$ *has a* $2\pi/w_j(r)$*-periodic solution* $\{x_j(t; r), y_j(t; r), z_j(t; r)\}$, $x_j(t; r) = r \sin w_j(r)\, t+h_j(w_j(r)\, t; r)$, *where* $h_j(t; r)$ *satisfies* (18) *and* $\|x_j(t; r)\|_C+\|y_j(t; r)\|_C+\|z_j(t; r)\|_C \to 0$ *as* $r \to +0$.

For example, suppose $d_*(\lambda, r)=\lambda^3 r^3-\lambda r^5+O(r^7)$ and $\beta=7$ (one can easily construct the corresponding function $F(x, y, \lambda)$ by the formulas used in Corollaries 5 or 6). Then relations (49) are valid for $\ell_0(r)=-2r$, $\ell_1(r)=-r/2$, $\ell_2(r)=r/2$, $\ell_3(r)=2r$.

For

$$d_*(\lambda, r)=r^3(\lambda-r^2)(2\lambda-r^2)+o(r^7), \qquad \beta=8, \tag{51}$$

relations (50) hold for $\ell_0(r) \equiv 0$, $\ell_1(r)=2r^2/3$, $\ell_2(r)=2r^2$.

### 5.5. *Continuous Branches of Solutions*

Suppose the function $d_*(\lambda, r)$ has form (39). Then simple additional conditions considered in Section 4 guarantee that system (3) has small periodic solutions *for all values* of $\lambda$ from one of the intervals $(\lambda_0-\varepsilon, \lambda_0)$ and $(\lambda_0, \lambda_0+\varepsilon)$.

THEOREM 5.2. *Let conditions* 1–5 *of Theorem* 2.1 *be satisfied. Suppose there exist the functions* $\rho(\lambda)$, $R(\lambda)$ *defined on some set* $\tilde{\Lambda} \subset \Lambda$ *with the limit point* $\lambda_0$ *such that*

$$0 < \rho(\lambda) < R(\lambda), \qquad R(\lambda) \to 0 \quad \text{as} \quad \lambda \to \lambda_0, \lambda \in \tilde{\Lambda}$$

*and either the relations*

$$\begin{aligned} &\lim_{\lambda \to \lambda_0,\, \lambda \in \tilde{\Lambda}} R^{-\beta}(\lambda)\, d_*(\lambda, \rho(\lambda)) = -\infty, \\ &\lim_{\lambda \to \lambda_0,\, \lambda \in \tilde{\Lambda}} R^{-\beta}(\lambda)\, d_*(\lambda, R(\lambda)) = +\infty, \end{aligned} \tag{52}$$

*or the relations*

$$\begin{aligned} &\lim_{\lambda \to \lambda_0,\, \lambda \in \tilde{\Lambda}} R^{-\beta}(\lambda)\, d_*(\lambda, \rho(\lambda)) = +\infty, \\ &\lim_{\lambda \to \lambda_0,\, \lambda \in \tilde{\Lambda}} R^{-\beta}(\lambda)\, d_*(\lambda, R(\lambda)) = -\infty \end{aligned} \tag{53}$$

*are valid. Then for every* $\lambda \in \tilde{\Lambda}$ *sufficiently close to* $\lambda_0$ *system* (3) *has a* $2\pi/w_\lambda$-*periodic solution* $\{x(t; \lambda), y(t; \lambda), z(t; \lambda)\}$ *such that* $\|x(t; \lambda)\|_C + \|y(t; \lambda)\|_C + \|z(t; \lambda)\|_C \to 0, w_\lambda \to 1$ *as* $\lambda \to \lambda_0$, $\lambda \in \tilde{\Lambda}$ *and the first component of the solution has the form* $x(t; \lambda) = r(\lambda) \sin w_\lambda t + h(w_\lambda t; \lambda)$, *where* $\rho(\lambda) < r(\lambda) < R(\lambda)$ *and* $h(t; \lambda)$ *satisfies* (18).

If the coefficients in (39) satisfy $a(\lambda_0) = 0$, $b(\lambda) \neq 0$ and the function $a(\lambda)$ takes the values of both sign in any vicinity of $\lambda_0$, then either relations (52) or (53) are valid for

$$\rho(\lambda) = [-a(\lambda)/b(\lambda)]^{\frac{1}{\beta_2 - \beta_1}}/2, \quad R(\lambda) = 2[-a(\lambda)/b(\lambda)]^{\frac{1}{\beta_2 - \beta_1}},$$
$$\tilde{\Lambda} = \{\lambda \in \Lambda : a(\lambda)\, b(\lambda) < 0\}.$$

Therefore, under the assumptions of Theorem 4.2 the small cycles of system (3) exist for all values of $\lambda$ from the appropriate interval $(\lambda_0 - \varepsilon, \lambda_0)$ or $(\lambda_0, \lambda_0 + \varepsilon)$.

The proofs of Theorems 3.1 and 5.2 are similar, the main difference is that in the proof of Theorem 3.1 below the variable $r$ is considered as a parameter and $\lambda$ is the unknown, while to prove Theorem 5.2 one should consider $r$ as the unknown and $\lambda$ as a parameter (like it is in the original problem).

Theorem 5.2 also gives sufficient conditions for the existence of multiple small periodic solutions to system (3). Suppose there exist functions

$$0 < \rho_1(\lambda) < R_1(\lambda) \leqslant \rho_2(\lambda) < R_2(\lambda) \leqslant \cdots \leqslant \rho_k(\lambda) < R_k(\lambda), \qquad \lambda \in \tilde{\Lambda}$$

such that $R_k(\lambda) \to 0$ as $\lambda \to \lambda_0$, $\lambda \in \tilde{\Lambda}$ and for every $j = 1, ..., k$ either the relations

$$\lim_{\substack{\lambda \to \lambda_0, \\ \lambda \in \tilde{\Lambda}}} R_j^{-\beta}(\lambda)\, d_*(\lambda, \rho_j(\lambda)) = -\infty, \qquad \lim_{\substack{\lambda \to \lambda_0, \\ \lambda \in \tilde{\Lambda}}} R_j^{-\beta}(\lambda)\, d_*(\lambda, R_j(\lambda)) = +\infty,$$

or the relations

$$\lim_{\substack{\lambda \to \lambda_0, \\ \lambda \in \tilde{\Lambda}}} R_j^{-\beta}(\lambda)\, d_*(\lambda, \rho_j(\lambda)) = +\infty, \qquad \lim_{\substack{\lambda \to \lambda_0, \\ \lambda \in \tilde{\Lambda}}} R_j^{-\beta}(\lambda)\, d_*(\lambda, R_j(\lambda)) = -\infty$$

hold. Then by Theorem 5.2, for every $\lambda \in \tilde{\Lambda}$ sufficiently close to $\lambda_0$ system (3) has $k$ distinct periodic solutions $\{x_j(t; \lambda), y_j(t; \lambda), z_j(t; \lambda)\}$, $j = 1, ..., k$ with the periods $2\pi/w_j(\lambda)$ such that $\|x_j(t; \lambda)\|_C + \|y_j(t; \lambda)\|_C + \|z_j(t; \lambda)\|_C \to 0$, $w_j(\lambda) \to 1$ as $\lambda \to \lambda_0$ and $x_j(t; \lambda) = r_j(\lambda) \sin w_j(\lambda)\, t + h(w_j(\lambda)\, t; \lambda)$, where $\rho_j(\lambda) < r_j(\lambda) < R_j(\lambda)$ and $h_j(t; \lambda)$ satisfies (18).

For example, if relations (51) are valid, then one can take

$$\rho_1(\lambda) = \sqrt{\frac{\lambda}{2}},\ R_1(\lambda) = \rho_2(\lambda) = \sqrt{\frac{3\lambda}{2}},\ R_2(\lambda) = 2\sqrt{\lambda},\ \tilde{\Lambda} = \{\lambda\colon \lambda > 0\},$$

hence system (3) has at least two different small cycles with the periods close to $2\pi$ for every sufficiently small $\lambda > 0$.

The equalities $d_*(\lambda, r) = r^3(\lambda - r^2)(\lambda^2 - r^2) + o(r^7)$, $\beta = 8$ imply the existence of at least two small cycles for every $\lambda \in (0, \varepsilon)$ and at least one small cycle for every $\lambda \in (-\varepsilon, 0)$, where $\varepsilon$ is sufficiently small.

### 5.6. *Systems with Symmetries*

Theorems 2.1 and 3.1 can not be used if $d_*(\lambda, r) \equiv 0$ for all $\lambda \in \Lambda$ and all small $r > 0$, since condition (29) is not satisfied.

Suppose relations (6), (7) hold. Then the identity $d_*(\lambda, r) \equiv 0$ means that a small vicinity of the origin in the phase plane $\{x, y\}$ of Eq. (15) for each $\lambda \in \Lambda$ consists of the cycles surrounding the zero equilibrium, hence all $\lambda \in \Lambda$ are bifurcation points for Eq. (15). This is true for some symmetric systems, e.g. for Eq. (15), where

$$F(x, y, \lambda) = F(x, -y, \lambda), \quad \lambda \in \Lambda. \tag{54}$$

Relation (54) implies that the zero equilibrium is surrounded by the cycles symmetric w.r.t. the axis $y = 0$, hence $d_*(\lambda, r) \equiv 0$ whenever $F(x, y, \lambda)$ is even in $y$.

One should take this into account when extracting the principal part $F(x, y, \lambda)$ from the nonlinearity $f(x, y, z, \lambda)$ in order to apply Theorems 2.1, and 3.1. For example, consider the equation

$$x'' + x = \psi(x, \lambda) + a(\lambda)(x')^{2n+1} \tag{55}$$

with some $n \in \mathbb{N}$. Here the function $F(x, y, \lambda)$ should include the term $a(\lambda)\, y^{2n+1}$ (otherwise $d_*(\lambda, r) \equiv 0$) and in fact, it is the only term in the right-hand side of (55) that determines the bifurcation points: equation (55) has a continuum of small cycles if $a(\lambda) = 0$ and no small cycles[15] if $a(\lambda) \neq 0$, i.e., the set of bifurcation points coincides with the set of zeroes of the function $a(\lambda)$.

In the following example, Theorems 2.1, 3.1 can not be used with any choice of the function $F(x, y, \lambda)$. The system

$$\begin{cases} x' = y, \\ y' = -x + \psi(x, \lambda) + a(\lambda)\, z x^n, \\ z' = -z - a(\lambda)\, y x^n \end{cases}$$

has a continuum of small cycles for $a(\lambda) = 0$ and has no small cycles for $a(\lambda) \neq 0$. To see this, one can rewrite the system as

$$x'' + x = \psi(x, \lambda) + a(\lambda)\, z x^n, \qquad z' = -z - a(\lambda)\, x' x^n$$

and multiply the first equation by $x'$ and the second one by $z$. After summation and integration along the period one obtains $\int z^2(t)\, dt = 0$, and $z = 0$ implies either $a(\lambda) = 0$ or $x = 0$. Therefore the bifurcation points are the zeroes of the function $a(\lambda)$, i.e., they are determined by the term $a(\lambda)\, z x^n$, whereas in Theorems 2.1, 3.1 bifurcation points are determined by equation (15) independent of $z$. Clearly, in this example

$$d_0(\lambda, r) = \int_0^{2\pi} \cos t\, \psi(r \sin t, \lambda)\, dt$$

is identically zero.

## 6. PROOF OF LEMMA

Denote by $C_0$ and $C_0^1$ the subspaces

$$C_0 = \{x(t) \in C : x(0) = x(2\pi)\},$$
$$C_0^1 = \{x(t) \in C^1 : x(0) = x(2\pi),\, x'(0) = x'(2\pi)\}$$

of the spaces $C$ and $C^1$. Set $\|\cdot\|_E = \|\cdot\|_C$, then $E$ is a subspace[16] of $C_0$.

[15] If $a(\lambda) = 0$, the cycles are the level lines of the function $V(x, y) = x^2 + y^2 - 2 \int_0^x \psi(\xi, \lambda)\, d\xi$. If $a(\lambda) \neq 0$, the function $a(\lambda)\, V(x, y)$ strictly increases along the trajectories of equation (55) in some vicinity of the origin, therefore small cycles do not exist.

[16] Let us recall that $E$ is the space of continuous $2\pi$-periodic $h(t)$, satisfying (18).

Since the operator $B(w)$ acts from $E$ to $E \cap C^2$, system (21)–(22) coupled with conditions (18), (23) is equivalent to the system

$$
\begin{aligned}
w &= \left(1-\frac{1}{\pi r}\int_0^{2\pi} \sin t\, F(r \sin t+h(t), wr \cos t+wh'(t), \lambda)\, dt\right)^{\frac{1}{2}} \\
&\stackrel{\text{def}}{=} W(w, h; \lambda, r), \\
h(t) &= B(w)\, QF(r \sin t+h(t), wr \cos t+wh'(t), \lambda) \\
&\stackrel{\text{def}}{=} H(w, h; \lambda, r),
\end{aligned}
$$

where $h \in C_0^1$.

Consider the space $\mathbb{R}\times C_0^1$ with the norm $\|(w, h)\|_r = \max\{r|w|, \|h\|_{C^1}\}$ depending on the parameter $r > 0$; by definition,

$$
\|\cdot\|_{r_1} \leqslant \|\cdot\|_{r_2} \leqslant r_2 r_1^{-1}\, \|\cdot\|_{r_1}, \qquad 0 < r_1 \leqslant r_2.
$$

We prove that for a sufficiently large $K$ and a sufficiently small $\varepsilon > 0$ the operator

$$
U_{\lambda, r}(w, h) = (W(w, h; \lambda, r), H(w, h; \lambda, r)), \qquad \lambda \in \Lambda, r \in (0, \varepsilon) \tag{56}
$$

maps the ball

$$
\Pi(r, K) = \{(w, h) \in \mathbb{R}\times C_0^1 : \|(w-1, h)\|_r \leqslant Kr^\alpha\}
$$

into its interior and contracts on this ball with the contraction coefficient $Kr^\nu < K\varepsilon^\nu < 1$, i.e., $\|(w_i-1, h_i)\|_r \leqslant Kr^\alpha$, $i = 1, 2$, implies that

$$
\begin{cases}
\|(W(w_i, h_i; \lambda, r)-1, H(w_i, h_i; \lambda, r))\|_r < Kr^\alpha, \\
\|U_{\lambda, r}(w_1, h_1)-U_{\lambda, r}(w_2, h_2)\|_r \leqslant Kr^\nu\, \|(w_1, h_1)-(w_2, h_2)\|_r.
\end{cases} \tag{57}
$$

First note that the operator $B(w)$ exists for any $w$ from any segment $\Omega = [1-\delta, 1+\delta]$ if $\delta < 1/2$. Let us fix such a $\delta$. The estimates

$$
\left.
\begin{aligned}
&\sup_{w \in \Omega} \|B(w)\|_{E \to C_0^1} \leqslant p_1, \\
&\|B(w_1)-B(w_2)\|_{E \to C_0^1} \leqslant p_2\, |w_1-w_2|,
\end{aligned}
\right. \qquad w_1, w_2 \in \Omega \tag{58}
$$

are valid. Also, $\|Q\|_{C \to E} = q_0 < \infty$. The constants $p_1$ and $p_2$ depend on $\delta$ only, $q_0$ is an independent constant.

Take a $K_1 > 0$. Relations (6), (7) imply for all $(w_i, h_i) \in \Pi(r, K_1)$ the estimates

$$\begin{aligned}
&\|F(r \sin t + h_i(t), w_i r \cos t + w_i h_i'(t), \lambda)\|_C \\
&\quad \leqslant c_1 (\|r \sin t + h_i(t)\|_C + w_i \|r \cos t + h_i'(t)\|_C)^\alpha \\
&\quad \leqslant c_1 (1 + w_i)^\alpha (r + \|h_i\|_{C^1})^\alpha \leqslant c_1 (2 + K_1 r^{\alpha-1})^\alpha (1 + K_1 r^{\alpha-1})^\alpha r^\alpha
\end{aligned}$$

and

$$\begin{aligned}
&\|F(r \sin t + h_1(t), w_1 r \cos t + w_1 h_1'(t), \lambda) \\
&\qquad - F(r \sin t + h_2(t), w_2 r \cos t + w_2 h_2'(t), \lambda)\|_C \\
&\quad \leqslant c_1 \max\{\|r \sin t + h_i\|_C^\nu, w_i^\nu \|r \cos t + h_i'\|_C^\nu\} \\
&\qquad \times (\|h_1 - h_2\|_C + \|(w_1 - w_2) r \cos t + w_1 h_1' - w_2 h_2'\|_C) \\
&\quad \leqslant c_1 (2 + K_1 r^{\alpha-1})^\nu (1 + K_1 r^{\alpha-1})^\nu r^\nu (3 + 2K_1 r^{\alpha-1}) \\
&\qquad \times \max\{\|h_1 - h_2\|_{C^1}, r |w_1 - w_2|\}.
\end{aligned}$$

Set $q_1 = 6^\alpha c_1$ and $q_2 = 5 \cdot 6^\nu c_1$. If $K_1 r^{\alpha-1} \leqslant 1$, then

$$\begin{aligned}
&\|F(r \sin t + h_i(t), w_i r \cos t + w_i h_i'(t), \lambda)\|_C \leqslant q_1 r^\alpha, \\
&\|F(r \sin t + h_1(t), w_1 r \cos t + w_1 h_1'(t), \lambda) \\
&\qquad - F(r \sin t + h_2(t), w_2 r \cos t + w_2 h_2'(t), \lambda)\|_C \\
&\quad \leqslant q_2 r^\nu \max\{r |w_1 - w_2|, \|h_1 - h_2\|_{C^1}\};
\end{aligned}$$

let us stress that $q_1$ and $q_2$ are independent of $K_1$. Suppose $K_1 r^{\alpha-1} < \delta$, then the operator $B(w)$ is well defined. Combining the last estimates with relations (58), one obtains

$$\begin{aligned}
&|W^2(w_i, h_i; \lambda, r) - 1|/2 \leqslant q_1 r^{\alpha-1}, \qquad \|H(w_i, h_i; \lambda, r)\|_{C^1} \leqslant p_1 q_0 q_1 r^\alpha, \\
&|W^2(w_1, h_1; \lambda, r) - W^2(w_2, h_2; \lambda, r)|/2 \leqslant q_2 r^{\nu-1} \|(w_1, h_1) - (w_2, h_2)\|_r, \\
&\|H(w_1, h_1; \lambda, r) - H(w_2, h_2; \lambda, r)\|_{C^1} \\
&\quad \leqslant p_1 q_0 q_2 r^\nu \|(w_1, h_1) - (w_2, h_2)\|_r + p_2 q_0 q_1 r^\alpha |w_1 - w_2| \\
&\quad \leqslant q_0 (p_1 q_2 + p_2 q_1 r^{\alpha-\nu-1}) r^\nu \|(w_1, h_1) - (w_2, h_2)\|_r.
\end{aligned}$$

Therefore, for any $K_1 > 0$ and

$$K > \max\{q_1, p_1 q_0 q_1, q_2, q_0(p_1 q_2 + p_2 q_1)\} \tag{59}$$

($K$ is independent of $K_1$) there is a sufficiently small $\varepsilon(K_1, K) > 0$ such that the relations $(w_i, h_i) \in \Pi(r, K_1)$, $i = 1, 2$, and $0 < r < \varepsilon(K_1, K)$ imply

$$|W(w_i, h_i; \lambda, r) - 1| < Kr^{\alpha-1}, \qquad \|H(w_i, h_i; \lambda, r)\|_{C^1} < Kr^{\alpha},$$
$$|W(w_1, h_1; \lambda, r) - W(w_2, h_2; \lambda, r)| \leqslant Kr^{\nu-1} \,\|(w_1, h_1) - (w_2, h_2)\|_r,$$
$$\|H(w_1, h_1; \lambda, r) - H(w_2, h_2; \lambda, r)\|_{C^1} \leqslant Kr^{\nu} \,\|(w_1, h_1) - (w_2, h_2)\|_r.$$

This proves (57) for every $K$ satisfying (59) and every $r \in (0, \varepsilon(K, K))$.

Relations (57) with $Kr^{\nu} < 1$ imply that operator (56) has a unique fixed point $(w_*, h_*)$ in the interior of the ball $\Pi(r, K)$ and that the iterations $(w_{n+1}, h_{n+1}) = U_{\lambda, r}(w_n, h_n)$ starting from the center of this ball converge to $(w_*, h_*)$, therefore

$$\|(w_n, h_n) - (w_*, h_*)\|_r \leqslant (Kr^{\nu})^n \,\|(w_* - 1, h_*)\|_r \leqslant K^{n+1} r^{\nu n + \alpha}.$$

This is equivalent to statements (i) and (iii) of Lemma 3.1.

To prove statement (ii) is to show that the point $(w_*(\lambda, r), h_*(\lambda, r)) \in \mathbb{R} \times C_0^1$ depends continuously on the variables $\lambda$, $r$, i.e.,

$$\lim_{\lambda_1 \to \lambda,\, r_1 \to r} \|(w_*(\lambda, r), h_*(\lambda, r)) - (w_*(\lambda_1, r_1), h_*(\lambda_1, r_1))\|_r = 0 \tag{60}$$

for all $\lambda \in \Lambda$, $r \in (0, \varepsilon(K, K))$. Since $U_{\lambda, r}$ is a contracting operator in a vicinity of the point $(w_*(\lambda, r), h_*(\lambda, r))$ in the space $\mathbb{R} \times C_0^1$ with the norm $\|\cdot\|_r$, relation (60) follows by the standard argument from the uniform continuity of operator (56) w.r.t. the set of all its variables $\lambda \in \Lambda$, $r \in [r_1, r_2]$, $w \in \Omega$, and $h \in C_0^1$, $\|h\|_{C^1} \leqslant c$. The uniform continuity of operator (56) follows from continuity of the function $F(x, y, \lambda)$ and Cantor theorem.

This completes the proof.

## 7. PROOF OF THEOREM

### 7.1. *Scheme of the Proof*

First rescale the time in system (3) and obtain the system

$$\begin{cases} w\, x' = y, \\ w\, y' = -x + f(x, y, z, \lambda), \\ w\, z' = A(\lambda)\, z + g(x, y, z, \lambda) \end{cases} \tag{61}$$

with the unknown $w>0$. The vector-valued function $\{x(t), y(t), z(t)\}$ is a $2\pi$-periodic solution of system (61) iff $\{x(wt), y(wt), z(wt)\}$ is a $2\pi/w$-periodic solution of system (3). We look for $2\pi$-periodic solutions of system (61) such that $x(t)$ has form (17) with some $r \geqslant 0$ and $h(t)$ satisfying (18). Formula (17) and the first equation of (61) give

$$x(t)=r\sin t+h(t), \quad y(t)=wr\cos t+wh'(t), \qquad r\geqslant 0, \quad h(t)\in E, \tag{62}$$

therefore the second equation of (61) reads as

$$r(1-w^2)\sin t+w^2h''+h=f(r\sin t+h, wr\cos t+wh', z(t), \lambda)$$

and, like in Section 3, we replace it with the equivalent system

$$\begin{aligned}\pi[1-w^2]\,r&=\int_0^{2\pi}\sin t\, f(r\sin t+h(t), wr\cos t+wh'(t), z(t), \lambda)\,dt,\\ 0&=\int_0^{2\pi}\cos t\, f(r\sin t+h(t), wr\cos t+wh'(t), z(t), \lambda)\,dt,\\ w^2h''+h&=Qf(r\sin t+h(t), wr\cos t+wh'(t), z(t), \lambda).\end{aligned}\tag{63}$$

The last equation of (61) is not changed:

$$w\,z'=A(\lambda)\,z+g(r\sin t+h, wr\cos t+wh', z, \lambda). \tag{64}$$

The system of four equations (63)–(64) contains three scalar unknowns $r$, $\lambda$, $w$ and two unknown functions $h=h(t)\in E$ and $z=z(t)$. Below the variable $r$ is considered as a parameter, we show that for every sufficiently small $r>0$ system (63)–(64) coupled with periodicity conditions (23) and

$$z(0)=z(2\pi) \tag{65}$$

has a solution $(\lambda, w, h, z)$ such that $\lambda\to\lambda_0$, $w\to 1$, $\|h\|_{C^1}\to 0$, $\|z\|_C\to 0$ as $r\to +0$. By construction, this solution determines the continuum of $2\pi$-periodic solutions $\{x(t+\varphi), y(t+\varphi), z(t+\varphi)\}$, $\varphi\in[0, 2\pi)$, of system (61), where $x$, $y$ are defined by (62) and $\lambda$, $w$ are the same for both systems. Conversely, if $\{x(t), y(t), z(t)\}$ is a $2\pi$-periodic solution of system (61) for some $\lambda$, $w$, then $(\lambda, w, Qx(t-\varphi), z(t-\varphi))$ is a solution of problem (63)–(65), (23) for $r=\|Px(t)\|_C$, where the phase $\varphi\in[0, 2\pi)$ is defined by the relation $r\sin(t+\varphi)=Px(t)$ if $r>0$ and it may be undefined[17] if $r=0$. By (62) the amplitudes $\|x\|_C$ and $\|y\|_C$ are small iff $r$ and $\|h\|_{C^1}$ are small.

[17] In fact, we shall see that $r=0$ implies $x\equiv y\equiv z\equiv 0$.

Let us stress that we change the roles of the variables $r$ and $\lambda$. In the original problem $\lambda$ is a parameter and $r = \|Px(t)\|_C$ is an unknown amplitude of the first harmonics in the Fourier expansion for $x(t)$; the amplitude $r$ is the same for all periodic solutions $\{x(t+\varphi), y(t+\varphi), z(t+\varphi)\}$. Now $r$ is a parameter, $\varphi$ is fixed in such a way that $Px(t) = r \sin t$, and $\lambda$ is an unknown. This choice of the parameter and the unknowns allows to prove Theorem 3.1 by standard topological methods. The main point is to extract the principal terms of Eqs. (63)–(64). The last two equations of system (63)–(64) have the principal nondegenerate linear part. The principal term of the first scalar equation is $\pi(1-w^2)\,r$, its *signum* is $\operatorname{sign}(1-w)$. The principal term of the second scalar equation is $d_*(\lambda, r)$, its sign is determined by condition (29): this term is greater than $r^\beta$, the other terms are $O(r^\beta)$.

The formal proof uses homotopy technique.

### 7.2. *Homotopy*

Everywhere we consider $w$ close to 1 and $\lambda$ close to $\lambda_0$.

Denote by $\mathbb{C}_0$ the space of continuous $2\pi$-periodic functions $z(t)$: $\mathbb{R} \to \mathbb{R}^{m_1}$ with the uniform norm $\|\cdot\|_{\mathbb{C}_0} = \|\cdot\|_C$. Denote by $B_1 = B_1(w, \lambda)$ the linear operator that maps any function $v(t) \in \mathbb{C}_0$ to a unique classical $2\pi$-periodic solution $z = B_1 v$ of the equation

$$wz' = A(\lambda)\, z + v(t),$$

the existence of this solution follows from condition 1 of Theorem 2.1. By the definitions of the linear operators $B$ (see Section 6) and $B_1$, problem (63)–(65), (23) is equivalent to the system

$$\begin{cases} \displaystyle\int_0^{2\pi} \cos t\, f(x(t), y(t), z(t), \lambda)\, dt = 0, \\ \displaystyle\int_0^{2\pi} \sin t\, f(x(t), y(t), z(t), \lambda)\, dt + \pi r(w^2 - 1) = 0, \\ h - B(w)\, Qf(x(t), y(t), z(t), \lambda) = 0, \\ z - B_1(w, \lambda)\, g(x(t), y(t), z(t), \lambda) = 0, \end{cases} \tag{66}$$

where $(\lambda, w, h, z) \in \mathbb{E} = \{\mathbb{R} \times \mathbb{R} \times C_0^1 \times \mathbb{C}_0\}$ and $x = x(t)$, $y = y(t)$ are functions (62). Note that the norm of the linear operator $B_1$ satisfies the uniform estimate

$$\|B_1(w, \lambda)\|_{\mathbb{C}_0 \to \mathbb{C}_0} \leqslant p \tag{67}$$

with the number $p$ independent of $w, \lambda$. Also, we use the fact that the operators $B_1(w,\lambda)\,v$ and $B(w)\,u$ with the values in the spaces $\mathbb{C}_0$ and $C_0^1$ are completely continuous w.r.t. the sets of their arguments $w, \lambda, v$ and $w, u$ respectively, where $v \in \mathbb{C}_0$, $u \in E$.

For any small $r > 0$ consider in the space $\mathbb{E}$ the completely continuous deformation

$$\Psi_\xi(\lambda, w, h, z) = \begin{pmatrix} (1-\xi)\displaystyle\int_0^{2\pi} \cos t\, f(x(t), y(t), z(t), \lambda)\, dt + \xi d_*(\lambda, r) \\ (1-\xi)\displaystyle\int_0^{2\pi} \sin t\, f(x(t), y(t), z(t), \lambda)\, dt + \pi r(w^2 - 1 - \xi(w_*^2 - 1)) \\ h - \xi h_* - (1-\xi)\, B(w)\, Qf(x(t), y(t), z(t), \lambda) \\ z - (1-\xi)\, B_1(w, \lambda)\, g(x(t), y(t), z(t), \lambda) \end{pmatrix} \tag{68}$$

of the vector field $\Psi_0 = \Psi_0(\lambda, w, h, z) \in \mathbb{E}$ to the vector field $\Psi_1 = \Psi_1(\lambda, w, h, z) \in \mathbb{E}$; here $\xi \in [0, 1]$ is the deformation parameter, $w_* = w_*(\lambda, r)$, $h_* = h_*(\lambda, r)$ are functions (24). Now system (66) can be written as $\Psi_0(\lambda, w, h, z) = 0$. That is, to prove Theorem 3.1 we need to show that for any $\varepsilon > 0$ the vector field $\Psi_0$ with some $r \in (0, \varepsilon)$ has a zero $(\lambda, w, h, z) \in \mathbb{E}$ such that

$$|\lambda - \lambda_0| \leqslant \varepsilon, \qquad |w - 1| \leqslant \varepsilon, \qquad \|h\|_{C^1} \leqslant \varepsilon, \qquad \|z\|_C \leqslant \varepsilon. \tag{69}$$

Take any $\lambda^-, \lambda^+$ such that relations (29) hold and $|\lambda^\pm - \lambda_0| \leqslant \varepsilon$; without loss of generality, suppose $\lambda^- < \lambda^+$. Consider the parallelepiped

$$S = \{\lambda \in [\lambda^-, \lambda^+],\ w \in [1-\varepsilon, 1+\varepsilon],\ \|h\|_{C^1} \leqslant \varepsilon,\ \|z\|_C \leqslant \varepsilon\} \subset \mathbb{E}. \tag{70}$$

In the next section we prove the following statement.

LEMMA 7.1. *For any sufficiently small $\varepsilon > 0$ there is a $r_0 = r_0(\varepsilon) > 0$ such that deformation* (68) *with any $r \in (0, r_0)$ is nondegenerate*[18] *on the boundary $\partial S$ of parallelepiped* (70)*, i.e.,*

$$\Psi_\xi(\lambda, w, h, z) \neq 0, \qquad (\lambda, w, h, z) \in \partial S, \quad \xi \in [0, 1]. \tag{71}$$

[18] A nondegenerate deformation is also called homotopy.

It follows from (71) that the rotation $\gamma(\Psi_\xi, \partial S)$ of the vector field $\Psi_\xi$ (see [14]) on the boundary of parallelepiped (70) does not depend on $\xi$, in particular, the rotations $\gamma(\Psi_0, \partial S)$ and $\gamma(\Psi_1, \partial S)$ are identical. Let us calculate this common value $\gamma_0$.

Consider another deformation

$$\tilde{\Psi}_\xi(\lambda, w, h, z) = \{d_*(\lambda, r), \pi r[w^2-1-(1-\xi)(w_*^2-1)], h-(1-\xi)\, h_*, z\}, \quad \xi \in [0, 1] \tag{72}$$

of the vector field $\tilde{\Psi}_0 = \Psi_1$ to the vector field

$$\tilde{\Psi}_1 = \{d_*(\lambda, r), \pi r(w^2-1), h, z\}. \tag{73}$$

If $(\lambda, w, h, z) \in \partial S$, then at least one of the equalities $\lambda = \lambda^\pm$, $w = 1 \pm \varepsilon$, $\|h\|_{C^1} = \varepsilon$, $\|z\|_C = \varepsilon$ is valid. By condition (29),

$$d_*(\lambda^-, r) < 0, \qquad d_*(\lambda^+, r) > 0 \tag{74}$$

for any small $r > 0$, i.e., the first component of deformation (72) is nondegenerate for $\lambda = \lambda^\pm$. Estimates (25) imply that the second and the third components of (72) are nondegenerate for $w = 1 \pm \varepsilon$ and $\|h\|_{C^1} = \varepsilon$ respectively for small $r$. Finally, the last component is nondegenerate for $\|z\|_C = \varepsilon$, hence

$$\tilde{\Psi}_\xi(\lambda, w, h, z) \neq 0, \qquad (\lambda, w, h, z) \in \partial S, \quad \xi \in [0, 1].$$

Therefore deformation (72) is also a homotopy, the rotation $\gamma(\tilde{\Psi}_{xi}, \partial S)$ is well defined and independent of $\xi$. The rotation $\gamma(\tilde{\Psi}_1, \partial S)$ of vector field (73) on $\partial S$ equals $\gamma(\tilde{\Psi}_0, \partial S) = \gamma_0$.

Each of the four components of vector field (73) depends on its own unique unknown. The rotation $\gamma(\tilde{\Psi}_1, \partial S)$ may be calculated by the rotation product formula ([14]). It equals $\gamma_1\gamma_2\gamma_3\gamma_4$, where $\gamma_1$ and $\gamma_2$ are the rotations of the scalar vector fields $d_*(\lambda, r)$ and $\pi r(w^2-1)$ on the boundaries of the segments $\lambda \in [\lambda^-, \lambda^+]$ and $w \in [1-\varepsilon, 1+\varepsilon]$; $\gamma_3$ and $\gamma_4$ are the rotations of the infinite-dimensional identical vector fields $h \in C_0^1$ and $z \in \mathbb{C}_0$ on the spheres $\|h\|_{C^1} = \varepsilon$ and $\|z\|_C = \varepsilon$. Relations (74) and

$$r\pi((1-\varepsilon)^2-1) < 0, \qquad r\pi((1+\varepsilon)^2-1) > 0$$

imply that $\gamma_1 = \gamma_2 = 1$. The rotations of the identical vector fields $\gamma_3$ and $\gamma_4$ also equal 1, hence $\gamma(\tilde{\Psi}_1, \partial S) = 1$. Therefore[19] $\gamma(\Psi_0, \partial S) = 1$, this means that the equation $\Psi_0(\lambda, w, h, z) = 0$ has a solution in the interior of the parallelepiped $S$.

[19] If $\lambda^- > \lambda^+$, then $\gamma(\tilde{\Psi}_1, \partial S) = -1$.

Thus, it remain to prove Lemma 7.1 to complete the proof of Theorem 3.1.

### 7.3. *Proof of Lemma* 7.1

Suppose that the point $(\lambda, w, h, z) \in \mathbb{E}$ lies in parallelepiped (69) for some $\varepsilon \in (0, 1/2)$; suppose $\Psi_\xi(\lambda, w, h, z) = 0$ at this point for some $\xi \in [0, 1]$ and arbitrarily small $r > 0$, i.e.,

$$d_*(\lambda, r) = (1-\xi)\left(d_*(\lambda, r) - \int_0^{2\pi} \cos t\, f(x(t), y(t), z(t), \lambda)\, dt\right), \tag{75}$$

$$\pi r(w^2 - w_*^2) = (1-\xi)\left(\pi r(1-w_*^2) - \int_0^{2\pi} \sin t\, f(x(t), y(t), z(t), \lambda)\, dt\right), \tag{76}$$

$$h - h_* = (1-\xi)(B(w)\, Qf(x(t), y(t), z(t), \lambda) - h_*), \tag{77}$$

$$z = (1-\xi)\, B_1(w, \lambda)\, g(x(t), y(t), z(t), \lambda). \tag{78}$$

From (69) it follows that

$$\begin{aligned} &\|x\|_C \leqslant r + \|h\|_{C^1} \leqslant r + \varepsilon, \\ &\|y\|_C \leqslant w(r + \|h\|_{C^1}) \leqslant 2(r+\varepsilon), \qquad \|z\|_C \leqslant \varepsilon, \end{aligned} \tag{79}$$

hence $\|x\|_C + \|y\|_C + \|z\|_C \leqslant 4(r+\varepsilon)$. Relations (67) and (78) imply that

$$\begin{aligned} \|z(t)\|_C &\leqslant p\, \|g(x(t), y(t), z(t), \lambda)\|_C \\ &\leqslant p\, \|G(x(t), y(t), \lambda)\|_C + p\, \|\Gamma(x(t), y(t), z(t), \lambda)\|_C. \end{aligned}$$

However, by condition 2 of Theorem 2.1, $\sup_{\lambda \in \Lambda} |\Gamma(x, y, z, \lambda)|/|z| \to 0$ as $|x| + |y| + |z| \to 0$, so if $\varepsilon$ and $r$ are sufficiently small, then estimates (79) imply that

$$\|\Gamma(x(t), y(t), z(t), \lambda)\|_C \leqslant (2p)^{-1}\, \|z(t)\|_C,$$

therefore $\|z(t)\|_C \leqslant 2p\, \|G(x(t), y(t), \lambda)\|_C$ and by condition (4),

$$\|z\|_C \leqslant 2pc_1(\|x\|_C + \|y\|_C)^\gamma. \tag{80}$$

Consider equalities (76)–(78). Set

$$x_*(t) = r \sin t + h_*(t), \qquad y_*(t) = w_* r \cos t + w_* h_*'(t).$$

It follows from (25) that

$$\|x_*\|_C \leqslant 2r, \qquad \|x_*'\|_C \leqslant 2r, \qquad \|y_*\|_C \leqslant 2r \tag{81}$$

for all small $r > 0$. By definition of the functions $d_*$, $w_*$, and $h_*$, the identities

$$d_*(\lambda, r) = \int_0^{2\pi} \cos t\, F(x_*(t), y_*(t), \lambda)\, dt,$$

$$\pi r(1 - w_*^2) = \int_0^{2\pi} \sin t\, F(x_*(t), y_*(t), \lambda)\, dt,$$

$$h_* = B(w_*)\, QF(x_*(t), y_*(t), \lambda)$$

hold; put them to the right-hand side of equalities (76)–(78) and obtain

$$|d_*(\lambda, r)| \leqslant 2\pi\Delta, \qquad r\,|w_* - w| \leqslant 2\Delta,$$
$$\|h_* - h\|_{C^1} \leqslant p_1 q_0 \Delta + p_2 q_0\, |w_* - w|\, \|F(x_*, y_*, \lambda)\|_C,$$

where

$$\Delta = \|f(x(t), y(t), z(t), \lambda) - F(x_*(t), y_*(t), \lambda)\|_C$$

and the factors $p_1$, $p_2$ come from (58), and $q_0 = \|Q\|_{C \to E}$. Relations (6) and (81) imply $\|F(x_*(t), y_*(t), \lambda)\|_C \leqslant c_1(4r)^\alpha$, therefore

$$\|h - h_*\|_{C^1} \leqslant p_1 q_0 \Delta + p_2 q_0\, |w - w_*|\, c_1 4^\alpha r^\alpha \leqslant p_1 q_0 \Delta + p_2 q_0 c_1 4^\alpha r^{\alpha - 1} \cdot 2\Delta$$

and hence

$$|d_*(\lambda, r)| \leqslant \sigma\Delta, \quad r\,|w - w_*| \leqslant \sigma\Delta, \qquad \|h - h_*\|_{C^1} \leqslant \sigma\Delta, \tag{82}$$

where $\sigma = 2\pi + p_1 q_0 + 2 \cdot 4^\alpha p_2 q_0 c_1 \rho^{\alpha - 1}$ for $r \in (0, \rho)$.

Consider the functions

$$x - x_* = h - h_*, \qquad y - y_* = wx' - w_* x_*' = w(x' - x_*') + (w - w_*)\, x_*'.$$

Since $w \leqslant 1 + \varepsilon < 2$, $\|x_*'\|_C \leqslant 2r$, the relation

$$\|x - x_*\|_C + \|y - y_*\|_C \leqslant 2r\,|w - w_*| + 2\,\|h - h_*\|_{C^1}$$

is valid. Therefore $\|x - x_*\|_C + \|y - y_*\|_C \leqslant 4\sigma\Delta$ and from the formula $f(x, y, z, \lambda) = F(x, y, \lambda) + \Phi(x, y, z, \lambda)$, it follows the estimate

$$\|x - x_*\|_C + \|y - y_*\|_C \leqslant 4\sigma\Delta_1 + 4\sigma\,\|\Phi(x(t), y(t), z(t), \lambda)\|_C, \tag{83}$$

where

$$\Delta_1 = \|F(x(t), y(t), \lambda) - F(x_*(t), y_*(t), \lambda)\|_C.$$

Now note that by the Lipschitz condition (7),

$$\Delta_1 \leqslant c_1 \max\{\|x\|_C^\gamma, \|y\|_C^\gamma, \|x_*\|_C^\gamma, \|y_*\|_C^\gamma\}(\|x-x_*\|_C+\|y-y_*\|_C).$$

If $\varepsilon$ and $r$ are sufficiently small, estimates (79) and (81) imply that

$$c_1 \max\{\|x\|_C^\gamma, \|y\|_C^\gamma, \|x_*\|_C^\gamma, \|y_*\|_C^\gamma\} < (8\sigma)^{-1},$$

hence $8\sigma\Delta_1 \leqslant \|x-x_*\|_C+\|y-y_*\|_C$ and combining this with (83), we obtain

$$\|x-x_*\|_C+\|y-y_*\|_C \leqslant 8\sigma\, \|\Phi(x(t), y(t), z(t), \lambda)\|_C,$$
$$\Delta_1 \leqslant \|\Phi(x(t), y(t), z(t), \lambda)\|_C.$$

It follows from (5), (80) that $\|\Phi(x(t), y(t), z(t), \lambda)\|_C \leqslant \sigma_0(\|x\|_C+\|y\|_C)^\beta$ with $\sigma_0 = c_1(1+(2pc_1)^{1/\gamma})^\beta$, therefore

$$\|x-x_*\|_C+\|y-y_*\|_C \leqslant 8\sigma\sigma_0(\|x\|_C+\|y\|_C)^\beta, \tag{84}$$
$$\Delta_1 \leqslant \|\Phi(x(t), y(t), z(t), \lambda)\|_C \leqslant \sigma_0(\|x\|_C+\|y\|_C)^\beta. \tag{85}$$

Estimate (84) implies

$$\begin{aligned}\|x\|_C+\|y\|_C &\leqslant \|x_*\|_C+\|y_*\|_C+8\sigma\sigma_0(\|x\|_C+\|y\|_C)^\beta\\ &\leqslant 4r+8\sigma\sigma_0(3(r+\varepsilon))^{\beta-1}\,(\|x\|_C+\|y\|_C),\end{aligned}$$

thus $\|x\|_C+\|y\|_C \leqslant 8r$ whenever $r, \varepsilon$ are sufficiently small. Hence relation (80) yields

$$\|z\|_C \leqslant 2pc_1 8^\gamma r^\gamma.$$

It follows from (85) that $\Delta_1 \leqslant \|\Phi(x(t), y(t), z(t), \lambda)\|_C \leqslant \sigma_0 8^\beta r^\beta$, therefore $\Delta \leqslant 2\cdot\sigma_0 8^\beta r^\beta$ and the estimate

$$\max\{|d_*(\lambda, r)|, r\,|w-w_*|, \|h-h_*\|_{C^1}\} \leqslant 2\cdot 8^\beta\sigma\sigma_0 r^\beta$$

follows from (82). That is, there are positive numbers $\sigma_1$ and $\varepsilon_1$, $r_1$ such that for every $\varepsilon \in (0, \varepsilon_1)$, $r \in (0, r_1)$ relations (69) and (75)–(78) imply the estimates

$$\begin{aligned}&\|z\|_C \leqslant \sigma_1 r^\gamma, && |w-w_*| \leqslant \sigma_1 r^{\beta-1},\\ &\|h-h_*\|_{C^1} \leqslant \sigma_1 r^\beta, && |d_*(\lambda, r)| \leqslant \sigma_1 r^\beta.\end{aligned} \tag{86}$$

Finally, take any $\varepsilon \in (0, \varepsilon_1)$ and any pair $\lambda^-$, $\lambda^+$ such that relations (29) hold and $[\lambda^-, \lambda^+] \subset [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$; then parallelepiped (70) is included in parallelepiped (69). Since at least one of the equalities

$$\lambda = \lambda^\pm, \qquad w = 1 \pm \varepsilon, \qquad \|h\|_{C^1} = \varepsilon, \qquad \|z\|_C = \varepsilon$$

is valid for any point $(\lambda, w, h, z)$ of the boundary $\partial S$ of parallelepiped (70) and

$$\lim_{r \to +0} r^{-\beta} d_*(\lambda^-, r) = -\infty, \qquad \lim_{r \to +0} r^{-\beta} d_*(\lambda^+, r) = \infty,$$

$$|w_* - 1| < K r^{\alpha - 1}, \qquad \|h_*\|_{C^1} < K r^\alpha,$$

it follows that

$$\lim_{r \to +0} \inf_{(\lambda, w, h, z) \in \partial S} \left( \frac{\|z\|_C}{r^\gamma} + \frac{|w - w_*|}{r^{\beta - 1}} + \frac{\|h - h_*\|_{C^1} + |d_*(\lambda, r)|}{r^\beta} \right) = \infty.$$

At the same time, for any $r \in (0, r_1)$, $(\lambda, w, h, z) \in S$, $\xi \in [0, 1]$ equalities (75)–(78) imply (86), i.e.,

$$\Psi_\xi(\lambda, w, h, z) = 0$$
$$\Rightarrow r^{-\gamma} \|z\|_C + r^{1-\beta} |w - w_*| + r^{-\beta} \|h - h_*\|_{C^1} + r^{-\beta} |d_*(\lambda, r)| \leqslant 4\sigma_1.$$

Therefore relation (71) holds for all sufficiently small $r > 0$. This completes the proof of Lemma 3.1.

## 8. PROOF OF THEOREMS 4.1 AND 4.2

As we know, every $2\pi/w$-periodic solution of system (3) with any $\lambda \in \Lambda$ has the form

$$\{r \sin(wt + \varphi) + h(wt + \varphi),\ wr \cos(wt + \varphi) + wh'(wt + \varphi),\ z(wt + \varphi)\}, \qquad (87)$$

where $r \geqslant 0$, $\varphi \in [0, 2\pi)$, and $(\lambda, w, h, z) \in \mathbb{E}$ is a solution of system (66) or, which is the same, $\Psi_0(\lambda, w, h, z) = 0$.

If $r = 0$, then the last two equations of system (66) can be written as

$$h = B(w)\, Qf(h, wh', z, \lambda), \qquad z = B_1(w, \lambda)\, g(h, wh', z, \lambda), \qquad (88)$$

therefore

$$\|h\|_{C^1} \leqslant p_1 q_0 \, \|f(h(\cdot), wh'(\cdot), z(\cdot), \lambda)\|_C, \ \|z\|_C = p\, \|g(h(\cdot), wh'(\cdot), z(\cdot), \lambda)\|_C.$$

However, by conditions 2–4 of Theorem 2.1,

$$\lim_{v \to 0} \sup_{\lambda \in \Lambda} \frac{|f(x, y, z, \lambda)|}{v} = \lim_{v \to 0} \sup_{\lambda \in \Lambda} \frac{|g(x, y, z, \lambda)|}{v} = 0, \qquad v := |x| + |y| + |z|.$$

Hence $h \equiv 0$, $z \equiv 0$ is the only solution of system (88) in the set $\|h\|_{C^1} \leqslant \varepsilon$, $\|z\|_C \leqslant \varepsilon$ with any sufficiently small $\varepsilon > 0$ for any $\lambda \in \Lambda$, $w \in \Omega$. This means that $r > 0$ for every sufficiently small nontrivial periodic solution (87) of system (3).

Let $r \neq 0$. Suppose that $\lambda$, $w$ are close to $\lambda_0$, 1, and the amplitude of solution (87) is sufficiently small. Then $r > 0$ is small and estimates (69) hold for a small $\varepsilon > 0$. It is shown in the proof of Lemma 3.1 above that the relation $\Psi_0(\lambda, w, h, z) = 0$ implies estimates (86), hence

$$w = w_*(\lambda, r) + O(r^{\beta - 1}), \qquad d_*(\lambda, r) = O(r^\beta).$$

If formula (37) is valid, we obtain

$$w = 1 + D_0(\lambda)\, r^{\alpha_1} + o(r^{\alpha_1}),$$

hence $\operatorname{sign}(w - 1) = \operatorname{sign} D_0(\lambda_0)$ for small $r$, $|\lambda - \lambda_0|$ whenever $D_0(\lambda_0) \neq 0$. This proves Theorem 4.1.

If representation (39) holds, then

$$a(\lambda)\, r^{\beta_1} + b(\lambda)\, r^{\beta_2} = o(r^{\beta_2})$$

and the relation $b(\lambda_0) \neq 0$ implies $a(\lambda)\, b(\lambda_0) < 0$ for all small $r$, $|\lambda - \lambda_0|$. This proves Theorem 4.2.

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