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Hopf Bifurcation Generated by Small Nonlinear Terms

We study the problem of generation of small cycles from the equilibrium in autonomous quasilinear systems depending on a parameter. In contrast to the usual situations, the linearized equation is degenerate for all parameter values (not only for the bifurcation point). Therefore the existence of small cycles is determined by small nonlinear terms. The main example is an equation where the principal degenerate linear part is independent of the parameter. We suggest sufficient conditions for the existence and stability of small cycles for higher-order scalar equations. The results are based on topological methods and methods of monotone operators.

1. Bifurcation points

Consider the equation

$$L\left(\frac{d}{dt}\right)x(t) = f(x(t), \lambda) \tag{1}$$

with the parameter $\lambda \in (0,1)$. Here $L(p)$ is a polynomial with constant real coefficients. The function $f(x,\lambda) : \mathbb{R} \times (0,1) \to \mathbb{R}$ is continuous and sublinear:

$$f(0,\lambda) \equiv 0, \qquad \lim_{x\to 0} \sup_{\lambda\in(0,1)} |x|^{-1}|f(x,\lambda)| = 0.$$

Therefore for every λ the origin is an equilibrium for equation (1).

A solution $x_*(t)$ of equation (1) is said to be ε-small if $0 < \max\{|x_*(t)| : t \in \mathbb{R}\} < \varepsilon$. We study the so-called Hopf bifurcation for this equation.

Definition 1. *The value λ_0 of the parameter is a* **Hopf bifurcation point** *with the frequency $w_0 > 0$ for equation* (1) *if for any $\varepsilon > 0$ there exists a $\lambda = \lambda_\varepsilon \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ such that equation* (1) *with this λ has at least one ε-small periodic solution $x(t) = x_\lambda(t)$ of a period $T = T_\lambda \in (2\pi/w_0 - \varepsilon, 2\pi/w_0 + \varepsilon)$.*

A necessary condition for the existence of Hopf bifurcation points with the frequency w_0 for equation (1) is that $L(iw_0) = 0$, i.e. the polynomial $L(p)$ has the form

$$L(p) = (p^2 + w_0^2)M(p).$$

In addition to this, we assume that the polynomial $M(p)$ satisfies the following condition.

(H1) The relations $\Im m\, M(iw_0) \neq 0$ and $M(ikw_0) \neq 0$, $k = 0, 1, 2, ...$ are valid.

In classical situations (see, e.g. [1,2,3]), Hopf bifurcation points are determined by the linear part of the system, the smaller nonlinear terms are of no importance. This is not true for equation (1) though. Here the linear part is independent of the parameter and therefore the Hopf bifurcation points are determined by the nonlinearity $f(x,\lambda)$. The odd and even parts

$$f_{odd}(x,\lambda) = [f(x,\lambda) - f(-x,\lambda)]/2, \qquad f_{even}(x,\lambda) = [f(x,\lambda) + f(-x,\lambda)]/2$$

of the nonlinearity play different roles in the results below. Suppose the following hypotheses are valid.

(H2) The odd part can be represented as $f_{odd}(x,\lambda) = a(\lambda)x|x|^{\alpha-1} + \varphi(x,\lambda)$ where $\alpha > 1$ and $\varphi(x,\lambda) = o(|x|^\alpha)$.

(H3) For some $\beta > 1$ the even part satisfies the estimates

$$|f_{even}(x,\lambda)| \leq C_1|x|^\beta, \qquad |f_{even}(x,\lambda) - f_{even}(y,\lambda)| \leq C_2 \max\{|x|^{\beta-1}, |y|^{\beta-1}\}\, |x-y|.$$

Theorem 1. *Let hypotheses* (H1), (H2), (H3) *be valid and $\alpha < 2\beta - 1$. Suppose $a(\lambda_0) = 0$ and the function $a(\lambda)$ takes the values of both sign in any vicinity of the point λ_0. Then λ_0 is a Hopf bifurcation point with the frequency w_0 for equation* (1).

2. Stability of small cycles

If the more terms are known in the representation of the odd part of the nonlinearity, then Theorem 1 can be supplied with the stability analysis of the small periodic solutions of equation (1). We can also determine, for which λ from a small vicinity of the Hopf bifurcation point λ_0 these periodic solutions exist.

If the polynomial $M(p)$ has a root with a positive real part, then all the small periodic solutions of equation (1) are *a priori* unstable in any natural sense, this case is not considered here. We assume that the polynomial $M(p)$ is Hurwitzian, i.e. all its roots are in the open left half-plane of the complex plane. Suppose that in place of (H2) the following condition is satisfied.

(H4) The odd part of the nonlinearity $f(x,\lambda)$ can be represented as $f_{odd}(x,\lambda) = a(\lambda)x|x|^{\alpha-1} + b(\lambda)x|x|^{\gamma-1} + \psi(x,\lambda)$ where $1 < \alpha < \gamma$ and the function $\psi(x,\lambda)$ satisfies

$$\psi(x,\lambda)|x|^{-\gamma} \to 0, \qquad |\psi(x,\lambda) - \psi(y,\lambda)| \le C \max\{|x|^{\gamma-1}, |y|^{\gamma-1}\}\,|x-y|.$$

Set $\mu(\lambda) = b(\lambda)\Im m\, M(iw_0)$.

T h e o r e m 2. *Let hypotheses* (H1), (H3), (H4) *be valid and*

$$\alpha < \gamma < 2\beta - 1, \qquad a(\lambda_0) = 0,\ b(\lambda_0) \neq 0.$$

Let the polynomial $M(p)$ be Hurwitzian. Then any limit point λ_0 of the set $\Lambda_0 = \{\lambda \in (0,1) : a(\lambda)b(\lambda_0) < 0\}$ is a Hopf bifurcation point with the frequency w_0 for equation (1). *Moreover, there exist a vicinity $\Lambda \ni \lambda_0$ and a number $\varepsilon_0 > 0$ such that the following statements hold.*

(i) *If $\mu(\lambda_0) > 0$, then equation* (1) *has at least one orbitally stable ε_0-small periodic solution for any $\lambda \in \Lambda \bigcap \Lambda_0$.*

(ii) *If $\mu(\lambda_0) < 0$, then equation* (1) *has at least one orbitally unstable ε_0-small periodic solution for any $\lambda \in \Lambda \bigcap \Lambda_0$.*

(iii) *Equation* (1) *has no ε_0-small periodic solutions of any period $T > 0$ for $\lambda \in \Lambda \setminus \Lambda_0$.*

3. Remarks and examples

Condition (H1) implies that the polynomial $L(p)$ is not even. Therefore Theorems 1 and 2 can not be used to study the equation $x'' + x = f(x,\lambda)$. The simplest equation satisfying condition (H1) is $x''' + x'' + x' + x = f(x,\lambda)$.

If the nonlinearity $f(x,\lambda)$ is sufficiently smooth, then α, β, and γ are integers. Since the estimate $2\beta - 1 > \alpha$ is not valid for $\beta = 2$ and any odd integer $\alpha > 1$, Theorems 1 and 2 are inapplicable to equation (1) with the smooth nonlinearity $f(x,\lambda)$ having nonzero quadratic principal terms at the origin.

As an example of applications of Theorems 1 one can consider the equations

$$x''' + x'' + x' + x = \lambda x^3 + o(x^3), \qquad x''' + x'' + x' + x = c(\lambda)x^4 + \lambda x^5 + o(x^5).$$

For both of them $\lambda_0 = 0$ is a Hopf bifurcation point with the frequency 1.

Theorem 2 is applicable to the equation $x''' + x'' + x' + x = a(\lambda)x^3 + c(\lambda)x^4 + b(\lambda)x^5 + o(x^5)$.

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4. References

1 Kozyakin, V.S., Krasnosel'skii, M.A.: The method of parameter functionalization in the Hopf bifurcation problem; Nonl. Analysis. TMA **11**, 2 (1987), 149–161.

2 Marsden, J., McCracken, M.: Hopf Bifurcation and its Applications; Springer-Verlag, New York 1982.

3 Hassard, B., Kazarinoff, N., Wan, Y.-H.: Theory and Applications of Hopf Bifurcations; Cambridge University Press, London 1981.

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