The index at infinity for some vector fields with oscillating nonlinearities

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Abstract

This paper is devoted to the computation of the index at infinity for some asymptotically linear completely continuous vector fields $x - T(x)$, when the principal linear part $x - Ax$ is degenerate (1 is an eigenvalue of A), and the sublinear part is not asymptotically homogeneous (in particular do not satisfy Landesman-Lazer conditions). In this work we consider only the case of a one-dimensional degeneration of the linear part, i.e.s 1 is a simple eigenvalue of A. For this case we formulate an abstract theorem and give some general examples for vector fields of Hammerstein type and for a two point boundary value problem.

1 Introduction

This paper concerns a computation of the index at infinity for some asymptotically linear completely continuous vector fields $x - T(x)$. If the principal linear part $x - Ax$ of the field is non-degenerate, i.e. if $I - A$ is invertible, then the index at infinity of $x - T(x)$ is well-defined by the spectrum of A [5]. If this principal linear part is degenerate, i.e. if 1 is an eigenvalue of the linear operator A, the index computation can be done only with the use of some properties of the sublinear part of $T(x)$.

If this sublinear part is asymptotically homogeneous (e.g. satisfies some Landesman-Lazer type conditions [6]), then the computation of the index can be reduced to that of the degree of some finite dimensional vector field Q (see [2, 3, 4]). If this vector field Q is also degenerate, one can use higher order terms.

In this paper we consider the situation where the sublinear part is not asymptotically homogeneous. We have then the following possibilities:

i) the index is defined and its computation can be reduced to that of a finite dimensional field;

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ii) the index is not defined : there exists some sequence x_n of zeros such that $\|x_n\| \to \infty$;

iii) the known information is insufficient to get any answer and we need to use higher order terms. This last possibility has in some natural sense "zero measure".

In this work we consider only the case of a one-dimensional degeneration of the linear part, i.e. the case where 1 is a simple eigenvalue of the linear operator A. For this case we formulate an abstract theorem and give some general examples for vector fields of Hammerstein type and for a two-point boundary value problem.

The paper is organized as follows : the basic abstract result is formulated in Sections 2 and 3 , the main condition (1) is discussed in Section 4, applications to two-point boundary value problems are proposed in Section 5, the computation of some asymptotic characteristics of the vector field is considered in Section 6, the proofs the main theorems are given in Section 7 – 9, and some remarks conclude the paper in Section 10.

2 Index at infinity

Consider a vector field $\Phi x = x - T(x)$ in a Banach space E. We suppose that the operator $T(x)$ is completely continuous, in which case the vector field $\Phi(x)$ is also called completely continuous, and that $x \neq T(x)$ for $\|x\| \geq \rho$. Then the rotation (see [5]) of this field on the boundary of every ball $B(r, 0) = \{x \in E, \|x\| \leq r\}$ (or the Leray-Schauder degree of the field on $B(r, 0)$), is defined for every $r \geq \rho$, and its value is independent of r. It is called the *index at infinity* of the field $\Phi(x)$, and is denoted by $\operatorname{ind}_\infty \Phi$.

The index at infinity can be used for the study of the equation $x = T(x)$: if $\operatorname{ind}_\infty \Phi$ is defined ($x \neq T(x)$ for $\|x\| \geq \rho$ for some ρ) and different from zero, then equation $x = T(x)$ has at least one solution.

The homotopic invariance of the index makes it applicable to the study of unbounded branches of solutions or to the study of asymptotic bifurcation points for equations with a parameter (see [5]). Consider the equation $x = T(x; \lambda)$ with a real parameter $\lambda \in \Lambda = [a, b]$. Let the operator $T(x; \lambda)$ be completely continuous with respect to both variables. If the index at infinity of the field $\Phi_\lambda = x - T(x; \lambda)$ is defined for two distinct values λ_1 and λ_2 and $\operatorname{ind}_\infty \Phi_{\lambda_1} \neq \operatorname{ind}_\infty \Phi_{\lambda_2}$, then, on the interval $[\lambda_1, \lambda_2] \in \Lambda$, equation $x - T(x; \lambda) = 0$ has an unbounded branch of solutions: for any $R > 0$ there exist $\lambda_R \in (\lambda_1, \lambda_2)$ and x_R, $\|x_R\| \geq R$ such that $x_R = T(x_R; \lambda_R)$. This branch is continuous in a natural sense (see [5]). In other words, if $\operatorname{ind}_\infty \Phi_{\lambda_1} \neq \operatorname{ind}_\infty \Phi_{\lambda_2}$, then there exists at least one asymptotic bifurcation point on $[\lambda_1, \lambda_2]$, which can be unique or not.

In this paper, we describe situations, where asymptotic bifurcation points fill some nontrivial interval $J \subset [\lambda_1, \lambda_2]$.

3 Abstract results

Consider a Banach space E and a completely continuous vector field $x - T(x)$. Suppose that $T(x) = Ax + F(x)$, where A is a linear operator and $F(x)$ is a bounded nonlinearity, i.e. there exists $r_1 > 0$ such that $\|F(x)\| \leq r_1$ for all $x \in E$.

Suppose that 1 is a simple eigenvalue of the linear operator A, and let $e \in E$ be such that $Ae = e$, $\|e\| = 1$. There exists a projector P on the one-dimensional subspace $E_1 = \{\alpha e,\ \alpha \in \mathbb{R}\}$, which commutes with A. Setting $Q = I - P$, $E_2 = QE$, the subspace E_2 has co-dimension one, is invariant for A, and, in E_2, the value 1 is regular for the operator A. Denote by $p : E \to \mathbb{R}$ the linear functional defined by the relation $Px = p(x)e$. For the applications we have in mind, it is useful to consider another Banach space $\tilde{E} \subset E$ with a stronger norm, namely $\|x\|_E \le c\,\|x\|_{\tilde{E}}$, for some $c > 0$ and all $x \in E$.

We can formulate the main assumption for the nonlinearity $F(x)$:

for any $c > 0$

$$\lim_{|\xi|\to\infty} \sup_{\|h\|_{\tilde{E}} \le c} |p(F(\xi e + h)) - p(F(\xi e))| = 0. \tag{1}$$

This property, which will be discussed below, holds for various classes of concrete nonlinear operators which appear in applications.

Let us introduce the scalar function Ψ defined on $\mathbb{R}$ by

$$\Psi(\xi) = p(F(\xi e)),$$

which is bounded and continuous, and its four upper and lower limits at $\pm\infty$

$$\psi_\pm = \liminf_{\xi\to\pm\infty} \Psi(\xi), \quad \psi^\pm = \limsup_{\xi\to\pm\infty} \Psi(\xi). \tag{2}$$

Theorem 1. *Let the operator A act continuously from E to $\tilde{E}$. Let the nonlinearity $F(x)$ also act from E to $\tilde{E}$ and let it be bounded in $\tilde{E}$, i.e.*

$$\|F(x)\|_{\tilde{E}} \le r_2,$$

for some $r_2 > 0$ and all $x \in E$. Let condition (1) hold for any $c > 0$. Then the following conclusion holds.

a) If either $\psi^+ < 0 < \psi_-$, or $\psi^- < 0 < \psi_+$, then the index at infinity of the field $\Phi(x)$ is defined and $|\operatorname{ind}_\infty \Phi| = 1$. In particular, equation $x - T(x) = 0$ has at least one solution.

b) If either $\psi_- > 0$ and $\psi_+ > 0$, or $\psi^- < 0$ and $\psi^+ < 0$, then the index at infinity of the field $\Phi(x)$ is defined and $\operatorname{ind}_\infty \Phi = 0$.

c) If either $\psi_- < 0 < \psi^-$, or $\psi_+ < 0 < \psi^+$, or both, then the index at infinity of the field $\Phi(x)$ can not be defined: there exists a sequence $x_n \in E$ such that $x_n = T(x_n)$ and $\|x_n\|_E \to \infty$.

If this theorem is inapplicable, then at least one of the numbers (2) has to be zero. Even if $\psi_- \cdot \psi^- = 0$ ($\psi_+ \cdot \psi^+ = 0$) Theorem 1 can be applicable if $\psi_+ < 0 < \psi^+$ (resp. $\psi_- < 0 < \psi^-$).

Under the assumptions of Theorem 1, if $\psi_+ < 0 < \psi^+$, the sequence x_n has the form $x_n = \xi_n e + h_n$, where the functions $h_n \in E_2$ are uniformly bounded and $\xi_n \to +\infty$. If $\psi_- < 0 < \psi^-$, then $\xi_n \to -\infty$; if $0 \in (\psi_-, \psi^-) \cap (\psi_+, \psi^+)$, then there are two sequences $x_n^\pm = \xi_n^\pm e + h_n^\pm$, with $\xi_n^\pm \to \pm\infty$.

If $\psi_- = \psi^-$ and $\psi_+ = \psi^+$, then the condition c) of Theorem 1 can not take place; this is the case, when the nonlinearity is asymptotically homogeneous in the sense of [4].

4 Hammerstein type nonlinearities

In this section we formulate some sufficient conditions for the validity of the assumption (1) and the corresponding results for the computation of the index at infinity of Hammerstein type vector fields in some function spaces.

Let $\Omega \subset \mathbb{R}^m$ be a domain and let $E = L^2(\Omega, \mathbb{R}^n) := L^2$.

Suppose that the nonlinearity $F(x)$ is a Hammerstein type operator generated by the linear operator A (let us recall that we study the vector field $\Phi(x) = x - T(x) = x - Ax - F(x)$ where A is the asymptotic derivative of $T(x)$) and some bounded continuous function $f(t,x) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, i.e. that

$$F(x) = Af(t,x).$$

Theorem 2. *Let $\tilde{E} = E$ and*

$$meas\{t \in \Omega : \ e(t) = 0\} = 0. \tag{3}$$

Let the function $f(t,x)$ satisfy the Lipschitz condition

$$\|f(t,x) - f(t,y)\|_{\mathbb{R}^n} \le L(r)\,\|x-y\|_{\mathbb{R}^n}, \qquad \|x\|_{\mathbb{R}^n},\ \|y\|_{\mathbb{R}^n} > r, \tag{4}$$

where

$$\lim_{r\to\infty} L(r) = 0. \tag{5}$$

Then condition (1) is valid for any $c > 0$. Moreover

$$\lim_{|\xi|\to\infty} \sup_{\|h\|_{L^2}\le c} \|f(\xi e(t) + h(t)) - f(\xi e(t))\|_{L^2} = 0. \tag{6}$$

For scalar functions $e(t)$, defined on $\Omega = [a,b] \subset \mathbb{R}$, it is possible to obtain a similar result under less restrictive conditions. Suppose the linear projector Px is generated by the function $g(t) \in L^2$, namely

$$(Px)(t) = e(t)\int_a^b g(s)x(s)\,ds.$$

Theorem 3. *Let $E = L^2 = L^2([a,b],\mathbb{R})$ and $\tilde{E} = C^1 = C^1([a,b],\mathbb{R})$. Let $e(t) \in C^1$ and*

$$meas\{t \in [a,b] : \ e'(t) = 0\} = 0. \tag{7}$$

Let the function $f(t,x)$ be uniformly continuous in t with respect to $x \in \mathbb{R}$ and let $g(t) \in L^2$. Then condition (1) holds for any $c > 0$, i.e.

$$\lim_{|\xi|\to\infty} \sup_{\|h(t)\|_{C^1}\le c} \left| \int_a^b g(t)[f(t,\xi e(t)+h(t)) - f(t,\xi e(t))]\,dt \right| = 0. \tag{8}$$

Theorem 3 does not contain any restrictive Lipschitz conditions like (4) – (5). For example, the bounded function $f(t,x)$ is uniformly continuous in t with respect to x if $f(t,x) = a(t) + b(t)f(x)$.

The proof given below shows that Theorem 3 can be generalized to the case $E = L^2(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^m$ where $m \geq n$. It seems that for $m < n$ we have essential difficulties and need additional hypotheses on $f(t,x)$.

Theorem 1 and Theorems 2 and 3 allow us to formulate statements for the computation of the index at infinity of Hammerstein type vector fields in some function spaces.

Corollary 1. *Let $E = L^2(\Omega, \mathbb{R}^n)$, where Ω is a domain in $\mathbb{R}^m$. Let the linear completely continuous operator A act in E, has 1 as a simple eigenvalue, $e(t)$ as corresponding eigenfunction, and let condition (3) hold. If $f(t,x): \ \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a bounded continuous function satisfying (4) and (5), then all the conclusions of Theorem 1 hold for the vector field $\Phi(x) = x - A(x + f(\cdot, x))$.*

Corollary 2. *Let $E = L^2([0,\pi], \mathbb{R})$, and let the linear completely continuous operator $A: E \to E$ act continuously from E to the space $C^1(0,\pi)$. Let 1 be a simple eigenvalue for A, with eigenfunction $e(t)$, and let condition (7) hold. If $f(t,x): \ \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded continuous function which is uniformly continuous in t with respect to x, then all the conclusions of Theorem 1 hold for the vector field $\Phi(x) = x - A(x + f(\cdot, x))$.*

Corollary 1 is a direct consequence of Theorems 1 and 2, and Corollary 2 a direct consequence of Theorems 1 and 3.

5 Two-point boundary value problems

In this section we give some results for two-point boundary value problems at resonance for second order ordinary differential equations. Consider the problem

$$x'' + n^2 x + f(t,x) = 0, \quad x(0) = x(\pi) = 0 \tag{9}$$

for some positive integer n and some bounded continuous function $f(t,x): \ [0,\pi] \times \mathbb{R} \to \mathbb{R}$. Suppose that $f(t,x)$ is uniformly continuous in t with respect to x. Put

$$e(t) = \sqrt{\frac{2}{\pi}} \sin nt, \qquad \Psi(\xi) = \int_0^\pi e(t)\, f\left(t, \xi e(t)\right)\, dt$$

and define the corresponding numbers (2).

Theorem 4. *If either $\psi^+ < 0 < \psi_-$, or $\psi^- < 0 < \psi_+$, then problem (9) has at least one solution, and the set K of its solutions is bounded. If either $\psi_- < 0 < \psi^-$, or $\psi_+ < 0 < \psi^+$, or both, then problem (9) has an infinite number of solutions, and the set K is unbounded.*

This theorem follows directly from Corollary 2.

We illustrate this theorem with the description of the set of solutions of the problem

$$x'' + n^2 x + f(t,x) - \lambda e(t) = 0, \quad x(0) = x(\pi) = 0 \tag{10}$$

with a scalar parameter λ. Define the numbers ψ_-, ψ^-, ψ_+, ψ^+ as in Theorem 4 and let $\psi_- < \psi^- < 0 < \psi_+ < \psi^+$. For every $\xi \in \mathbb{R}$ we can find at least one λ and a nonempty set $H(\xi)$ of functions $h(t)$ such that $x(t) = \xi e(t) + h(t)$ is a solution of problem (10).

In the plane $\{\lambda, \xi\}$ the graph of the mapping $\xi \mapsto \lambda$ can be seen at Fig.1.

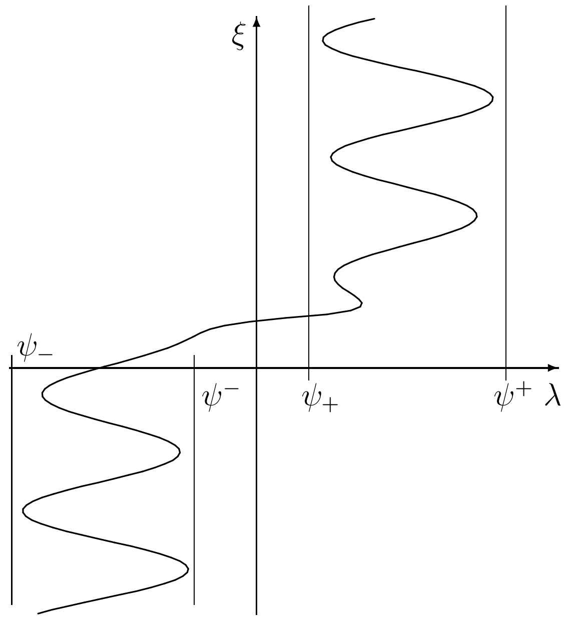

Fig.1

Here the set $[\psi_-, \psi^-] \cup [\psi_+, \psi^+]$ is the set of asymptotic bifurcation points for the problem (10). Of course, the behaviour of the graph for small values of ξ can be much more cumbersome, as we made no assumption concerning the behaviour of the function $f(t,x)$ for small x. If $f'_x(t,x)$ is small enough uniformly in t, (the constant depends only on x), then for any ξ there exist a unique $h(t)$ and a unique λ such that $x(t) = \xi e(t) + h(t)$ is a solution of problem (10).

6 The computation of $\psi_\pm$, $\psi^\pm$

A natural question which arises is how to compute the numbers (2). The following remarks are of interest.

If $f(t,x) = f_1(t,x) + f_2(t,x)$ and

$$\lim_{|\xi|\to\infty} \int_\Omega g(t)\, f_2(t, \xi e(t))\, dt = 0, \tag{11}$$

then the numbers (2) computed for the functions $f(t,x)$ and $f_1(t,x)$ coincide.

If the limits

$$\lim_{\xi\to\pm\infty} \int_\Omega g(t)\, f_2(t, \xi e(t))\, dt = \tilde{\psi}(\pm) \tag{12}$$

exist, then they are equal to the difference between the numbers (2), computed for the functions $f(t,x)$ and the same numbers, computed for $f_1(t,x)$.

This simple idea allows us to split the initial function $f(t,x)$ into a sum of functions such that it is possible either to see that (11) is valid or that limits (12) exist, and for one last function, we need to do computations of different upper and lower limits.

Relation (11) holds for various functions $f_2(t,x)$. Let us give some examples.

1. Let $f_2(t,x) \to 0$ as $|x| \to \infty$ uniformly with respect to t. Let $e(t)$ satisfy (3). Then (11) holds.

2. Let $f_2(t,x)$ satisfy some Landesman-Lazer type conditions $f_2(t,x) \to \varphi^{\pm}(t)$ as $x \to \pm\infty$. Let $e(t)$ satisfy (3). Then the limits (12) exist and

$$\tilde{\psi}(\pm) = \int_{\{\pm e(t) \geq 0\}} g(t)\,\varphi^{\pm}(t)\,dt + \int_{\{\pm e(t) \leq 0\}} g(t)\,\varphi^{\mp}(t)\,dt.$$

3. Let $f_2(t,x) = f(x)$ be an even function, $t \in [0,\pi]$, $e(t) = \sqrt{2/\pi}\sin nt$ and n be even. Then for any ξ

$$\int_0^{\pi} e(t)\,f(\xi e(t))\,dt = 0.$$

Of course (11) holds for this case.

4. Let $e(t) \in C^1$ and $e'(t)$ satisfy (7). Let again $f_2(t,x) = f(x)$ and let its primitive $\mathcal{F}(x)$ be sublinear:

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x f(u)\,du = 0. \tag{13}$$

Then (11) holds.

Some analogous statement is formulated in [1] in the context of the existence of unbounded solutions, and can be proved in the following way. The set where $e'(t)$ is small has a small measure due to (7). This means that without loss of generality we can consider only the case, where the function $e(t)$ is monotone (for example, it increases), $e'(t) > \varepsilon$ and $g(t) \in C^1$. Then

$$\lim_{\xi\to\infty} \int_a^b g(t)\,f_2(t,\xi e(t))\,dt = \lim_{\xi\to\infty} \int_a^b g(t)\,f(\xi e(t))\,dt = \lim_{\xi\to\infty} \int_a^b \frac{g(t)}{\xi e'(t)}\,d\mathcal{F}(\xi e(t)) =$$

$$= \lim_{\xi\to\infty} \left(\frac{g(t)}{e'(t)}\,\frac{\mathcal{F}(\xi e(t))}{\xi}\,\Bigg|_{t=a}^{t=b} \right) - \lim_{\xi\to\infty} \int_a^b \left[\frac{d}{dt}\left(\frac{g(t)}{e'(t)}\right)\right] \frac{\mathcal{F}(\xi e(t))}{\xi}\,dt = 0.$$

For example $f(x)$ has a sublinear primitive if $f(x)$ is periodic or almost periodic with zero average, and the functions $\sin\sqrt{|x|}$ and $\sin(x^2)$ also have sublinear primitives. The function $\sin(\ln(\ln(3+|x|)))$ has no sublinear primitive and the limit in (11) can be nonzero. If a primitive of a function $g(x)$ is bounded and

$$\lim_{x\to\infty} \frac{f''(x)}{(f'(x))^2} = 0,$$

then $g(f(x))$ has a sublinear primitive.

As an example consider the problem

$$x'' + x + \sin(\ln(\ln(3+|x|))) + \frac{3}{\pi}\arctan x + 100\cos x = \lambda \sin t, \quad x(0) = x(\pi) = 0.$$

Here $p(x) = \int_0^{\pi} e(t)\,x(t)\,dt$. The function $100\cos x$ has a bounded primitive and satisfies (11), limits (12) exist for the function $(3/\pi)\arctan x$, namely $\tilde{\psi}(\pm) = \pm 3\sqrt{2/\pi}$, and, for the function $\sin(\ln(\ln(3+|x|)))$, the numbers (2) take the values $\psi_{\pm} = -2\sqrt{2/\pi}$,

$\psi^{\pm} = 2\sqrt{2/\pi}$. This implies that for the complete nonlinearity $\sin(\ln(\ln(3+|x|))) + 3/\pi \arctan x + 100\cos x$ the numbers (2) take four different values $\psi_- = -5\sqrt{2/\pi}$, $\psi^- = -\sqrt{2/\pi}$, $\psi_+ = \sqrt{2/\pi}$ and $\psi^+ = 5\sqrt{2/\pi}$.

Consequently, if $2/\pi < |\lambda| < 10/\pi$, then the set of solutions is unbounded; if $|\lambda| < 2/\pi$, then the set of solutions is nonempty and bounded; if $|\lambda| > 10/\pi$, then the set of solutions is bounded (may be empty).

7 Proof of Theorem 1

The proof of Theorem 1 will make use of Theorem 3 from [4], that we formulate, for the reader's convenience, in a form adapted to our particular case.

Let X be a Banach space, and some completely continuous operator $\mathcal{A} = \{A_1, A_2\}$ be defined on the set $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 = B(0,r) \subset X$ and $\Omega_2 = [\xi_*, \xi^*]$. For any $\xi \in \Omega_2$, let the vector field $x_1 - A_1(x_1, \xi)$ be non-degenerate on $\partial\Omega_1 = \{\|x\|_X = r\}$ and let the rotation $\gamma_1 = \gamma(x_1 - A_1(x_1,\xi), \partial\Omega_1)$ be non-zero (this γ_1 does not depend on ξ). For each $\xi \in \Omega_2$, denote by $K(\xi)$ the non-empty set of solutions x_1 of the equation $x_1 = A_1(x_1, \xi)$.

Lemma 1. *If $\xi_* - A_2(x_1, \xi_*) < 0$ for $x_1 \in K(\xi_*)$ and $\xi^* - A_2(x_1, \xi^*) > 0$ for $x_1 \in K(\xi^*)$, then $\gamma(\mathcal{A}, \partial\Omega) = \gamma_1$.*

We now prove separately the different conclusions of Theorem 1.

a) Let $\psi^+ < 0 < \psi_-$ (the case $\psi^- < 0 < \psi_+$ can be treated in a similar way). Put

$$s(\xi) = \begin{cases} 1, & \text{if } \xi \geq 1, \\ \xi, & \text{if } |\xi| < 1, \\ -1, & \text{if } \xi \leq -1. \end{cases} \tag{14}$$

Consider for $\lambda \in [0,1]$ the homotopy

$$\Xi_1(\lambda, x) = Qx - AQx - \lambda F(x) + (1-\lambda)s(p(x))e.$$

Obviously, $\Xi_1(1,x) = \Phi(x)$ and $\Xi_1(0,x) = Qx - AQx + s(p(x))e$. To prove the conclusion a) of Theorem 1 it is sufficient to see that $|\operatorname{ind}_\infty \Xi_1(0,\cdot)| = 1$ and to state an a priori estimate $\|x\| \leq c$ for all the possible zeros x of the homotopy $\Xi_1(\lambda, x)$ and all λ's, $0 \leq \lambda \leq 1$.

The first step is rather simple. The equation $\Xi_1(0,x) = 0$ has a unique solution $x = 0$, whose index (in any sense) equals ± 1. This follows, using the product index formula, from the fact that the index of the origin in E_2 of the vector field $Qx - AQx$ is equal to $(-1)^\beta$, with β the sum of multiplicities of all real eigenvalues of A strictly greater than 1, and from the fact that the index of the origin in E_1 of $s(\xi)$ is one.

Now let us prove an a priori estimate. Let $x = \xi e + h,\ h \in E_2$ and $\Xi_1(\lambda, x) = 0$ for some $\lambda \in [0,1]$. This implies the two equalities

$$Qx = AQx + \lambda QF(x) \tag{15}$$

and

$$p(\Xi_1(\lambda, x)) = -\lambda PF(x) + (1-\lambda)s(\xi)e = 0. \tag{16}$$

Equality (15), rewritten as $h = (I - AQ)^{-1}QF(x)$, implies the estimate

$$\|h\|_{\tilde{E}} \leq \|(I - AQ)^{-1}\|_{\mathcal{L}(\tilde{E}\cap E_2)} \sup_{x\in E} \|QF(x)\|_{\tilde{E}} \overset{\Delta}{:=} c < \infty. \tag{17}$$

Consider equality (16). If $|\xi|$ is sufficiently large, then $s(\xi) = \operatorname{sign}\xi$. We consider only the case $\xi > 1$, the case $\xi < -1$ being analogous. Since

$$p(\Xi_1(\lambda, x)) = -\lambda p(F(x)) + (1 - \lambda) =$$

$$= -\lambda(p(F(\xi e + h)) - p(F(\xi e))) - \lambda\Psi(\xi) + (1 - \lambda) \geq o(1) - \lambda\psi^+ + 1 - \lambda$$

and $-\lambda\psi^+ + 1 - \lambda \geq \max\{1, -\psi^+\} > 0$ then for $\xi \to +\infty$ we have $p(\Xi_1(\lambda, x)) > 0$. This gives the a priori estimate for ξ, which, together with (17), proves the conclusion a).

b) Let $\psi_- \geq \psi_+ > 0$. The case $\psi_+ > \psi_- > 0$ and the case of negative ψ^- and ψ^+ are analogous to the case considered. Consider for $\lambda \in [0, 1]$ the homotopy

$$\Xi_2(\lambda, x) = Qx - AQx - \lambda F(x) - (1 - \lambda)\psi_+ e.$$

Now $\Xi_2(1, x) = \Phi(x)$, $\Xi_2(0, x) = Qx - AQx - \psi_+ e$, the equation $\Xi_2(0, x) = 0$ has no solutions and $\operatorname{ind}_\infty \Xi_2(0, \cdot) = 0$. To prove the conclusion b) of Theorem 1 it is sufficient to prove an a priori estimate $\|x\| \leq c$ for all the possible zeros x of the homotopy $\Xi_2(\lambda, x)$ when $0 \leq \lambda \leq 1$. This a priori estimate follows from the estimate (17) (valid here too) and the relations

$$p(\Xi_2(\lambda, x)) = -\lambda p(F(x)) - (1 - \lambda)\psi_+ = -\lambda(p(F(\xi e + h)) - p(F(\xi e))) - \lambda\Psi(\xi) - (1 - \lambda)\psi_+$$

$$\leq o(1) - \lambda\psi_+ - \psi_+ + \lambda\psi_+ = o(1) - \psi_+ < 0.$$

c) Let $\psi_+ < 0 < \psi^+$ (the case $\psi_- < 0 < \psi^-$ can be considered in a similar way). We prove that there exists a sequence of zeros $x_n = \xi_n e + h_n$ of the vector field $\Phi(x)$, satisfying $\|x_n\|_E \to \infty$. Put

$$\varepsilon = \frac{1}{3}\min\{\psi^+, -\psi_+\}, \quad c = \left\|(I - A)^{-1}\right\|_{\mathcal{L}(\tilde{E}\cap E_2)} \sup_{x\in E} \|QF(x)\|_{\tilde{E}},$$

$$c_1 = \left\|(I - A)^{-1}\right\|_{\mathcal{L}(E_2)} \sup_{x\in E} \|QF(x)\|_E.$$

According to (1), choose a $\xi_0 > 0$ such that for $\xi \geq \xi_0$

$$|p(F(\xi e + h)) - \Psi(\xi)| < \varepsilon, \quad \|h\|_{\tilde{E}} \leq c. \tag{18}$$

¿From the definition of the numbers ψ_+ and ψ^+, there exist numbers $\xi_* > \xi_0$ and $\xi^* > \xi_*$ such that

$$\Psi(\xi_*) < -2\varepsilon, \quad \Psi(\xi^*) > 2\varepsilon. \tag{19}$$

Consider the cylinder $\Omega = \{\|Qx\|_E \leq c_1 + 1,\ p(x) \in [\xi_*, \xi^*]\}$. Let us prove that the vector field $x - T(x)$ is non-zero on $\partial\Omega$. The boundary $\partial\Omega$ consists in three parts:

$$G_* = \{\|Qx\|_E \leq c_1 + 1,\ p(x) = \xi_*\}, \quad G^* = \{\|Qx\|_E \leq c_1 + 1,\ p(x) = \xi^*\},$$

$$G = \{\|Qx\|_E = c_1 + 1,\ p(x) \in [\xi_*, \xi^*]\}.$$

The vector field $\Phi(x)$ is non-zero on G because of the estimate $\|Qx\|_E \leq c_1$ for all possible solutions x of $Q\Phi(x) = 0$. The vector field $\Phi(x)$ is non-zero on G_* due to the relations

$$p(\Phi(x)) = p(F(x)) - \Psi(p(x)) + \Psi(p(x)) < \varepsilon - 2\varepsilon = -\varepsilon < 0,$$

The vector field $\Phi(x)$ is non-zero on G^* due to the relations

$$p(\Phi(x)) = p(F(x)) - \Psi(p(x)) + \Psi(p(x)) > -\varepsilon + 2\varepsilon = \varepsilon > 0.$$

In the last two formulas, we used the fact that, if $\Phi(x) = 0$, then $\|Qx\|_{\tilde{E}} \leq c$ and therefore

$$|p(F(x)) - \Psi(p(x))| < \varepsilon, \quad \Psi(p(x)) < -2\varepsilon, \qquad x \in G_*$$

and

$$|p(F(x)) - \Psi(p(x))| < \varepsilon, \quad \Psi(p(x)) > 2\varepsilon, \qquad x \in G^*.$$

We additionally proved that $p(\Phi(x)) < 0$ if $Q\Phi(x) = 0$ and $p(x) = \xi_*$ and $p(\Phi(x)) > 0$ if $Q\Phi(x) = 0$ and $p(x) = \xi^*$.

Now let us calculate the rotation $\gamma(\Phi, \partial\Omega)$ of non-degenerate vector field $\Phi(x)$ on $\partial\Omega$. Lemma 1 implies the relations

$$\gamma(\Phi, \partial\Omega) = \gamma(Q\Phi, \{\|Qx\|_E = c_1 + 1\}) = \pm - 1 \neq 0.$$

We proved that the set Ω contains at least one zero of the vector field Φ. Denote it as x_1 and choose other numbers ξ_* and ξ^*, both greater than both previous ones. Again, the corresponding cylinder contains at least one zero of Φ. Since we can choose arbitrary large numbers ξ_* and ξ^*, satisfying (18), we obtain a sequence x_n of zeros with $\|x_n\|_E \to \infty$, and the proof is complete.

8 Proof of Theorem 2

Let us choose a function $\mu : [\xi_0, \infty) \to \mathbb{R}$ satisfying the following conditions

$$\mu(\xi) \to \infty, \quad \text{for } \xi \to \infty, \qquad \mu(\xi) \leq \sqrt{\xi}, \qquad \mu(\xi) \leq \frac{1}{\sqrt{L(\sqrt{\xi})}},$$

where $L(r)$ is the function from (4), satisfying (5). Since

$$\|f(t, \xi e(t) + h(t)) - f(t, \xi e(t))\|_{L^2}^2 =$$

$$= \int_\Omega \|f(t, \xi e(t) + h(t)) - f(t, \xi e(t))\|_{\mathbb{R}^n}^2 \, dt \leq \int_{\Omega_1} \ldots + \int_{\Omega_2} \ldots + \int_{\Omega_3} \ldots,$$

where

$$\Omega_1 = \{t \in \Omega : \|e(t)\|_{\mathbb{R}^n} \leq \frac{2}{\sqrt{|\xi|}}\}, \quad \Omega_2 = \{t \in \Omega : \|h(t)\|_{\mathbb{R}^n} \geq \mu(|\xi|)\},$$

$$\Omega_3 = \{t \in \Omega: \ \|e(t)\|_{\mathbb{R}^n} \geq \frac{2}{\sqrt{|\xi|}}, \ \|h(t)\|_{\mathbb{R}^n} \leq \mu(|\xi|)\},$$

it is sufficient to estimate the last three integrals. According to (3)

$$\left| \int_{\Omega_1} \ldots \right| \leq 4 \sup |f^2(t,x)| \cdot \operatorname{meas} \Omega_1 \to 0,$$

according to Chebyshev inequality

$$\left| \int_{\Omega_2} \ldots \right| \leq 4 \sup |f^2(t,x)| \cdot \operatorname{meas} \Omega_2 \leq 4 \sup |f^2(t,x)| \cdot \frac{\|h(t)\|^2_{L^2}}{\mu^2(|\xi|)} \to 0.$$

The estimate of the third integral follows from the inequality

$$\|\xi e(t) + h(t)\|_{\mathbb{R}^n} \geq |\xi| \, \|e(t)\|_{\mathbb{R}^n} - \|h(t)\|_{\mathbb{R}^n} \geq 2\sqrt{|\xi|} - \mu(|\xi|) \geq \sqrt{|\xi|},$$

which holds for $t \in \Omega_3$. Now according to (4)

$$\left| \int_{\Omega_3} \ldots \right| \leq \int_{\Omega_3} L^2(\sqrt{|\xi|}) \, \|h(t)\|^2_{\mathbb{R}^n} \, dt \leq \operatorname{meas} \Omega \; L^2(\sqrt{|\xi|}) \; \mu^2(|\xi|) \leq \operatorname{meas} \Omega \, L(\sqrt{|\xi|}) \to 0.$$

Theorem 2 is proved.

9 Proof of Theorem 3

We have to prove that for any $\varepsilon > 0$ the supremum in (8) is less than ε for sufficiently large $|\xi|$. First let us choose a continuous function $\tilde{g}(t)$ such that

$$2 \sup |f(t,x)| \int_a^b |g(t) - \tilde{g}(t)| dt < \varepsilon/3.$$

We need to prove that

$$\sup_{\|h(t)\|_{C^1} \leq c} \left| \int_a^b \tilde{g}(t)[f(t, \xi e(t) + h(t)) - f(t, \xi e(t))] \, dt \right| < \frac{2}{3}\varepsilon. \tag{20}$$

Let us split the interval $[a,b]$ into a finite number of subintervals $[a_i, b_i]$ and $[b_i, a_{i+1}]$ in the following way. The intervals $[b_i, a_{i+1}]$ contain the set $\{t \in [a,b]: \ e'(t) = 0\}$; according to (7) the union of these intervals can have arbitrary small measure. Let

$$2 \sup |f(t,x)| \int_{\cup [b_i, a_{i+1}]} |\tilde{g}(t)| \, dt < \varepsilon/3. \tag{21}$$

Suppose that the points a_i and b_i are fixed up to the end of the proof. For any $[a_i, b_i]$ the estimate

$$\inf_{t \in [a_i, b_i]} |e'(t)| \geq \delta > 0$$

holds. This means that the function $e(t)$ is strictly monotone on every $[a_i, b_i]$; for sufficiently large $|\xi|$ ($|\xi| > 2c\delta^{-1}$), the function $\xi e(t) + h(t)$ is also strictly monotone, as $|\xi e'(t) + h'(t)| > 1/2\ |\xi|\delta$. Consider the integrals

$$\mathcal{J}_i = \int_{a_i}^{b_i} \tilde{g}(t) f(t, \xi e(t) + h(t))\, dt.$$

In any of them, make the change of variables $t = t_\xi(\tau)$, defined by the formula $\xi e(\tau) = \xi e(t) + h(t)$. This gives

$$\mathcal{J}_i = \int_{t_\xi^{-1}(a_i)}^{t_\xi^{-1}(b_i)} \frac{\tilde{g}(t_\xi(\tau))\, f(t_\xi(\tau), \xi e(\tau))}{t'_\xi(\tau)} d\tau.$$

The function $t_\xi(\tau)$ is one-to-one, $t_\xi(\tau) \to \tau$ and $t'_\xi(\tau) \to 1$ uniformly in τ as $|\xi| \to \infty$. Now

$$t_\xi^{-1}(a_i) \to a_i, \quad t_\xi^{-1}(b_i) \to b_i,$$

$\tilde{g}(t_\xi(\tau)) \to \tilde{g}(\tau)$ due to the continuity of $\tilde{g}(\cdot)$, and

$$f(t_\xi(\tau), \xi e(\tau)) \to f(\tau, \xi e(\tau))$$

due to the assumption of the uniform continuity of $f(t,x)$ in t. Consequently, for each i, one has

$$\mathcal{J}_i - \int_{a_i}^{b_i} \tilde{g}(\tau) f(\tau, \xi e(\tau) + h(\tau))\, d\tau \to 0$$

when $|\xi| \to \infty$. This together with (20) and (21) proves Theorem 3.

10 Remarks

1. The conclusions of Theorem 1 can also be proved for some asymptotically linear degenerate vector fields with unbounded nonlinearities.

2. If we cannot compute the exact values of the numbers (2), we can use some estimates of those numbers.

3. The main condition (1) is valid if

$$|p(F(x)) - p(F(y))| \leq L(r)\, \|x - y\|_E\,, \quad \|x\|_E\,, \|y\|_E \geq r \tag{22}$$

and (5) holds. Unfortunately we do not know any natural examples of nonlinearities satisfying (22).

4. Analogs of Theorem 1 can be formulated for vector fields with non-compact operators (e.g. with condensing ones, see [5]).

5. The conditions for conclusion c) of Theorem 1 give us some information about the localization of solutions x_n .

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