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HOPF BIFURCATIONS AT INFINITY, GENERATED BY BOUNDED NONLINEAR TERMS *

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Abstract. In the paper we present new sufficient conditions for existence of large-amplitude periodic solutions for autonomous equations with a parameter. In contrast to the usual situations, the linear degenerate part of the equation does not depend on the parameter. Therefore the existence of periodic solutions is determined by the asymptotic behavior of bounded nonlinear terms at infinity. We present a new simple method to reduce the original degenerate problem to topologically nondegenerate one. This infinite-dimensional problem is studied by degree theory methods.

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1. Introduction.

Consider the differential equation [1]

$$L\left(\frac{d}{dt},\lambda\right)x = M\left(\frac{d}{dt},\lambda\right)F(x,\lambda). \tag{1}$$

Here $L(p,\lambda)$ and $M(p,\lambda)$ are coprime polynomials of degrees ℓ and m, $\ell > m$, with real coefficients, which depend on the scalar parameter $\lambda \in \Lambda = (a,b)$.

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[1] A definition of solution for equation (1) is given in most books on control theory, see, e.g., [2] and [8]. If $M(p,\lambda) \equiv 1$, then (1) is a usual quasilinear ODE.

The continuous nonlinearity $F(x,\lambda)$ is uniformly bounded. We shall use bounded nonlinearities of various types: functional nonlinearities $f(x(t),\lambda)$, where $f(x,\lambda):\mathbb{R}\times\ngeq\to\mathbb{R}$ is a continuous function, nonlinearities $f(x(t-h),\lambda)$ with the delay $h>0$, hysteresis nonlinearities. We study sufficient conditions for existence of large-amplitude periodic solutions $x(t)$ of equation (1).

DEFINITION 1. *The number* λ_0 *is called* [2] a Hopf bifurcation point at infinity *(shortly,* a Hopf bifurcation point*) for equation* (1) *with the frequency* w_0 *if for every* $\varepsilon>0$ *there is a parameter value* $\lambda_\varepsilon\in(\lambda_0-\varepsilon,\lambda_0+\varepsilon)$ *such that equation* (1) *with* $\lambda=\lambda_\varepsilon$ *has a periodic solution* $x_\varepsilon(t)$ *of a period* T_ε *satisfying* $|T_\varepsilon-2\pi/w_0|<\varepsilon$ *with the amplitude* $\max|x_\varepsilon(t)|>\varepsilon^{-1}$.

In other words, λ_0 is a Hopf bifurcation point with the frequency w_0 if for arbitrarily close to λ_0 values of the parameter λ equation (1) has periodic solutions of arbitrarily large amplitudes with periods arbitrarily close to $2\pi/w_0$.

The following result is formulated in [3]. Suppose the polynomial $L(p,\lambda)$ has a pair of simple conjugate roots $\sigma(\lambda)\pm w(\lambda)i$ depending continuously on λ, where $\sigma(\lambda_0)=0$ and the function $\sigma(\lambda)$ takes values of both sign in every neighborhood of the point λ_0. Suppose $L(kw(\lambda_0)i,\lambda_0)\neq 0$ for $k=0,2,3,\dots$ Then λ_0 is a Hopf bifurcation point for equation (1) with the frequency $w(\lambda_0)$.

These sufficient conditions for bifurcation point existence use only the information on the linear part of (1). The result holds for equation (1) with any continuous bounded nonlinearity, moreover, it holds for unbounded sublinear[3] nonlinearities. So the nonlinear part of the equation is of no importance under the assumptions above.

In this paper we study equations of the form (1), where the linear part is independent of a parameter and is degenerate, i.e., $L(p,\lambda)=L(p)$, $M(p,\lambda)=M(p)$ and where the polynomial $L(p)$ has a pair of pure imaginary roots $\pm w_0 i$. For such equations, the asymptotic behavior of nonlinear terms at infinity is a criterion for λ_0 to be a Hopf bifurcation point.

The paper is organized as follows. Theorem 1 of Section 2 gives sufficient conditions for existence of a Hopf bifurcation point for equations with the delayed term. These conditions are formulated in a simpler form for equations without delays in Theorem 2.

In Section 3 equations with the stop hysteresis nonlinearity ([5]) are studied. The stop is used for simplicity, similar results are valid for equations

[2] See [3].

[3] The nonlinearity $F(x,\lambda):E\times\Lambda\to E_1$ is *sublinear* if $\|F(x,\lambda)\|_{E_1}=o(\|x\|_E)$ as $\|x\|_E\to\infty$.

with scalar hysteresis nonlinearities of various types: the Prandtl-Ishlinskii nonlinearities, the Preisach nonlinearities etc.

In Section 4 we give some remarks on the results obtained. The proofs (Section 5) are based on a new method to reduce the original degenerate problem to a topologically nondegenerate one. The method can also be used to study some classical bifurcation problems with nondegenerate linear part depending on a parameter, in particular to prove the result from [3] above.

2. Equations with delay. Consider the differential equation

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)\left[f(x(t),\lambda) + g(x(t-h),\lambda)\right] \tag{2}$$

with the real coprime polynomials $L(p)$, $M(p)$ of degrees $\ell > m$. The functions $f(x,\lambda) : \mathbb{R} \times \ngeq \to \mathbb{R}$ and $g(x,\lambda) : \mathbb{R} \times \ngeq \to \mathbb{R}$ are continuous with respect to the set of their arguments and uniformly bounded.

Let $L(\pm w_0 i) = 0$. Define

$$\beta \stackrel{\text{def}}{=} \lim_{w\to w_0} \frac{\Im m[L(wi)M(-wi)]}{\Re e[L(wi)M(-wi)]}, \qquad \alpha \stackrel{\text{def}}{=} \operatorname{arctg}\beta. \tag{3}$$

The limit in (3) (finite or infinite) always exists. To be definite, we put $\alpha = \pi/2$ if it is infinite (for example, $\alpha = \pi/2$ if the denominator in (3) is the identical zero).

Denote by f_{odd} and g_{odd} the odd components

$$f_{odd}(x,\lambda) = (f(x,\lambda) - f(-x,\lambda))/2, \qquad g_{odd}(x,\lambda) = (g(x,\lambda) - g(-x,\lambda))/2$$

of the functions f and g. Set

$$\begin{aligned}\Psi(\xi,\lambda) &= \int_0^{2\pi} \sin t\, f(\xi\sin t,\lambda)\,dt = 4\int_0^{\pi/2} \sin t\, f_{odd}(\xi\sin t,\lambda)\,dt,\\ \Gamma(\xi,\lambda) &= \int_0^{2\pi} \sin t\, g(\xi\sin t,\lambda)\,dt = 4\int_0^{\pi/2} \sin t\, g_{odd}(\xi\sin t,\lambda)\,dt\end{aligned} \tag{4}$$

and

$$\Phi(\xi,\alpha,\lambda) = \sin\alpha\Psi(\xi,\lambda) + \sin(\alpha + w_0 h)\Gamma(\xi,\lambda).$$

Functions (4) are rather usual for the control theory as *describing functions* (see, e.g., [7]).

Theorem 1. *Let the following assumptions hold:*

1. *The number w_0 is a root of odd multiplicity K for the polynomial $L(wi)$.*
2. *The relation $L(kw_0 i) \neq 0$ holds for every $k = 0, 2, 3, 4, \dots$*
3. *In every neighborhood of the point λ_0 there are points λ_1 and λ_2 such that*

$$\limsup_{\xi\to+\infty} \Phi(\xi,\alpha,\lambda_1) < 0, \qquad \liminf_{\xi\to+\infty} \Phi(\xi,\alpha,\lambda_2) > 0, \tag{5}$$

where α is given by (3).

Then λ_0 is a Hopf bifurcation point for equation (2) *with the frequency w_0.*

Consider the application of Theorem 1 to the problem without the delayed term. Let equation (2) have the form

$$L\left(\frac{d}{dt}\right) x = M\left(\frac{d}{dt}\right) f(x,\lambda). \tag{6}$$

Theorem 2. *Let the following conditions hold:*

1. *The number w_0 is a root for both polynomials $\Im m[L(wi)M(-wi)]$ and $L(wi)$ of the same odd multiplicity K.*
2. *The relation $L(kw_0 i) \neq 0$ holds for every $k = 0, 2, 3, 4, \dots$*
3. *For every λ there exists the limit*

$$\psi(\lambda) \stackrel{\text{def}}{=} \lim_{\xi\to+\infty} \Psi(\xi,\lambda). \tag{7}$$

4. *Equation $\psi(\lambda) = 0$ has a solution λ_0 such that the function $\psi(\lambda)$ takes the values of both sign in every neighborhood of the point λ_0.*

Then λ_0 is a Hopf bifurcation point for equation (6) *with the frequency w_0.*

Under assumption 1 of Theorem 2 the polynomial $\Im m[L(wi)M(-wi)]$ is nonzero, so at least one of the polynomials $L(p)$ and $M(p)$ is not even. Under this assumption the limit in (3) is distinct from zero, hence $\alpha \neq 0$. Since $\Phi(\xi,\alpha,\lambda) = \sin\alpha\,\Psi(\xi,\lambda)$, conditions 3 and 4 of Theorem 2 imply condition 3 of Theorem 1 and Theorem 2 follows from Theorem 1.

If $\Im m[L(wi)M(-wi)] \equiv 0$, then $\alpha = 0$ and hence $\Phi(\xi,\alpha,\lambda) \equiv 0$. Therefore condition 3 of Theorem 1 is not satisfied for equation (6). In fact, it is an exceptional case. One can show that the identity $\Im m[L(wi)M(-wi)] \equiv 0$ holds iff both polynomials $L(p)$ and $M(p)$ are even. Furthermore, if $L(p) = L(-p)$, $M(p) = M(-p)$, where the polynomial $L(p)$ satisfies conditions 1 and 2 of Theorem 1, then equation

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)f(x)$$

with any bounded continuous function $f(x)$ has the continuum of periodic cycles $x(t;\xi)$, $\xi \geq \xi_0$ such that $\|x(t;\xi)\|_C \to \infty$ and $T(\xi) \to 2\pi/w_0$ as $\xi \to \infty$, where $T(\xi)$ is the period of the cycle $x(t;\xi)$ and ξ is a parameter. Therefore the identity $\Im m[L(wi)M(-wi)] \equiv 0$ implies that all the values $\lambda \in \Lambda$ are Hopf bifurcation points for equation (6) with the frequency w_0.

A simple example of equation (6) that can be studied with Theorem 2 is $x''' + x'' + x' + x = f(x,\lambda)$.

Sufficient conditions for existence of limit (7) are discussed in Section 4.

3. System with hysteresis. Here we consider the equation

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)\big(a(\lambda)U(\mu_0)x + f(x,\lambda)\big) \tag{8}$$

with a continuous bounded function $f(x,\lambda) : \mathbb{R} \times \ngeqq \to \mathbb{R}$ and a continuous function $a(\lambda)$. By $U(\mu_0)$ we denote the stop nonlinearity with the initial state $\mu_0 \in [-1,1]$. The definition of the stop is given shortly below; for more details and for the general mathematical theory of hysteresis operators, see [5].

For a given initial state μ_0 and for every continuous input $x(t)$, $t \geq t_0$ the operator $\mu(t) = U(\mu_0)x(t)$ determines the state of the stop at each moment $t \geq t_0$. The continuous function $\mu(t)$, $t \geq t_0$ with the values in $[-1,1]$ is at the same time the output of the stop. For monotone continuous inputs,

$$U(\mu_0)x(t) = \begin{cases} \min\{\ \ 1,\ \mu_0 + x(t) - x(t_0)\} & \text{if } x(t) \text{ increases,} \\ \max\{-1,\ \mu_0 + x(t) - x(t_0)\} & \text{if } x(t) \text{ decreases.} \end{cases}$$

For each piecewise monotone continuous input the output is calculated with the help of the semigroup identity $U(U(\mu(t_0))x(t_1))x(t) = U(\mu(t_0))x(t)$, $t \geq t_1 \geq t_0$. To define the outputs for any continuous inputs, the operator $U(\mu_0)$ is extended by continuity in the space $C[t_0,t_1]$ of continuous functions from the dense set of piecewise monotone inputs $x(t)$ to the whole space. The correctness of this procedure is proved in [5].

Figure 1 shows the trajectories of the point $\{x(t), U(\mu)x(t)\}$ in the plane $\{x, Ux\}$. The point is always in the closed band $|Ux| \leq 1$, which is the join of the two boundary horizontal lines $Ux = \pm 1$ and continual number of slanting lines $Ux = x - \theta$ with $x \in (\theta - 1, \theta + 1)$ (where $\theta \in \mathbb{R}$ is a parameter). If the initial state μ is not ± 1 the point $\{x(t), U(\mu)x(t)\}$ goes along a slanting

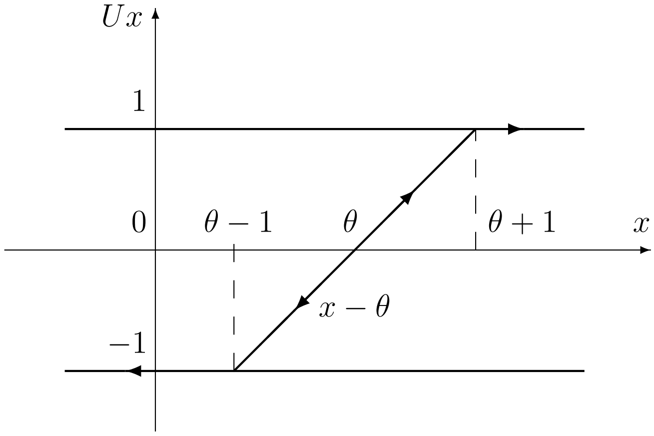

Fig. 1. *Stop* nonlinearity

line: upwards right if $x(t)$ increases and downwards left if $x(t)$ decreases. As the point reaches the horizontal line, it switches to it and goes to the right along the line $Ux = 1$ if $x(t)$ increases and to the left along the line $Ux = -1$ if $x(t)$ decreases. The point switches again to a slanting line as soon as the input $x(t)$ switches from increasing to decreasing or conversely.

The stop $U(\mu_0)x$ is a continuous operator from $[-1, 1] \times C[t_0, t_1]$ to $C[t_0, t_1]$. Moreover, this operator is Lipschitz continuous in both arguments μ_0, $x(t)$ and monotone in the natural sense.

In the following, the initial state μ_0 of the stop is not fixed. A solution $x(t)$ of equation (8) is periodic if both the function $x(t)$ and the variable state $\mu(t) = U(\mu_0)x(t)$ of the stop are periodic with the same period.

Define the number α and the function $\Psi(\xi, \lambda)$ by formulas (3) and (4).

Theorem 3. *Let assumptions* 1 *and* 2 *of Theorem* 1 *hold and limit* (7) *exist for each* λ. *Suppose the equation*

$$0 = \psi(\lambda) \sin \alpha - 4a(\lambda) \cos \alpha \stackrel{\text{def}}{=} \phi(\lambda)$$

has a solution λ_0 *such that the function* $\phi(\lambda)$ *takes the values of both sign in every neighborhood of the point* λ_0. *Then* λ_0 *is a Hopf bifurcation point for equation* (8) *with the frequency* w_0.

4. Remarks.

4.1. Equations with variable linear part. Consider the equation

$$L\left(\frac{d}{dt},\lambda\right)x = M\left(\frac{d}{dt},\lambda\right)[f(x(t),\lambda)+g(x(t-h),\lambda)],$$

where the polynomial $L(p,\lambda)$ has the same imaginary roots $\pm w_0 i$ of the same odd multiplicity K for all parameter values and $L(kw_0 i,\lambda)\neq 0$ for every $k=0,2,3\ldots$ and every λ. If the nonlinearity satisfies the conditions of Theorem 1, then the conclusion of the theorem holds for this equation. The same is true for Theorem 2 applied to the equation

$$L\left(\frac{d}{dt},\lambda\right)x = M\left(\frac{d}{dt},\lambda\right)f(x,\lambda).$$

These facts follow from the proof of Theorem 1 (see Section 5) without any additional argument.

Values (3) may depend on λ in this case.

4.2. Computation of limit (7)**.** For applications of Theorems 2 and 3 it is important to know if limit (7) exists for a given function $f(x,\lambda)$. Consider some sufficient conditions for the limit existence for nonlinearities without parameters (for nonlinearities depending on the parameter we suppose these conditions for every parameter value).

Let $f(x)=f_1(x)+f_2(x)+f_3(x)$, where the function $f_1(x)$ satisfies the Landesman-Lazer conditions, i.e., the finite limits

$$\lim_{x\to+\infty} f_1(x)=f^+,\qquad \lim_{x\to-\infty} f_1(x)=f^-$$

exist; the function $f_2(x)$ is even; the primitive of the function $f_3(x)$ is sublinear:

$$\lim_{x\to\infty} x^{-1}\int_0^x f_3(u)\,du = 0. \tag{9}$$

Then limit (7) exists and

$$\lim_{\xi\to\infty}\int_0^{2\pi}\sin t\, f(\xi\sin t)\,dt = 2(f^+-f^-).$$

For example, equality (9) holds for all periodic and almost periodic functions $f_3(x)$ with zero average value, for the functions $\sin x^3$, $\sin\sqrt{|x|}$, for every function $f_3(x)$ vanishing at infinity etc. The sum of the functions satisfying (9) also satisfies (9). Equality (9) is not valid for the function $\sin\ln(1+|x|)$.

4.3. More general result. Natural analogs of Theorems 1 – 3 hold for equations with several delays, for example

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)\sum_{k=0}^{N} f_k(x(t-h_k),\lambda),$$

and for equations containing both delayed and hysteresis terms.

Consider the more complicated than (2) equation

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)f(x(t),x(t-h),\lambda) \tag{10}$$

with a bounded continuous function $f(x,y,\lambda): \mathbb{R}\times\mathbb{R}\times \not\gtrsim \to \mathbb{R}$. Set

$$\Phi_0(\xi,\alpha,w,\lambda) \stackrel{\text{def}}{=} -\int_0^{2\pi} \cos(t+\alpha)\, f(\xi\sin t, \xi\sin(t-wh),\lambda)\,dt, \tag{11}$$

where α is given by (3).

We say that the function $f(x,y,\lambda)$ satisfies the proper Lipschitz condition in x if

$$|f(x_1,y,\lambda)-f(x_2,y,\lambda)| \le \zeta(|x_1|+|x_2|+|y|)|x_1-x_2|, \qquad \lambda\in\Lambda$$

with $\zeta(r)\to 0$ as $r\to\infty$. Similarly, $f(x,y,\lambda)$ satisfies the proper Lipschitz condition in y if

$$|f(x,y_1,\lambda)-f(x,y_2,\lambda)| \le \zeta(|x|+|y_1|+|y_2|)|y_1-y_2|, \qquad \lambda\in\Lambda$$

with $\zeta(r)$ vanishing at infinity. The following result can be proved by the slightly modified method of the proof of Theorem 1.

THEOREM 4. *Let assumptions* 1 *and* 2 *of Theorem* 1 *hold. Let*

$$f(x,y,\lambda) = f_1(x,y,\lambda) + f_2(x,y,\lambda),$$

where the function f_1 satisfies the proper Lipschitz condition in x and the function f_2 satisfies the proper Lipschitz condition in y. Suppose in every neighborhood of the point λ_0 there are points λ_1 and λ_2 such that for each $R>0$ the relations

$$\limsup_{\xi\to\infty} \sup_{|w-w_0|\le R\xi^{-1/K}} \Phi_0(\xi,\alpha,w,\lambda_1) < 0,$$
$$\liminf_{\xi\to\infty} \inf_{|w-w_0|\le R\xi^{-1/K}} \Phi_0(\xi,\alpha,w,\lambda_2) > 0$$

hold, where Φ_0 is function (11) *and K is the multiplicity of the root w_0 of the polynomial $L(wi)$. Then λ_0 is a Hopf bifurcation point for equation* (10) *with the frequency w_0.*

5. Proofs.

5.1. Change of variables. We look for periodic solutions of equations (2) and (8) of the periods $2\pi/w$ with unknown w close to w_0. Let us change the time scaling and replace equations (2) and (8) with the equations

$$L\left(w\frac{d}{dt}\right)x = M\left(w\frac{d}{dt}\right)[f(x(t),\lambda) + g(x(t-wh),\lambda)] \tag{12}$$

and

$$L\left(w\frac{d}{dt}\right)x = M\left(w\frac{d}{dt}\right)\big(a(\lambda)U(\mu_0)x(t) + f(x(t),\lambda)\big). \tag{13}$$

Evidently, $x(wt)$ is a $2\pi/w$-periodic solution of equation (2) (respectively, (8)) iff $x(t)$ is a 2π-periodic solution of equation (12) (respectively, (13)).

By assumption, the polynomial $L(wp)$ of the variable p has the roots $\pm i$ for $w = w_0$; furthermore, the relations $M(\pm wi) \neq 0$ and $L(\pm kwi) \neq 0$, $k = 0, 2, 3, \ldots$ hold for $w = w_0$, so they hold also for every w from a small neighborhood Ω of the point w_0.

We show that each of equations (12) and (13) has 2π-periodic solutions of the form

$$x(t) = \xi \sin t + z(t), \tag{14}$$

where $z(t)$ is orthogonal in $L^2 = L^2(0, 2\pi)$ to the functions $\sin t$ and $\cos t$. More precisely, for every sufficiently large positive ξ there are the numbers w and λ and the function $z(t)$ such that (14) is a solution of the equation considered. Since ξ is arbitrarily large, so is the amplitude of (14).

Let us stress the following.

First, the shift of time generates the continuum $\{x(t+\varphi),\ \varphi \in \mathbb{R}\}$ of periodic solutions for every given nonconstant periodic solution $x(t)$ of any autonomous equation. That is, together with solution (14) equations (12) and (13) have solutions of the form $\xi \sin(t+\varphi) + z(t+\varphi)$ with any φ. By fixing the phase φ, we choose a unique solution from the continuum $x(t+\varphi)$ (namely, it is the solution orthogonal to $\cos t$ with the positive Fourier coefficient ξ by $\sin t$).

Secondly, the original problem depends on the parameter λ. Unknown solutions of the problem are functions $x(t) = \xi \sin wt + \eta \cos wt + z(wt)$ of unknown period $2\pi/w$ with unknown Fourier coefficients ξ, η and unknown component $z(\cdot)$. Thus, originally we have a problem with the parameter λ and the four unknowns ξ, η, w, $z(\cdot)$; each solution of the problem is included in the continuum of solutions with shifted time. Now the Fourier coefficient ξ is considered as a parameter, the phase φ is fixed (that is, we put $\eta = 0$), and the unknowns are w, λ, $z(t)$. This choice of a parameter and unknowns leads to a problem that can be studied without much difficulty by standard topological methods.

5.2. Topological lemma. For the sequel, we need the following lemma on solution existence for a system of two scalar equations and an equation in the Banach space E.

LEMMA 1. *Consider the system*

$$B_1(w,\lambda,z) = 0, \quad B_2(w,\lambda,z) = 0, \quad z = B_3(w,\lambda,z) \tag{15}$$

with $z \in E$ and scalar $w \in \Omega$, $\lambda \in \Lambda$, where the operators $B_1, B_2 : \Omega\times\Lambda\times E \to \mathbb{R}$ are continuous and the operator $B_3 : \Omega \times \Lambda \times E \to E$ is completely continuous with respect to the set of their arguments. Suppose the operator B_3 maps its domain into a bounded set $Z \subset E$. Suppose there are segments $[w_1, w_2] \subset \Omega$, $[\lambda_1, \lambda_2] \subset \Lambda$ such that $B_1(w_1, \lambda, z) \cdot B_1(w_2, \lambda, z) < 0$ for every $\lambda \in [\lambda_1, \lambda_2]$, $z \in Z$ and $B_2(w, \lambda_1, z) \cdot B_2(w, \lambda_2, z) < 0$ for every $w \in [w_1, w_2]$, $z \in Z$. Then system (15) *has a solution $w \in [w_1, w_2]$, $\lambda \in [\lambda_1, \lambda_2]$, $z \in Z$.*

The proof of Lemma 1 is based on the product formula for vector field rotations (see [4], [6]). Under the assumptions of Lemma 1 the rotation γ_1 of the infinite-dimensional vector field $z - B_3(w, \lambda, z)$ with fixed w, λ on every sphere $\{\|z\|_E = \rho\}$ of a sufficiently large radius ρ equals 1. The rotation γ_2 of the two-dimensional vector field $\{B_1(w, \lambda, z),\ B_2(w, \lambda, z)\}$ with fixed z on the boundary of the rectangular $T = \{w \in (w_1, w_2), \lambda \in (\lambda_1, \lambda_2)\}$ is either 1 or -1. The rotation γ_0 of the vector field

$$\{B_1(w,\lambda,z),\ B_2(w,\lambda,z),\ z - B_3(w,\lambda,z)\}$$

on the boundary of the domain $T \times \{\|z\|_E < \rho\}$ in the space $\mathbb{R} \times \mathbb{R} \times \mathbb{E}$ equals $\gamma_1\gamma_2$, i.e., $|\gamma_0| = 1$. Hence there exists a solution of system (15) in the this domain.

Now we replace equations (12) and (13) with 2π-periodic boundary conditions with systems of form (15). For both equations (12) and (13) system (15) can be constructed in a common way, consider equation (12). We multiply the equation by $\sin t$ and integrate over the segment $[0, 2\pi]$ to obtain the first of equalities (15). Multiplying the equation by $\cos t$ and integrating over the segment $[0, 2\pi]$, we obtain the second scalar equality of (15) (the details are in the next subsections). The equation in a Banach space is constructed as follows.

Denote by $E \subset C$ the space of functions $z(t) : [0, 2\pi] \to \mathbb{R}$, satisfying

$$z(0) = z(2\pi), \qquad \int_0^{2\pi} \sin t \; z(t) \; dt = \int_0^{2\pi} \cos t \; z(t) \; dt = 0$$

and set $\|z\|_E = \|z\|_C$. Consider the linear operator $A(w)$ that maps each function $u(t) \in E$ to a unique 2π-periodic solution $x(t) = A(w)u(t) \in E$ of the equation

$$L\left(w\frac{d}{dt}\right) x(t) = M\left(w\frac{d}{dt}\right) u(t). \tag{16}$$

The operator $A(w)$ existence follows from assumption 2 of Theorem 1. The operator $A(w)$ maps E to $E \cap C^1$. It is a completely continuous operator in the space E and a bounded operator from E to C^1. Moreover, the norms of the operators $A(w)$ are uniformly bounded for all $w \in \Omega$ and the operator $A(w)u : \Omega \times E \to E$ is completely continuous with respect to the set of its arguments w, u.

Denote by $C_0 \subset C$ the space of functions $u(t) : [0, 2\pi] \to \mathbb{R}$, satisfying $u(0) = u(2\pi)$ with the norm $\|u\|_{C_0} = \|u\|_C$. Set

$$Pu(t) \stackrel{\text{def}}{=} \frac{1}{\pi} \int_0^{2\pi} \cos(t-s)\, u(s)\, ds.$$

The operators P and $I - P$ project the space C_0 on the plane $\Pi = \{\xi \sin t + \eta \cos t\}$ and on the subspace E of the space C_0 respectively. Let us extend the operator $A(w)$ to the whole space C_0 by the formula $A(w)u = A(w)(I-P)u$. Each of the projectors P and $I - P$ commutes with the extended operator $A(w)$.

Now the last equation of system (15) can be written as

$$z = A(w)[f(\xi \sin t + z(t), \lambda) + g(\xi \sin t_{wh} + z(t_{wh}), \lambda)];$$

here and henceforth $t_\sigma = t - \sigma$ for $t \geq \sigma$ and $t_\sigma = t - \sigma + 2\pi$ for $t < \sigma$. By construction, $A(w)[f(\xi \sin t + z(t), \lambda) + g(\xi \sin t_{wh} + z(t_{wh}), \lambda)]$ is a completely continuous operator from $\Omega \times \Lambda \times E$ to E for every fixed ξ.

The system constructed can be easily transformed to the system satisfying the conditions of Lemma 1. The main point of the proof of Theorem 1 is to determine the segments $[w_1, w_2]$ and $[\lambda_1, \lambda_2]$ for every large ξ. In the proof of Theorem 3 we also determine the initial stop state μ_0 such that $\mu(t) = U(\mu_0)x(t)$ is a 2π-periodic function for the 2π-periodic solution $x(t)$ of (13).

5.3. Auxiliary lemmas. Let $e(t)$ be a Lipschitz continuous function.

LEMMA 2. *For every $c > 0$ the equality*

$$\lim_{\xi\to\infty} \sup_{z\in C^1,\, \|z\|_{C^1}\le c,\, \lambda\in\Lambda} \left| \int_0^{2\pi} e(t)\big(f(x(t),\lambda) - f(\xi\sin t,\lambda)\big)dt \right| = 0 \tag{17}$$

holds, where $x(t) = \xi \sin t + z(t)$.

LEMMA 3. *For every $c > 0$ the equality*

$$\lim_{\xi\to\infty} \sup_{z\in C^1,\, \|z\|_{C^1}\le c,\, \mu_0\in[-1,1]} \left| \int_0^{2\pi} e(t)\,\big(U(\mu_0)x(t) - \operatorname{sign}(\cos t)\big)dt \right| = 0 \tag{18}$$

holds, where $x(t) = \xi \sin t + z(t)$.

Below we use Lemmas 2 and 3 with $e(t) = \sin t$ and $e(t) = \cos t$.

Lemma 3 is proved in [1]. Let us prove Lemma 2.

Take an arbitrary $\varepsilon > 0$. We need to show that the supremum in (17) is smaller than ε for all sufficiently large ξ, i.e.,

$$\left| \int_0^{2\pi} e(t)\,\big(f(x(t),\lambda) - f(\xi\sin t,\lambda)\big)\,dt \right| < \varepsilon \tag{19}$$

for every $\lambda \in \Lambda$ and $z \in C^1$, $\|z\|_{C^1} \le c$. For this purpose, consider the partition $\cup I_i$ of the segment $[0,2\pi]$, where $I_1 = [0,\pi/2-\delta]$, $I_2 = [\pi/2-\delta,\pi/2+\delta]$, $I_3 = [\pi/2+\delta,3\pi/2-\delta]$, $I_4 = [3\pi/2-\delta,3\pi/2+\delta]$, and $I_5 = [3\pi/2+\delta,2\pi]$ with a small $\delta > 0$. The join $I_2 \cup I_4$ contains the set $\{t \in [0,2\pi] :\, \cos t = 0\}$. Choose $\delta > 0$ so that

$$2\sup|f(x,\lambda)| \int_{I_2\cup I_4} |e(t)|\,dt < \varepsilon/2 \tag{20}$$

and fix this δ up to the end of the proof. From

$$\inf_{t\in I_1\cup I_3\cup I_5} |\cos t| = \sin\delta > 0$$

it follows that

$$\inf_{t\in I_1\cup I_3\cup I_5} |\xi\cos t + z'(t)| > 1/2\,\xi\sin\delta$$

whenever ξ is sufficiently large, hence the functions $\xi\sin t$ and $\xi\sin t + z(t)$ are strictly monotone in some neighborhood $\tilde I_i$ of the segment I_i, $i = 1,3,5$. Therefore the formula $\xi\sin\tau = \xi\sin t + z(t)$, $\tau \in \tilde I_i$ defines a strictly monotone function $\tau = \tau(\xi, t)$ of the argument $t \in I_i$ for every large ξ. Let $t = t(\xi, \tau)$ be the inverse function. Consider the integrals

$$\mathcal{J}_i = \int_{I_i} e(t) f(\xi \sin t + z(t), \lambda)\, dt, \qquad i = 1, 3, 5.$$

Changing the variable, we obtain

$$\mathcal{J}_i = \int_{\tau(\xi, a_i)}^{\tau(\xi, b_i)} e(t(\xi, \tau))\, f(\xi \sin \tau, \lambda)\, t'_\tau(\xi, \tau)\, d\tau,$$

where a_i, b_i are the ends of the segment I_i. By construction, $t(\xi, \tau) \to \tau$ and $t'_\tau(\xi, \tau) \to 1$ as $\xi \to \infty$ uniformly in τ. So, the Lipschitz continuity of $e(\cdot)$ implies $e(t(\xi, \tau)) \to e(\tau)$. Also,

$$\tau(\xi, a_i) \to a_i, \quad \tau(\xi, b_i) \to b_i.$$

Hence,

$$\mathcal{J}_i - \int_{a_i}^{b_i} e(\tau) f(\xi \sin \tau, \lambda)\, d\tau \to 0, \qquad i = 1, 3, 5.$$

Together with (20) this proves (19). ■

5.4. Scalar equations. Let us multiply equation (16) by $\sin t$ (resp., $\cos t$) and integrate over $[0, 2\pi]$. The following lemma writes explicitly the resulting scalar equalities, which allows to write explicitly the scalar equations of system (15).

LEMMA 4. *Suppose the functions $x(t) = \xi \sin t + z(t)$, $z \in E$, and $u(t) \in C_0$ satisfy* (16). *Then*

$$\pi\, \Re e\, \frac{L(wi)}{M(wi)} \xi = \int_0^{2\pi} \sin t\, u(t)\, dt, \quad \pi\, \Im m\, \frac{L(wi)}{M(wi)} \xi = \int_0^{2\pi} \cos t\, u(t)\, dt. \tag{21}$$

Proof. It follows from (16) that

$$L\left(w \frac{d}{dt}\right) (\xi \sin t) = M\left(w \frac{d}{dt}\right) Pu(t).$$

Equivalently,

$$\begin{aligned}\pi\, \Re e[L(wi)] \xi \sin t + \pi\, \Im m[L(wi)] \xi \cos t = {}& \\ (\Re e[M(wi)] \cos t - \Im m[M(wi)] \sin t) & \int_0^{2\pi} \cos s\, u(s) ds + \\ (\Re e[M(wi)] \sin t + \Im m[M(wi)] \cos t) & \int_0^{2\pi} \sin s\, u(s) ds,\end{aligned}$$

that is

$$\pi\,\Re e[L(wi)]\xi = -\,\Im m[M(wi)]\int_0^{2\pi}\cos s\; u(s)\,ds + \Re e[M(wi)]\int_0^{2\pi}\sin s\, u(s)\,ds,$$

$$\pi\,\Im m[L(wi)]\xi = \quad \Re e[M(wi)]\int_0^{2\pi}\cos s\; u(s)\,ds + \Im m[M(wi)]\int_0^{2\pi}\sin s\, u(s)\,ds.$$

Multiplying the first of these equalities by $\Re e[M(wi)]$, the second one by $\Im m[M(wi)]$ and summing, we obtain the first of equations (21). Summing the first of the equalities multiplied by $-\,\Im m[M(wi)]$ with the second one multiplied by $\Re e[M(wi)]$, we obtain the second of equations (21). ■

5.5. Proof of Theorem 1. Consider the system

$$\begin{aligned}
&\pi\,\Re e\,\frac{L(wi)}{M(wi)}\xi = \int_0^{2\pi}\sin t\,[f(\xi\sin t + z(t),\lambda) + g(\xi\sin t_{wh} + z(t_{wh}),\lambda)]\,dt,\\
&\pi\,\Im m\,\frac{L(wi)}{M(wi)}\xi = \int_0^{2\pi}\cos t\,[f(\xi\sin t + z(t),\lambda) + g(\xi\sin t_{wh} + z(t_{wh}),\lambda)]\,dt,\\
&\qquad z = A(w)[f(\xi\sin t + z(t),\lambda) + g(\xi\sin t_{wh} + z(t_{wh}),\lambda)].
\end{aligned} \tag{22}$$

It follows from the definition of the operator $A(w)$ and from Lemma 4 that the function $x(t) = \xi\sin wt + z(wt)$ is a $2\pi/w$-periodic solution of equation (2) whenever the triple $\{w,\lambda,z\}\in\Omega\times\Lambda\times E$ is a solution of system (22) for some $\xi > 0$. Therefore to prove Theorem 1 it is sufficient to show that system (22) has a solution $\{w,\lambda,z\}$ with w and λ arbitrarily close to w_0 and λ_0 for every sufficiently large ξ. Let us transform (22) to the equivalent system satisfying the conditions of Lemma 1.

Rewrite the first two equations of (22) as

$$\pi\frac{\Re e[L(wi)M(-wi)]}{|M(wi)|^2}\xi = \int_0^{2\pi}[\sin t f(x(t),\lambda) + \sin(t+wh)g(x(t),\lambda)]\,dt \tag{23}$$

and

$$\pi\frac{\Im m[L(wi)M(-wi)]}{|M(wi)|^2}\xi = \int_0^{2\pi}[\cos t f(x(t),\lambda) + \cos(t+wh)g(x(t),\lambda)]\,dt, \tag{24}$$

where $x(t) = \xi\sin t + z(t)$. Consider separately the following two situations: first, limit (3) is either infinite or zero; second, limit (3) is a finite number $\beta \neq 0$.

Suppose $\beta = 0$ (the case $\beta = \infty$ is similar and we do not consider it here). Then $\alpha = 0$ and

$$\Phi(\xi,0,\lambda) = \sin(w_0 h)\int_0^{2\pi}\sin t\, g(\xi\sin t,\lambda)\,dt. \tag{25}$$

It follows from

$$\lim_{w\to w_0}\frac{\Im m[L(wi)M(-wi)]}{\Re e[L(wi)M(-wi)]}=0$$

that

$$\begin{aligned}\Im m[L(wi)M(-wi)]&=(w-w_0)^{K+N}Q_1(w),\\ \Re e[L(wi)M(-wi)]&=(w-w_0)^{K}\quad Q_2(w),\end{aligned}\tag{26}$$

where $Q_2(w_0)\neq 0$ and either $Q_1(w_0)\neq 0$ or $Q_1(w)$ is the identical zero; N is a positive integer; K is the multiplicity of the root $w_0 i$ of the polynomial $L(p)$. Now equations (23), (24) can be written as

$$\pi(w-w_0)^K\frac{\xi Q_2(w)}{|M(wi)|^2}-\int_0^{2\pi}[\sin t f(x(t),\lambda)+\sin(t+wh)g(x(t),\lambda)]dt=0,\tag{27}$$

$$\pi(w-w_0)^{K+N}\frac{\xi Q_1(w)}{|M(wi)|^2}-\int_0^{2\pi}[\cos t f(x(t),\lambda)+\cos(t+wh)g(x(t),\lambda)]dt=0.\tag{28}$$

Let us express the term $(w-w_0)^K$ from equation (27) and substitute in (28) to obtain

$$\begin{aligned}(w-w_0)^N\frac{Q_1(w)}{Q_2(w)}\int_0^{2\pi}\Big(\sin t\, f(x(t),\lambda)+\sin(t+wh)g(x(t),\lambda)\Big)\,dt-\\ \int_0^{2\pi}\Big(\cos t\, f(x(t),\lambda)+\cos(t+wh)g(x(t),\lambda)\Big)\,dt=0.\end{aligned}\tag{29}$$

We use (27) and (29) as the scalar equations of system (15). The equation in a Banach space is the last equation of (22).

Since $\sup|f(x,\lambda)|+\sup|g(x,\lambda)|<\infty$ and the norms of the operators $A(w):C_0\to C^1$ are uniformly bounded, it follows that the nonlinear operator

$$A(w)[f(\xi\sin t+z(t),\lambda)+g(\xi\sin t_{wh}+z(t_{wh}),\lambda)]$$

maps its domain $\Omega\times\Lambda\times E$ onto a bounded subset Z of C^1, so Z is also bounded in E. Thus, the last equation of (22) satisfies the conditions of Lemma 1. The conditions concerning the scalar equations should be verified for $z\in Z$. We take z from the fixed ball $\{\|z\|_{C^1}\le c\}$ containing Z.

Set

$$w_1=w_1(\xi)\overset{\text{def}}{=}w_0-R^{1/K}\xi^{-1/K},\qquad w_2=w_2(\xi)\overset{\text{def}}{=}w_0+R^{1/K}\xi^{-1/K},\tag{30}$$

where

$$R = 4 \sup_{w\in\Omega} \frac{|M(wi)|^2}{|Q_2(w)|}[\sup|f(x,\lambda)| + \sup|g(x,\lambda)|].$$

Since the number K is odd, the sign of the left-hand side of equation (27) is $(-1)^j \operatorname{sign} Q_2(w_0)$ for $w = w_j$ whenever ξ is sufficiently large and $\lambda \in \Lambda$.

Take any pair of parameter values satisfying the condition 3 of Theorem 1 as the numbers λ_1, λ_2 used in Lemma 1. It follows from Lemma 2 that all the terms in (29) except (25) vanish as $\xi \to \infty$ and $w \to w_0$, hence $\operatorname{sign} \Phi(\xi, 0, \lambda)$ is the sign of the left-hand side of equation (29). By condition 3, the left-hand side of (29) is negative for $\lambda = \lambda_1$ and positive for $\lambda = \lambda_2$ whenever $w \in [w_1(\xi), w_2(\xi)]$ with large enough ξ, which completes the proof for $\beta = 0$.

Now suppose $\beta \neq 0$. This case is similar to the case above. The only difference is that $N = 0$ and $Q_j(w_0) \neq 0$ for both $j = 1, 2$ in formulas (26). Therefore equality (29) can be written in the form

$$\cos\alpha \left[\frac{Q_1(w)}{Q_2(w)} - \beta\right] \int_0^{2\pi} [\sin t\ f(x(t),\lambda) + \sin(t+wh)\, g(x(t),\lambda)]\, dt -$$
$$\int_0^{2\pi} [\cos(t+\alpha)\, f(x(t),\lambda) + \cos(t+wh+\alpha)\, g(x(t),\lambda)]\, dt = 0,$$

and the sign of the left-hand side coincides with $\operatorname{sign}\Phi(\xi,\alpha,\lambda)$. The further arguments are exactly like above. ■

5.6. Proof of Theorem 3. The proof follows the line of the proof of Theorem 1. The analog of system (22) for equation (8) is

$$\pi\,\Re e\,\frac{L(wi)}{M(wi)}\xi = \int_0^{2\pi} \sin t \big(a(\lambda)U(\mu_0)[\xi\sin t + z(t)] + f(\xi\sin t + z(t),\lambda)\big)\, dt,$$

$$(31)\qquad \pi\,\Im m\,\frac{L(wi)}{M(wi)}\xi = \int_0^{2\pi} \cos t \big(a(\lambda)U(\mu_0)[\xi\sin t + z(t)] + f(\xi\sin t + z(t),\lambda)\big)\, dt,$$

$$z = A(w)\big(a(\lambda)U(\mu_0)[\xi\sin t + z(t)] + f(\xi\sin t + z(t),\lambda)\big).$$

The same transformation as used in the proof of Theorem 1 brings system (31) to the form where the term $(w - w_0)^K\xi$ is principal for the first scalar equation, and the principal term of the second scalar equation is

$$-\int_0^{2\pi} \cos(t+\alpha)\,\big(a(\lambda)U(\mu_0)[\xi\sin t + z(t)] + f(\xi\sin t + z(t),\lambda)\big)\, dt.$$

By Lemmas 2 and 3, this expression goes to $\phi(\lambda) = \psi(\lambda)\sin\alpha - 4a(\lambda)\cos\alpha$ as $\xi \to \infty$. Thus, by Lemma 1 system (31) has a solution $\{w(\mu_0), \lambda(\mu_0), z(t;\mu_0)\}$ for each initial stop state $\mu_0 \in [-1,1]$ whenever $\xi > 0$ is sufficiently large.

It remains to determine a value μ_0 such that the function $\mu(t;\mu_0) = U(\mu_0)(\xi \sin t + z(t;\mu_0))$ is 2π-periodic, i.e., $\mu(2\pi;\mu_0) = \mu_0$; then $x(t;\mu_0) = \xi \sin t + z(t;\mu_0)$ is a 2π-periodic solution of equation (13) for $\lambda = \lambda(\mu_0)$, $w = w(\mu_0)$ and hence $x(wt;\mu_0)$ is a $2\pi/w$-periodic solution of equation (8).

It follows from the semigroup property of the operator $U(\mu)$ that

$$\mu(t;\mu_0) = U(\mu(7\pi/4;\mu_0))x(t;\mu_0), \qquad t \geq 7\pi/4.$$

Since $z(t;\mu_0) \in Z$, where the set Z is bounded in C^1, the relations

$$x'(t;\mu_0) = \xi \cos t + z'(t;\mu_0) \geq 0, \qquad 7\pi/4 \leq t \leq 2\pi,\ \mu_0 \in [-1,1],$$

and

$$x(2\pi;\mu_0) - x(7\pi/4;\mu_0) \geq 2, \qquad \mu_0 \in [-1,1], \tag{32}$$

hold for any large ξ. That is, the input $x(t;\mu_0)$ increases on the segment $7\pi/2 \leq t \leq 2\pi$, hence

$$\mu(t;\mu_0) = \min\{1, \mu(7\pi/2;\mu_0) + x(t;\mu_0) - x(7\pi/2;\mu_0)\}, \qquad 7\pi/2 \leq t \leq 2\pi$$

and (32) implies that $\mu(2\pi;\mu_0) = 1$ for every $\mu_0 \in [-1,1]$. Therefore $x(wt;\mu_0)$ is a $2\pi/w$-solution of equation (8) iff $\mu_0 = 1$. ■

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