On existence of cycles for quasilinear higher order ODE*

Krasnosel'skii A.M., Rachinskii D.I.

Simple conditions for the existence of cycles are suggested for general quasilinear higher order ODEs. We use sector estimates of nonlinearities and their linear asymptotics at zero and at infinity.

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1. Introduction

Consider the equation

$$L\Big(\frac{d}{dt}\Big)x = qf(x, x', \ldots, x^{(\ell-1)}). \tag{1}$$

Here q is a parameter,

$$L(p) = p^\ell + a_1 p^{\ell-1} + \cdots + a_\ell \tag{2}$$

is a real polynomial with constant coefficients. It is supposed that $f(\cdot) : \mathbb{R}^\ell \to \mathbb{R}$ is a continuous function and $f(0, \ldots, 0) = 0$, i.e., the origin is an equilibrium of equation (1). We shall use both notations $f(x_0, \ldots, x_{\ell-1})$ and $f(z)$, $z = (x_0, \ldots, x_{\ell-1}) \in \mathbb{R}^\ell$.

We present sufficient conditions for the existence of nonstationary periodic solutions; their periods are defined by the polynomial $L(p)$ and some robust properties of the nonlinearity.

Let the nonlinearity $f(\cdot)$ satisfy the sector estimate

$$|f(z)| \le |z|. \tag{3}$$

Here $|z|$ is the Euclidean norm of the vector $z \in \mathbb{R}^\ell$, generated by some scalar product $\langle \cdot, \cdot \rangle$. If the polynomial $L(p)$ has no roots on the imaginary axis and $|q|$ is sufficiently small, then equation (1) has no nontrivial periodic solutions.

We suppose that the polynomial $L(p)$ has at least one pair of pure imaginary conjugate roots. Then under some appropriate asymptotic conditions at zero and at infinity nontrivial periodic solutions (cycles) always exist if $|q|$ is sufficiently small.

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Our results are not applicable if both the polynomial $L(p)$ is even and the function $f(\cdot)$ depends on even derivatives only. In this case, the results should be different, for example for Hamiltonian equations the existence of integral manifolds consisting of cycles is natural.

We do not use geometrical constructions in the phase space or any analogs of the torus principle (see, e.g. [4], [5]) in the proofs. Instead, operator equations in infinite dimensional functional spaces are considered.

This paper develops Theorem 1 from [1], where we presented sufficient conditions for the existence of cycles and estimated $|q|$ for control theory equations with scalar nonlinearities $f(x)$ without derivatives.

2. Main results

Suppose that the polynomial $L(p)$ has a pair of imaginary roots $\pm w_0 i$ ($w_0 > 0$). Denote their multiplicity by K. Let $L(nw_0 i) \neq 0$ for all integer $n \neq \pm 1$. In other notation,

$$L(p) = (p^2 + w_0^2)^K L_1(p)$$

and

$$L_1(nw_0 i) \neq 0, \qquad n \in \mathbb{Z}. \tag{4}$$

Suppose that the function $f(z)$ is differentiable at zero and at infinity, i.e., for some $b = (b_0, b_1, \ldots, b_{\ell-1})$ and $c = (c_0, c_1, \ldots, c_{\ell-1})$ the relations

$$\lim_{|z|\to\infty} \frac{f(z) - \langle b, z\rangle}{|z|} = \lim_{z\to 0} \frac{f(z) - \langle c, z\rangle}{|z|} = 0$$

hold. Define the polynomials

$$B(p) = b_{\ell-1}p^{\ell-1} + b_{\ell-2}p^{\ell-2} + \cdots + b_0, \quad C(p) = c_{\ell-1}p^{\ell-1} + c_{\ell-2}p^{\ell-2} + \cdots + c_0. \tag{5}$$

Theorem 1. *Let the multiplicity K of the roots $\pm w_0 i$ of $L(p)$ be odd. Let*

$$\Im m[L_1(-w_0 i)B(w_0 i)]\ \Im m[L_1(-w_0 i)C(w_0 i)] < 0. \tag{6}$$

Then there exists a $q_0 > 0$ such that for any $|q| < q_0$ equation (1) has at least one nontrivial cycle.

Under the conditions of this theorem the cycles are neither small nor large, their amplitudes can be estimated both from below and from above. For small q the periods of cycles are close to $T_0 = 2\pi/w_0$.

As an example, consider the equation

$$x''' + x'' + x' + x = qf(x). \tag{7}$$

Let the function $f(x)$ be differentiable at zero and at infinity and let $f'(0) \cdot f'(\infty) < 0$ (here $f'(\infty) = \lim_{x\to\infty} f(x)/x$). Let $|f(x)| \leq |x|$ and $q < .745$. Then equation (7) has at least one nontrivial cycle with a period $T \in [6.283, 7.652]$.

This statement follows from Theorem 3 from [1]. The relation $f'(0)\cdot f'(\infty)<0$ is equivalent to (6), since $B(p)\equiv f'(\infty), C(p)\equiv f'(0)$ for the function $f(x)$.

Now we formulate an analog of Theorem 1 for equations with delays. Consider the equation

$$L\Big(\frac{d}{dt}\Big)x(t)=qf\big(x(t),x'(t),\ldots,x^{(\ell-1)}(t);x(t-\tau),x'(t-\tau),\ldots,x^{(\ell-1)}(t-\tau)\big). \tag{8}$$

We suppose that the nonlinearity $f(x_0,\ldots,x_{\ell-1};y_0,\ldots,y_{\ell-1})=f(z_1;z_2)$ is continuous, satisfies the estimate

$$|f(z_1;z_2)|\le|z_1|+|z_2|$$

and is differentiable at zero and at unfinity. By the latter assumption, there are vectors b, c and $b^*=(b_0^*,b_1^*,\ldots,b_{\ell-1}^*)$, $c^*=(c_0^*,c_1^*,\ldots,c_{\ell-1}^*)$ such that

$$\lim_{|z_1|+|z_2|\to\infty}\frac{f(z_1;z_2)-\langle b,z_1\rangle-\langle b^*,z_2\rangle}{|z_1|+|z_2|}=\lim_{|z_1|+|z_2|\to 0}\frac{f(z_1;z_2)-\langle c,z_1\rangle-\langle c^*,z_2\rangle}{|z_1|+|z_2|}=0.$$

Define the polynomials (5) and

$$B^*(p)=b_{\ell-1}^*p^{\ell-1}+b_{\ell-2}^*p^{\ell-2}+\cdots+b_0^*,\quad C^*(p)=c_{\ell-1}^*p^{\ell-1}+c_{\ell-2}^*p^{\ell-2}+\cdots+c_0^*.$$

Theorem 2. *Let the multiplicity K of the roots $\pm w_0 i$ of $L(p)$ be odd. Let*

$$\Im m\Big[L_1(-w_0i)\big(B(w_0i)+B^*(w_0i)e^{-\tau w_0 i}\big)\Big]\ \Im m\Big[L_1(-w_0i)\big(C(w_0i)+C^*(w_0i)e^{-\tau w_0 i}\big)\Big]<0. \tag{9}$$

Then there exists a $q_0>0$ such that for any $|q|<q_0$ equation (8) has at least one nontrivial cycle.

This theorem is more general than Theorem 1. Both Theorems 1 and 2 can be proved in the same way. We present the proof of Theorem 1 only.

3. Proof

3.1. The choice of unknowns. First of all let us rescale the time in (1). Let $w>0$. Evidently, any 2π-periodic solution $x(t)$ of the equation

$$L\Big(w\frac{d}{dt}\Big)x=q\,f(x,wx',\ldots,w^{\ell-1}x^{(\ell-1)}) \tag{10}$$

determines the $2\pi/w$-periodic solution $x(wt)$ of equation (1). The frequency w is *a priori* unknown.

Instead of (1), we analyze equation (10) and look for its 2π-periodic solutions of the form

$$x(t)=r\sin t+h(t),\qquad r>0, \tag{11}$$

where the Fourier-series expansion of the 2π-periodic function $h(t)$ does not contain the first harmonics, i.e.,

$$\int_0^{2\pi} \cos t \; h(t)\, dt = \int_0^{2\pi} \sin t \; h(t)\, dt = 0.$$

Equivalently, $Ph = 0$, where

$$Px(t) \overset{\text{def}}{=} \frac{1}{\pi} \int_0^{2\pi} \cos(t-s)\, x(s)\, ds.$$

We prove that if $|q|$ is sufficiently small, then under the assumptions of Theorem 1 there exist a number w from some interval $\Omega = [w_1, w_2] \ni w_0$, a number $r > 0$, and a 2π-periodic function $h(t)$ such that the function (11) is a solution of equation (10).

Let us note that any nonstationary periodic solution $x(t)$ of autonomous equation (10) belongs to the continuum of periodic solutions $x(t+\alpha)$ with arbitrary $\alpha \in \mathbb{R}$. Among the shifts $x(t+\alpha)$ we choose a unique function (11) that has the zero projection onto $\cos t$ and a positive projection onto $\sin t$, by this we fix the phase of the first harmonics. Now the unknowns are the amplitude r of the first harmonics, the function $h(t)$, and the frequency w. This choice of unknowns will allow to reduce the problem to well-defined operator equations and to study them by topological methods.

3.2. Linear spaces and operators. Set $\Omega = [w_1, w_2]$ where $0 < w_1 < w_0 < w_2$. Suppose that $L(nwi) \neq 0$ for all integer $n \neq \pm 1$ and all $w \in \Omega$. Suppose that both the polynomials $\Im m[L_1(-wi)B(wi)]$ and $\Im m[L_1(-wi)C(wi)]$ have constant sign on Ω. According to conditions (4) and (6) these assumptions are fulfilled for any sufficiently small interval $\Omega \ni w_0$. These assumptions imply $L_1(wni) \neq 0$ for any $n \in \mathbb{Z}$ and $w \in \Omega$.

Consider[1] in the space L^2 the projector P, the plane $\Pi = PL^2$, and its orthogonal complement $\Pi^* = QL^2$, where $Q = I - P$. The subspace Π^* has co-dimension 2.

Denote by $A(w)$ ($w \in \Omega$) the linear operator that maps any function $u(t) \in \Pi^*$ to a unique solution $x(t) \in \Pi^*$ of the linear ODE

$$L\left(w\frac{d}{dt}\right) x = u(t). \tag{12}$$

The existence of $x(t)$ follows from $L(wni) \neq 0$ for $n \neq \pm 1$ and from $u(t) \in \Pi^*$, the uniqueness follows from $x(t) \in \Pi^*$. The operators $A(w)$ may be extended to the whole space L^2 for $w \neq w_0$, but the norms of the extensions tend to infinity as $w \to w_0$. In the subspace Π^* the norms of the operators $A(w)$ are uniformly bounded for all $w \in \Omega$, i.e.,

$$\big\|A(w)\big\|_{\Pi^* \to \Pi^*} \leq d_* < \infty.$$

Let $C_{\ell-1}$ be the space of $\ell - 1$ times continuously differentiable functions with the usual norm. Each operator $A(w)$ acts from Π^* to $C_{\ell-1}$ and is completely continuous. Moreover, the operator $A(w)u$ is completely continuous with respect to the set of the variables w, u.

[1] All the functional spaces consist of scalar functions defined on the segment $[0, 2\pi]$.

The operators $A(w)Q$ are well-defined on the whole space L^2 and have uniformly bounded norms:

$$\|A(w)Q\|_{L^2\to L^2}, \|A(w)Q\|_{L^2\to C_{\ell-1}} \le d^* < \infty. \tag{13}$$

Lemma 1. *Let $w \in \Omega$. The functions $x(t) = r\sin t + h(t)$ ($h \in \Pi^*$) and $u(t) \in L^2$ satisfy* (12) *if and only if $h = A(w)Qu$ and*

$$-i\pi L(wi)r = \int_0^{2\pi} e^{-it}\, u(t)\, dt. \tag{14}$$

Formula (14) follows from the equality $L\left(w\dfrac{d}{dt}\right) r\sin t = Pu(t)$.

3.3. Deformation of the vector field. Due to Lemma 1, the 2π-periodic problem for equation (10) is equivalent to the system

$$\begin{aligned} -i\pi(w_0^2 - w^2)^K L_1(wi) &= \frac{q}{r}\int_0^{2\pi} e^{-it}\, f(x, wx', \dots, w^{\ell-1}x^{(\ell-1)})\, dt, \\ h &= qA(w)Qf(x, wx', \dots, w^{\ell-1}x^{(\ell-1)}). \end{aligned}$$

Let us multiply the first equation by $iL_1(-wi)$ (this value is nonzero for $w \in \Omega$ by assumption) and rewrite the system in the real form:

$$\begin{aligned} 0 &= \frac{q}{r}\int_0^{2\pi} \Im m\big(ie^{-it}L_1(-wi)\big)\, f(x, wx', \dots, w^{\ell-1}x^{(\ell-1)})\, dt, \\ (w_0^2 - w^2)^K |L_1(wi)|^2\pi &= \frac{q}{r}\int_0^{2\pi} \Re e\big(ie^{-it}L_1(-wi)\big)\, f(x, wx', \dots, w^{\ell-1}x^{(\ell-1)})\, dt, \\ h &= qA(w)Qf(x, wx', \dots, w^{\ell-1}x^{(\ell-1)}). \end{aligned} \tag{15}$$

We look for solutions $\{r, w, h\} \in \mathbb{R} \times \Omega \times C_{\ell-1}$ of system (15) in the set

$$G = G(\rho, R) \stackrel{\text{def}}{=} \{\rho \le r \le R,\ w_1 \le w \le w_2,\ \|h\|_{C_{\ell-1}} \le R\}, \tag{16}$$

where $\rho > 0$ is sufficiently small and R is sufficiently large; what “small” and “large” means we explain below. Let us stress that the operator $A(w)Qf(\dots)$ acts from the space $\mathbb{R} \times \Omega \times C_{\ell-1}$ to the space $C_{\ell-1}$ and is completely continuous with respect to the set of the variables $\{r, w, h\}$.

Without loss of generality we suppose that $q \ne 0$ and divide by q the first equation of system (15) (for $q = 0$ the linear homogeneous equation (1) has periodic solutions $\sin w_0 t$

and $\cos w_0 t$). To prove the theorem, it suffices to show that for each sufficiently small $|q|$ the rotation $\gamma = \gamma(\Phi, G)$ of the vector field

$$\Phi(r,w,h) = \begin{cases} \dfrac{1}{r}\displaystyle\int_0^{2\pi} \Im m\big(ie^{-it}L_1(-wi)\big)\, f(x, wx', \dots, w^{\ell-1}x^{(\ell-1)})\, dt, \\ (w_0^2 - w^2)^K |L_1(wi)|^2\pi - \dfrac{q}{r}\displaystyle\int_0^{2\pi} \Re e\big(ie^{-it}L_1(-wi)\big)\, f(x, wx', \dots, w^{\ell-1}x^{(\ell-1)})\, dt, \\ h - qA(w)Qf(x, wx', \dots, w^{\ell-1}x^{(\ell-1)}) \end{cases}$$

on the boundary ∂G of the set G is well-defined and nonzero ([2, 3]).

Set

$$w_\xi = w_0 + \xi(w - w_0), \qquad x_\xi = r\sin t + \xi h(t), \qquad \xi \in [0,1],$$

and consider the deformation

$$\Phi_\xi(r,w,h) = \begin{cases} \dfrac{1}{r}\displaystyle\int_0^{2\pi} \Im m\big(ie^{-it}L_1(-w_\xi i)\big)\, f(x_\xi, w_\xi x_\xi', \dots, w_\xi^{\ell-1}x_\xi^{(\ell-1)})\, dt, \\ (w_0^2 - w^2)^K |L_1(wi)|^2\pi - \dfrac{\xi q}{r}\displaystyle\int_0^{2\pi} \Re e\big(ie^{-it}L_1(-wi)\big)\, f(x, wx', \dots, w^{\ell-1}x^{(\ell-1)})\, dt, \\ h - \xi qA(w)Qf(x, wx', \dots, w^{\ell-1}x^{(\ell-1)}). \end{cases}$$

If this deformation is nondegenerate on the boundary ∂G of the set G, then it is sufficient to prove that the rotation $\gamma = \gamma(\Phi_0, G)$ of the vector field

$$\Phi_0(r,w,h) = \begin{cases} \dfrac{1}{r}\displaystyle\int_0^{2\pi} \Im m\big(ie^{-it}L_1(-w_0 i)\big)\, f(r\sin t, w_0 r\cos t, -w_0^2\sin t, \dots)\, dt, \\ (w_0^2 - w^2)^K |L_1(wi)|^2\pi, \\ h \end{cases} \tag{17}$$

on ∂G is nonzero.

The rotation of the vector field (17) on the boundary of the set (16) is equal to the product of the rotations of the components of this field on the boundaries of the sets $[\rho, R]$, $[w_1, w_2]$, and $\|h\|_{C_{\ell-1}} \le R$ (see the rotation product formula, e.g., in [3]). The rotation of the third component h on the sphere $\|h\|_{C_{\ell-1}} = R$ equals 1. The first and the second scalar components (denote them by $\varphi_1(r)$ and $\varphi_2(w)$ respectively) have the rotations ± 1 if $\varphi_1(\rho)\varphi_1(R) < 0$ and $\varphi_2(w_1)\varphi_2(w_2) < 0$ (if the opposite inequality holds for any of these components, then its rotation is zero).

Due to the differentiability of the function $f(\dots)$ at zero, the first component satisfies

$$\varphi_1(r) = \frac{1}{r}\int_0^{2\pi} \Im m\big(ie^{-it}L_1(-w_0 i)\big)\big(c_0 r\sin t + c_1 w_0 r\cos t - c_2 w_0^2 \sin t + \cdots\big)\, dt + o(1)$$

$$= \int_0^{2\pi} \Im m\big(ie^{-it}L_1(-w_0 i)\big) C\Big(w_0\frac{d}{dt}\Big)\sin t\, dt + o(1) = \pi\, \Im m[L_1(-w_0 i)C(w_0 i)] + o(1)$$

as $r \to 0$. Similarly, the differentiability of the function $f(\ldots)$ at infinity implies the relation $\varphi_1(r) = \pi\,\Im m[L_1(-w_0 i)B(w_0 i)] + o(1)$ as $r \to \infty$. Therefore for all sufficiently small $\rho > 0$ and large R relation (6) implies $\varphi_1(\rho)\varphi_1(R) < 0$. We consider such ρ,R and the corresponding set $G = G(\rho, R)$.

By assumption the number K is odd. It follows from the definition of the interval Ω that $0 < w_1 < w_0 < w_2$ and $|L_1(iw_j)| > 0$. Therefore the second component of the vector field (17) satisfies $\varphi_2(w_1)\varphi_2(w_2) < 0$. Consequently, the rotation of this field on ∂G is either 1 or -1.

It remains to prove that the deformation Φ_ξ is nondegenerate on ∂G for any small $|q|$.

If the third component of the deformation is zero, then from relations (3) and (13) it follows that

$$\|h\|_{C^{\ell-1}} \le d_0 r|q|, \tag{18}$$

where d_0 is the same for all $r > 0$, $w \in \Omega$ and all sufficiently small $|q|$. Therefore the third component is nondegenerate on the part of ∂G where $\|h\|_{C_{\ell-1}} = R$ for small $|q|$.

If the second component is zero and (18) is valid, then $(w_0^2 - w^2)^K |L_1(wi)|^2 \pi = O(q)$. Therefore the deformation is nondegenerate for $w = w_j$ if $|q|$ is sufficiently small.

Finally, if h satisfies (18), then due to the differentiability of $f(\ldots)$ the first component of the deformation goes to $\pi\,\Im m[L_1(-w_\xi i)C(w_\xi i)]$ as $r \to 0$, $q \to 0$ and goes to $\pi\,\Im m[L_1(-w_\xi i)B(w_\xi i)]$ as $r \to \infty$, $q \to 0$. By assumption, both these limits are nonzero for all $w_\xi \in \Omega$. Therefore the deformation is nondegenerate for $r = \rho$ and $r = R$ if $|q|$ and ρ are sufficiently small and R is sufficiently large.

Since at each point $\{r, w, h\} \in \partial G$ at least one of the equalities $\|h\|_{C_{\ell-1}} = R$, $w = w_j$, $r = \rho$, $r = R$ is valid, the deformation is nondegenerate on ∂G.

This completes the proof.

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