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A time-dependent Preisach model

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Abstract

We study a new type of hysteresis nonlinearity: that arising in time-dependent Preisach systems. The Preisach model is amended to allow for state or time dependent threshold values. The paper presents a simple time-dependent Preisach model. The basic properties of this model, including finite approximations are considered. © 2001 Published by Elsevier Science B.V.

Keywords: Preisach model; Hysteresis; Time variability

1. Introduction

In the standard Preisach model the threshold values α, β ($\alpha \leqslant \beta$) do not vary over time. This specification is not appropriate when applied to some of the problems that arise in non-laboratory disciplines such as economics.

Consider, for example, investment decisions involving sunk costs, taken under conditions of uncertainty regarding the net returns on investment projects (see Ref. [4]). Here α represents the hurdle rate of return required for a firm to go ahead with an investment project, and β the return at which the project would be abandoned. In a monetary union, such as the USA or t he European EMU, all firms will be faced by a common input affecting their investment decisions in the form of the interest rate set by the central bank, the Federal Reserve or the ECB. But the threshold values of particular firms will vary with shocks affecting the particular market, region or, in the case of the EMU, the country in which the firm operates. The threshold values are thus state or time dependent, varying with the market, region or, in the case of EMU, country-specific factors that affect business confidence.

To analyse this type of problem, the Preisach model needs to be amended to allow for state or time-dependent threshold values. This paper presents a simple time-dependent Preisach model. After introducing the standard Preisach non-linearity the following sections deal with variable threshold values, present a time variable Preisach model, consider finite approximations of this model, and provide examples of closed systems.

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2. Preisach non-linearity

Denote as

$$\eta(t) = R_{\alpha,\beta}[t_0, \eta_0]x(t) \quad t \geqslant t_0, \tag{1}$$

the variable state of *non-ideal relay* with threshold values α, β $(\alpha < \beta)$ with the input $x(t)$ $(t \geqslant t_0)$ and with the initial state η_0. Here the input is an arbitrary continuous scalar function; η_0 takes the values 1 and -1; the scalar function $\eta(t)$ satisfies $\eta(t_0) = \eta_0$ and $|\eta(t)| = 1$ for any t and has at most a finite number of jumps on any finite interval $t_0 \leqslant t \leqslant t_1$. The values of operator (1) may be naturally defined for $\eta_0 = -1$, if $x(t_0) \leqslant \alpha$, for $\eta_0 = 1$, if $x(t_0) \geqslant \beta$, and both for $\eta_0 = -1$ and $\eta_0 = 1$, if $\beta > x(t_0) > \alpha$. The equalities $\eta(t) = 1$ for $x \times (t) \geqslant \beta$ and $\eta(t) = -1$ for $x(t) \leqslant \alpha$ always hold for $t \geqslant t_0$.

For various applications, it is convenient to define operator (1) for any initial states (e.g. for $\eta = 1$ if $x(t_0) < \alpha$). Define

$$R_{\alpha,\beta}[t_0, \eta_0]x(t) = R_{\alpha,\beta}[t_0, \eta_1]x(t), \ \ t \geqslant t_0,$$

where

$$\eta_1 = \begin{cases} -1 & \text{if } x(t_0) \leqslant \alpha, \\ 1 & \text{if } x(t_0) \geqslant \beta, \\ \eta_0 & \text{if } \alpha < x(t_0) < \beta. \end{cases} \tag{2}$$

Now Eq. (1) is defined for any continuous input $x(t)$ and any initial state η_0.

Consider a family $\mathcal{R}$ of relays $R^w = R_{\alpha_w,\beta_w}$ with threshold values α_w, β_w, $w \in \Omega$. The set Ω may be finite or infinite, we call such family a *bundle* of relays. Suppose that the set Ω is endowed with a probability measure μ. We always suppose that both functions α_w, β_w are measurable with respect to μ.

We call any measurable function $\eta(w) : \Omega \to \{-1, 1\}$ the *initial state* of the bundle $\mathcal{R}$. For any initial state $\eta_0(w)$ and any continuous input $x(t)$, $t \geqslant t_0$ define the function

$$\xi(t) = \xi[t_0, \eta_0](t) = \int_\Omega R^w[t_0, \eta_0(w)]x(t)\,\mathrm{d}\mu, \quad t \geqslant t_0.$$

We call the described model *Preisach non-linearity*, $\xi(t)$ is the output of the Preisach model with the initial state η_0 and the input $x(t)$.

Often the set Ω is considered as a two-dimensional half-plane $\Pi = \{(\alpha, \beta) : \beta > \alpha\}$ (see details in Ref. [1]). Such an approach does not fit our purposes.

If the measure μ has a finite support $w_1, \ldots, w_N$, then this non-linearity may be rewritten as a parallel connection of the relays $R^{w_j} = R_{\alpha_{w_j},\beta_{w_j}}$ with weights $\mu(w_j)$

$$\xi(t) = \xi[t_0, \eta_0](t) = \sum_{j=1}^{N} \mu(w_j) R^{w_j}[t_0, \eta_0(w_j)]x(t), \quad t \geqslant t_0.$$

3. Non-ideal relay with variable thresholds

Denote

$$R^* x(t) = R^*[\eta_0]x(t) = R_{-1,1}[\eta_0]x(t).$$

Let $\alpha(t)$, $\beta(t)$ $(t \geqslant t_0)$ be continuous scalar functions.

Let us define the variable state of the *time variable non-ideal relay* with variable threshold values $\alpha(t)$, $\beta(t)$ ($\alpha(t) < \beta(t)$ for all $t \geqslant t_0$) with the input $x(t)$ $(t \geqslant t_0)$ and with the initial state $\eta_0 \in \{-1, 1\}$ by the formula

$$R_{\alpha(t),\,\beta(t)}[t_0, \eta_0]x(t) = R^*[\eta_0]y(t), \tag{3}$$

where

$$y(t) = \left(x(t) - \frac{\alpha(t) + \beta(t)}{2}\right) \Big/ \left(\frac{\beta(t) - \alpha(t)}{2}\right). \tag{4}$$

Here, again the input is an arbitrary continuous scalar function; η_0 takes the values 1 and -1; the scalar function $\eta(t)$ satisfies $\eta(t_0) = \eta_0$ and $|\eta(t)| = 1$ for any t and has at most a finite number of jumps on any finite interval $t_0 \leqslant t \leqslant t_1$. The equalities $\eta(t) = -1$ for $x(t) \leqslant \alpha(t)$ and $\eta(t) = 1$ for $x(t) \geqslant \beta(t)$ always hold for $t \geqslant t_0$.

Let us list the simplest properties of operators (3).

1. Anti-monotonicity with respect to the threshold values $\alpha(t)$ and $\beta(t)$ for any fixed input $x(t)$ (again in the sense of cone theory): if $\alpha_0(t) \leqslant \alpha_1(t)$ and $\beta_0(t) \leqslant \beta_1(t)$, then

$$R_{\alpha_0(t),\,\beta_0(t)}[t_0, \eta_0]x(t) \geqslant R_{\alpha_1(t),\,\beta_1(t)}[t_0, \eta_0]x(t)\text{i}.$$

2. Monotonicity with respect to the input and initial state: for any fixed $\alpha(t)$ and $\beta(t)$: if $\eta_{0,1} \geqslant \eta_{0,2}$ and $x_1(t) \geqslant x_2(t)$ for $t \in [t_0, t_1]$, then

$$R_{\alpha(t),\,\beta(t)}[t_0,\ \eta_{0,1}]x_1(t) \geqslant R_{\alpha(t),\,\beta(t)}[t_0,\ \eta_{0,2}]x_2(t).$$

This property means that the relay operator is monotone in the space of continuous functions in the sense of cone theory.

3. Periodicity: if an input $x(t)$, $t \geqslant t_0$ and both functions $\alpha(t)$ and $\beta(t)$ are periodic with a common period $T > 0$, then the output is also T-periodic again for $t \geqslant t_0 + T$.

4. Time variable Preisach model

Consider a family $\mathscr{R}$ of time variable relays $R^w = R_{\alpha_w(t),\,\beta_w(t)}$ with threshold functions $\alpha_w(t)$, $\beta_w(t)$, $w \in \Omega$. The set Ω again may be finite or infinite, we call such a family a bundle of time variable relays. The set Ω again is endowed with a measure μ, both functions $\alpha_w(t)$, $\beta_w(t)$ are measurable with respect to μ for any t.

We call any measurable function $\eta(w) : \Omega \to \{-1,\ 1\}$ the initial state of the bundle $\mathscr{R}$.

Proposition 1. *For any fixed t the function* $R^w[t_0,\ \eta_0(w)]x(t)$ *is measurable in w.*

Proof. See in Ref. [3].

Therefore, for any initial state $\eta_0(w)$ and any input $x(t) \in C$, $t \geqslant t_0$ define the function

$$\xi(t) = \Xi x(t) = \xi[t_0,\ \eta_0](t) = \int_\Omega R^w[t_0,\ \eta_0(w)]x(t)\,\mathrm{d}\mu. \qquad (5)$$

We call the model described *the time variable Preisach non-linearity*; here $\xi(t)$ is its output with the initial state η_0 and the input $x(t)$.

It is possible to consider the universal set Ω of pairs $\{\alpha(t),\ \beta(t)\}$ of continuous functions of t: we do not discuss this approach in the present paper.

The simplest properties of time variable Preisach models again are monotonicity and periodicity: they follow directly from the corresponding properties of time variable relays.

Proposition 2. *For any fixed measure μ if $\eta_{0,1} \geqslant \eta_{0,2}$ and $x_1(t) \geqslant x_2(t)$ for $t \in [t_0,\ t_1]$, then $\xi[t_0,\ \eta_{0,1}]x_1(t) \geqslant \xi[t_0,\ \eta_{0,2}]x_2(t)$.*

If an input $x(t)$, $t \geqslant t_0$ is periodic with some period $T > 0$, then the output $\xi(t)$ is also T-periodic but for $t \geqslant t_0 + T$.

Any individual time variable relay is discontinuous as an operator in any standard functional space. The time variable Preisach model often has 'better' properties.

Proposition 3. *The output $\xi(t)$ is a continuous function for any initial state η_0 and any input $x(t)$ iff the measure μ satisfies*

$$\forall \alpha^*, \beta^* : \quad \mu\{w :\ \alpha_w = \alpha^*\} = 0,$$
$$\mu\{w :\ \beta_w = \beta^*\} = 0.$$

The conditions of continuity of the operator $x(t) \mapsto \xi(t)$ is more difficult than for a standard Preisach model. We formulate only a couple of simple assertions in this direction.

Proposition 4. *Let*

$$\mu\left\{w :\ \min_{\tau < t < \sigma} \{x(t) - \alpha_w(t)\} = 0\right\} = 0,$$
$$\mu\left\{w :\ \min_{\tau < t < \sigma} \{\beta_w(t) - x(t)\} = 0\right\} = 0$$

hold for any continuous $x(\cdot) \in X$ and any $t_0 < \tau < \sigma < t_1$.

Then the operators are continuous in C.

This proposition does not guarantee any uniform continuity even on bounded sets of functions $x(t)$. The uniform continuity requires additional assumptions. Let us formulate a simple assertion in this direction.

Proposition 5. *Let X be a compact subset of $C[t_0,\ t_1]$. Let for a given $\delta > 0$ the equality $\mu\{\beta_w(t) - \alpha_w(t) > \delta\} = 1$ holds for all $t \in [t_0,\ t_1]$. Suppose finally that*

$$\mu\{w :\ |\min_{\tau < t < \sigma} \{x(t) - \alpha_w(t)\}| < \varepsilon\} < \lambda\varepsilon,$$
$$\mu\{w :\ |\min_{\tau < t < \sigma} \{\beta_w(t) - x(t)\}| < \varepsilon\} < \lambda\varepsilon$$

for some fixed $\lambda > 0$ for all $t_0 < \tau < \sigma < t_1$, all $x(\cdot) \in X$ and any positive ε.

Then the operator satisfies a Lipschitz condition on X.

Proposition 6. *For any fixed input $x(t)$ and any different initial states $\eta_{0,1}(w)$ and $\eta_{0,2}(w)$ the function $\left|\xi[t_0, \eta_{0,1}(w)](t) - \xi[t_0, \eta_{0,2}(w)](t)\right|$ decreases.*

5. Finite approximations

The natural question arises: how it is possible to compute the described Preisach model numerically? One of the possible answers is given in this section.

Suppose we have some initial state $\eta_0(w)$.

Denote by $\Omega_\mu \subset \Omega$ the support of measure μ. Let the set $\mathscr{W}$ of functions $[t_0, t_1] \mapsto (\alpha_w(t), \beta_w(t))$, $w \in \Omega_\mu$ be compact in the space $C = C([t_0, t_1])$ of continuous functions.

Let us choose a finite ε-net $(\alpha_j(t), \beta_j(t))$, $j = 1, \ldots, N$ in the set $\mathscr{W}$. Denote $\mathscr{B}_{|}$ the ball in C of the radius ε centered at $(\alpha_j(t), \beta_j(t))$. Split the set $\mathscr{W}$ into a partition $\mathscr{P}_{|}$ such that $\mathscr{P}_{|} \subset \mathscr{B}_{|}$. Without loss of generality, we can suppose that for any w satisfying $(\alpha_w(t_0), \beta_w(t_0)) \in \mathscr{P}_{|}$ the function $\eta_0(w)$ takes the same value $\eta_{0,j}$ (otherwise we choose a finer subpartition).

Consider the Preisach operator

$$\Xi_\varepsilon x(t) = \sum_{j=1}^{N} \mu_j R_{\alpha_j, \beta_j}[t_0, \eta_{0,j}]x(t),$$

where $\mu_j = \mu\left(\left\{w : (\alpha_w, \beta_w) \in \mathscr{P}_{|}\right\}\right)$.

This operator is a natural approximation for the basic Preisach operator ξ. Below, we discuss how to estimate the precision of this approximation.

Consider the functions

$$\alpha_j^-(t) = \min\{\alpha(t) : (\alpha(t), \beta(t)) \in \mathscr{P}_{|}\},$$
$$\alpha_j^+(t) = \max\{\alpha(t) : (\alpha(t), \beta(t)) \in \mathscr{P}_{|}\},$$
$$\beta_j^-(t) = \min\{\beta(t) : (\alpha(t), \beta(t)) \in \mathscr{P}_{|}\},$$
$$\beta_j^+(t) = \max\{\beta(t) : (\alpha(t), \beta(t)) \in \mathscr{P}_{|}\}$$

and the corresponding Preisach operators

$$\Xi_\varepsilon^- x(t) = \sum_{j=1}^{N} \mu_j R_{\alpha_j^-, \beta_j^-}[t_0, \eta_{0,j}]x(t),$$

$$\Xi_\varepsilon^+ x(t) = \sum_{j=1}^{N} \mu_j R_{\alpha_j^+, \beta_j^+}[t_0, \eta_{0,j}]x(t).$$

Proposition 7. *The estimation*

$$\Xi_\varepsilon^+ x(t) \leqslant \Xi x(t) \leqslant \Xi_\varepsilon^- x(t)$$

holds. If the conditions of Proposition 4 hold, then $\varepsilon \to 0$ implies $\|\Xi_\varepsilon^- x(t) - \Xi_\varepsilon^+ x(t)\| \to 0$.

6. The simplest closed systems

Below, we present some propositions on closed-loop systems with time variable Preisach nonlinearities. The choice of problems considered was dictated by their simplicity only.

It is possible to consider much more general systems. For example, it is possible to specify time variable Preisach models where the functions $\alpha_w(t)$ and $\beta_w(t)$ depend on the phase variables. It is also possible to study other problems for such systems, such as bifurcation problems, and problems on bounded solutions.

Consider the equation

$$z' = f(t, z, y), \quad y(t) = \Xi x(t), \quad x(t) = \langle c, z(t) \rangle. \tag{6}$$

Here $z, c \in \mathbb{R}^n$, $x, y \in \mathbb{R}$, $t \geqslant t_0$, $\langle \cdot, \cdot \rangle$ is a inner product in $\mathbb{R}^n$. The initial state $\eta_0(w)$ for the operator $\Xi x(t)$ is supposed to be fixed.

Proposition 8. *Let the condition of Proposition 5 hold for any compact set X. Let a compact set $K \subset \Pi$ exist such that for any t the support of the measure $\mu^*(t; \cdot)$ belongs to K. Let the function $f(t, z, y)$ satisfy Lipschitz condition with respect to z* and *y. Then the Cauchy problem $z(t_0) = z_0$ for* (6) *has a unique solution.*

The proof follows from Proposition 5 and the contracting mapping principle.

Consider the equation

$$L\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)x = \Xi x(t) + \varphi(t). \tag{7}$$

Here $L(p)$ is a real polynomial with constant coefficients, $t \geqslant t_0$, $\varphi(\cdot) : [t_0, t_1] \to \mathbb{R}$ is a continuous function. The initial state $\eta_0(w)$ for the operator $\Xi x(t)$ is supposed to be fixed.

Proposition 9. *Let the polynomial $L(p)$ have only real roots. Then the Cauchy problem $x(t_0) = x_0$, $x'(t_0) = x_1$, …, for* Eq. (7) *has at least one solution.*

The proof follows from the first part of Proposition 2 and the Birkhoff–Tarsky principle [2].

Consider again Eq. (7) with T-periodic φ. Suppose that all functions $\alpha_w(t)$ and $\beta_w(t)$ are also periodic with the same period T.

Proposition 10. *Let the conditions of Proposition 4 hold. Let the polynomial $L(p)$ have no roots of the type* $0, \pm 2\pi i/T, \ldots, 2k\pi i/T, \ldots$. *Then* Eq. (7) *has at least one T-periodic solution.*

The proof is based on Proposition 3 and the classical Schauder principle.

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