Oscillations in Systems with Asymptotically Even Nonlinearities *

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Abstract

In the paper we consider the 2π-periodic problem for the scalar equation $x''+n^2x = g(|x|)+f(t,x)+b(t)$ with bounded $g(u)$ and $f(t,x) \to 0$. New conditions of solvability based on a general theorem on index at infinity calculation for vector fields which have degenerate principal linear part as well as degenerate "next order" terms are obtained. This theorem is also applied to the solvability of a two-point boundary value problem and to resonant problems for equations arising in control theory.

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Contents

1 Introduction

Consider the problem of forced 2π-periodic oscillation for the second order quasilinear equation $x'' = w(t, x)$. Suppose that its right-hand side is asymptotically linear, i.e. the following limit exists:

$$\lim_{x \to \infty} \frac{w(t, x)}{x} = -k.$$

If k is not a square of integer, then the problem is not resonant and it can be easily studied by, for example, the Schauder principle. Suppose instead that $k = n^2$ and $w(t, x) = -n^2 x + v(t, x)$ with bounded $v(t, x)$. The usual way to study this case is to assume the Landesman-Lazer conditions ($v(t, x) \to V^{\pm}(t)$ as $x \to \pm\infty$). Then, under some appropriate conditions of non-degeneration of the limits $V^{\pm}(t)$, our problem can be studied by, for example, topological methods. If these non-degeneracy conditions fail, then it is necessary to use properties of terms vanishing at infinity (see [5]).

In this paper we consider a situation without the Landesman-Lazer property: the 2π-periodic problem for the equation $x'' + n^2 x = g(|x|) + f(t, x) + b(t)$ where the function $g(u)$ has no any limit at infinity and $f(t, x)$ tends to zero if $|x| \to \infty$.

The 2π-periodic problem for linearized equation $x'' + n^2 x = 0$ is degenerate for an integer n and the properties of perturbed equations depend on some delicate properties of the bounded nonlinearities $g(u)$ and $f(t, x)$. If $g(u) \equiv 0$ and $f(t, x) \equiv 0$ then the Fredholm alternative gives a complete answer on the solvability of the 2π-periodic problem of $x'' + n^2 x = b(t)$, which is completely defined in terms of the value of the complex number

$$b_n = \int_0^{2\pi} e^{int} b(t)\, dt.$$

If $b_n = 0$ then there exists a 2-dimensional linear set of 2π-periodic solutions, while if $b_n \neq 0$ then such solutions do not exist.

Nonlinear terms change the situation. We suppose that we know only the asymptotics of nonlinearities at infinity (behaviour of nonlinearities for sufficiently large $|x|$). Consider the equation $x'' + n^2 x = f(t, x) + b(t)$ with $f(t, x) \to 0$ as $|x| \to \infty$. If $b_n \neq 0$, then the topological properties of the equations can not guarantee the solvability of the 2π-periodic problem: indices at infinity of the corresponding vector fields are equal to 0. If $b_n = 0$, then, under some appropriate hypotheses, the index differs from 0 and topological methods are applicable. Related problems (for $g(u) \equiv 0$) were studied in [1, 2], in series of papers by J.Mawhin with co-authors (see [1, 3, 7, 8] and the references therein). In addition, in [4] the equation $x'' + n^2 x = g(|x|) + v(t, x)$ with a non-degenerate Landesman-Lazer type term $v(t, x)$ was considered.

In the next section we present a new theorem on the solvability of our equation. This theorem follows from a general theorem, Theorem 2, on the index at infinity calculation given in section 3. Sections 4 and 5 contain proofs. Theorem 2 can also be applied to various problems on the solvability and bifurcation at infinity, etc, for some other boundary value problems. Examples of possible applications to two-point boundary value problems and problems of forced oscillations in control systems are presented in section 6.

2 Main result for second order equation

Consider the equation

$$x'' + n^2 x = g(|x|) + f(t,x) + b(t) \tag{1}$$

with non-zero integer n and functions $f(t,x)$ and $b(t)$ which are 2π-periodic in t. These functions together with the function $g(u)$ supposed to be bounded and continuous with respect to all their variables.[1]

The following hypotheses are supposed to be valid.

(A) One of the following one-side estimates holds true:

$$f(t,x) \cdot \operatorname{sign} x \geq \varphi(|x|), \quad |x| \geq u_0 \tag{2}$$

or

$$f(t,x) \cdot \operatorname{sign} x \leq -\varphi(|x|), \quad |x| \geq u_0 \tag{3}$$

for some $u_0 > 0$, where $\varphi(u): \ \{u \geq u_0\} \to I\!R^+$ is positive continuous nonincreasing function.

(B) The asymptotical Lipschitz condition holds for some $\alpha \in (0,1)$:

$$|g(u) - g(v)| \leq c r^{-\alpha} |u - v|, \qquad r = \min\{u, v\}, \quad u, v \geq u_0. \tag{4}$$

Hypothesis (B) is valid for example for $g(u) = \sin(u^{1-\alpha})$. If the function $g(u)$ is differentiable then this hypothesis is equivalent to

$$|g'(u)| \leq c u^{-\alpha}, \qquad u \geq u_0.$$

Theorem 1. *Let $b_n = 0$, suppose that both hypotheses (A) and (B) hold, and let*

$$\lim_{u \to \infty} \varphi(u)\, u^{\alpha} = \infty. \tag{5}$$

Then (1) *has at least one 2π-periodic solution.*

From the proof of Theorem 1 in section 5 one can see that the function $\varphi(|x|)$ in conditions (2) and (3) can be replaced by the function $\varphi_0(t,|x|)$ where

$$\varphi_0(t,u) = \begin{cases} \varphi(u), & t \in \Omega_0, \\ 0, & t \notin \Omega_0 \end{cases}$$

and $\operatorname{mes} \Omega_0 > 0$.

[1] This assumption can be weaken. It is possible to prove the assertion of Theorem 1 for Carathéodorian functions $f(t,x)$ and integrable functions $b(t)$.

3 Abstract theorem

In this section we give an abstract generalization of Theorem 1.

Consider Ω be some set of positive finite measure and consider the Hilbert space L^2 of scalar square integrable functions $x(t): \ \Omega \to I\!R$ with the usual scalar product $(\cdot, \cdot)$ and norm $\| \cdot \|$.

Consider a Hammerstein type vector field

$$\Upsilon x(\cdot) = x(\cdot) - A\left(x(\cdot) + f(\cdot, x(\cdot)) + g(|x(\cdot)|) + b(\cdot)\right),$$

in this space, where A is a linear completely continuous in L^2 operator, and f, g and b are as above.

We are interested in the calculation of so-called *index at infinity* for this vector field (see e.g. [6]).

If 1 is not an eigenvalue of the operator A then this index is defined and is equal $(-1)^\sigma$, where σ is the sum of multiplicities of all real eigenvalues of A which are greater than 1.

We will consider the case in which 1 is an eigenvalue of a normal ($AA^* = A^*A$) linear operator A. Denote $E_0 = Ker\,(I - A)$ the corresponding linear finite dimensional subspace of eigenvectors (the normality of A guarantees that there are no generalized eigenvectors corresponding to the eigenvalue 1).

The main restriction on the operator A, which allows vector fields with an even term $g(|x|)$ to be considered, is the following identity:

$$\int_\Omega e(t) g(|e(t)|)\, dt = 0, \qquad e(t) \in E_0. \tag{6}$$

This identity, which was mentioned first in [4], is valid for various important applications.

If $\Omega = [0, 2\pi]$ and E_0 is the 2-dimensional subspace which contains functions $\sin nt$ and $\cos nt$, then (6) is valid. This example appears in the study of the 2π-periodic problem, considered in section 1.

Identity (6) is valid for $\Omega = [0, \pi]$ and the 1-dimensional subspace E_0 which contain the function $\sin nt$ if n is even. This case appear in the study of degenerate two-point boundary value problems.

Suppose that

(A') One of the following one-side estimates holds true:

$$f(t, x) \cdot \operatorname{sign} x \geq \varphi(t, |x|), \quad |x| \geq u_0, \ t \in \Omega \tag{7}$$

or

$$f(t, x) \cdot \operatorname{sign} x \leq -\varphi(t, |x|), \quad |x| \geq u_0, \ t \in \Omega \tag{8}$$

for some $u_0 > 0$, where $\varphi(t, u) : \Omega \times \{u \geq u_0\} \to I\!R^+$ is a nonnegative Carathéodorian nonincreasing function which is strictly positive for $t \in \Omega_0$ with $\operatorname{mes} \Omega_0 > 0$.

(B') The asymptotical Lipschitz condition holds for some $d(r)$:

$$|g(u) - g(v)| \leq d(r)|u - v|, \qquad r = \min\{|u|, |v|\}, \ u, v \geq u_0 \tag{9}$$

where $d(r):\ I\!R^+ \to I\!R^+$ is some positive non-increasing function satisfying

$$\lim_{r\to\infty} d(r) = 0. \tag{10}$$

In the formulation and proof of the following theorem a leading role is played by the distribution function

$$\chi(\delta) = \chi(\delta; e) = \text{ mes}\,\{t \in \Omega:\ |e(t)| \le \delta\} \tag{11}$$

of a non-zero function $e(t) \in E_0$.

Denote by P the orthogonal projector onto E_0. In the formula (13) below we put $d(u) = d(u_0)$ for $0 \le u < u_0$.

Theorem 2. *Let $Pb(t) = 0$. Suppose that both hypotheses (A') and (B') hold and that*

$$\chi(0) = \text{ mes}\,\{t \in \Omega:\ e(t) = 0\} = 0 \tag{12}$$

for any non-zero function $e(t) \in E_0$. Suppose also that the operator A maps square integrable functions into essentially bounded ones and that A is continuous as an operator from the space L^2 to the space L^∞. Let for any $R > 0$ and $u_* \ge u_0$

$$\lim_{\xi\to\infty}\ \sup_{e\in E_0,\ \|e\|=1}\ \frac{\int_\Omega |e(t)| d(\xi|e(t)|)\, dt}{\int_\Omega |e(t)|\varphi(t, u_* + R\xi|e(t)|)\, dt} = 0 \tag{13}$$

and

$$\lim_{\xi\to\infty}\ \sup_{e\in E_0,\ \|e\|=1}\ \frac{\chi(\xi^{-1}, e)}{\int_\Omega |e(t)|\varphi(t, u_* + R\xi|e(t)|)\, dt} = 0. \tag{14}$$

Then

$$ind_\infty \Upsilon = (-1)^{\sigma_0}$$

where $\sigma_0 = \sigma + dimE_0$ for the case (7) *and $\sigma_0 = \sigma$ for the case* (8) (*here σ is the sum of multiplicities of all real eigenvalues of the operator A which are greater than* 1).

Condition (12) was used by many authors, functions (11) were considered in the related context in Chapter 25 of [1] and were later systematically used in [3]. The combination of (6) with the possibility of $f(t, x) \to 0$ has never considered before.

The assumption about the operator $A:\ L^2 \to L^\infty$ is technical and can be omitted, but this makes proof much more combersome. In many applications it is usually valid.

In concrete cases conditions (13) and (14) are not so combersome (see [3]), e.g. in the case of Theorem 1 relation (5) guarantees both (13) and (14). If the function $d(u)$ has the form $d(u) = cu^{-\alpha}$ for some $\alpha \in (0, 1)$ and (14) holds, then (13) is equivalent to

$$\lim_{\xi\to\infty}\ \sup_{e\in E_0,\ \|e\|=1}\ \frac{\xi^{-\alpha}}{\int_\Omega |e(t)|\varphi(t, u_* + \xi|e(t)|)\, dt} = 0.$$

4 Proof of Theorem 2

Consider the homotopy

$$\Phi(x,\lambda) = \Upsilon x + \lambda A x \tag{15}$$

with λ having a fixed sign. Let us prove that for $|\lambda| \le \lambda_0$ with positive λ_0 small enough the vector field $\Phi(x,\lambda)$ is non-zero for sufficiently large values of $\|x\|$ under the assumptions of Theorem 2. We consider only non-positive values of λ for the case where (7) holds and non-negative λ for the opposite case.

This *a priori* estimate proves Theorem 2: for small $\lambda \neq 0$ the linear part $I-(1-\lambda)A$ of the field $\Phi(x,\lambda)$ is non-degenerate and its index at infinity has exactly the value given in Theorem 2; the value $\mu = (1-\lambda)^{-1}$ is the eigenvalue of the operator $(1-\lambda)A$, $\mu > 1$ iff $\lambda < 0$. The general properties of index complete the proof. The proof of a common estimate will be given only for the case (7), $\lambda \in [-\lambda_0, 0]$.

Suppose that $\Phi(x,\lambda) = 0$ for some $x \in L_2$ and $\lambda \in [0,\lambda_0]$. Denote by $E_1 \subset L^2$ the orthogonal to E_0 subspace and let $Q = I - P$. The projectors P and Q commute with the operator A, the projector P can be easily represented as $Px = \sum (e_j, x) e_j$. Here $\{e_j\}$ is a orthogonal normed basis in E_0 and $AP = P$.

For $|\lambda|$ small enough the linear operators $B(\lambda) = Q(I-(1-\lambda)A)$ are continuously invertible in E_1 for any λ. Moreover these inverse operators have uniformly bounded norms

$$\|B(\lambda)^{-1}\|_{E_1 \to E_1} \le c_1, \qquad |\lambda| \le \lambda_0. \tag{16}$$

Here and below c_j denote some constants, their exact values do not play any role, only their existence in meaningful. These constants do not depend on λ and x.

The equality $\Phi(x,\lambda) = 0$ can be rewritten as the pair of equations: $Q\Phi(x,\lambda) = 0$ and $P\Phi(x,\lambda) = 0$.

The first equality, rewritten as $B(\lambda)x = AQ(f(t,x) + g(|x|) + b(t))$, together with the continuity of the operator $A:\ L^2 \to L^\infty$ imply the estimate

$$\|Qx\|_{L^\infty} \le c_2. \tag{17}$$

The second equality can be rewritten as

$$\lambda P x = P(f(t,x) + g(|x|)) \tag{18}$$

($Pb = 0$ is an assumption of Theorem 2). This last formula will now be studied in detail.

Let $Px = \xi e(t)$ where $\|e\| = 1,\ \xi \ge 0$ and $h = Qx$. According to (17) we have to obtain an *a priori* estimate for the scalar positive ξ.

Let us multiply (18) by $e(t)$. Then the scalar equality obtained has the form

$$\lambda \xi = \int_\Omega e(t)(f(t,x) + g(|x|))\, dt,$$

which implies that

$$\int_\Omega e(t)\left(f(t, \xi e(t) + h(t)) + g(|\xi e(t) + h(t)|)\right)\, dt \le 0. \tag{19}$$

The scheme of the proof below is as following. The left-hand side of (19) contains two terms. The first term can be estimated from below, the estimate having the form

$$\int_\Omega e(t)f(t,\xi e(t)+h(t))\,dt \ge \int_\Omega |e(t)|\varphi(t,u_*+\xi|e(t)|)\,dt - c_3\chi(c_3\xi^{-1},e). \qquad (20)$$

The second term can be estimated from above:

$$\left|\int_\Omega e(t)g(|\xi e(t)+h(t)|)\,dt\right| \le c_4\int_\Omega |e(t)|d(c_5\xi|e(t)|)\,dt + c_6\chi(c_6\xi^{-1},e). \qquad (21)$$

The combination of these two estimates together with (19) and conditions of Theorem 2 imply the estimate $\xi \le c_7$.

Estimate (20) was considered in [3] for various general cases. Here we sketch the proof. Let us split the set Ω into two parts:

$$\Omega_1 = \{t\in\Omega : |e(t)| > \frac{c_2+u_0}{\xi}\}$$

and

$$\Omega_2 = \{t\in\Omega : |e(t)| \le \frac{c_2+u_0}{\xi}\},$$

where constant c_2 comes from (17). Since $\operatorname{sign} x(t) = \operatorname{sign} e(t)$ for $t\in\Omega_1$ we have

$$\int_{\Omega_1} e(t)f(t,x(t))\,dt = \int_{\Omega_1} |e(t)|f(t,x(t))\operatorname{sign} e(t)\,dt = \int_{\Omega_1} |e(t)|f(t,x(t))\operatorname{sign} x(t)\,dt$$

$$\ge \int_{\Omega_1} |e(t)|\varphi(t,|x(t)|)\,dt \ge \int_{\Omega_1} |e(t)|\varphi(t,u_*+\xi|e(t)|)\,dt$$

$$\ge \int_\Omega |e(t)|\varphi(t,u_*+\xi|e(t)|)\,dt - c_8\operatorname{mes}\Omega_2.$$

Inequality (20) now follows from the estimate $\operatorname{mes}\Omega_2 \le \chi(c_9/\xi,e)$.

Let us now obtain the estimate (21). Since

$$\int_\Omega e(t)g(|\xi e(t)|)\,dt = 0$$

by assumption, then

$$\left|\int_\Omega e(t)g(|\xi e(t)+h(t)|)\,dt\right| = \left|\int_\Omega e(t)g(|\xi e(t)+h(t)|)\,dt - \int_\Omega e(t)g(|\xi e(t)|)\,dt\right|$$

$$\le \int_\Omega |e(t)|\cdot\Big|g(|\xi e(t)+h(t)|)-g(|\xi e(t)|)\Big|\,dt \le c_{10}\operatorname{mes}\Omega_2 + c_{11}\int_\Omega |e(t)|d(c_{12}\xi|e(t)|)\,dt$$

$$\le c_{10}\chi(\frac{c_9}{\xi},e) + c_{11}\int_\Omega |e(t)|d(c_{12}\xi|e(t)|)\,dt.$$

5 Proof of Theorem 1

Rewrite equation (1) as $x'' + (n^2+1)x = x + g(|x|) + f(t,x) + b(t)$ and invert the differential operator $\mathcal{L}$ in the left-hand side with 2π-periodic boundary conditions, putting $A = \mathcal{L}^{-1}$. The operator A satisfies all the assumptions of Theorem 2 and, moreover, is self-adjoint instead of normal. Equation (1) has the equivalent form $\Upsilon x = 0$. Any zero $x \in L^2$ of the vector field Υx is obviously twice continuously differentiable. If a vector field has non-zero index at infinity then it has at least one zero. Now Theorem 1 follows directly from Theorem 2: under assumptions of this theorem the index equals ± 1.

We need to check that conditions of Theorem 2 are fulfilled under the assumptions of Theorem 1: to check that condition (5) guaranties both assumptions (13) and (14) for $E_0 = \{e(t) = A\sin(nt+\theta)\}$ and that (6) is valid.

First of all note the estimates

$$c_1\delta \le \chi(\delta, e) \le c_2\delta, \qquad e \in E_0,\ \|e\| = 1, \quad \delta \le \delta_0,$$

which mean that condition (14) can be rewritten as

$$\int^{\infty} u\varphi(u) = \infty. \tag{22}$$

By assumption $d(r) = cr^{-\alpha}$, condition (13) has the form of condition (5) which is more restrictive than (22).

Exact calculations are rather simple and we omit them.

Identity (6) for $e(t) = \sin(nt+\theta)$ follows from the relationships

$$\int_0^{2\pi} e(t)g(|e(t)|)\,dt = \frac{1}{n}\int_0^{2n\pi} \sin\tau g(|\sin\tau|)\,d\tau =$$

$$= \int_0^{2\pi} \sin\tau g(|\sin\tau|)\,d\tau = \int_0^{\pi} \sin\tau g(|\sin\tau|)\,d\tau + \int_{\pi}^{2\pi} \sin\tau g(|\sin\tau|)\,d\tau =$$

$$= \int_0^{\pi} \sin\tau g(|\sin\tau|)\,d\tau - \int_0^{\pi} \sin\tau g(|\sin\tau|)\,d\tau = 0.$$

6 Other applications of Theorem 2

The first application of Theorem 2 was Theorem 1. In this section we give two other kinds of examples: solvability results and a theorem on asymptotic bifurcation points.

Consider for some integer $n \ne 0$ the boundary value problem

$$x'' + 4n^2x = f(t,x) + g(|x|) + b(t), \qquad x(0) = x(\pi) = 0. \tag{23}$$

Theorem A1. *Let functions $f(t,x)$, $g(u)$ and $b(t)$ be bounded and continuous, let one of the one-side estimates* (2) *or* (3) *be valid, let*

$$\int_0^{\pi} \sin nt\, b(t)\,dt = 0,$$

let the function $g(u)$ satisfy Hypothesis (B) for some $\alpha \in (0,1)$ and let $\varphi(u)$ satisfy (5) *with this α. Then problem* (23) *has at least one solution.*

Note that Theorem A1 is not valid for equation (23) with $(2n-1)^2$ instead of $4n^2$ in the left-hand side.

In the final part of this section we consider a special type of ordinary differential equations which arise in the control theory:

$$L\left(\frac{d}{dt}\right) x(t) = M\left(\frac{d}{dt}\right)(f(t,x) + g(|x|) + b(t)). \tag{24}$$

Here $L(p)$ and $M(p)$ are real coprime polynomials with $l = deg\, L(p) > m = deg\, M(p)$. If $M(p) \equiv 1$ then equation (24) is a usual quasilinear ODE of higher order, but all results below are new even for this case.

The sense of such equations for nonsmooth functions in the right-hand side of (24) can be interpreted in different ways. Among the most usual ones there is the possibility to consider an equivalent differential equation of first order or an equivalent integral equation (for periodic problem). Details can be found in almost any manual on control theory.

Suppose that both functions f and b are continuous, bounded and 2π-periodic in t. If the polynomial $L(p)$ has no roots of the form ni for integer n, then (nonresonant case) equation (24) has at least one 2π-periodic solution. This again can be proved by the Schauder principle. The other case is much more difficult.

Theorem A2. *Suppose that the polynomial $L(p)$ has exactly one pair of roots of the type ni, i.e. $+ni$ and $-ni$. Under the assumptions of Theorem 1 equation* (24) *has at least one 2π-periodic solution.*

Now consider equation (24) with a parameter:

$$L\left(\frac{d}{dt}\right) x(t) = M\left(\frac{d}{dt}\right)(f(t,x;\lambda) + g(|x|;\lambda) + b(t;\lambda)). \tag{25}$$

The following result concerns with the so-called *asymptotic bifurcation points*. The general definition, given by Mark Krasnosel'skii in the early 50s can be found in [6]. We reformulate the definition for the 2π-periodic problem for equation (25).

The value λ_0 is called *the point of nonlinear resonance* for equation (25) if for any $\varepsilon > 0$ there exists a λ with $\|\lambda - \lambda_0| \le \varepsilon$ such that equation (25) for this λ has at least one 2π-periodic solution with amplitude greater than ε^{-1}.

Theorem A3. *Let $f(t,x;\lambda) \to 0$ as $|x| \to \infty$ for any λ sufficiently close to λ_0. Suppose that for λ sufficiently close to λ_0 the right-hand side of* (25) *satisfies all the assumptions of Theorem 1. Consider the value*

$$b_n(\lambda) = \int_0^{2\pi} e^{int} b(t;\lambda)\, dt$$

and suppose that $b_n(\lambda_0) = 0$ with $b_n(\lambda) \ne 0$ for $\lambda \ne \lambda_0$. Then λ_0 is a point of nonlinear resonance for equation (25)*, for $\lambda = \lambda_0$ there exists at least one 2π-periodic* *solution of* (25)*, while for* $\lambda \neq \lambda_0$ *and* λ *sufficiently close to* λ_0 *there exist at least two* 2π*-periodic solutions.*

We give here a brief sketch of the proof. Suppose (without loss of generality) that the polynomial $L(p) + M(p)$ has no roots of the type ki. Consider the equivalent integral equation

$$x = A(x + f(t, x; \lambda) + g(|x|; \lambda) + b(t; \lambda)) \equiv AF(t, x; \lambda).$$

Here A is an integral operator, its kernel is generated by the impulse response function of the linear link with transfer function $W(p) = M(p)/(M(p) + L(p))$. For any integrable function $z(t)$ the image $x(t) = Az(t)$ is the unique 2π-periodic solution of the linear equation

$$L\left(\frac{d}{dt}\right) x(t) = M\left(\frac{d}{dt}\right) z(t).$$

The index at infinity of the vector field $\Psi x = x - AF(t, x; \lambda)$ for $\lambda = \lambda_0$ is equal ± 1 and the same index for $\lambda \neq \lambda_0$ is zero. The last fact is non-trivial, it follows from $f(t, x; \lambda) \to 0$ as $|x| \to \infty$, it was mentioned and proved in [4].

The first statement follows directly from the changing index principle [6], the second one from the usual idea. For $\lambda = \lambda_0$ the index is non-zero, this means the existence of solutions for sufficiently close to λ_0 values of λ in a common ball. The rotation of vector field Ψx on the boundary of this ball is nonzero. But the index at infinity of this field is zero, therefore the second solution exists.

Theorem A3 can be developed for the case where the polynomial $L(p)$ has $2n$ ($n > 1$) roots of the type ki. In this case instead of (5) it is necessary to suppose more restrictive relationship

$$\lim_{u \to \infty} u^{\gamma} \varphi(u) = \infty, \quad \text{with} \quad \gamma = \min\{\frac{1}{2n-1}, \alpha\}.$$

More details about the estimate of distribution functions for this case can be found in [3].

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