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ON SUBHARMONICS BIFURCATION IN EQUATIONS WITH HOMOGENEOUS NONLINEARITIES

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Dedicated to our friend V.S. Kozyakin in the occasion of his 50th birthday.

Abstract. The bifurcation of subharmonics for resonant nonautonomous equations of the second order is studied. The set of subharmonics is defined by principal homogeneous parts of the nonlinearities *provided these parts are not polynomials.* Analogous statements are proved for bifurcations of p-periodic orbits of a planar dynamical system. The analysis is based on topological methods and harmonic linearization.

1. **Introduction.**

Consider the equation

$$x'' + \lambda x' + \beta x = f(t, x, x'; \lambda) \tag{1.1}$$

with a scalar parameter $\lambda \in [-1, 1]$ and a constant $\beta > 0$ independent of λ. Here $f(t, 0, 0; \lambda) \equiv 0$, the nonlinearity $f(t, x, y; \lambda)$ is continuous and sublinear at zero:

$$\lim_{|x|+|y|\to 0} \sup_{\lambda,t} \frac{|f(t, x, y; \lambda)|}{|x| + |y|} = 0.$$

Equation (1.1) is usual Liénard equation $x'' + g(t, x, x'; \lambda)x' + G(t, x, x'; \lambda)x = 0$ rewritten in a special form. We assume that the functions $g(t, x, y; \lambda)$ and $G(t, x, y; \lambda)$ are linearized in the point $x = 0$, $y = 0$ and that $g(t, 0, 0; \lambda)$ and $G(t, 0, 0; \lambda)$ do not depend on t.

We suppose that the nonlinearity is periodic with the period 2π:

$$f(t, x, y; \lambda) \equiv f(t + 2\pi, x, y; \lambda), \qquad t, x, y \in \mathbb{R},\ \lambda \in [-1, 1].$$

We are interested in the existence of small periodic solutions $x_\lambda(t)$ at arbitrary small λ with minimal period $2\pi n$, when $n > 1$ is a given integer. These oscillations are usually called *subharmonics*. Of course, such subharmonics may exist only if

$$\beta = \left(\frac{m}{n}\right)^2 \tag{1.2}$$

where the positive integers m, n are coprime. It is known, however, that representation (1.2) does not guarantee that small subharmonics exist for arbitrary small λ.

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Moreover, if $n > 4$, then existence of such oscillations is known to be 'unlikely' *if the main homogeneous part of the nonlinearity* $f(t, x, y; \lambda)$ *in variables* x, y *is a quadratic or cubic polynomial.* The word 'unlikely' means here that the subharmonics oscillations do not exist, unless some algebraic equalities with respect to higher coefficients in the Taylor expansion for f are valid. Instead, in a generic situation, we find the *subfurcation* phenomenon: for the values of parameter λ approaching λ_0 there arise (and then disappear) sporadically some oscillations of infinitely increasing periods. This effect was discovered by Kozyakin [4, 1]. We emphasize, that subfurcation is qualitatively different from the subharmonic bifurcation as considered below: we fix a period of small solutions and consider periodic solution of this fixed period only for various λ. Loosely speaking, the situation can be summarized as follows.

For smaller n*, in particular for* $n \leq 4$ *one have so called strong resonance case. The existence of subharmonics may be defined by the principal homogeneous part, subharmonics may exist for some open sets of system parameters. For* $n > 4$ *we have the weak resonant situation: the quadratic part does not dictate whether the subharmonics exist and the subharmonics may exist only for some degenerated situations.*

The first part of the paper, Section 2, is devoted mainly to careful analysis of equation (1.1) in the situation when f has a main homogeneous part F in x, y, but this part is not polynomial in x, y. Here we find the situation completely transformed. The following two observations are valid:

- **I.** Convenient sufficient conditions for the existence of small subharmonics can be given in terms of the main homogeneous part F. These conditions reduce essentially to the sign-alternating property of a scalar function, which is explicitly written via the function F. Moreover, for reasonably small m (for a given n) the subharmonics often exist for open sets of system parameters. See Theorem 2.2 and subsequent examples for the rigorous statements.
- **II.** If the number m is large enough then under some appropriate general conditions subharmonics do no exist, unless F satisfies a special integral equality. The exact meaning of the words "large enough" depends on the properties of F: how fast the integral sums of some integral tend to this integral. Exact formulations can be found in Subsection 2.3, Proposition 2.4.

Roughly speaking, the summary is as follows.

Let the main homogeneous part be not polynomial. Then, for reasonable small m *(not* n *as it could be expected!) the situation is similar to the strong resonant case (see the previous italicized paragraph): subharmonics may exist for some 'fat' sets of system parameters, and the answer can be given in terms of the principal homogeneous term of the nonlinearity. For bigger* m *we have a kind of weakly resonant situation: the subharmonics either do not exist (which is here the 'generic' case), or an answer can not be given in terms of the principal homogeneous part.*

Equation (1.1) of the second order is considered for the sake of simplicity only. It is possible to obtain similar results for much more cumbersome equations, for equations of higher order, for general system of Nth order, for equations with delays, it is possible to study bifurcations from infinity. Some corresponding results are formulated in Subsections 2.4 and 2.5.

In Section 3 we consider analogs of our results for the discrete dynamical systems. That is we consider the mapping $f_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that $f_\lambda(0) \equiv 0$ and that the linearization $A(\lambda)$ at $\lambda = \lambda_0$ of this mapping at the origin has a pair of eigenvalues on the unit circle in the complex plane of the form $\mu_\pm = \exp(\pm iq/p)$ with coprime p, q. The question is *whether the mapping has some p-periodic orbits close to zero* at λ close to λ_0. If the main homogeneous part $F = F_{\lambda_0}$ (which is now a vector valued mapping $\mathbb{R}^2 \to \mathbb{R}^2$) of f_{λ_0} is a pair of quadratic polynomials then the situation was studied in details [4]. In particular, for a weakly resonant case $p > 4$ the corresponding quadratic polynomial does not define an answer and, further, the answer is negative unless some algebraic conditions hold, generically the subfurcation phenomenon presents: some periodic orbits with infinitely increasing periods can be sporadically observed as $\lambda \to \lambda_0$ [4].

In the present paper we consider the situation when F is not a polynomial, examples are given in Subsection 3.2. It turns out that in this case some effective sufficient conditions for existence of small p-periodic orbits can be given in terms of F (more precisely, in terms of rotation of a two-dimensional vector field, which is explicitly constructed via the mapping F). We formulate these conditions in Subsection 3.2 and also demonstrate how these conditions work by some examples. Further, in Subsection 3.3 we illustrate that for very large p one can say some features of 'weakly-resonant' behaviour. In the case of discrete dynamical systems this reduces to an observation that for a very large p our method *cannot* be used to establish the period p bifurcation, unless the main homogeneous part of nonlinearity would satisfy a special integral equality. This equality holds, however, for any even function F. (In contrast to analogous discussions in Section 2 we do not know yet whether the bifurcation could occur; we have proved only that it cannot be discovered using a particular method.)

The discrete dynamical systems appear, in particular, as the shift operator for a given time along the trajectories of an ODE. Thus we can consider the discrete system that is generated by equation (1.1). In Subsection 3.4 we compare two approaches to analyze this equation: straightforward, as in Section 2, and via discrete dynamical system. Note, that discrete dynamical systems arise also in many problems of mathematical biology, economy etc. Also such systems arise naturally as a Poincare mapping for an autonomous differential equation. In the later case we actually investigate resonances in the Hopf bifurcation of a cycle. Some nonpolynomial sublinear terms could occur here due asymmetric feedbacks such as mentioned in [3].

Just as in Section 2, we focus on the two-dimensional case mainly for convenience. The analogous results can be formulated for the mappings $f : \mathbb{R}^N \to \mathbb{R}^N$ where the linearization A of this mapping at zero has just one pair of eigenvalues on the unit circle in the complex plane, and these eigenvalues are the roots of unit.

Finally, this paper is related to the paper [5] on the problem of weakly resonant Hopf bifurcation. In that paper the role of nonlinearities with 'non-polynomial' principal homogeneous part in synchronization of double-frequency oscillations in control systems was considered, and the 'flavor' of results is similar to those from the present article.

2. Main results for Liénard equation.

2.1. Definitions.

Definition 2.1. *The value* $\lambda = \lambda_0$ *is called* ***n-subharmonics bifurcation point*** *for equation* (1.1)*, if for any* $\varepsilon > 0$ *there exist an* $\varepsilon_0 \in (0, \varepsilon)$ *and a half-neighborhood* $\Lambda = (\lambda_0 - \varepsilon_0, \lambda_0)$ *or* $\Lambda = (\lambda_0, \lambda_0 + \varepsilon_0)$ *of the point* λ_0 *such that for any* $\lambda \in \Lambda$ *equation* (1.1) *has a nontrivial* $(2n\pi)$*-periodic solution* $x_\lambda(t)$ *satisfying* $\max\{|x_\lambda(t)|, |x'_\lambda(t)|\} \le \varepsilon$.

Let the nonlinearity $f(t,x,y;\lambda):\ [0,2\pi]\times\mathbb{R}\times\mathbb{R}\times[-1,1]$ can be represented as

$$f(t,x,y;\lambda)=F(t,x,y;\lambda)+\psi(t,x,y;\lambda). \tag{2.3}$$

Here the term $F(t,x,y;\lambda)$ is the principal term, which is positively homogeneous of the order $\alpha>1$:

$$F(t,\xi x,\xi y;\lambda)=\xi^{\alpha}F(t,x,y;\lambda),\qquad \xi>0$$

and the rest term $\psi(t,x,y;\lambda)$ is of a smaller order:

$$\lim_{|x|+|y|\to 0}\frac{|\psi(t,x,y;\lambda)|}{|x|^{\alpha}+|y|^{\alpha}}=0.$$

Let function $F(t,x,y;\lambda)$ satisfy the following Lipschitz condition at zero:

$$\bigl|F(t,x_1,y_1;\lambda)–F(t,x_2,y_2;\lambda)\bigr|\le d(\xi)(|x_1–x_2|+|y_1–y_2|),\quad \xi=\max\{|x_j|,|y_j|\} \tag{2.4}$$

where $d(\xi)\to 0$ as $\xi\to 0$. This property looks very natural for homogeneous functions with $\alpha>1$. It follows from the usual Lipschitz condition with respect to x and y on the circle $x^2+y^2=1$.

Consider a function of a variable $\varphi\in[0,2\pi]$:

$$\Psi(\varphi,\lambda)\overset{\text{def}}{=}\int_0^{2\pi}\sin(mt+\varphi)\,F(nt,\sin(mt+\varphi),\tfrac{m}{n}\cos(mt+\varphi);\lambda)\,dt. \tag{2.5}$$

This function is continuous with respect to both variables. The formula (2.5) for this function may be rearranged, in the next subsection some possibilities are given. We only mention that the function $\Psi(\varphi,\lambda)$ is periodic in φ with a period $2\pi/n$. This fact follows from the periodicity of this function simultaneously with the period 2π (this is obvious) as well as with the period $2m\pi/n$. The last periodicity follows from the formula

$$\Psi(\varphi,\lambda)=\frac{1}{m}\int_0^{2m\pi}\sin\tau\,F(\tfrac{n}{m}(\tau-\varphi),\sin\tau,\tfrac{m}{n}\cos\tau;\lambda)\,d\tau$$

and from 2π-periodicity of the function $F(t,x,y;\lambda)$ in t.

Consider another function of the variable $\varphi\in[0,2\pi]$:

$$\Psi^*(\varphi,\lambda)\overset{\text{def}}{=}\int_0^{2\pi}\cos(mt+\varphi)\,F(nt,\sin(mt+\varphi),\tfrac{m}{n}\cos(mt+\varphi);\lambda)\,dt. \tag{2.6}$$

This function is also periodic with a period $2\pi/n$.

We shall say that a scalar continuous function has a *proper zero*, if this zero is isolated and if the function locally takes positive values on one side of the zero and takes negative values on another side. Of course, a proper zero has nonzero topological index [6].

Obviously, if a non-constant function is periodic, then generically there exist an even number of proper zeros. If the function $F(t,x,y;\lambda)$ is differentiable in t, then $\Psi(\varphi,\lambda)$ is differentiable in φ. If φ^* is a zero and $\Psi'_{\varphi}(\varphi^*,\lambda_0)\neq 0$, then this zero is proper.

2.2. **Principal theorem and examples.** Consider the equation

$$x''+\lambda x'+\bigl(\tfrac{m}{n}\bigr)^2x=f(t,x,x';\lambda). \tag{2.7}$$

Theorem 2.2. *Let function* (2.5) *for* $\lambda=\lambda_0=0$ *have a proper zero* φ^* *and let* $\Psi^*(\varphi^*,\lambda_0)\neq 0$. *Then* $\lambda_0=0$ *is n-subharmonics bifurcation point for* (2.7), *for any* λ *sufficiently close to* λ_0 *such that* $\lambda\Psi^*(\varphi^*,\lambda_0)>0$ *there exists a subharmonic*

$$x(t)=r\sin(\tfrac{m}{n}t+\varphi^*)+h(t)\qquad \textit{where}\qquad r=\Bigl(\frac{\lambda\pi m}{n\Psi^*(\varphi^*,\lambda_0)}\Bigr)^{\frac{1}{\alpha-1}}+o(|\lambda|^{\frac{1}{\alpha-1}}).$$

The proof is a bit cumbersome, so it is relegated to Section 4 at the end of the paper. Now we will discuss the theorem and illustrate how it can be used.

Proposition 2.3. *Let function* (2.5) *be nonzero for* $\lambda = \lambda_0 = 0$. *Then* $\lambda_0 = 0$ *is not an n-subharmonics bifurcation point for* (2.7).

The proof is given at the end of Section 4. At the first glance the proposition means that the conditions of the theorem are "almost necessary" for the bifurcation. However, the situation is not as simple. The point is that in some natural situations function (2.5) equals to zero identically.

Now let us turn to examples. Let $F(t,x,y;\lambda) = F(x,y;\lambda)$. Then obviously $\Psi(\varphi,\lambda)$ does not depend on φ and Theorem 2.2 is unapplicable.

The simplest example if F depends on t is $F(t,x,y;\lambda) = \sin t\, F(x,y)$. For this case $m > 1$ implies $\Psi(\varphi,\lambda) \equiv 0$ and our approach does not work. For $m = 1$ examples are possible. Both functions $\Psi(\varphi)$ and $\Psi^*(\varphi)$ may be computed in evident form:

$$\Psi(\varphi) = r_1 \sin\Big(\frac{n}{m}\varphi + s_1\Big), \qquad \Psi^*(\varphi) = r_2 \sin\Big(\frac{n}{m}\varphi + s_2\Big),$$

where r_j, s_j are constants. But $m \neq 1$ implies $r_1 = r_2 = 0$! This follows from the identity

$$J = \int_0^{2\pi} \sin nt\; g(mt)\, dt = 0,$$

which is valid for $m > 1$ for any 2π-periodic $g(t)$. If $m = 1$, then, generically, $J \neq 0$. For $F(t,x,y;\lambda) = x^2 \sin t$ Theorem 2.2 is applicable iff $n = 3$ and $\Psi \equiv 0$ for other n. If $F(t,x,y;\lambda) = x^4 \sin t$, Theorem 2.2 is applicable for $n = 3, 5$ and $\Psi \equiv 0$ for other n.

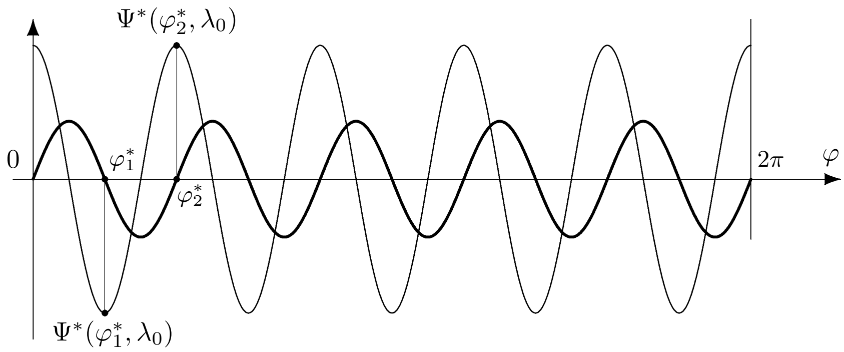

Figure 1: Functions $\Psi(\varphi,\lambda_0)$ and $\Psi^*(\varphi,\lambda_0)$ for $\alpha = 1.7,\ n = 5,\ m = 3$.

Consider more complicated example $F(t,x,y;\lambda) = \sin t\, |x\cos t + y|^\alpha$. If $\alpha = 2$, then again $\Psi(\varphi) = \Psi(\varphi,\lambda) = 0$ for $n > 3$ (the same is true for any positive integer α and $n > \alpha + 1$). This is the key point, let us stress again that the Liénard equation with quadratic nonlinearities can be studied with various other methods (see, e.g., [4]), the answers are very different from the case considered here. But if α is not an integer number, then everything is OK ($\Psi(\varphi) \not\equiv 0$) and Theorem 2.2 may be applied. On Fig. 1 one can see the graphs of the functions $\Psi(\varphi,\lambda_0)$ (thick line) and $\Psi^*(\varphi,\lambda_0)$ (thin line) for $\alpha = 1.7,\ n = 5,\ m = 3$. On a period (on Fig. 1 it equals $2\pi/5$) the function $\Psi(\varphi,\lambda_0)$ has two proper zeros: φ_1^* and φ_2^*.

Obviously, $\Psi^*(\varphi_1^*, \lambda_0) \cdot \Psi^*(\varphi_2^*, \lambda_0) < 0$. It means that in the case considered we have subharmonics both for $\lambda < 0$ and for $\lambda > 0$.

2.3. **Conditions of nonexistence.** Let

$$\int_0^{2\pi} \sin t\, \Phi(\sin t, \tfrac{m}{n}\cos t; \lambda_0)\, dt \neq 0, \qquad \Phi(x,y;\lambda) \stackrel{\text{def}}{=} \int_0^{2\pi} F(t,x,y;\lambda)\, dt. \quad (2.8)$$

Proposition 2.4. *If* (2.8) *is valid, then there exists a* M *such that for* $m > M$ *sufficiently small cycles of the period* $2n\pi$ *do not exist for* λ *sufficiently close to* λ_0.

Proof. By Proposition 2.3 it suffices to establish the following assertion.

Lemma 2.5. *If* (2.8) *is valid, then there exist a* M *such that function* (2.5) *is nonzero for* $m > M$.

This follows from the next chain of relations:

$$\begin{aligned}
\Psi(\varphi, \lambda_0) &= \int_0^{2\pi} \sin(mt+\varphi)\, F(nt, \sin(mt+\varphi), \tfrac{m}{n}\cos(mt+\varphi); \lambda_0)\, dt \\
&= \frac{1}{m}\int_0^{2m\pi} \sin\tau\, F(\tfrac{n}{m}(\tau-\varphi), \sin\tau, \tfrac{m}{n}\cos\tau; \lambda_0)\, d\tau \\
&= \frac{1}{m}\sum_{k=0}^{m-1}\int_{2k\pi}^{2(k+1)\pi} \sin\tau\, F(\tfrac{n}{m}(\tau-\varphi), \sin\tau, \tfrac{m}{n}\cos\tau; \lambda_0)\, d\tau \\
&= \frac{1}{m}\sum_{k=0}^{m-1}\int_0^{2\pi} \sin\tau\, F(\tfrac{n}{m}(\tau-\varphi) + \tfrac{2kn\pi}{m}, \sin\tau, \tfrac{m}{n}\cos\tau; \lambda_0)\, d\tau \\
&= \frac{1}{m}\sum_{k=0}^{m-1}\int_0^{2\pi} \sin\tau\, F(\tfrac{n}{m}(\tau-\varphi) + \tfrac{2k\pi}{m}, \sin\tau, \tfrac{m}{n}\cos\tau; \lambda_0)\, d\tau.
\end{aligned}$$

For sufficiently large m the last sum is sufficiently close to the integral

$$\frac{1}{2\pi}\int_0^{2\pi} dt \int_0^{2\pi} \sin\tau\, F(\tfrac{n}{m}(\tau-\varphi) + t, \sin\tau, \tfrac{m}{n}\cos\tau; \lambda_0)\, d\tau.$$

After changing the order of integration in it we obtain

$$\begin{aligned}
&\int_0^{2\pi} d\tau \int_0^{2\pi} \sin\tau F(\tfrac{n}{m}(\tau-\varphi) + t, \sin\tau, \tfrac{m}{n}\cos\tau; \lambda_0)\, dt \\
&\qquad\qquad = \int_0^{2\pi} \sin t\, \Phi(\sin t, \tfrac{m}{n}\cos t; \lambda_0)\, dt.
\end{aligned}$$

The last term is different from zero and does not depend on φ.

We have shown that for sufficiently large m the main function is arbitrary close to nonzero constant. The lemma is proved and so is the proposition. ■

2.4. **Bifurcations at infinity.**

Definition 2.6. *The value* $\lambda = \lambda_0$ *of the parameter is called* ***n-subharmonics bifurcation point at infinity*** *for equation* (2.7), *if for any* $\varepsilon > 0$ *there exist an* $\varepsilon_0 \in (0, \varepsilon)$ *and a half-neighborhood* $\Lambda = (\lambda_0 - \varepsilon_0, \lambda_0)$ *or* $\Lambda = (\lambda_0, \lambda_0 + \varepsilon_0)$ *of the point* λ_0 *such that for any* $\lambda \in \Lambda$ *equation* (1.1) *has a nontrivial* $(2n\pi)$*-periodic solution* $x_\lambda(t)$ *satisfying* $\max\{|x_\lambda(t)|, |x'_\lambda(t)|\} \geq \varepsilon^{-1}$.

Suppose that the nonlinearity in (1.1) is bounded and has the principal part that is positively homogeneous of the order 0: expansion (2.3) is valid,

$$F(t,\xi x,\xi y;\lambda)=F(t,x,y;\lambda),\qquad \xi>0\qquad\text{and}\qquad \lim_{|x|+|y|\to\infty}\ \sup_{t,\lambda}|\psi(t,x,y;\lambda)|=0.$$

Of course, such function $F(t,x,y;\lambda)$ is discontinuous at zero, this must be compensated with the discontinuity at zero of the small term $\psi(t,x,y;\lambda)$.

Suppose that the function $F(t,x,y;\lambda)$ is continuous at all other points, or, what is the same, it is continuous for $x^2+y^2=1$. This is rather strict assumption: we can not consider the nonlinearities like $f(t,x,y;\lambda)=a(t)\arctan x$. Its principal homogeneous part $\frac{\pi}{2}a(t)\,\mathrm{sign}\,x$ is discontinuous on the line $\{x=0, y\in\mathbb{R}\}$ in the plane $\{x,y\}$. It is also possible to study equations with such nonlinearities but it is necessary to use special additional machinery presented in [2].

Theorem 2.7. *Let function* (2.5) *for* $\lambda=\lambda_0=0$ *have a proper zero* φ^* *and let* $\Psi^*(\varphi^*,\lambda_0)\neq 0$. *Then* $\lambda_0=0$ *is n-subharmonics bifurcation point at infinity for* (2.7)*, for* λ *sufficiently close to* λ_0 *such that* $\lambda\Psi^*(\varphi^*,\lambda_0)>0$ *there exists a large subharmonics*

$$x(t)=R\sin(\tfrac{m}{n}t+\varphi^*)+h(t)\qquad\text{where}\qquad R=\frac{n\Psi^*(\varphi^*,\lambda_0)}{\lambda\pi m}+o(|\lambda|^{-1}).$$

2.5. **Control theory equation with a delay.** If the nonlinearity has more cumbersome form, for instance, it includes delays, or the linear part is more complicated, the similar result can be proved with the same methods.

We consider here the equation with a delay:

$$L\Big(\frac{d}{dt};\lambda\Big)x(t)=M\Big(\frac{d}{dt};\lambda\Big)f\big(x(t),x(t-\theta);\lambda\big). \tag{2.9}$$

Here

$$L(p;\lambda)=p^\ell+a_1(\lambda)p^{\ell-1}+\dots+a_\ell(\lambda),\ M(p;\lambda)=b_0(\lambda)p^m+b_1(\lambda)p^{m-1}+\dots+b_m(\lambda)$$

are coprime polynomials of degrees ℓ and m. Let $\ell>m$ and let coefficients of the polynomials be continuous in λ. Systems (2.9) are usual in control theory (see, e.g. [7]). Readers who are not very familiar with such type of equations may assume that $M(p)\equiv 1$ and consider usual higher order ODE.

Consider the functions of the variable $\varphi\in[0,2\pi]$:

$$\Psi(\varphi,\lambda)\stackrel{\text{def}}{=}\int_0^{2\pi}\sin(mt+\varphi)\,F\big(nt,\sin(mt+\varphi),\sin(mt+\varphi-\tfrac{m}{n}\theta);\lambda\big)\,dt \tag{2.10}$$

and

$$\Psi^*(\varphi,\lambda)\stackrel{\text{def}}{=}\int_0^{2\pi}\cos(mt+\varphi)\,F\big(nt,\sin(mt+\varphi),\sin(mt+\varphi-\tfrac{m}{n}\theta);\lambda\big)\,dt. \tag{2.11}$$

These functions are similar to functions (2.5) and (2.6).

Theorem 2.8. *Suppose that* $L(p;\lambda)=(p^2+\lambda p+m^2/n^2)L_1(p;\lambda)$ *where the polynomial* $L_1(p,\lambda)$ *has no roots of the type* $\frac{k}{n}i$, $k\in\mathbb{Z}$ *for* $\lambda=\lambda_0=0$. *Let the function*

$$\Re e\big[M(\tfrac{m}{n}i;\lambda_0)L_1(-\tfrac{m}{n}i;\lambda_0)\big]\Psi^*(\varphi,\lambda_0)+\Im m\big[M(\tfrac{m}{n}i;\lambda_0)L_1(-\tfrac{m}{n}i;\lambda_0)\big]\Psi(\varphi,\lambda_0)$$

have a proper zero φ^* *and let*

$$\Re e\big[M(\tfrac{m}{n}i;\lambda_0)L_1(-\tfrac{m}{n}i;\lambda_0)\big]\Psi(\varphi^*,\lambda_0)+\Im m\big[M(\tfrac{m}{n}i;\lambda_0)L_1(-\tfrac{m}{n}i;\lambda_0)\big]\Psi^*(\varphi^*,\lambda_0)\neq 0.$$

Then $\lambda_0=0$ *is n-subharmonics bifurcation point for* (2.9).

Let us emphasize that $M(\frac{m}{n}i;\lambda_0) \neq 0$ since $L(\frac{m}{n}i;\lambda_0) = 0$ and since the polynomials $L(p,\lambda_0)$ and $M(p,\lambda_0)$ supposed to be coprime.

3. **Iterations of operators.**

3.1. **Definitions.** Consider a mapping $f_\lambda(x)$, which is defined for $x \in \mathbb{R}^2$, $\lambda \in \mathbb{R}^1$ and takes values in $\mathbb{R}^2$. The number λ is a parameter, and for a given λ we are interested in periodic orbits of the discrete dynamical system generated by the mapping $x \mapsto f_\lambda(x)$, $x \in \mathbb{R}^2$. Recall that a point x is periodic with the (minimal) period p if $f^p(x) = x$ and $f^k(x) \neq x$ for $1 \leq k < p$. We will always suppose that f is continuous and $f_\lambda(0) \equiv 0$. Here k, p are positive integers and f^p denotes the iterated mapping. For a p-periodic point x the set $\{x, f(x), \dots f^{p-1}(x)\}$ is the *orbit* of x with the notation $\mathrm{Or}(x)$.

Let $p > 1$ be a positive integer. We say that λ_0 is a period p bifurcation point if for any $\varepsilon > 0$ there exist λ arbitrary close to λ_0 and a p-periodic point of $f_\lambda(x)$ with the minimal period p satisfying

$$\mathrm{Or}(x) \subset B_\varepsilon = \{x \in \mathbb{R}^2 : |x| < \varepsilon\}.$$

For $p = 1$ this is called usually branching-bifurcation, for $p = 2$ this is period doubling or flip bifurcation (many other names are also in use).

We suppose that f is represented as

$$f_\lambda(x) = A_\lambda x + F_\lambda(x) + \psi_\lambda(x) \tag{3.12}$$

(cf (2.3)). Here A_λ is a matrix, $F_\lambda(x)$ is the principal nonlinear term, which is positively homogeneous of the order $\alpha > 1$:

$$F_\lambda(\xi x) = \xi^\alpha F_\lambda(x), \qquad \xi > 0,$$

and the rest term $\psi_\lambda(x)$ has a smaller order:

$$\lim_{|x|\to 0} \frac{|\psi_\lambda(x)|}{|x|^\alpha} = 0.$$

We will be interested in the case $p > 4$. The necessary condition for period p bifurcation is that the matrix A_{λ_0} has the eigenvalues of the form $\exp(\pm iq/p)$, $q, p \in \mathbb{Z}$, without loss of generality suppose that $q \in (0, p/2)$ is coprime with p. Let us repeat that the period p bifurcation is 'unlikely' if the principal homogeneous part F_λ is a polynomial of the low degree (say 2 or 3). In this case actually happens the so-called *subfurcation*, discovered by Kozyakin: there arise (and then disappear) sporadically some oscillations of infinitely increasing periods. Our main point is that *for more general classes of functions F_λ the period p bifurcation becomes quite natural and it is easy to catch them by standard topological methods.*

Before to proceed further we remind some standard notation definitions, see [6] for details. Let D be a bounded open set and g be a continuous mapping $\mathbb{R}^2 \to \mathbb{R}^2$, which is defined onto $\partial(D)$ and is non-degenerated at $\partial(D)$ (that is $g(x) \neq 0$ for $x \in \partial(D)$). The symbol $\gamma(g, D)$ denotes then the rotation (or winding number, or topological degree) of the vector field g at the boundary $\partial(D)$. If $x = 0$ is an isolated zero of g, then for small positive ε the rotations $\gamma(g, B_\varepsilon)$ coincide and their common value is the Kronecker index of 0 with the notation $\mathrm{Ind}(0, g)$.

3.2. **Main results.** The following assertion is a modification of the general Index Changing Principle [6].

We suppose for the simplicity that the entries of A_λ are smooth in λ. Denote by $\mu_\lambda, \bar\mu_\lambda$ the eigenvalues of $A(\lambda)$ and by I the identity mapping.

Lemma 3.1. *Let*

$$\mu_\lambda \neq e^{\pm iq/p} \qquad (\lambda \neq \lambda_0), \qquad\qquad \mathrm{Ind}(0, f^p_{\lambda_0} - I) \neq 1. \tag{3.13}$$

Then the period p bifurcation occurs. Moreover, small p-periodic solutions exist for any λ that is sufficiently close to λ_0 and different from λ_0.

Proof. The first relation (3.13) implies that the eigenvalues of the matrix A_λ are different from p-roots of unity for any λ sufficiently close to λ_0, $\lambda \neq \lambda_0$. Thus the mapping $f^p_\lambda - I$ for $\lambda \neq \lambda_0$ has a nondegenerate linear part $A^p_\lambda - I$. In particular, $\mathrm{Ind}(0, f^p_\lambda - I) = \mathrm{Ind}(0, A^p_\lambda - I)$, $\lambda \neq \lambda_0$.

The eigenvalues of A_λ are adjoint complex numbers for λ sufficiently close to λ_0. Therefore, the determinant $\det(A^p_\lambda - I)$ is positive. Using the formula $\mathrm{Ind}(0, A^p_\lambda - I) = \mathrm{sign}(\det(A^p_\lambda - I))$ (see [6]) we obtain

$$\mathrm{Ind}(0, f^p_\lambda - I) = 1.$$

Combining this relation with the second inequality (3.13) we conclude that small p-periodic solutions exist for any λ that is sufficiently close to λ_0 and different from λ_0 (there exists a small nonzero fixed point of the operator f^p_λ). Using the fact that the numbers p, q are coprime, we conclude that this fixed point must be a p-periodic point of the operator f_λ with the minimal period p. Therefore the lemma is proved. ■

Clearly, in our settings the linear part of $f^p_{\lambda_0}$ equals identically to zero; thus, under simple technical restrictions, the 'natural leading term' of $f^p(x) - I$ is

$$G(x) = A^{p-1}F(x) + A^{p-2}F(Ax) + \ldots + AF(A^{p-2}x) + F(A^{p-1}x) \tag{3.14}$$

where $F(x) = F_{\lambda_0}(x)$ and $A = A_{\lambda_0}$. For instance this is true if the restriction of F_{λ_0} to the circle $B_1 = \{|x| = 1\}$ is Lipschitz continuous. Combining this representation with Lemma 3.1 we get

Proposition 3.2. *Let* (3.13) *be valid. Let the restriction of F_{λ_0} to the circle B_1 be Lipschitz continuous and $G(x) \neq 0$ for $x \neq 0$. Let, finally,* $\mathrm{Ind}(0, G) \neq 1$ *or, what is the same, $\gamma(G, B_1) \neq 1$. Then the period p bifurcation occurs. Moreover, small p-periodic solutions exist for any $\lambda \neq \lambda_0$ that is sufficiently close to λ_0.*

Note a simple corollary, it is valid since the function G is even together with F, the rotation $\gamma(G, B_1)$ of any even vector field is even, and therefore differs from 1.

Corollary 3.3. *Let* (3.13) *be valid. Let the restriction of F_{λ_0} to the circle B_1 be Lipschitz continuous and even: $F_{\lambda_0}(x) = F_{\lambda_0}(-x)$, $x \in \mathbb{R}^2$. Let, finally, $G(x) \neq 0$ for $x \neq 0$. Then the period p bifurcation occurs. Moreover, small p-periodic solutions exist for any λ that is sufficiently close to λ_0 and different from λ_0.*

When applying the method it is worthy to have in mind the following simple statement, it follows the usual pattern [6] and so is omitted.

Proposition 3.4. *Let the restriction of F_{λ_0} to the circle $|x| = 1$ be Lipschitz continuous and $G(x) \neq 0$ for $x \neq 0$. The congruence* $\mathrm{Ind}(0, G) = 1 \pmod{p}$ *holds.*

The integer number $K = \big(\mathrm{Ind}(0, G) - 1\big)p^{-1}$ contains some additional information about the bifurcation. In particular, the value $|K|$ provides an algebraic number of different branches of p-periodic orbits.

Note again, that this method does not work, if F is a quadratic polynomial: the catch is that in this case the mapping G equals identically to zero. However, experimental calculations suggest that the simple assertions formulated above are quite effective in analysis of bifurcations if F is not a polynomial. There is no troubles at all in calculating the rotation $\gamma(G, B_1)$ of an explicitly give two-dimensional vector field: one can use the classical Poincare formula [6] or appropriate computer-aided geometrical constructions (which are actually more convenient).

Some numerics are presented below in Table 1. We interpret here $\mathbb{R}^2$ as the complex plain with the elements $x + iy$. In all experiments $q = 1$, $p = 5$ and the matrix A_{λ_0} is the multiplying by $\exp(2\pi i/5)$.

$F(x,y)$	$\frac{x^2-y^2-5xyi}{\lvert x\rvert^{0.3}+\lvert y\rvert^{0.3}}$	$\frac{x^2-y^2+3xyi}{\lvert x\rvert^{0.3}+\lvert y\rvert^{0.3}}$	$\frac{2x^2+y^2+2xyi}{\lvert x\rvert^{0.3}+\lvert y\rvert^{0.3}}$
$\mathrm{Ind}(0,G)$	-14	6	-4
$F(x,y)$	$\frac{x^4}{x^2+2y^2}$	$\frac{(x^2-y^2+xyi)^2}{x^2+2y^2}$	$x(\lvert x+y\rvert-\lvert x-y\rvert)(\lvert x\rvert-\lvert y\rvert)$
$\mathrm{Ind}(0,G)$	-4	6	11
$F(x,y)$	$\lvert xy\rvert$	$\lvert x\rvert^{1.4}-\lvert y\rvert^{1.4}$	$x(\sqrt{\lvert x+y\rvert}-\sqrt{\lvert x-y\rvert})(\lvert x\rvert-\lvert y\rvert)$
$\mathrm{Ind}(0,G)$	-4	6	-9

TABLE 1. Experimental results

Sometimes for rational $F(x,y)$ it is possible to calculate $\mathrm{Ind}(0,G)$ "by hands". Let $p = 5$ and $F(z) = |z|^{\alpha+s}/(2|z|^s + z^s)$ ($s \in \mathbb{Z}$, $\alpha > 1$). In this case

$$\begin{cases} \text{if} & s = 5k, & \text{then } G \equiv 0 \text{ and our method is unapplicable,} \\ \text{if} & s = 5k+1, & \text{then } \mathrm{Ind}(0,G) = s, \\ \text{if} & s = 5k+2, & \text{then } \mathrm{Ind}(0,G) = 3s, \\ \text{if} & s = 5k+3, & \text{then } \mathrm{Ind}(0,G) = 2s, \\ \text{if} & s = 5k+4, & \text{then } \mathrm{Ind}(0,G) = 4s. \end{cases}$$

Analogous formulas can be written for other p.

Let us mention also, that the function G does not depend on q, if the matrix A_{λ_0} is a multiplying by $\exp(2\pi i/5)$, the function G is the same and results are the same.

3.3. **Large *p*.** In this subsection we present some ideas, close to Subsection 2.3. Let us transform formula (3.14). Put $\theta = \exp(2\pi iq/p)$ and let on the complex plane the operator A be the multiplication by θ. Then

$$\begin{aligned} G(x) &= A^{p-1}F(x) + A^{p-2}F(Ax) + \ldots + AF(A^{p-2}x) + F(A^{p-1}x) \\ &= \theta^{-1}\sum_{0}^{p-1}\theta^{-k}F(\theta^k x) = \theta^{-1}\sum_{0}^{p-1}e^{-2k\pi iq/p}F\Big(e^{2k\pi iq/p}x\Big). \end{aligned}$$

Now let us reorder the set $\{e^{2k\pi iq/p}\},\ k=0,1,\dots,p-1$ on the unit circle B_1 as $\{e^{2k\pi i/p}\},\ k=0,1,\dots,p-1$. Since p and q are coprime, these sets coincide. Therefore

$$G(x)=\theta^{-1}\sum_{k=0}^{p-1}e^{-2k\pi iq/p}F\Big(e^{2k\pi iq/p}x\Big)$$
$$=\theta^{-1}\sum_{k=0}^{p-1}e^{-2k\pi i/p}F\Big(e^{2k\pi i/p}x\Big)\approx\frac{p}{2\pi\theta i}\int_0^{2\pi}e^{-zi}F(e^{zi}x)\,dz$$

(we changed integral sums with the corresponding integral). Now put $x=e^{\varphi i}$, then

$$G(x)\approx\frac{p}{2\pi\theta i}\int_0^{2\pi}e^{-zi}F(e^{(z+\varphi)i})\,dz=e^{i\varphi}\frac{p}{2\pi\theta i}\int_{B_1}z^{-1}F(z)\,dz.$$

We have proved the following assertion.

Proposition 3.5. *Let*

$$\int_{B_1}z^{-1}F(z)\,dz\neq 0. \tag{3.15}$$

Then for sufficiently large[1] *numbers p the equality $\gamma(G,B_1)=1$ is valid.*

By this proposition inequality (3.15) implies for large p the equality $\operatorname{Ind}(0,f^p_{\lambda_0}-I)=1$ and for such F Proposition 3.2 *can be never* used to establish the period p bifurcation. (We do not know whether the bifurcation could occur, we proved only that it can not be discovered using a particular method.) Note, finally, that inequality (3.15) is always wrong when F is even, thus Proposition 3.5 "does not contradict" Proposition 3.4.

3.4. **The dynamical system generated by Liénard equation.** Now let us define the connection between this section and the rest part of the paper.

Suppose that Cauchy problem for equation (1.1) has a unique solution locally at zero. Then in a vicinity of zero we can define an operator of translation along the trajectories of equation (1.1) at the time 2π. This operator can be considered as the mapping $f_\lambda(x)$ and the period p bifurcations are exactly n-subharmonics bifurcations in the sense of Definition 1.

Of course, results about n-subharmonics bifurcations are more exact: we suppose much more initial information.

To present exact formulae, we rewrite equation (1.1) at $\lambda=\lambda_0=0$ in a complex form

$$z'=i\frac{m}{n}z+i\frac{n}{m}f(t,z),\qquad y=x'\frac{n}{m},\qquad z=x-iy.$$

Here we denote $f(t,x,x';\lambda_0)$ as $f(t,z)$. This means that

$$z(t)=e^{i\frac{m}{n}t}z(0)+i\frac{n}{m}\int_0^t e^{i\frac{m}{n}(t-s)}f\big(s,z(s)\big)\,ds. \tag{3.16}$$

The corresponding dynamical system is generated by the mapping

$$z\mapsto e^{i\frac{m}{n}2\pi}z+i\frac{n}{m}\int_0^{2\pi}e^{i\frac{m}{n}(2\pi-s)}F(s,e^{i\frac{m}{n}s}z)\,ds+\text{'smaller terms'}$$

with $F(t,z)=F(t,x,x';\lambda_0)$ (cf (3.12)). The nth iteration of this mapping has the form

$$z\mapsto z+i\frac{n}{m}\int_0^{2n\pi}e^{-i\frac{m}{n}s}F(s,e^{i\frac{m}{n}s}z)\,ds+\text{'smaller terms'}.$$

[1] If the integral sums are close enough to the integral.

We wrote this formula without any references to formula (3.14), for the translation operator it is simpler to put $t = 2n\pi$ in (3.16). The most interesting term in the last formula is the principal nonlinear term

$$F^* = \int_0^{2n\pi} e^{-i\frac{m}{n}s} F(s, e^{i\frac{m}{n}s} z)\, ds = n \int_0^{2\pi} e^{-ims} F(ns, e^{ims} z)\, ds.$$

Put $z = \exp(\varphi i)$. Then

$$\begin{aligned} F^* &= n \int_0^{2\pi} e^{ims} F(ns, e^{i(ms+\varphi)})\, ds = n\, e^{-i\varphi} \int_0^{2\pi} e^{-i(ms+\varphi)} F(ns, e^{i(ms+\varphi)})\, ds \\ &= n\, e^{-i\varphi} \Big(\Psi(\varphi, \lambda_0) + i\Psi^*(\varphi, \lambda_0) \Big) \qquad \text{(since } F \text{ is a real function).} \end{aligned}$$

Now we see that functions (2.5) and (2.6) define the value $\mathrm{Ind}(0, G)$. Let us emphasize that for the case, considered in Subsection 2.2, $\mathrm{Ind}(0, G) = 6$, the number 6 here equals as 5 generated by the field $\{\Psi(\varphi, \lambda_0), \Psi^*(\varphi, \lambda_0)\}$ plus 1 generated by the multiplier $\exp(-i\varphi)$. Theorem 1 guarantees the existence of 10 solutions: 5 for $\lambda > 0$ and 5 for $\lambda < 0$. Any 5 of them (for fixed λ) form an orbit of any of them.

4. **Proofs of Theorem 2.2 and Proposition 2.3.**

4.1. **Time rescaling.** First of all let us rescale the time. Instead of equation (2.7) we consider the equation

$$x'' + n\lambda x' + m^2 x = n^2 f(nt, x, n^{-1}x'; \lambda). \tag{4.17}$$

Every 2π-periodic solution of equation (4.17) is a $2n\pi$-periodic solution of equation (2.7).

4.2. **Linear maps.** Consider the projector

$$Pu(t) = \frac{1}{\pi} \int_0^{2\pi} \cos m(t-s)\, u(s)\, ds.$$

It projects $L^2 = L^2(0, 2\pi)$ onto the plane Π, spanned onto the functions $\sin mt$, $\cos mt$. Consider also a projector $Q = I - P$, it projects L^2 onto the subspace Π^* ($\operatorname{codim} \Pi^* = 2$) of 2π-periodic functions, which do not contain the mth harmonics in their Fourier expansions.

Consider for any λ the linear operator $A(\lambda)\colon \Pi^* \to \Pi^*$ that maps any $u(t) \in \Pi^*$ into the 2π-periodic solution $x(t) = A(\lambda)u(t)$ of the linear equation $x'' + n\lambda x' + m^2 x = u(t)$. For $\lambda = 0$ the existence follows from $u \in \Pi^*$, the uniqueness follows from $x \in \Pi^*$.

The operator $A(\lambda)h$ is completely continuous with respect to both variables λ, h for h belonging to the intersection of Π^* and the most usual spaces: L^2, C, C^1. It is also completely continuous as an operator from $C \cap \Pi^*$ to C^1. The operators $A(\lambda)$ commute with P and Q.

The operators $A(\lambda)$ are uniformly (with respect to λ) bounded. In L^2 any $A(\lambda)$ is normal and it is possible to write the exact formula

$$\|A(\lambda)\|_{\Pi^* \to \Pi^*} = \sup_{k \in \mathbb{Z}, z \neq \pm m} \frac{1}{|m^2 - k^2 + nk\lambda i|}, \qquad \sup_{\lambda} \|A(\lambda)\| = c^* \le 1.$$

We use the estimate

$$\|A(\lambda)Q\|_{C \to C^1} \le c < \infty.$$

The operators $A(\lambda)$ for $\lambda \neq 0$ can be defined for any $u \in L^2$, but the norms of such operators will not be uniformly bounded.

4.3. **Equivalent system.** Now we are ready to rewrite 2π-periodic problem for equation (4.17) in some convenience equivalent form.

We shall find 2π-periodic solutions of our equation in the form

$$x(t) = r\sin(mt+\varphi) + h(t) \tag{4.18}$$

where $h(t) \in \Pi^*$. Here $r > 0, \varphi \in [0, 2\pi]$ and $h \in \Pi^*$ are unknown.

Put this representation in equation (4.17) and project the equation onto Π and onto Π^*. The infinite dimensional projection onto Π^* has the form

$$h = A(\lambda)Qn^2 f(nt, x, n^{-1}x'; \lambda) \tag{4.19}$$

and 2-dimensional projection has the form of two scalar equations

$$\begin{aligned} 0 &= \int_0^{2\pi} \sin(mt+\varphi) f(nt, x, \frac{x'}{n}; \lambda)\, dt \\ &= \pi\lambda r m - n\int_0^{2\pi} \cos(mt+\varphi) f(nt, x, \frac{x'}{n}; \lambda)\, dt. \end{aligned} \tag{4.20}$$

System (4.19) – (4.20) is equivalent to 2π-periodic problem for equation (4.17): any 2π-periodic solution $x(t)$ can be represented in the form (4.18) and the corresponding r, φ, h satisfy the system, and any solutions r, φ, h of this system define 2π-periodic solution $x(t)$ of equation (4.17).

4.4. **The principal part of equations** (4.19)**.** Now for a moment let us omit small terms in equations (4.20). We replace λ with λ_0 in the integrals, we put everywhere the principal part $F(t, x, y; \lambda_0)$ instead of the complete nonlinearity $f(t, x, y; \lambda_0)$, finally we replace the function $x(t)$ with $r\sin(mt+\varphi)$. The resulting equations have the form

$$0 = \int_0^{2\pi} \sin(mt+\varphi) F(nt, \sin(mt+\varphi), \tfrac{m}{n}\cos(mt+\varphi); \lambda_0)\, dt = \Psi(\varphi, \lambda_0),$$

$$\pi\lambda r^{1-\alpha}\tfrac{m}{n} = \int_0^{2\pi} \cos(mt+\varphi) F(nt, \sin(mt+\varphi), \tfrac{m}{n}\cos(mt+\varphi); \lambda_0)\, dt = \Psi^*(\varphi, \lambda_0).$$

Define the unknown φ from the first equation, put it in the second one, and find r.

We do not explain, why we suppose that $\|h\|$ is of smaller oder then r, in what space we consider our equations etc. We formulate this subsection for convenience of the reader only to explain the genesis of the assumptions of Theorem 2.2.

4.5. **Homotopy.** Now let us consider a space $\mathbb{E}$, each element $\{\varphi, r, h\}$ of this space has three components: a real φ, a real positive r, and 2π-periodic C^1 function $h(t) \in \Pi^*$.

Consider the deformation $\mathcal{F}(\varphi, r, h; \xi) : \mathbb{E} \times [0,1] \to \mathbb{E}$ where $\xi \in [0,1]$ is the parameter of the deformation. The components of the deformation are defined as follows. The first component $\mathcal{F}_\varphi$ of the deformation is defined by the equality

$$\mathcal{F}_\varphi(\varphi, r, h; \xi) \stackrel{\text{def}}{=} \xi r^{-\alpha} \int_0^{2\pi} \sin(mt+\varphi) f(nt, x, n^{-1}x'; \lambda)\, dt + (1-\xi)\Psi(\varphi, \lambda_0),$$

the second component $\mathcal{F}_r$ of the deformation is defined by the equality

$$\begin{aligned} \mathcal{F}_r(\varphi, r, h; \xi) &\stackrel{\text{def}}{=} \pi\lambda \frac{m}{n} r^{1-\alpha} \\ &\quad - r^{-\alpha}\xi \int_0^{2\pi} \cos(mt+\varphi) f(nt, x, n^{-1}x'; \lambda)\, dt - (1-\xi)\Psi^*(\varphi, \lambda_0), \end{aligned}$$

and the third component $\mathcal{F}_h$ of the deformation is defined by the equality

$$\mathcal{F}_h(\varphi, r, h; \xi) \stackrel{\text{def}}{=} h - \xi A(\lambda) Q n^2 f(nt, x, n^{-1}x'; \lambda).$$

In all three formulae $x(t)$ is function (4.18). This deformation is completely continuous: the operator $x \mapsto f(nt, x, n^{-1}x'; \lambda)$ is continuous as an operator from C^1 to C, and $A(\lambda)Q$ is completely continuous as an operator from C to C^1.

For $\xi = 0$ our deformation has a form of very simple vector field

$$\mathcal{F}(\varphi, r, h; 0) = \Big\{\Psi(\varphi; \lambda_0),\ \pi\lambda\frac{m}{n}r^{1-\alpha} - \Psi^*(\varphi; \lambda_0),\ h\Big\}.$$

From the general theory of the rotation of vector fields [6] and our assumptions it directly follows that the zero

$$\Big\{\varphi^*, r^* = \Big(\frac{\lambda\pi m}{n\Psi^*(\varphi^*, \lambda_0)}\Big)^{\frac{1}{\alpha-1}}, 0\Big\}$$

of $\mathcal{F}(\cdot, \cdot, \cdot; 0)$ for any small λ is isolated and its rotation on the boundary ∂G_ε of the set

$$G_\varepsilon = \{|\varphi - \varphi^*| \le \varepsilon, |r - r^*| \le \varepsilon, \|h\|_{C^1} \le \varepsilon\} \subset \mathbb{E}$$

is equal either 1 or -1. Hear $\varepsilon > 0$ is small enough.

For $\xi = 1$ the equation $\mathcal{F}(\varphi, r, h) = 0$ coincide with system (4.19) – (4.20). This means that now it is sufficient to prove nondegeneracy of our deformation on ∂G_ε to complete the proof of Theorem 2.2.

4.6. **Finalizing the proof of Theorem 2.2.**

Lemma 4.1. *For any zero $\{\varphi, r, h\}$ of the deformation $\mathcal{F}(\varphi, r, h; \xi)$ the estimate*

$$\|h\|_{C^1} \le cr^\alpha$$

is valid for some $c > 0$ independent of $\xi \in [0, 1]$.

Lemma 4.1 follows from the equalities $h = \xi Q A(\lambda) n^2 f(nt, x, n^{-1}x'; \lambda)$ and (2.3).

Now let us prove that $\mathcal{F}(\varphi, r, h; \xi) \ne 0$ for $\{\varphi, r, h\} \in \partial G_\varepsilon$ and $\xi \in [0, 1]$. This nondegeneracy follows from the continuity of all the functions considered, Lipschitz condition (2.4) and Lemma 4.1.

Two first components of our deformation can be rewritten as

$$\mathcal{F}_\varphi(\varphi, r, h; \xi) = \Psi(\varphi, \lambda_0) + \xi\Big[r^{-\alpha}\int_0^{2\pi}\sin(mt + \varphi) f(nt, x, n^{-1}x'; \lambda)\,dt - \Psi(\varphi, \lambda_0)\Big]$$

and

$$\begin{aligned}\mathcal{F}_r(\varphi, r, h; \xi) &= \pi\lambda\frac{m}{n}r^{1-\alpha} - \Psi^*(\varphi, \lambda_0)\\ &\quad - \xi\Big[r^{-\alpha}\int_0^{2\pi}\cos(mt + \varphi) f(nt, x, n^{-1}x'; \lambda)\,dt - \Psi^*(\varphi, \lambda_0)\Big]\end{aligned}$$

The terms in the large square brackets in both relations are small (we write formulae for the first one only):

$$\begin{aligned}r^{-\alpha}\int_0^{2\pi}&\sin(mt+\varphi) f(nt, x, \frac{x'}{n}; \lambda)\,dt - \Psi(\varphi, \lambda_0) = r^{-\alpha}\int_0^{2\pi}\sin(mt+\varphi)\psi(nt, x, \frac{x'}{n}; \lambda)\,dt\\ &+ \int_0^{2\pi}\sin(mt+\varphi)\Big(F(nt, \frac{x(t)}{r}, \frac{x'(t)}{nr}; \lambda_0) - F(nt, \sin(mt+\varphi), n^{-1}\cos(mt+\varphi); \lambda_0)\Big)dt\\ &+ \int_0^{2\pi}\sin(mt+\varphi)\Big(F(nt, \frac{x(t)}{r}, \frac{x'(t)}{nr}; \lambda) - F(nt, \frac{x(t)}{r}, \frac{x'(t)}{nr}; \lambda_0)\Big)dt.\end{aligned}$$

If $|\lambda|$ is small, then r is also small: this follows from $\mathcal{F}_r(\varphi, r, h; \xi) = 0$. Therefore the terms in square brackets tends to zero as $\lambda \to 0$.

Now if $|r - r^*| = \varepsilon$, then $\mathcal{F}_r(\varphi, r, h; \xi) \neq 0$; if $|\varphi - \varphi^*| = \varepsilon$, then $\mathcal{F}_\varphi(\varphi, r, h; \xi) \neq 0$; if $\|h\| = \varepsilon$, then $\mathcal{F}_h(\varphi, r, h; \xi) \neq 0$. It means that $\mathcal{F}(\varphi, r, h; \xi) \neq 0$ for $\{\varphi, r, h\} \in \partial G_\varepsilon$ for any $\xi \in [0, 1]$ and this completes the proof. ∎

4.7. **Proof of Proposition 2.3.** If $x(t)$ is $2n\pi$-periodic solution of equation (2.7), then it is a zero of the vector field $\mathcal{F}(\varphi, r, h; 1)$. Under conditions of the proposition the principal part of the first component of the field $\mathcal{F}(\varphi, r, h; 1)$ is uniformly nonzero: this follows from Lemma 2.5. This means, that for sufficiently small r and $|\lambda|$ the first component of the field $\mathcal{F}(\varphi, r, h; 1)$ is nonzero.

This contradiction completes the proof of the proposition. ∎

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